Pricing of Mortality-Contingent Claims

Diploma Thesis

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1 Introduction

One of the most recent and most serious problems in the modern world is the aging of its population. Given that unpredictable longevity risk is present as the medical treatments are improving, governments and financial sectors like insurance companies, investment banks and pension providers, are put under pressure. As people get more and more conscious, they intend to buy annuities to finance their retirement. However pricing these, not enough attention was paid on the evaluation of future mortality impacts. Nevertheless the majority of these products offer long-term guarantees which lock-in future mortality and therefore behave like an option on mortality.

Motivation of this diploma thesis was to develop a strategy applied to the Hungarian mortality table – that allows us to price mortality-contingent claims. We treat the underlying life annuity as a defaultable coupon-bearing bond, where the default occurs at the exogenous time of death. Article [6] introduces a method to price options based on two underlying stochastic variables, future interest rates and future mortality rates.

It was highly important to us to create a simple, consistent model, which can be used in real life. Usability is defined as to be deployable in an existing company and to be presentable to the management of this company.

Our data was a series of Hungarian mortality tables and the actual yearly discount rates. As we want to hedge mortality improvement risk it is crucial to examine the development, not only for the past, but to give a forecast for the future. It is comprehensible that the calculation of precise future mortality rates is not possible, forasmuch as the amount of data is too small – it contains only 60 some years – to be effectively used in a practical model. In our case, this does not appear to be a problem, due to our aim to present a way of option pricing and not to provide precise future mortality estimates. Interested readers can find more information on Hungarian mortality projection in [1]. We propose to handle this challenge by presenting the Azbel-model, which gives a satisfactory solution for this issue. This assumption is based on [5].

As it has been mentioned, we attempt to focus on the option price calculation. Therefore we have to face the problem, that the distribution of the contingent claim is unknown to us. Accordingly no explicit formula exist. In order to provide the consistency of this calculation, we revert to Monte Carlo simulation. Mathematically this involves the simulation of future mortality rates, which are applied to make further calculation.

To calculate our option price using these data we can take the general formula presented in [6]. This is based on the thesis, that an option on an annuity can be considered as equivalent to a basket of options on pure endowments if given conditions are satisfied. Obviously these conditions have to be verified to our data first.

Now, after having calculated the option price, and before drawing premature conclusions it is reasonable to make a sensitivity analysis, to determine how "sensitive" our model is to changes in the underlying parameters. If the model behaves as expected from real world observations, it gives some indication that the parameter values reflect, at least partially the "real world".

To achieve this goal we are first going to discuss the basics of classical option pricing framework in section 2. The third section links the classical financial options to the mortality based contingent claims. Application to the Hungarian mortality data and the sensitivity analysis can be found in section 4 as well as the conclusion of this thesis.

2 Option pricing theory

2.1 The option:

An option contract gives the buyer the right to purchase/sell an underlying asset for a strike price. The price of an option corresponds to the difference between the strike price and the value of the underlying asset (commonly a stock, a bond, a currency, etc.), or zero.

Option types:

- call option (right to purchase): gives the holder the right to purchase the underlying asset, while the option writer commits to sale.
- put option (right to sell): gives the holder the right sale the underlying asset, while the option writer commits to buy.

Main option styles:

- European-option: In case of an European-option the right may only be exercised at the maturity date.
- American-option: In case of an American-option the right may be exercised at any time up until the maturity date. The value of this option, due to the possibility of early exercise, is greater than or equal to the value of the European-type option.

2.2 Option pricing

Arbitrage:

Arbitrage means taking advantage of price differences between various markets without taking any risk ("free lunch"). For example the immediate sale of a security or a foreign currency on another market, than having bought on, using the exchange rate differences. The existence of "arbitrage guards" ensures, that the price of an asset will not deviate significantly from the reasonable over a long period. The price modelling of derivatives should therefore assume arbitrage free market conditions.

Option pricing:

The challenge in option pricing is to determine the value of the option in a given instant (for example T=0). It is obvious, that the purchaser can make riskfree profit, if they down have to pay an “entrance fee” for the opportunity to exercise the option at maturity. On the other hand, if this price is too high, and the value of the underlying asset is likely to stay close to the strike price, no rational market participant would buy the option for this price. The option price is the amount of money which would be accepted of both sides.

The option price shall be determined in an objective manner, without dependence on the risk bearing propensity of market participants. In our case, risk means mortality, which is clearly independent from an individuals attitude of bearing risk.

Our aim is to price an option, which gives a now x year old individual the right to purchase an annuity option in k years from now, based on current mortality knowledge, paying 1 unit a year, assuming the individual is alive then.

Replicating portfolio:

One of the possible techniques of pricing an option, is to price a replicating portfolio. This is a portfolio of assets whose value is equal to the value of a liability portfolio under today’s market conditions.

price of the option = price of a basket of risky and risk-free assets, which has the same pay-off at maturity as the option

For an extended discussion about options and option pricing , we refer to [7] and [8].

3 Pricing of mortality based contingent claims

3.1 Replicating with deterministic interest in discrete time

For an introduction of a method pricing options based on two underlying stochastic variables, future interest rates and future mortality rates we refer to article [6]. For the interpretation of parameters and basic assumptions please see [6]. We presuppose deterministic interest rates.

The main objective of this section is to link the classical financial options to the mortality based contingent claims. Therefore a simple example is shown containing two different approaches showing how a purchaser can replicate such an option, or on the other side how the insurer can hedge it.

In this instance we consider underlying n year pure endowment contracts as fundamental assets. These pure endowment contracts, denoted by $\boldsymbol{E}_n$, pay 1 unit at time n if the purchaser is then alive, and pay zero if this is not the case.

Our ultimate objective is to price an option on an annuity based on current mortality knowledge, paying 1 unit a year, assuming the individual is alive then. Such an option can be considered as a basket of options on pure endowments. Naturally under some circumstances – like opposite movements in interest and mortality – it could happen, that some pure endowment options would be in the money at expiry, and some would not. Therefore the annuity option price could be overvalued by the sum of option prices for all these pure endowments. Nevertheless, the assumption that the option holder exercises either all or none of the pure endowment options allows us to consider annuity options to have the same price as the sum of option prices of a basket of pure endowments.

3.1.1 General notation

Before getting to the example, it is necessary to make some basic indications.

First we introduce the variable $p_x(k, n)$ $(k \leq n)$, which denotes the probability that an currently x year old individual will survive to time n conditional upon surviving to time k. This F_k measurable random variable is, for the riskfree probability measure Q, given by the following formula:

$$p_x(k, n) = p_x(k, k+1) E_Q \left[p_x(k+1, n) \mid F_k \right] \qquad \forall k < n \tag{1}$$

where F_k is a filtration on a set Ω denoting the information flow. This in our case, where probabilities are constants, and expectation signs can be removed, reduces to a standard identity shown in section 4.

We presuppose the independence of interest rates – $D(k,n)$, and survival rates – $p_x(r,s)$, for all $k \leq n$, $r \leq s$. We thus obtain the formula for the price of our pure endowment contract purchased by an individual currently aged x, paying 1 unit at time n being alive then, conditional upon surviving to time k:

$$\Lambda_x(k,n) = D(k,n)\, p_x(k,n) \qquad \forall k \leq n, \tag{2}$$

which is also an F_k measurable random variable. The outcome formula (1) and (2) and the independence is,

$$\Lambda_x(k,n) = \Lambda_x(k,k+1) E_Q[\Lambda_x(k+1,n) \mid F_k] \qquad \forall k < n. \tag{3}$$

The call option on a pure endowment which may be exercised at time k, and pays 1 unit at time n, if the now x year old holder is then alive, is denoted by $C_x(k,n \mid \Lambda)$, where Λ is the strike price. To gain further insight into pricing this, consider the following example.

3.1.2 Example

At time 0 an individual, now age 32, would like to receive a single payment of 1 unit at age 75 being alive then. This is a two period pure endowment, with a first interval of 30 years, form the age of 32-62, and a second one of 13 years, from the age of $62 - 75$. Suppose this person purchases this endowment contract from an insurance company, which is assuming mortality as follows.

The probability that this person will live to the age of 62 is known, and given by: $p_{32}(0,30) = 0.7$. However the probability that this person will live to age 75, conditional upon reaching age 62, could depend on random events. Suppose that there are two states of nature that could materialize. In case of a medical breakthrough the survival probability will increase to the value of $p_{32}^{(1)}(30,43) = 0.7$, otherwise it will have a lower value of $p_{32}^{(2)}(30,43) = 0.5$. We suppose, that the insurer is effectively assigning equal chance to each possibility, which leads to:$E_Q\left[p_{32}(30,43)\right] = 0.6$.

From now on we assume zero interest rate for simplicity. Hence the purchase price using Eq. (2), can be calculated:

$$\Lambda_{32}(0,30) = (0.7)(0.6) = 0.42$$

Option replication by the purchaser

For instance the now 32-year-old individual could consider to wait, and purchase the pure endowment contract at the age of 62, if they are then alive. This makes sense, because so they can avoid to waste the contract price in case of an premature death. The cost of the contract will be therefore higher, depending on which state of nature – we mentioned above – occurs. The new price is either $\Lambda_{32}^{(1)}(30,43) = 0.7$ or $\Lambda_{32}^{(2)}(30,43) = 0.5$, instead of 0.42.

Now suppose that the insurer offers them the option, which ensures that the purchaser can acquire the desired pure endowment contract for a fixed price, based on current mortality knowledge. The strike price of this option is in our case 0.6.

