On a Generalization of the Koch Curve Built from n-gons

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Abstract

We study a generalization of the von Koch Curve, which has two parameters, an integer \( n \) and a real number \( c \) on the interval \((0, 1)\). This von Koch type curve is constructed as the limit of a recursive process that starts with a regular \( n \)-gon (or line segment) and repeatedly replaces the middle \( c \) portion of an interval by the \( n - 1 \) other sides of a regular \( n \)-gon placed contiguous to the interval. We show that there are values of \( n \) such that the set of \( c \) for which the \((n, c)\)-von Koch Curve is simple, i.e. does not intersect itself, is not an interval.

1 Introduction

Consider the following generalization of the classical triadic von Koch snowflake. Let \( 0 < c < 1 \) and an integer \( n \geq 3 \) be given. Starting from a regular \( n \)-gon, repeatedly replace the middle portion \( c \) of each interval by the \( n - 1 \) other sides of a regular \( n \)-gon placed contiguous to the interval (see Figure 1.) The limit curve is the \((n, c)\)-snowflake curve. The closure of the union of all the above regular \( n \)-gons is the \((n, c)\)-snowflake domain. The curve we get if we start from a segment is the \((n, c)\)-von Koch curve.

In [3] the special case \( n = 3, c \leq \frac{1}{3} \) is called “modified von Koch curve.” A specific \((3, c)\)-snowflake curve with \( c \) slightly less than \( \frac{1}{2} \) appears in [5, Plate 56]. Note that \( n = 3, c = \frac{1}{3} \) gives the
Figure 2: These figures demonstrate the meshing phenomenon.

classical case. Note also that the \((n, c)\)-von Koch curve is a self-similar set and the \((n, c)\)-snowflake consists of \(n\) copies of the \((n, c)\)-von Koch curve.

There are two ways to relate the self-intersection of these various constructions (which are elementary to prove). First, if the \((n, c)\)-snowflake curve is a simple curve (in other words if it is \textit{self-avoiding} / \textit{non-self-intersecting}) then it is the boundary of the \((n, c)\)-snowflake domain. Second, the \((n, c)\)-snowflake curve is self-intersecting if and only if the \((n, c)\)-von Koch curve is self-intersecting.

M. van den Berg studied the heat equation for planar regions similar to the \((n, c)\)-snowflake domain (see e.g. [1, 2]). He noticed that the \((4, c)\)-snowflake curve is self-avoiding if and only if \(c < \frac{1}{3}\) and asked for analogous result for the \((3, c)\)-snowflake. In [4] it was proved that the \((3, c)\)-snowflake curve is non-self-intersecting if and only if \(c < \frac{1}{2}\). Then, M. van den Berg asked what the critical \(c\) is for other \(n\).

The critical \(c\) phenomenon is a peculiarity of the low order cases, where a certain symmetry effects a self-intersection. A shorter proof of the result that the \((3, c)\)-snowflake curve self-intersects for \(c \geq \frac{1}{2}\) is presented to illustrate this symmetry.

However, the main goal of the present paper is showing that in general there is no such critical \(c\). For some \(n\) there exists \(c_1 < c_2\) such that the \((n, c_1)\)-snowflake curve is self-intersecting but the \((n, c_2)\)-snowflake curve is not (Theorem 4.15).

Figure 2 shows how this phenomenon can happen. Table 3 shows a list of triplets \((n, c_1, c_2)\) with the above property.
2 Terminology

The starting segment of an \((n, c)\)-von Koch curve is the base of the curves.

By self-similarity an \((n, c)\)-von Koch curve consists of smaller \((n, c)\)-von Koch curves. We make use of the language of genealogy in describing these von Koch Curves. An \((n, c)\)-von Koch Curve is said to have \(n + 1\) children. Two lie on either off-center \((1 - c)/2\) segment. The other \(n - 1\) have bases that, together with the center \(c\) segment of its parent, form a regular \(n\)-gon. These \(n - 1\) children are primary. Naturally, a von Koch Curve may be described as the union of its children. By descendants, we refer to the set curves comprised of a curve’s children, its grandchildren, its great grandchildren, etc.