This means, that if the mortality improvement occurs, they will exercise the option and purchase the contract for 0.6 instead of 0.7. If this is not the case they will not exercise the option and pay the prevailing price of 0.5. So the intrinsic value of the option is either 0.1, or 0.

Our main objective is to calculate the price of this option, denoted by $C_{60}(1,2 \mid 0.6)$. As it has been mentioned in section 2., one of the possible techniques, is to price a replicating portfolio.

In sake of clarity let our option, described above, be denoted by $\boldsymbol{P}$.

We will show that the value of this option will be the time zero cost of a portfolio which consists of 0.5 units of a 43 year pure endowment connected with a short-sale of 0.25 units of a 30 year pure endowment, as they are equivalent. Expressed as a formula this looks as follows:

$$\boldsymbol{P} = 0.5\boldsymbol{E}_{43} - 0.25\boldsymbol{E}_{30} \tag{4}$$

To prove this, suppose an individual holds the portfolio and is alive at time 1.

As they have sold short a 1 year pure endowment on their own life, and are alive at time 1, they must pay 0.25 to discharge the liability.

As we have seen above, reaching the age 62, two possible scenarios exist, either an expected medical breakthrough materializes, or not. In the first case they can purchase the remaining 0.5 units of income for $\frac{1}{2} \cdot \Lambda_{32}^{(1)}(30,43) = \frac{1}{2} \cdot 0.7 = 0.35$, making a total outlay of 0.6. In the latter case this cost is lower, $\frac{1}{2} \cdot \Lambda_{32}^{(2)}(30,43) = \frac{1}{2} \cdot 0.5 = 0.25$, making a total outlay of 0.5. As it can be seen, this replicates the option.

The time zero cost of this portfolio, and therefore the price of the option is

$$0.5 \cdot \Lambda_{32}(0,43) - 0.25 \cdot \Lambda_{32}(0,30) = (0.5)(0.42) - 0.25(0.7) = 0.4.$$

In practice, this is not yet operational, for as much it is not possible to sell short an pure endowment on ones life. This problem can be solved in the following manner.

Let us consider a 1 year, 1 unit zero-coupon bond, denoted by $\boldsymbol{B}_n$. This is a bond without periodic interest payments. It can be purchased for a price lower than its face value, with the face value repaid at maturity.

Moreover we need a life insurance contract, denoted by $\boldsymbol{I}_n$, paying 1 unit at time n if the individual dies prior to time n. This is at deterministic interest equivalent to a n year term insurance policy paying the price of a zero-coupon bond maturing at time n, provided that the individual dies within n years. Now, as the difference of a 1 unit zero-coupon bond and a life insurance contract is a contingent asset which pays 1 unit if and only if the individual is alive at the end of n years, the following equation holds:

$$\boldsymbol{E}_n = \boldsymbol{B}_n - \boldsymbol{I}_n \tag{5}$$

Observe, that a short position of $\boldsymbol{E}_{30}$ is equivalent with a long position in $\boldsymbol{I}_{30} - \boldsymbol{B}_{30}$, which is simply an insured loan.

$$\boldsymbol{P} = 0.5\boldsymbol{E}_{43} + 0.25(\boldsymbol{B}_{30} - \boldsymbol{I}_{30}) \tag{6}$$

The basic strategy in this case is therefore to borrow money at the age of 32 and additionally buy an insurance policy, which pays off the loan, if they die prior 62. Suppose the individual borrows 0.25 on a 30 year loan, this makes with the option price a total cash-flow of 0.29. Thus the price of this policy is:

$$0.25 \cdot q_{32}(0,30) = 0.25 \cdot (1 - p_{32}(0,30)) = 0.25 \cdot (1 - 0.7) = 0.8$$

This leaves a proceeds of 0.21, enough to purchase a 0.5 unit pure endowment contract paying 0.5 at age 75. If the individual is alive at age 62, they can pay off the loan, and depending on which state of nature occurs buy the remaining 0.5 units of income for $\{0.35\, or\, 0.25\}$, spending a total of $\{0.6\, or\, 0.5\}$

Option hedging by the issuer

In the second part of the example, to gain insight in every aspect of the problem, we look at matters from the point of view of the insurer. It is obvious, that the risk for the insurer is represented by possible mortality improvements in the pure endowments they have sold. It follows from Eq. (5), that insurers can hedge against these option, by selling a ratio of H units of insurance for each 1 unit of option described in the first part of the example.

It is intuitively obvious that in our example the hedge parameter is $H = \frac{1}{2}$. Therefore 0.5 units of a 43 year life insurance hedge the risks of selling $\boldsymbol{P}$. This results in a portfolio $-\boldsymbol{P} - 0.5\boldsymbol{I}_2$ (assuming zero interest rates) and by substituting this and (5) in formula (4), this equals:

$$\text{-}\boldsymbol{P} - 0.5\boldsymbol{I}_2 = -0.5\boldsymbol{E}_2 - 0.25(\boldsymbol{B}_1 - \boldsymbol{I}_1)\text{-}0.5\boldsymbol{I}_2 = \text{-}0.5\boldsymbol{B}_2 - 0.25\boldsymbol{E}_1$$

By eliminating the term $\boldsymbol{E}_2$ we have hedged the risk arising from the unpredictable mortality in the second interval.

Note that these contracts are not on the same lives. Nevertheless, the small sample risk is hedged in the usual way, by selling a sufficiently large number of contracts.

General formulas Assuming zero interest rate, a general formula can be given for the hedge parameter:

$$H = \begin{cases} \frac{\Lambda_{32}^{(1)}(30,43)\text{-}\Lambda}{\Lambda_{32}^{(1)}(30,43)-\Lambda_{32}^{(2)}(30,43)} & if \quad \Lambda_{32}^{(2)}(30,43) \le \Lambda \le \Lambda_{32}^{(1)}(30,43) \\ 1 & if \quad \Lambda < \Lambda_{32}^{(2)}(30,43) \\ 0 & if \quad \Lambda > \Lambda_{32}^{(1)}(30,43) \end{cases} \tag{7}$$

($H = 0$ if $\Lambda > \Lambda_{32}^{(1)}(30,43)$ – This condition is not possible in our present example) and the option portfolio:

$$\boldsymbol{P} = H\,[\boldsymbol{E}_{43} - \Lambda_{32}^{(2)}(30,43)\boldsymbol{E}_{30}]. \tag{8}$$

The call option price arises from taking expectations:

$$C_{32}(30,43|\Lambda) = H\,[\Lambda_{32}(0,30) - \Lambda_{32}^{(2)}(30,43)\Lambda_{32}(0,30)]. \tag{9}$$

Writing down Eq. (3) for this case

$$\Lambda_{32}(0,43) = \Lambda_{32}(0,30)E_Q[\Lambda_{32}(30,43)],$$

and substituting H, the price of the call option is given by:

$$C_{32}(30,43|\Lambda) = \left(\frac{E_Q[\Lambda_{32}(30,43)]\text{-}\Lambda_{32}^{(2)}(30,43)}{\Lambda_{32}^{(1)}(30,43) - \Lambda_{32}^{(2)}(30,43)}\right)\Lambda_{32}(0,30)\,(\Lambda_{32}^{(1)}(30,43) - \Lambda). \tag{10}$$

3.2 General formula for the option in discrete time

As we want to price mortality-contingent claims, based on the Hungarian mortality table, we need a general formula for the option to purchase at time 0, with a present age of x, paying 1 unit from the age of k, each year until death. The basic idea is to price this option as the sum of pure endowments with a maturity date n, $n = k+1...T$ (T denotes the end point $-$ the age when population dies out).

As mentioned before an option on an annuity, cannot in general be considered as equivalent to a basket of options on pure endowments.

For this purpose some conditions have to be met. The n-th contract covers the option for purchasing a pure endowment contract with maturity date n at time k. Let the strike price be denoted by K_n. This can naturally be exercised, if $K_n \leq \Lambda(k,n)$, ergo the strike price is smaller than the pure endowment price is at time k. The annuity option $K = \sum_{n=k+1}^{T} K_n$ will obviously being exercised if and only if $K \leq \sum_{n=k+1}^{T} \Lambda(k,n)$.

In practice this would be performed the other way around, the price of the annuity contract would be given. In this case the question occurs whether there is a series of K_n, summing to K, that either all or none of the individual pure endowments will be exercised. If this is achievable, the pricing of annuity options can be substituted with the pricing of pure endowment options.

To derive conditions for the assumption to hold is simpler soluble in case of a finite number of outcomes. Let us fix k, and suppose the annuity beginning at time $k+1$, further s possible states of nature. Now $\Lambda^i(k,n)$ denotes the price of an pure endowment option acquired at time k, paying one unit at time n if the individual is alive then, supposing outcome i. On the other hand, the price of the annuity option is indicated by $a^i(k)$. Therefore our goal to be achieved is described by the following equation:

$$a^i(k) = \sum_{n=k+1}^{T} \Lambda^i(k,n)$$

Supposing, that

$$i \leq j \; if \; \Lambda^i(k,n) \leq \Lambda^j(k,n)$$

for all $n = k+1, ..., T$, gives a partially order on possible outcomes.

3.2.1 Theorem

The price of an annuity option with exercise date k can be determined by pricing pure endowment options

$$\Updownarrow$$

$$\forall\, i \leq j,\, \forall\, n = k+1, ..., T \qquad \Lambda^i(k,n) \leq \Lambda^j(k,n)$$

Proof.

$\Longrightarrow$First suppose a linear ordering.
Let i be the minimal subscript such that $a^i(k) \geq K$, and then choose K_n so that $K_n \leq \Lambda^i(k,n) \qquad \forall\, n = k+1...T$ and $\sum_{k=1}^{T} K_n = K$.
Observe that in the case of an index j, so that

$$a^j(k) \geq K \quad \Rightarrow \quad j \geq i$$

In this instance the annuity option will be exercised as well as all pure endowment options will.
Otherwise, if j is such an index, that

$$a^j(k) < K \quad \Rightarrow \quad j < i$$

In this instance the annuity option wont be exercised and so will none of the pure endowment options. □
$\Longleftarrow$
Conversely let us consider the antithesis that the ordering is not linear. Mathematically this means, that

$$\exists\, i, j, m, n,\, such\, that \qquad \Lambda_i(k,n) > \Lambda_j(k,n), \qquad and \qquad \Lambda_j(k,m) > \Lambda_i(k,m)$$

Suppose that $a_i(k) \leq a_j(k)$, and let $K = a_i(k)$. Let K_n be a series of strike prices of pure endowment options, which we are assuming to satisfy the condition, $K = \sum_{n=k+1}^{T} K_n$. Observe that both the options payable at time m and n would be exercised for both outcomes, if

$$K_m \leq \Lambda_i(k,m),$$

and

$$K_n \leq \Lambda_j(k,n) < \Lambda_i(k,n).$$

Now, taking our assumptions in account, it follows that:

$$\sum_r K_r = \sum_{r=n+1}^{T} \Lambda_i(k,r) \Rightarrow \sum_{r \neq n,m} K_r > \sum_{r \neq n,m} \Lambda_i(k,r)$$

and therefore we conclude that the antithesis is wrong, as at least one pure endowment option for times different from m, n will not be exercised in outcome i. □

3.2.2 Formula for the option price

The pure endowment is in our case a contract purchased at time k, paying 1 unit at time n if the individual is alive then, for a strike price of Λ. The price is given by

$$C_x(\,k,n \mid \Lambda) = E_Q\left[\prod_{i=1}^{k} \Lambda_x(i-1,i) \cdot |\Lambda(k,n) - \Lambda|_+\right]$$

this will just be the expected value of the with interest and mortality discounted difference between the market price and the strike price at maturity (if positive).

This in our case, where probabilities are constants, and expectation signs can be removed, reduces to

$$C_x(\,k,n \mid \Lambda) = \prod_{i=1}^{k} \Lambda_x(i-1,i) \cdot |\Lambda(k,n) - \Lambda|_+ \tag{11}$$

Now if the above given conditions are satisfied we can indeed reduce the pricing of annuity options, and calculate the sum of pure endowment option prices instead. This results in the general formula for an annuity option to be purchased at time 0, at an present age of x, paying 1 unit from the age of k each year until death. Our notation is $\boldsymbol{P}_x(k|\Lambda)$, given by the following formula:

$$\boldsymbol{P}_x(\,k\,|\Lambda) = \sum_{n=k+1}^{T} C_x(\,k,n|\Lambda) = \sum_{n=k+1}^{T} \prod_{i=1}^{k} \Lambda_x(i-1,i) \cdot |\Lambda(k,n) - \Lambda|_+ \tag{12}$$

4 Application to Hungarian mortality table

4.1 Forecast the development of the given mortality table

After having discussed underlying theories it is time to develop a strategy – applied to the Hungarian mortality table – that allows us to price mortality-contingent claims. We treat the underlying life annuity as a defaultable coupon-bearing bond, where the default occurs at the exogenous time of death.

Our data is a series of Hungarian mortality tables (1949-2006) and the actual yearly discount rates.

We suppose that at time 0 an individual, now age 32, would like to receive a payment of 1 unit from the age of 62 each year until death. As we want to price mortality improvement risk it is crucial to examine the development of mortality and give a forecast for the future.

Our first step is to project mortality to the following 60 years. For sake of simplicity it is suitable to apply the Azbel-model. Our reasoning is as follows:

1. An alternative method would be to project mortality rates for each age and each year. This means a data of approximately 100 $ages \cdot 60$ $years$, which is far too much to make this possible.

2. Therefore we decided to fit a parametric model. This can be done, because we are not interested in precise data, but in the development of mortality. Now the goal is to choose a model which is readily interpretable and easy to deal with.

3. The obvious choice could be the to use the well known Gompertz-Makeham-model, but the Azbel-model is in most cases as good and even simpler than this.

4.1.1 The Azbel-model

Azbel construed a mortality law that can successfully describe the behaviour of different death rates like human and medfly mortality curves. The basic assumption of his proposal is that at least one Gompertz region exists in the death rate, which means the existence of a predominantly exponential region in the mortality curve. The mortality $q_x(t)$ at age x and time t, can be written as

$$q_x(t) = -\frac{N_{x+1}(t+1) - N_x(t)}{N_x(t)}$$

where $N_x(t)$ is the number of individuals alive with age x at time t. Azbel considered time as a continuous variable, and assumed that $q_x(t)$ is an exponential function of time. Than following the assertion that every death rate has its Gompertz region, where the logarithm of mortality rate is close to its linear regression he deduced the following equation: $lnq_x = a + bx$, where a and b are parameters to be determined by fitting the mortality curves of demographic data. Each curve provides a pair (a, b). Azbel studied the demographic data of Japanese and Swedish population extensively for reasons of having a low premature mortality. He showed that the points determined by a and b follow a straight line and so can be related by $a = lnA - bX$. With this the Azbel-formula of death rates is: $q_x = Ab\,exp[b(x - X)]$, where A, b and X are the parameters. [1]

Although this formula is substantially simpler than the Gompertz-Makeham model, a further improvement was made in [5].

The reparametrization $T = X - ln(Ab)/b$ leads to a supplemental simplification. The result is a two parametric equation

$$q_x = exp(b(x - T)), \tag{13}$$

with b the shape parameter and T the end point (the age when population dies out).

Using the idea of applying a loglinear regression-type estimator, given the fact that $lnq_x = b(x - T)$, we performed a simple linear regression with the logarithm of q_x as the dependent and the age as independent variable. How these regression coefficients can be calculated and used to obtain the parameters of the model in order to calculate projected mortality rates, can be seen after the verification of the model.

4.1.2 The verification of the model

To get a feeling about the suitability of the Azbel-model we have to refer to [5].

Using the two so-called A/E and ERL statistics and fitting different models to the ages 60-90, the article brings to light essential similarity in goodness-of-fit. As it can be read, the weighted fit with expected remaining lifetime and the simple loglinear regression turned out to give very similar results, while the weighted loglinear regression is slightly worse than the above ones. It furthermore can be seen that the Azbel-model with a loglinear regression-type estimator is reasonable.

4.1.3 Calculation of mortality development

To carry through a loglinear regression on q_x means, that after having taken the logarithm of every component of our table we need to preform a linear regression. Let our new data be $y_x^{z_i} = logq_x^{z_i}$ where x is the age and z the vector of calendar years $z = [1949...2006]$. As we believe q to have a loglinear y will have a linear relationship, which is governed by the familiar equation $y^i \sim a_i x + b_i$, where x is still the vector of ages. Now the linear regression will be applied in order to determine the two parameters a_i and b_i. In practice given a set of data $(x_j, y_j^{z_i})$ with n data points, a_i and b_i can be reflected by the following formula:

$$\begin{aligned} a &= \frac{cov(x,y)}{D^2(x)} = \frac{n\sum(x_j y_j) - \sum x_j \sum y_j}{n\sum({x_j}^2) - (\sum x_j)^2} \\ b &= \bar{y} - a\bar{x} = \frac{\sum y_j - a\sum x_j}{n} \end{aligned}$$

Now we can estimate the future development of the mortality table. As it is not our aim to give an exact forecast of mortality rates, but to forecast the option price, the well-known Lee-Carter method can be applied.

Our a_i and b_i coefficients, determind by the linear regression, can be expressed by following approximation:

$$a_i \sim Ai + B$$

$$b_i \sim Ci + D$$

The coefficients A, B, C, D are given through the following formulas:

$$z = vector\ of\ calendar\ years$$

$$A = \frac{cov(z,a)}{D^2(z)},\ B = \bar{a} - A\bar{z},\ C = \frac{cov(z,b)}{D^2(z)},\ D = \bar{b} - C\bar{z}$$

Forecasted mortality rates can now be calculated with the Azbel model using Eq. (13).

$$q_x^i := \begin{cases} e^{(Ai+B)x+(Ci+D)} & if\ e^{(Ai+B)x+(Ci+D)} < 1 \\ 1 & if\ e^{(Ai+B)x+(Ci+D)} > 1 \end{cases} \qquad x = 0\ldots100,\ i = 2007\ldots2066$$