In the same way, this terminology describes the piecewise-linear curves that are used to construct a von Koch Curve. Namely, a line segment has \(n + 1\) children, the \(n - 1\) primary children that form a primary \(n\)-gon and two non primary, off-center children.

Here, we define the vertices of an \((n, c)\)-von Koch Curve which we employ in our proof (see Figure 3). The \((n, c)\)-von Koch Curve \(AC\) is built upon the closed horizontal line segment \(AC\) with length 1 and with \(C\) on the right. Unless otherwise specified, all references to an \((n, c)\)-von Koch Curve refer to curve \(AC\). Call \(M\) the vertex of the primary \(n\)-gon touching the right non primary child. Call \(S\) the vertex of the primary \(n\)-gon adjacent to \(M\) on the right, and lastly call \(T\) the vertex adjacent to \(S\) on the right.

![Figure 3: The wedge-shaped region between segments MS and MC is singularly important in the search for self-intersection.](image)

We employ an orthonormal coordinate system centered at \(M\). The first coordinate extends along ray \(MC\). The second coordinate, naturally, extends upwards. Distance is Euclidean, and again line segment \(AC\) has length 1.
Figure 4: The value of $c$ is $\frac{1}{2}$. The dotted line segment is contained in the closure of the children of curve $MS$. This is sufficiently long to effect self-intersection.

3 Low Order Symmetry

In [4] Theorem 3.2 and its converse were proved. The proof of Theorem 3.2 presented here is substantially simpler than the proof in [4], where the converse has a simple proof.

Remark 3.1. The proof of the converse of Theorem 3.2 in [4] can be generalized to show that any $(n,c)$-snowflake is self-avoiding in an interval of $c$ around 0.

**Theorem 3.2.** The $(3,c)$-snowflake curve self-intersects when $c \geq \frac{1}{2}$.

*Proof.* The proof can be quite simply made once a few definitions are made. Let $DF$ be the inward facing primary child of curve $MC$. Let $EM$ be the lower non primary child of $SM$. Figure 4 illustrates these.

The key realization is that $DF$ and $EM$ are the same size for any $c$. Thus, consider a lineage of triangles emanating from $EM$ towards $DF$, which alternates between right and left primary children (so to become as close as possible to the midpoint of $DF$). In the same fashion, consider the symmetric (about the planar reflection that sends $D$ to $M$) lineage of triangles coming from $DF$ to $EM$. A quick calculation shows that these two sequences of triangles overlap when $c \geq \frac{1}{2}$.

The horizontal distance separating the midpoints of $EM$ and $DF$ is calculated to be

$$\left(\frac{1-c}{2}\right)^2 + 2 \frac{c(1-c)}{8} = \frac{1}{4} - \frac{c}{4}.$$ 

The horizontal length of each lineage, taking the midpoints of $EM$ and $DF$ to be the starting points, is

$$\frac{c^2(1-c)}{4} \left(1 + c + c^2 + c^3 + \cdots\right) = \frac{c^2}{4}.$$
This leads to the inequality
\[ \frac{c^2}{2} + \frac{c}{4} \geq \frac{1}{4}. \]
This inequality is satisfied so long as \( c \geq \frac{1}{2} \), establishing that the two sequences of triangles must overlap, and so the curve self-intersects.

An analogous result can be formulated for the \((4, c)\)-von Koch Curve using essentially the same method of proof. In larger \( n \)-gons, the lineages can still be used to provide a value of \( c \) above which the curve certainly self-intersects. However, the curve can self-intersect long before the symmetric lineages overlap, making the lineages somewhat irrelevant. For larger \( n \)-gons, there is an interesting mesoscopic region of \( c \), below which the curve is certainly self-avoiding and above which there is certain self-intersection. The self-intersection pattern in this mesoscopic region is quite complex.