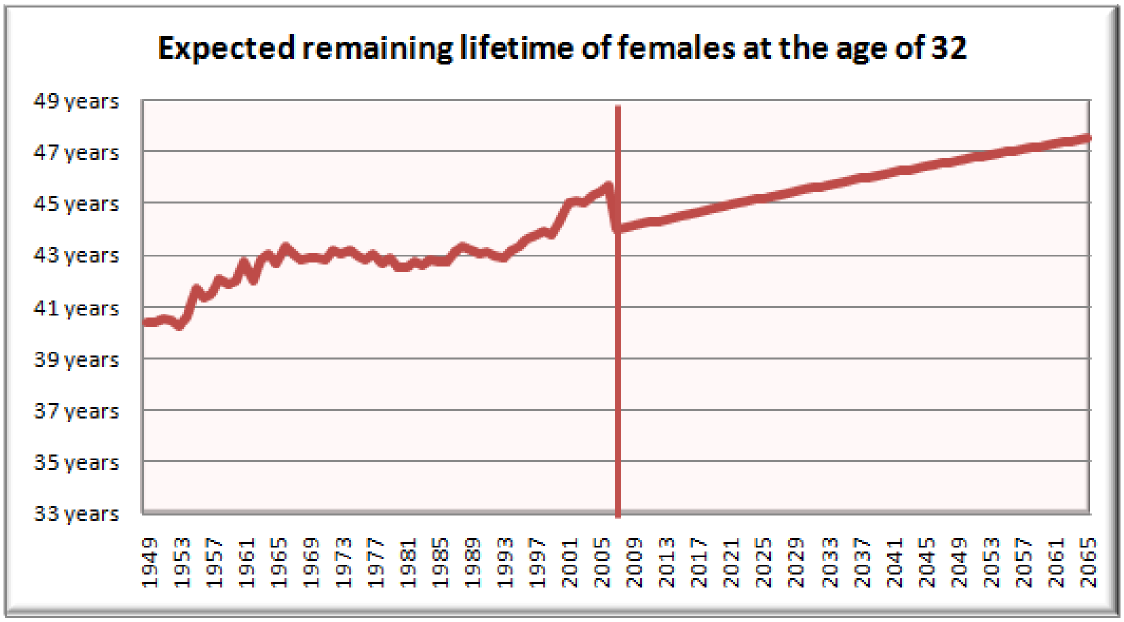


Fig. 4. Forecast of expected remaining lifetime at the age of 32 (female case)

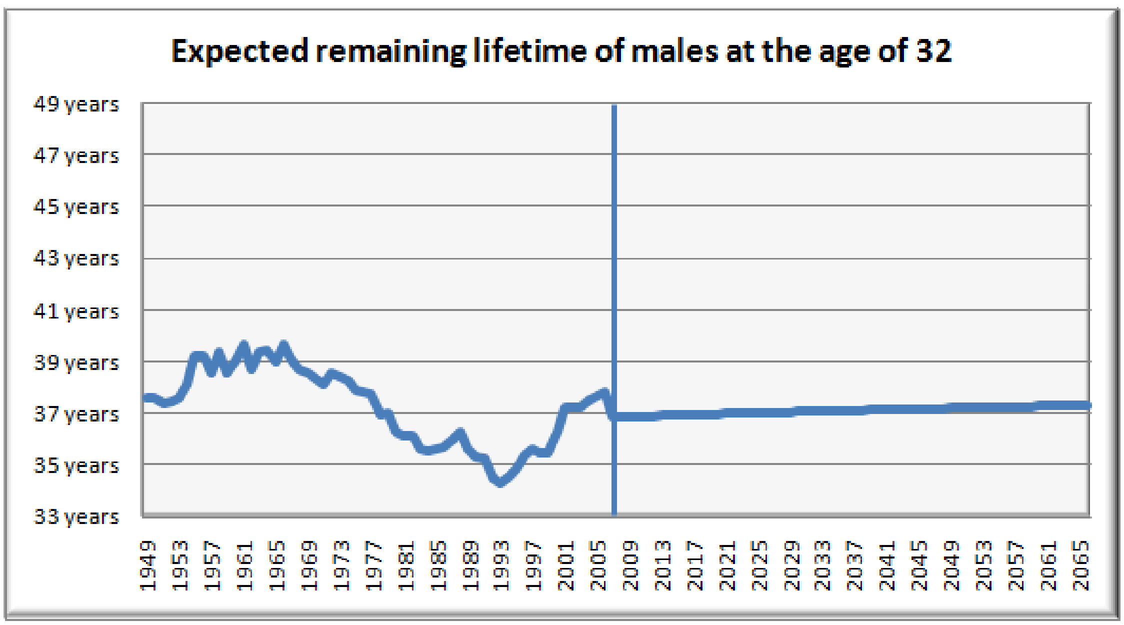


Fig. 5. Forecast of expected remaining lifetime at the age of 32 (male case)

4.2 Calculation of life expectancy

To get an impression how mortality changed over the years it is reasonable to calculate and plot the expected remaining lifetime for some selected ages. The mortality rate q_x denotes the probability of dying in the next year after having reached the year x.

$$p_x = 1 - q_x$$

$$q_x = P(X < x + 1 \mid X \geq x)$$

$$p_x = P(X \geq x + 1 \mid X \geq x)$$

The probability of reaching a certain age, l_x, is given by the formula below:

$$l_x = P(X \geq x) = P(X \geq x \mid X \geq x - 1)P(X \geq x - 1) = p_{x-1}l_{x-1}$$

Assuming that $l_0 = 1$, l_x can be expressed by survival probabilities:

$$l_x = \prod_{i=0}^{x-1} p_i$$

The life expectancy at birth for an individual:

$$E(X) = \sum_{i=0}^{T=100} iP(X = i) = \sum_{i=0}^{T=100} P(X \geq i) = \sum_{i=0}^{T=100} l_i,$$

where T is the last duration at which members of the particular cohort will be living.

4.3 Calculation of expected remaining lifetime

To get the expected remaining lifetime of an individual with the present age of x, we need a variable to express the probability of reaching a certain age assuming to be alive at the given age. Let this be l_n^x and it is given by the following formula:

$$\begin{aligned}
l_{x+k+1}^x &= P(X \geq x + k + 1 \mid X \geq x) \\
&= P(X \geq x + k + 1 \mid X \geq x + k)\, P(X \geq x + k \mid X \geq x) = p_{x+k} l_{x+k}^x \\
l_x^x &= 1 \\
l_{x+k+1}^x &= \prod_{i=x}^{x+k} p_i
\end{aligned}$$

Now the expected remaining lifetime of an individual with the present age of x can be calculated:

$$E(X - x \mid X \geq x) = \sum_{i=x}^{T=100} P(X \geq i) = \sum_{i=x}^{T=100} l_i^x$$

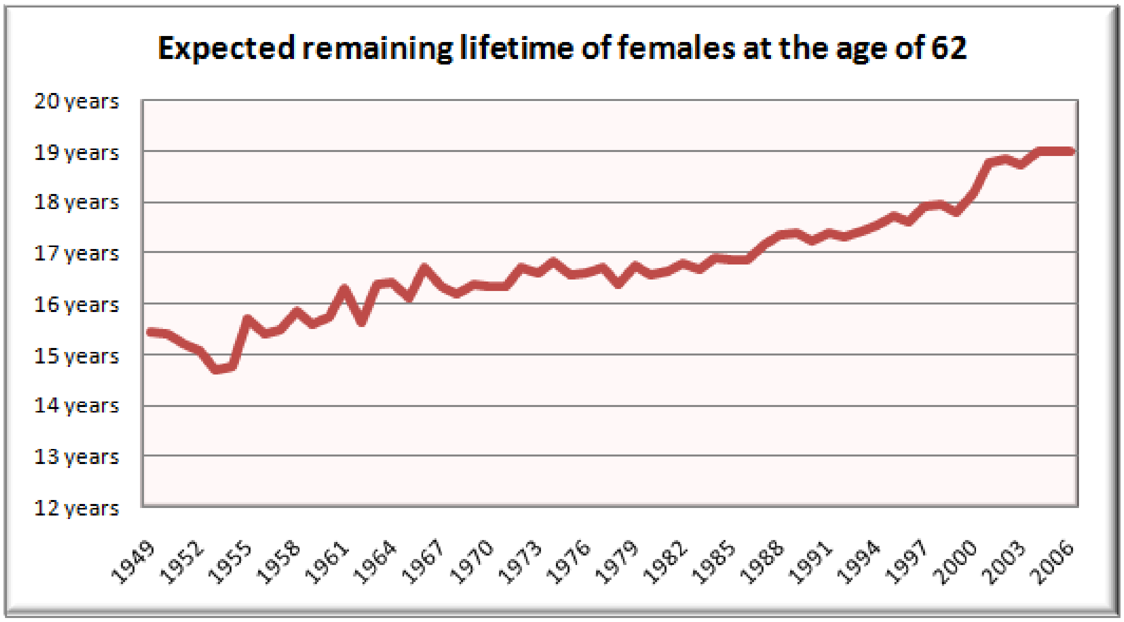


Fig. 6. Expected remaining lifetime at the age of 62 (female case)

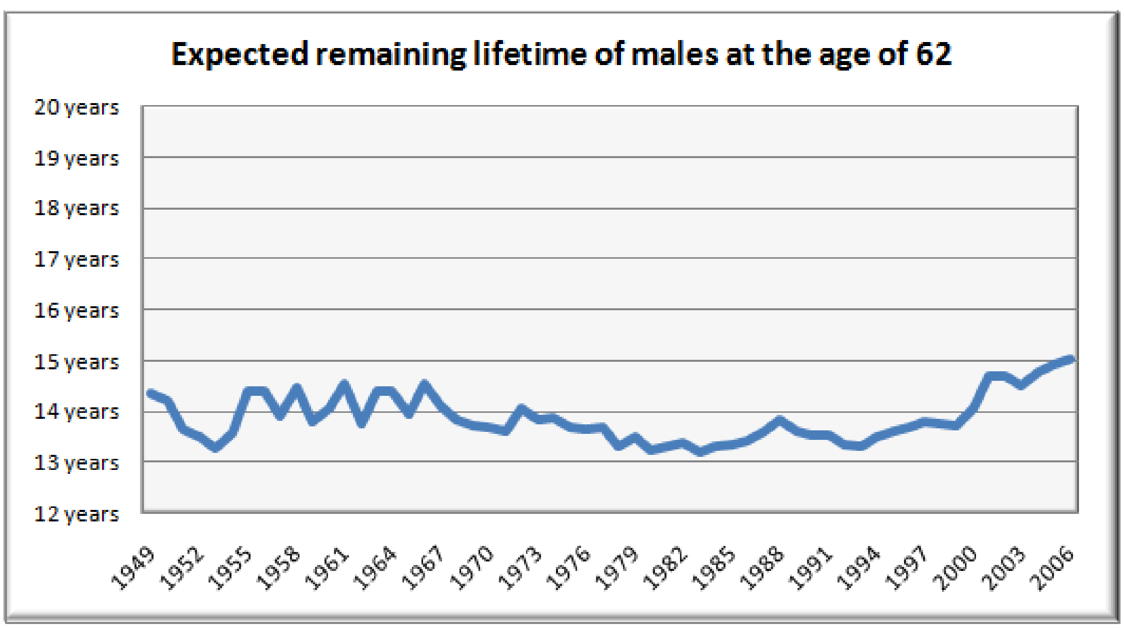


Fig. 7. Expected remaining lifetime at the age of 62 (male case)

4.4 Calculation of expected remaining lifetime using our forecast

We have seen how the expected remaining lifetime for a certain age is calculated with our given mortality table. For more proper results we have to take mortality changes into account, this means to define a generation mortality table for our forecast. In our case it would be interesting to know how long an individual lives after reaching the age of 62. This is described by following formulas:

- Let q_n^r be the forecasted mortality for the age n given by our regression for the year r.
- $q_n^r = q_n^{2006}$, when $n \leq 62$
- $q_n^r = q_{62+x}^{2006+x}$, when $n = 62 + x$

Now $l_n^{r,62}$, which is our regression variable to express the probability of reaching age n ($n \geq 62$) assuming to be alive at 62, is given by the following formula.

$$l_{62+k+1}^{r,62} = \prod_{i=62}^{62+k} p_i^r$$

To get the expected remaining lifetime of an individual with the present age of 62

$$E(X - 62 \mid X \geq 62) = \sum_{i=62}^{T=100} l_i^{r,62} = \begin{cases} 13.1 & Male \\ 18.2 & Female \end{cases}$$

4.5 Simulation

As mentioned in the beginning, our main aim is to price the option. In this case we have to face the problem that the distribution of q_x is unknown to us, so there is no explicit formula. In order to provide the consistent calculation of the option price, we revert to Monte Carlo simulation.

4.5.1 The Monte Carlo simulation

Monte Carlo simulation means in our case to simulate future mortality rates. For this purpose we have to randomize our regression. As we do not know the distribution of q_x, the first logical step is to calculate the distribution of our regression coefficients a and b. Presuming they are normally distributed, we can calculate the expected value and variance, and generate random numbers $a^s$and b^s, where s stands for simulation.

4.5.2 The verification of normality

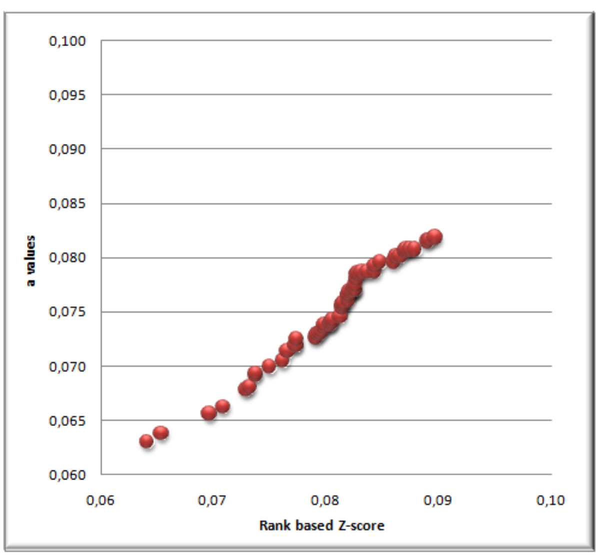
To use a normal distribution has two main reasons

1. consistency with article [5],

2. even when a distribution may not be exactly normal, it may still be convenient to assume that a normal distribution is a good approximation, as it is considered to be the most "basic" continuous probability distribution. In addition it is not only a well-known but an accepted method, and so practicable in insurance market which might be in the interest of a companies management.

Before presuming the normality of our regression coefficients a and b, we should apply a Quantile-Quantile plot (Q-Q plot). This is an exploratory graphical tool, in which the quantiles of two statistical variables are plot against each other to compare their distributions. The observed values of two features are sorted in ascending order and are combined into pairs to be carried into a coordinate system. Resulting points lying approximately on the line $y = x$, indicate a similar basic distribution of the two compared characters.

In our case we take our regression coefficients a and b, having used for the creation of random numbers, and sort the components in ascending order. The same is done to $a^s$and b^s being the randomly created vectors. Now for instance a $point(x, y)$ diagram can be used, which plots one of the quantiles of the second distribution against the same quantile of the first distribution. Results are showing satisfactory pictures.



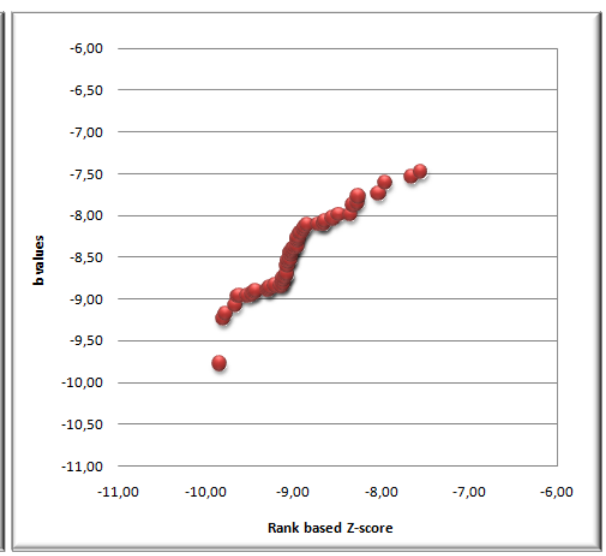


Fig. 8. Q-Q plot of a and a^s (female case) *Fig. 9. Q-Q plot of b and b^s (female case)*

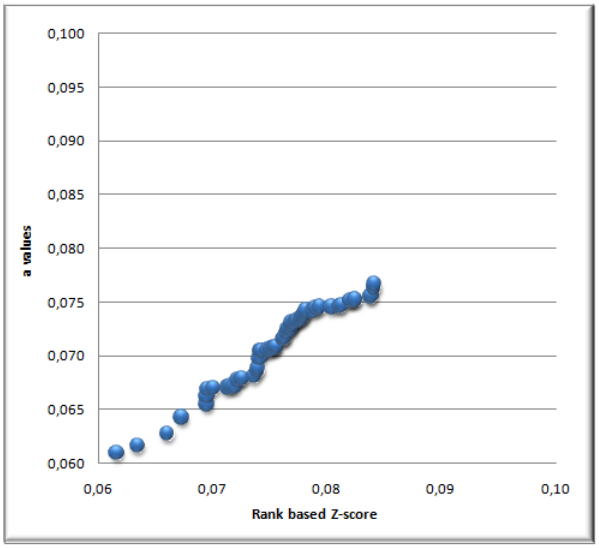


Fig. 10. Q-Q plot of a and a^s (male case)

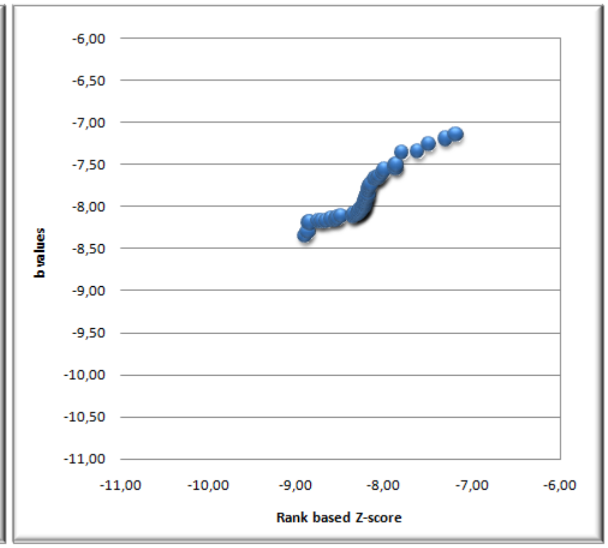


Fig. 11. Q-Q plot of b and b^s (male case)

4.5.3 Simulation of mortality rates

As shown in subsection 4.1 we can calculate q_x^s for every simulated $a^s$and b^s in the same way.

$$z = vector\ of\ calendar\ years$$

$$A^s = \frac{cov(z, a^s)}{D^2(z)},\ B^s = \bar{a^s} - A^s\bar{z},\ C^s = \frac{cov(z, b^s)}{D^2(z)},\ D^s = \bar{b^s} - C^s\bar{z}$$

$$q_x^i : \; = \begin{cases} e^{(A^s i + B^s)x + (C^s i + D^s)} & if\ e^{(A^s i + B^s)x + (C^s i + D^s)} < 1 \\ 1 & if\ e^{(A^s i + B^s)x + (C^s i + D^s)} > 1 \end{cases}$$

$$x \; = \; 0 \ldots 100,\ i = 2007 \ldots 2066$$

Now we have simulated this q_x^s a 1000 times both in female and male cases.