4 Main Result

We show that the set of \( c \) for which the \((n, c)\)-von Koch Curve self-intersects is not always interval.

It is superfluous to search for intersections everywhere on a curve \( AC \). Instead, a wedge-shaped region between segments \( MS \) and \( MC \) is all that needs to be inspected. This region is illustrated in Figures 3 and 5. We must expound that by self-similarity, it is superfluous to check for intersections between descendants of curve \( MS \) with other descendants of curve \( MS \) (and likewise between descendants of curve \( MC \) with other descendants of curve \( MC \)), as this is equivalent to searching for self-intersections of curve \( AC \).

In the wedge shaped region between segments \( MS \) and \( MC \), there is a considerable amount of repetition. In order to simplify the search for self-intersection, this repetition needs to be eliminated, and so we introduce the following map.

Definition 4.1. The rescaling map is defined as \( \bar{x} \mapsto \bar{x}(1-c)/2 \). The rescaling map sends curve \( MS \) into itself, and it sends curve \( MC \) into itself. The rescaled image is the image of curve \( AC \) under the rescaling map.

Lemma 4.2. Curve \( AC \) self-intersects if and only if there is an intersection between curve \( MS \) and curve \( MC \).

Proof. Suppose the curve \( MC \) is not strictly between rays \( MS \) and \( MC \). This implies that curve \( MC \) must intersect ray \( MS \) at some point besides \( M \). Iterated applications of the rescaling map bring this point arbitrarily close to \( M \). Hence there is a curve descended from curve \( MC \) having endpoint \( M \) whose other endpoint is arbitrarily close to \( M \), and so it can be ensured that part of curve \( MC \) crosses line segment \( MS \). As a result, there must be an intersection between curves \( MS \) and \( MC \).

Suppose the curve \( MC \) is strictly between rays \( MS \) and \( MC \). By similarity, curve \( ST \) is strictly between rays \( ST \) and \( MS \), and thus curve \( ST \) is separated
from $MC$. All other children of curve $AC$ are strictly separated from curve $MC$ by line $MS$. Furthermore, all primary children of curve $AC$ are pairwise disjoint, and thus if an intersection occurs, it must be between curve $MS$ and curve $MC$ (or between the left counterparts of curve $MS$ and $MC$, but by the natural left-right symmetry of curve $AC$, it is unnecessary to search both the left and right sides for intersection).

Figure 5: $n = 14$, $c = 0.037$, with circular approximations. The grayed circles do not need to be checked by self-similarity.

Using the rescaling map, Lemma 4.2 can be bettered. Only a properly chosen trapezoid needs to be checked, since by iterated applications of the rescaling map, the whole wedge is covered.

In order to isolate parts of the snowflake domain from others, we circumscribe a circle around the $n - 1$ primary children of a curve. For a curve of base length 1, we define $r(c)$ to be the radius of the smallest circle containing (not strictly) all this curve’s primary children. In this way, a curve gives rise to exactly one circle.

Only some of the circles are relevant to showing self-avoidance, and these are introduced here.

**Definition 4.3.** The relevant circles are the following (see Figure 5).

- $\zeta$ is the circle associated to curve $MS$
- $\alpha$ is the largest circle arising between $\zeta$ and $S$
- $\xi_n$ is a family of circles defined recursively. $\xi_1$ is the circle associated to curve $MC$, and $\xi_{n+1}$ is the rescaled image of $\xi_n$.
- $\beta_n$ is a family of circles called the first order intermediaries. Each $\beta_n$ is defined as the largest circle between $\xi_n$ and $\xi_{n+1}$.
Only a few of the circles from the two infinite families listed are needed. We limit our search for intersections between \( \zeta \), \( \alpha \), and the two families. Only the circles from these families that are closest to \( \zeta \) and \( \alpha \) are relevant to the search, and they are contiguous members of the sequence. This motivates the following definition.