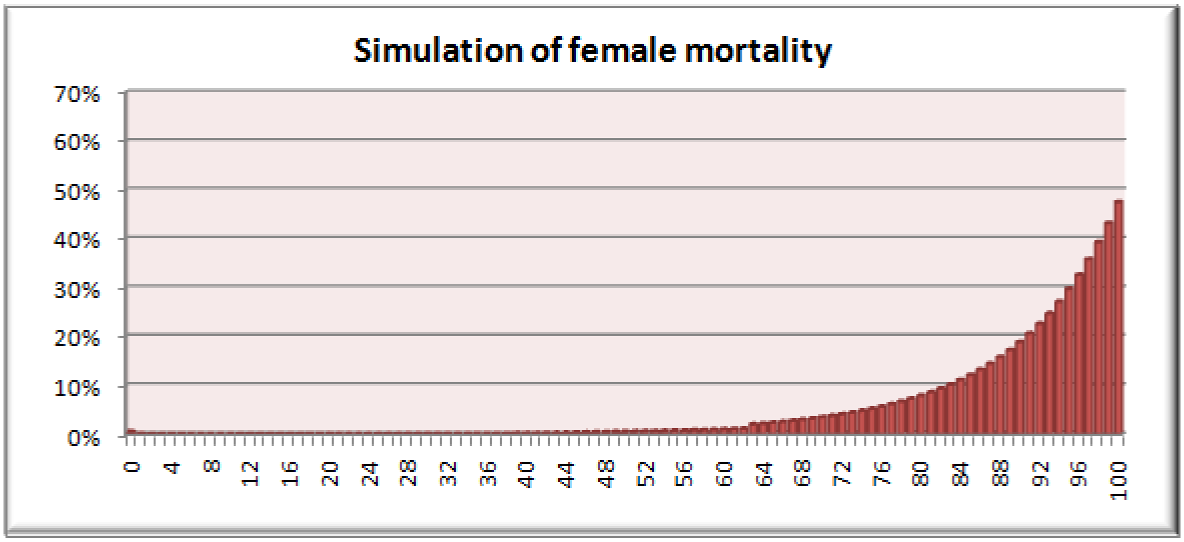


Fig. 12. Average of simulated mortality rates (female case)

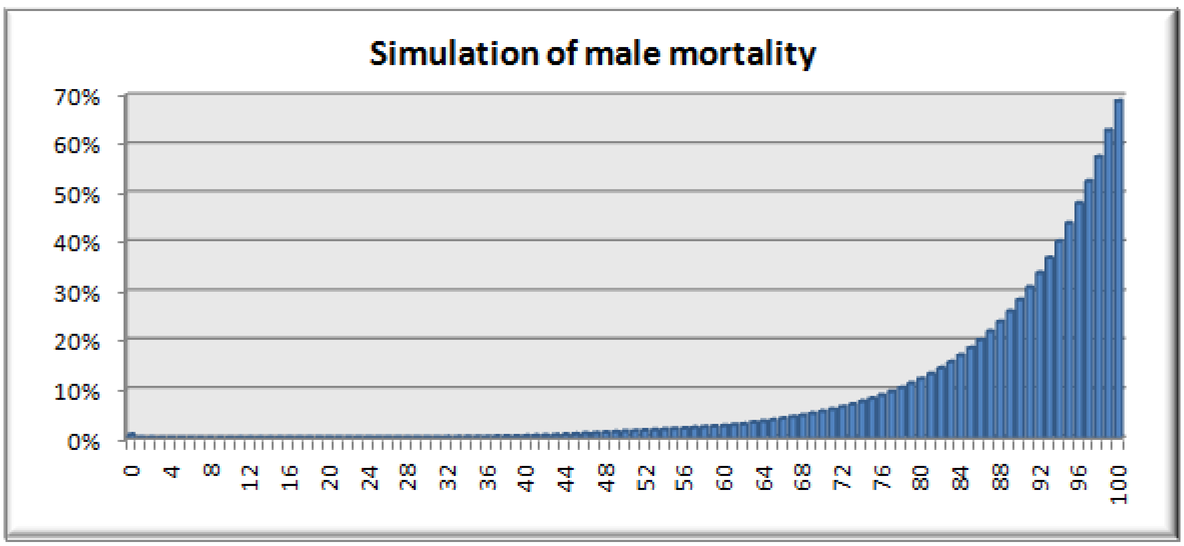


Fig. 13. Average of simulated mortality rates (male case)

4.5.4 Simulation of option price

Assuming that the Theorem (3.2.1) holds, annuity options can be priced by pure endowment options, so in this case we can use the general formula (12) to calculate the simulated option prices for every q_x^s. Taking the average of these we will receive the option price.

We first have to calculate survival probabilities:

$$p_x^s = 1 - q_x^s$$

Now follows the calculation of the probability that an individual in our cohort - who is currently aged x - will survive to time n conditional upon surviving to time k.

As mentioned in section 3. now, where probabilities are constants, the expectation sign can be removed. Therefore the formula number (1) can be written as follows:

$$p_x^s(k,n) = \prod_{i=k}^{n-1} p_x^s(i,i+1) = \prod_{i=k}^{n-1} p_{x+i}^s$$

The next step is to calculate the market price of our pure endowment contract using Eq. (2)

$$\Lambda_x^s(k,n) = D(k,n)\, p_x^s(k,n)$$

$$\Lambda_x^s(i-1,i) = D(i-1,i)\, p_{x+i}^s \qquad i = 1...k$$

Simulated option prices can be calculated using Eq. (11) and (12):

$$C_x^{s_l}(k,n|\Lambda) = \prod_{i=1}^{k} \Lambda^{s_l}(i-1,i) \cdot |\Lambda_x^{s_l}(k,n) - \Lambda|_+$$

$$\boldsymbol{P}_x^{s_l}(k|\Lambda) = \sum_{n=k+1}^{N=100} C_x^{s_l}(k,n|\Lambda)$$

Our option price will be the average of all simulated option prices.

$$\boldsymbol{P}_x(k|\Lambda) = \frac{\sum_{l=1}^{1000} \boldsymbol{P}_x^{s_l}(k|\Lambda)}{1000}$$

To verify Theorem (3.2.1), we have to calculate the price of a pure endowment option basket. This will be the same as the annuity option price if conditions are satisfied.

$$K_{32}^{s_l}(30 \mid \Lambda) = \sum_{n=31}^{N=100} \left(\prod_{i=1}^{k} \Lambda^{s_l}(i-1,i)\, max(\Lambda_{32}^{s_l}(30,n) - \Lambda, 0) \right)$$

Now we have to compare each $K^{s_l}$with $\sum_{n=31}^{100} \Lambda^{s_l}(30,n)$, to prove that:

$$K^{s_l} \leq \sum_{n=31}^{100} \Lambda^{s_l}(30,n)$$

As this holds for our data, the option price can be calculated:

$$\boldsymbol{P}_{32}(30|\Lambda) = \begin{cases} \boldsymbol{P}_{32}(30|0.6306) = 0,399 & Male \\ \boldsymbol{P}_{32}(30|0.8381) = 0,118 & Female \end{cases}$$

4.6 Sensitivity Analysis

Now, after having calculated the option price, it is reasonable to make a sensitivity analysis, to determine how "sensitive" our model is to changes in the underlying parameters. By doing series of tests setting different parameter values we can see how a change in the parameter causes an alteration in the value of interest. If the model behaves as expected from real world observations, it gives some indication that the parameter values reflect, at least in part, the reality.

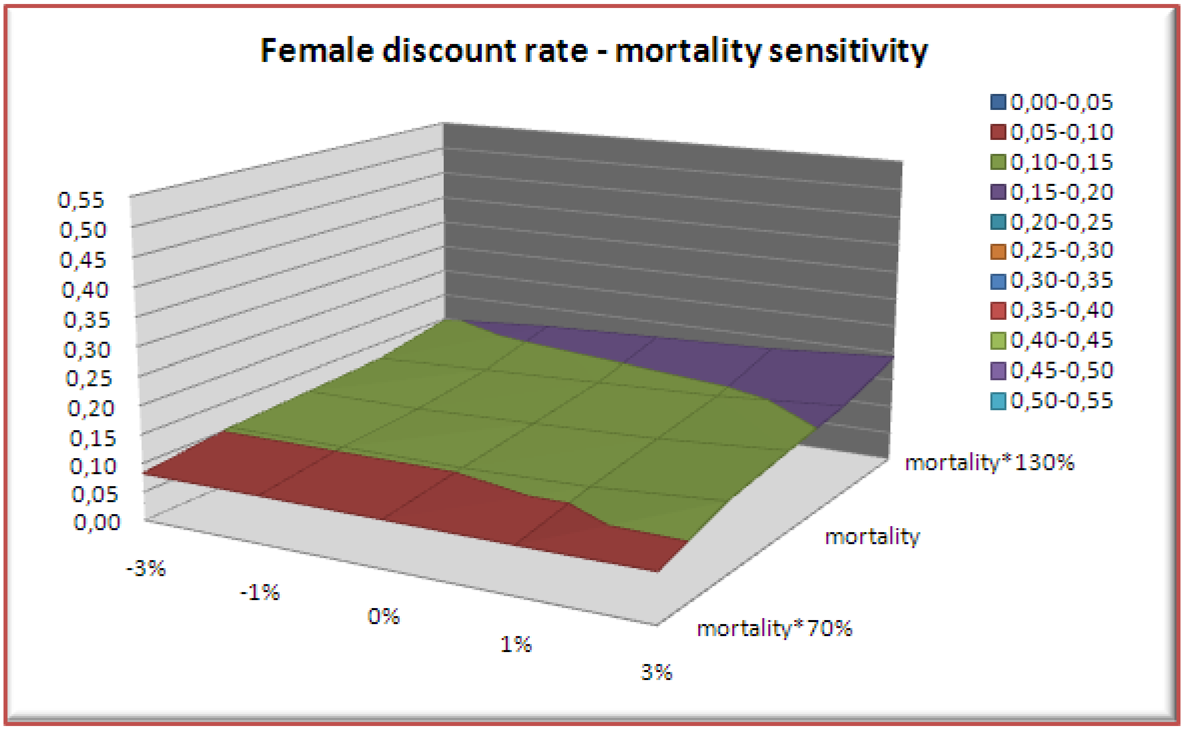
What we are interested in is the change in option price if circumstances vary. Our parameters are mortality, discount rates, and the age we may exercise the option:

- To illustrate mortality change we have to take the given percentage of every simulated mortality rate.
- Differences in discount rates can be treated in the same way.
- The age of exercise is a little bit more difficult, because it has to be simulated for each case.

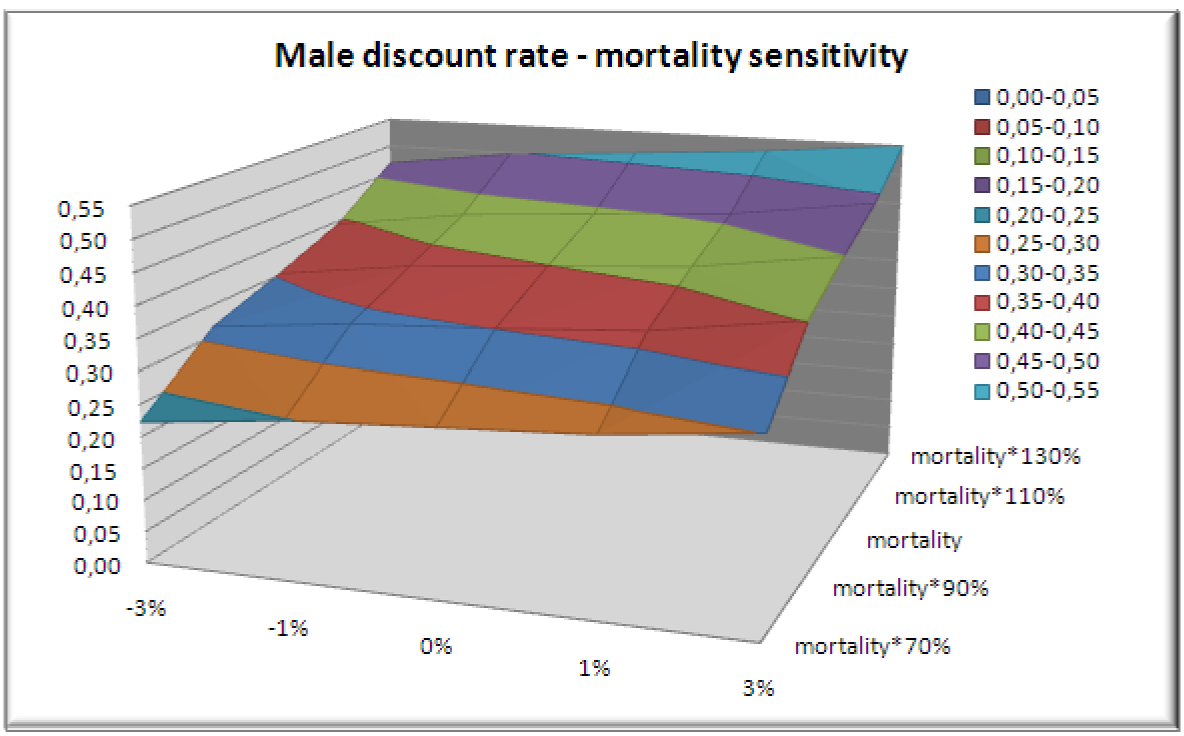
After calculating all cases and verification of Theorem (3.2.1) is performed in each, we get a 3-dimensional datatable. As this is not possible to plot we can split it in 3 different 2-dimensional datatables.

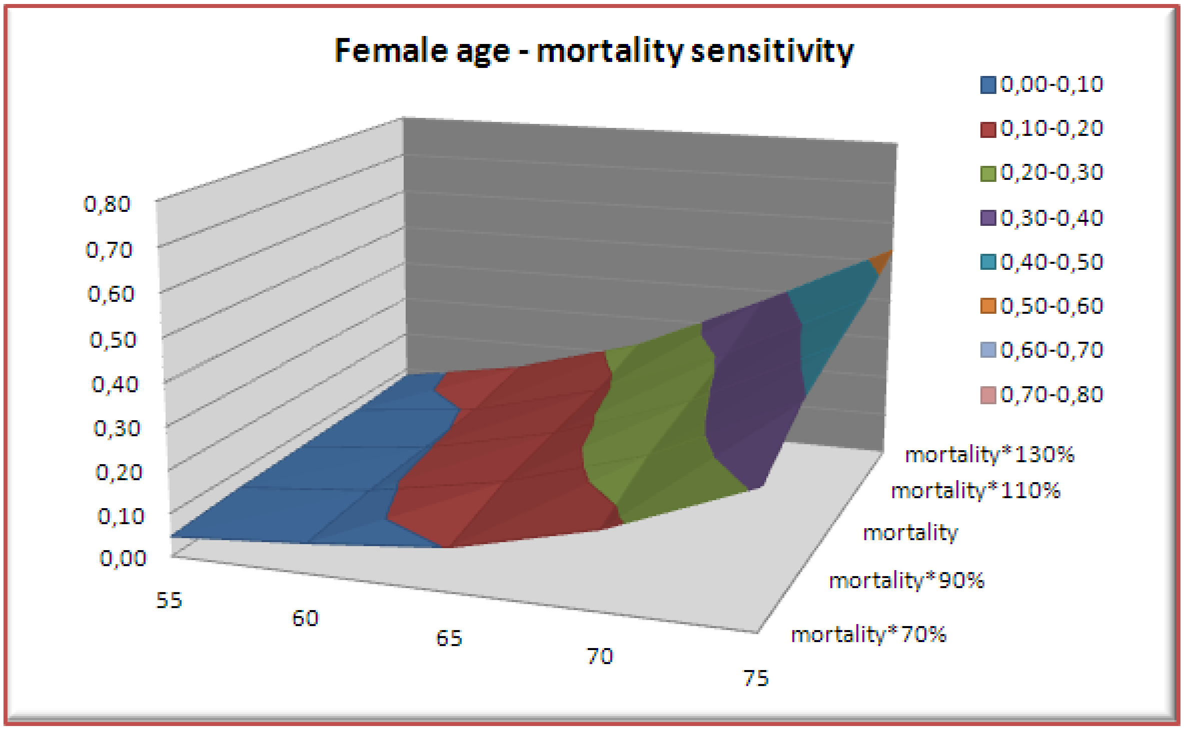
As we used future mortality and interest rates for discounting, we are anticipating, that the option price will change in the same direction as these two parameters do.

For a better understanding let us see it from another perspective: growth of mortality has the effect, that expected remaining lifetime decreases. This results in more money to be disbursed each year, which means a higher risk. The impacts on price of alteration in age of exercise can be explained in the same way, except that the expected remaining lifetime decreases exponentially with aging. Fig. 12 and fig. 13 reveal that male mortality grows more exponential than female mortality does.