**Definition 4.4.** Let \( k \) be the natural number such that the horizontal coordinate of the center of \( \zeta \) is between the horizontal coordinates of the centers \( \xi_k \) and \( \xi_{k+1} \).

We are now able to define the critical trapezoid in which intersections are sought. The trapezoid is the region of the plane bounded by line segments \( MS \) and \( MC \), between two parallel lines \( l_k \) and \( l_{k+1} \), which are defined as follows.

**Definition 4.5.** Let \( \{l_m\} \), with \( m \geq k \), be a family of line segments, defined recursively with \( l_{m+1} \) being the rescaled image of \( l_m \). Furthermore, \( l_k \) should be chosen so that \( l_{k+1} \) is a line segment, which:

- begins on segment \( MS \) and ends on segment \( MC \)
- if extended to a line, strictly separates \( \xi_{k+1} \) and the rescaled image of \( \alpha \) from \( \zeta \) and \( \beta_k \).

**Remark 4.6.** These line segments do not exist for all \( c \), and so for every \( c \) for which they are used, it must be shown first that they exist.

**Definition 4.7.** Let the critical trapezoid be the region of the plane bounded by \( l_k \), \( l_{k+1} \), segment \( MS \), and segment \( MC \).

Whenever the critical trapezoid exists, there are 4 tractable circles inside of it. However, these circles do not contain all portions of the curves \( MS \) and \( MC \) inside the critical trapezoid. There are two final regions that need to be defined to account for all parts of the curve between \( MS \) and \( MC \). These are the upper and lower strips.

**Definition 4.8.** Let the upper strip be the minimum constant-radius tubular neighborhood of segment \( MS \) containing those descendants of curve \( MS \) within the critical trapezoid that are not contained in \( \alpha \) or \( \zeta \). Analogously let the lower strip be the minimum constant-radius tubular neighborhood of segment \( MC \) containing those descendants of curve \( MC \) within the critical trapezoid that are not contained in either \( \xi_k \) or \( \beta_k \). We designate \( \mu \) to be the radius of the upper strip and \( \lambda \) to be the radius of the lower strip.

Lemma 4.9 collects the importance of the preceding definitions and remarks.

**Lemma 4.9.** If for a given \( c \), the line segments \( \{l_m\} \) exist, and the upper objects are disjoint from the lower, (i.e. \( \alpha \), \( \zeta \), and the upper strip are disjoint from \( \xi_k \), \( \beta_k \), the lower strip) then curve \( AC \) does not self-intersect.
Proof. Because the line segments $l_k$ and $l_{k+1}$ exist, the critical trapezoid, denoted $T$, exists. Because the rescaling map takes $l_m$ to $l_{m+1}$ for any $m \geq k + 1$, the iterated rescaled images of the $T$ cover the entire wedge between $M$ and $l_m$. There is a portion of the wedge to the right of $l_m$ not covered by these iterated rescaled images. Once the self-avoidance of the wedge portion between $M$ and $l_m$ is established, the rest of the wedge can be covered by the inverse rescaled image applied to wedge between $M$ and $l_m$. Thus, it suffices to check $T$ for intersections, and by the definition of the upper and lower strips, curve $MS$ is disjoint from curve $MC$ inside $T$. \hfill \square

Thus, once the existence of $l_k$ has been shown, it is straightforward to show that the curve is self-avoiding. One sufficient and computationally manageable criterion is the following.

**Definition 4.10.** Define $\vec{v}$ to be the unit vector pointing along segment $MS$.

**Lemma 4.11.** Define a linear functional $v^*$ by $\vec{x} \mapsto \vec{x} \cdot \vec{v}$; then, denoting by $\tilde{\alpha}$ the rescaled image of $\alpha$, the line segment $l_k$ exists if

$$\sup_{x \in \tilde{\alpha}} v^*(x) \leq \sup_{x \in \xi_{k+1}} v^*(x) < \inf_{x \in \zeta} v^*(x) \leq \inf_{x \in \beta_k} v^*(x).$$

Proof. The functional $v^*$ is in fact a semi-norm on the wedge shaped area. Thus, choose some $y$ between $\sup_{x \in \xi_{k+1}} v^*(x)$ and $\inf_{x \in \zeta} v^*(x)$, and let $l_{k+1}$ be defined as the preimage of $y$ under the restriction of $v^*$ to the wedge. \hfill \square

**Remark 4.12.** This choice of $\vec{v}$ is arbitrary, based on what seems to work in some cases. Any unit vector between segments $MS$ and $MC$ would do.

All of these lemmas together give a method for checking that the curve is self-avoiding. On the other hand, there also needs to be a method for checking if the curve is self-intersecting.

The base segments of the primary children of a curve form all but one sides of a regular $n$-gon. Inside of this $n$-gon, inscribe a circle. This inscribed circle is concentric with the outer approximation constructed earlier.

**Definition 4.13.** Let $\rho(c)$ be the radius of the circle inscribed inside of the $n$-gon formed by the bases of the primary children of a curve with unit base length.

**Lemma 4.14.** If the inscribed circles associated to two curves overlap, then the von Koch Curve self-intersects.

Proof. In particular, if the inscribed circles overlap, then the domains that the two curves bound overlap. Thus, the Snowflake domain overlaps, which is equivalent to the von Koch Curve self-intersecting. \hfill \square

We now have the machinery to prove the main result.
Table 1: These values establish the existence of the \( \{l_m\} \).

<table>
<thead>
<tr>
<th>( \sup_{x \in \alpha} v^*(x) )</th>
<th>( \sup_{x \in \xi_{k+1}} v^*(x) )</th>
<th>( \inf_{x \in \zeta} v^*(x) )</th>
<th>( \inf_{x \in \beta_k} v^*(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.014275</td>
<td>0.014828</td>
<td>0.015270</td>
<td>0.016623</td>
</tr>
</tbody>
</table>

Table 2: Here, \( \mu \) refers to the upper strip, and \( \lambda \) refers to the lower strip.

<table>
<thead>
<tr>
<th>( \beta_4 )</th>
<th>( \xi_4 )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.009329</td>
<td>0.000352</td>
<td>0.008846</td>
</tr>
<tr>
<td>0.000162</td>
<td>0.001038</td>
<td>0.001609</td>
</tr>
<tr>
<td>0.005625</td>
<td>0.002349</td>
<td>0.006012</td>
</tr>
</tbody>
</table>

**Theorem 4.15.** The set of \( c \) for which the \((n, c)\)-von Koch Curve is self-avoiding is not in general an interval.

**Proof.** We present numerical proof that when \( n \) is 14, the set of self-avoiding \( c \) is not an interval. In particular, when \( c = 0.037 \), the curve is self-avoiding, but when \( c = 0.032 \), the curve self-intersects (see Figure 2). It is assumed clear that for sufficiently small \( c (c \ll 0.032) \), the curve is self-avoiding.

For both of these values of \( c \), the value of \( k \) is computed to be 4. When \( c = 0.037 \), Lemma 4.9 holds that the curve is self-avoiding, so long as the critical trapezoid exists. Figure 1 shows that the criteria of Lemma 4.11 are satisfied, and hence the critical trapezoid exists.

The upper objects are shown to be separated from the lower objects. Table 2 shows the minimal distances between each of the objects. Formulae to compute these results are provided in the Appendix. Note that the minimal distance between the upper strip and lower strip is along the line \( l_{k+1} \). For the value in Table 2, the line \( l_{k+1} \) is chosen so that it bisects the \( v^* \)-distance between \( \xi_{k+1} \) and \( \zeta \).

As the upper objects are disjoint from the lower objects, with a separating distance that is well beyond numerical error, we conclude that at \( c = 0.037 \), the curve is self-avoiding.

To show that the curve self-intersects when \( c = 0.032 \), we inscribe a circle inside of the \( n \)-gon as remarked previously. The radius of the inscribed circle of curve \( AC \) is

\[
\frac{c}{2} \tan \left( \frac{\theta}{2} \right). \tag{1}
\]

We can then compute the distance between the centers of \( \zeta \) and \( \xi_5 \) minus the sum of inscribed radii of \( \zeta \) and \( \xi_5 \)

\[
\approx -0.0003895. \tag{2}
\]

And so by Lemma 4.14 von Koch Curve certainly self-intersects when \( c = 0.032 \).
Table 3: These are values of \( n \) up to 50 where the result given here may be repeated using precisely the same methodology. At \( c_1 \) the curve can be shown to self-intersect, and at \( c_2 \) the curve can be shown to be self-avoiding.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>0.032</td>
<td>0.037</td>
</tr>
<tr>
<td>19</td>
<td>0.014670</td>
<td>0.018424</td>
</tr>
<tr>
<td>20</td>
<td>0.014571</td>
<td>0.018028</td>
</tr>
<tr>
<td>26</td>
<td>0.0074988</td>
<td>0.0092013</td>
</tr>
<tr>
<td>27</td>
<td>0.0074905</td>
<td>0.0091380</td>
</tr>
<tr>
<td>28</td>
<td>0.0074653</td>
<td>0.0090610</td>
</tr>
<tr>
<td>29</td>
<td>0.0074555</td>
<td>0.0089734</td>
</tr>
<tr>
<td>30</td>
<td>0.0074584</td>
<td>0.0089705</td>
</tr>
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<tr>
<td>37</td>
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</tr>
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<td>38</td>
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<td>0.0044116</td>
</tr>
<tr>
<td>39</td>
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<td>0.0043830</td>
</tr>
<tr>
<td>40</td>
<td>0.0037966</td>
<td>0.0043671</td>
</tr>
<tr>
<td>41</td>
<td>0.0038046</td>
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<tr>
<td>42</td>
<td>0.0037935</td>
<td>0.0043392</td>
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<tr>
<td>43</td>
<td>0.0037928</td>
<td>0.0043135</td>
</tr>
<tr>
<td>44</td>
<td>0.0037883</td>
<td>0.0042984</td>
</tr>
</tbody>
</table>

5 Conclusion

The methodology used here also may be used to show that the set of self-avoiding \( c \) is not an interval for higher \( n \). In Table 3, some values of \( n \), \( c_1 \), and \( c_2 \) with \( c_1 < c_2 \) are given such that the \((n, c)\)-von Koch Curve can be shown to self-intersect at \( c_1 \) and can be shown to be self-avoiding at \( c_2 \) using the machinery developed here.

While the set of self-avoiding \( c \) is not necessarily an interval, it is unknown how complex the set might be. For example, a challenging open question is whether or not the set of self-avoiding \( c \) can be written as a finite union of intervals, or for that matter a countable collection of intervals (note that generalized Cantor sets appear in various places in the Snowflake domain).

Particularly, for this question, the aid of computer tools is limited, which have been indispensable in studying the von Koch Curve. Java software for exploring the fractal and generating figures (such as Figure 1) is available on request, as is an MPEG animation linking the stills in Figure 2.
Appendix

All the formulae for the computations are presented here. In what follows, the variable $\theta$ is the interior angle of a regular $n$-gon, as in Formula 3.

The radius $r(c)$ of the minimal circle enclosing all of a unit base-length curve’s primary children is given by Formula 4. Formula 4 may be derived by analyzing the relation between the radius of the minimal circle enclosing a von Koch Curve’s primary child and the minimal circle enclosing a primary child of a von Koch Curve’s primary child. The grandchild’s circle is tangent to and contained in the child’s circle. Thus, by writing the radius of the circle enclosing the grandchild in two ways, the following relation is derived

$$cr(c) = r(c) - \frac{c}{2} \tan \left( \frac{\theta}{2} \right) (1 + c).$$

This immediately yields the presented formula.

The distance $h(c)$ from the center of the minimal circle to the base segment of a unit base-length curve is given by Formula 5. To derive this, note that this minimal circle is concentric with the center of the polygon formed by this curve’s primary children.

Table 4: These are the relevant formulae for the circles shown to be disjoint.

<table>
<thead>
<tr>
<th>Horizontal Coordinate</th>
<th>Vertical Coordinate</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_k$</td>
<td>$(1-c^2)^k\frac{1}{2}$</td>
<td>$(1-c^2)^k h(c)$</td>
</tr>
<tr>
<td>$\beta_k$</td>
<td>$(1-c^2)^{k+1}(\frac{3+c}{4})$</td>
<td>$(1-c^2)^{k+2} h(c)$</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>$-\frac{c}{2} \cos (\theta) + c \sin (\theta) h(c)$</td>
<td>$\frac{c}{2} \sin (\theta) + c \cos (\theta) h(c)$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\frac{c}{4} (3 + c) \cos (\theta)$</td>
<td>$\frac{c}{4} (3 + c) \sin (\theta)$</td>
</tr>
<tr>
<td></td>
<td>$+\frac{c}{2} (1 - c) \sin (\theta) h(c)$</td>
<td>$+\frac{c}{2} (1 - c) \cos (\theta) h(c)$</td>
</tr>
</tbody>
</table>

Recall that $\mu$ and $\lambda$ are defined to be the radii of the upper and lower strips (see Definition 4.8). These can be computed presupposing the existence of the critical trapezoid, which are given as Equations 7 and 8. To derive these, observe that the approximating circle of a non primary child rises exactly as high as the center of the approximating circle of its parent. On curve $MS$, the next largest circle inside the critical trapezoid after $\alpha$ is a non primary child of the curve that $\alpha$ approximates. On curve $MC$, the next largest circle inside the critical trapezoid after $\beta_k$ is a non primary child of the curve that $\beta_k$ approximates.

The definition of $k$ holds that $k$ should satisfy the following inequality, which
Table 5: These are all the formulae, besides the circle formulae in Table 4, required to employ Theorem 4.15.

<table>
<thead>
<tr>
<th>Formula Description</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>The interior angle of an n-gon</td>
<td>$\theta = \frac{(n-2)\pi}{n}$ (3)</td>
</tr>
<tr>
<td>The radius of an approximating circle</td>
<td>$r(c) = \frac{c}{2} \left(1 + \frac{1}{1-c} \tan \left(\frac{\theta}{2}\right)\right)$ (4)</td>
</tr>
<tr>
<td>The height of the center of a circle above its base</td>
<td>$h(c) = \frac{c}{2} \tan \left(\frac{\theta}{2}\right)$ (5)</td>
</tr>
<tr>
<td>The index of the $\xi$ immediately to the right of $\zeta$</td>
<td>$k = \left\lfloor \frac{\ln \left[c(2\sin(\theta/2)^2(1+c)-1)\right]}{\ln \left[(1-c)(1/2)\right]} \right\rfloor$ (6)</td>
</tr>
<tr>
<td>The thickness of the upper strip</td>
<td>$\mu = c \left(\frac{1-c}{2}\right)^2 h(c)$ (7)</td>
</tr>
<tr>
<td>The thickness of the lower strip</td>
<td>$\lambda = \left(\frac{1-c}{2}\right)^{k+3} h(c)$ (8)</td>
</tr>
</tbody>
</table>

can be assembled from the formulae in Tables 4 and 5

\[ \left(\frac{1-c}{2}\right)^{k+1} \leq c(-\cos \theta + c \tan \left(\frac{\theta}{2}\right) \sin \theta) \leq \left(\frac{1-c}{2}\right)^k. \]

This inequality may be solved for $k$, yielding Formula 6.

The circle formulae in Table 4 can be deduced directly from their definitions. The coordinates for the centers of $\alpha$ and $\zeta$ are easiest to deduce if they are first computed in the rotated coordinate system where segment $MS$ is rotated to be the negative direction on the horizontal axis.

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