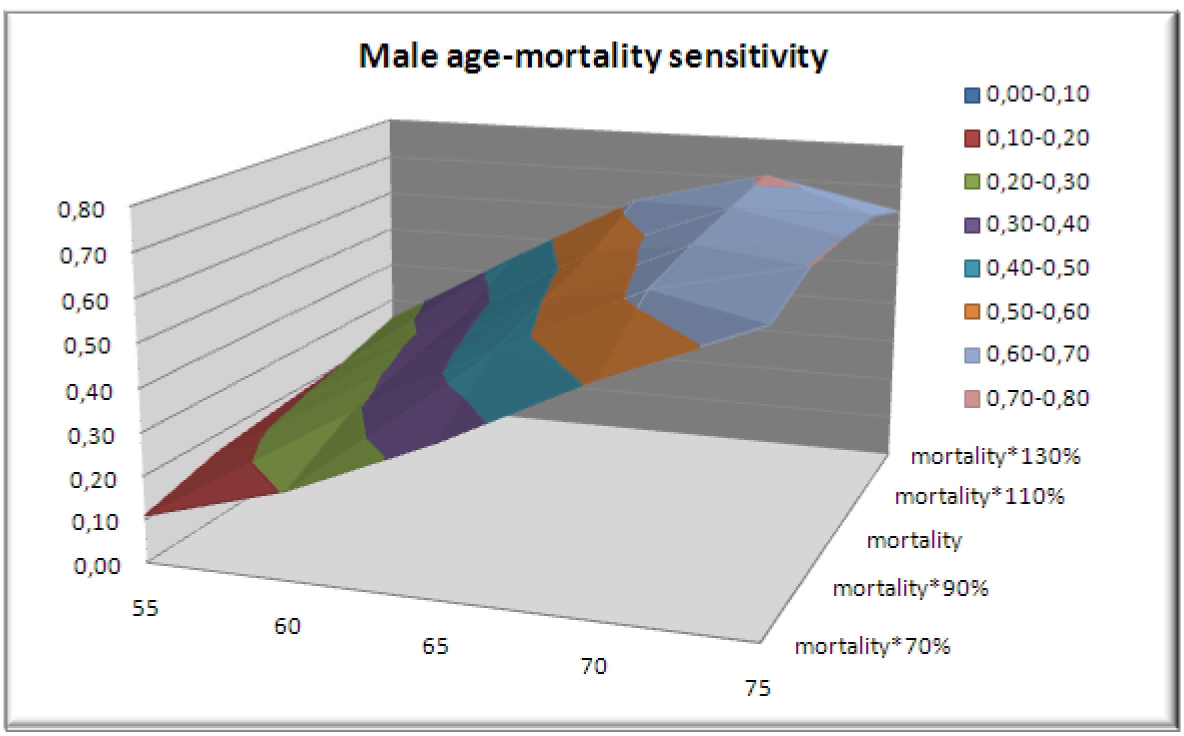


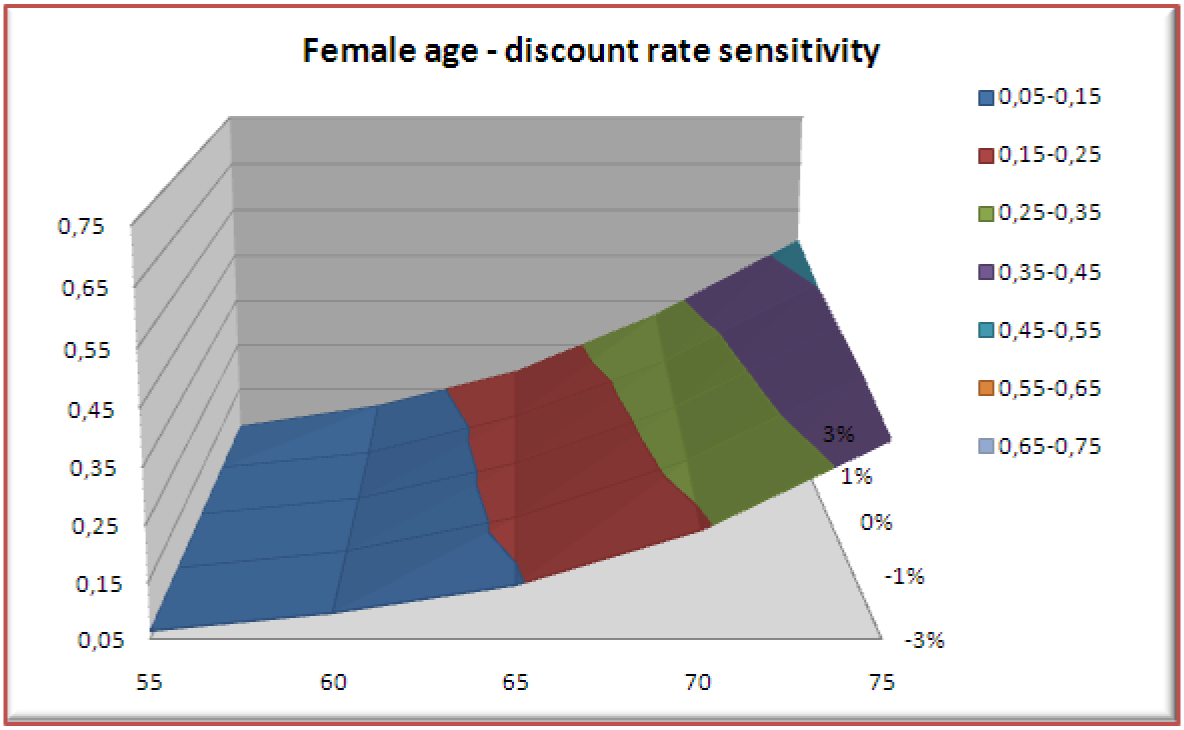
Our expectations are met, as the price of the option grows in both parameters both in female as in male case.



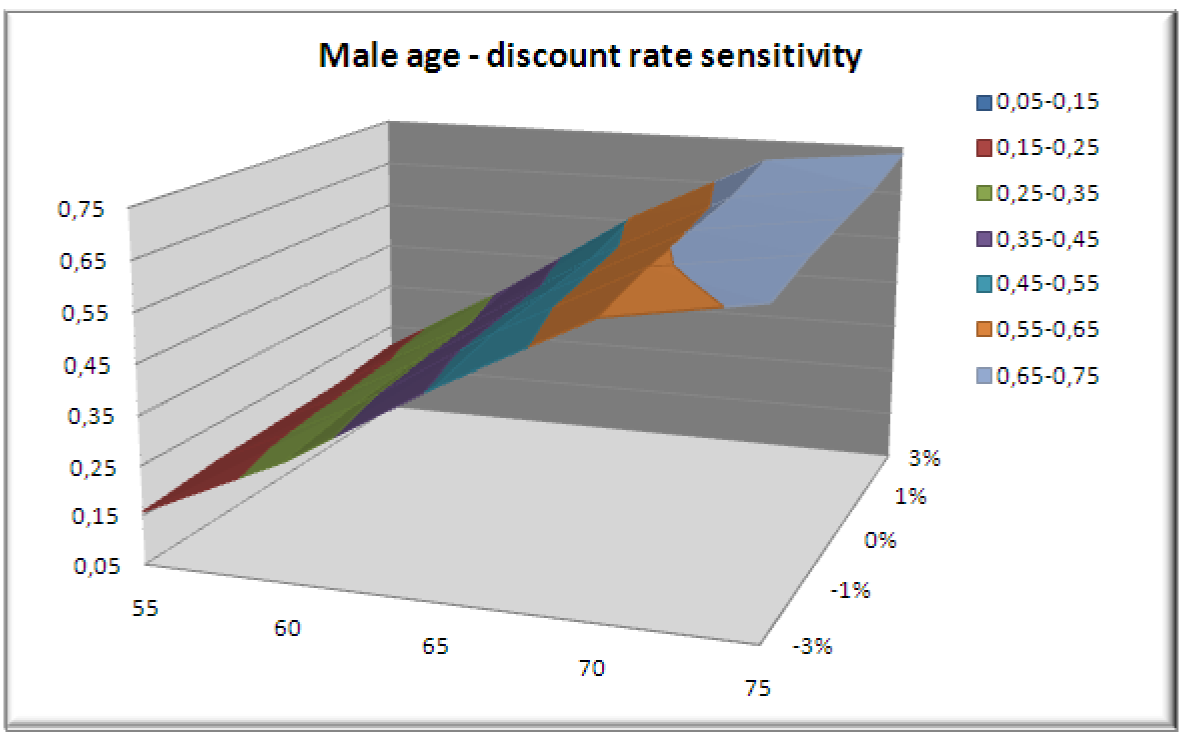


The price of the option grows in both parameters in female case, as this is not the case when a male individual purchases the option at 75 years, we have to look after the purposes. Before drawing any conclusions let us have a look at our third pair of parameters.





Like in the female cases above everything looks fine.



It can be seen, that alteration of mortality and discount rates cause no problem. The real reason is the third parameter. As mentioned, when the age of exercise is high – now 75 – expected remaining lifetime is very low as it decreases exponentially with aging. This has an effect on the male case because of the strong exponentiality of mortality rates.

5 Conclusion

We have proposed a model for pricing options on future mortality (and interest) rates. These options currently exist in the market, but very little has been written on how they should be priced or be reserved against. These options 'pay off' if life annuity (pure endowment) prices end up at a higher than some pre-specified strike price, at the time the contract was issued. We have presented a discrete time model and after having demonstrated how to hedge these options using pure endowments, default free bonds and life insurance contracts, we applied these knowledges to the Hungarian mortality table. Our results showed that 3,3% in case of male and 1% in case of female, respectively, of the annuity price is the price of this embedded option. In case of male, as it was expected, this amount is quite substantial and is likely to draw attention of annuity providers to further investigate mortality contingent claims. An even more realistic case could be examined such as stochastic interest rate and continuous time.

6 References

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7 Appendices

7.1 Forecast of the development of the hungarian mortality table

Calculation of the option price, in the female case, with age 62 to purchase, with sensitivity analysis, for this age

```
qx ←read.csv2("C:/femalemort.csv")
    # read male mortality rates
px←1-qx
    # survival rates
```

7.1.1 Loglinear regression

```
lnqx←log(qx)
z←seq(1,58)
a←rep(0,58)
for(i in 1:58) a[i]←cov(x,lnqx[,i])/var(x,x)
b←rep(0,58)
for(i in 1:58) b[i]←mean(lnqx[,i])-a[i]* mean(x)
A←cov(z,a)/var(z,z)
B←mean(a)-A*mean(z)
C←cov(z,b)/var(z,z)
D←mean(b)-C*mean(z)
e← matrix(0, nrow=101, ncol=60)
qx←cbind(qx,e)
for(i in 1:101) for(j in 59:118) if (exp((A*j+B)*x[i]+(C*j+D))>1) qx[i,j]=1
+ else qx[i,j]=exp((A*j+B)*x[i]+(C*j+D))
```

7.1.2 Generation mortality table

```
rp62← rep(0,38)
for(i in 1:38) rp62[i]← 1-qx[62+i,57+i]
    # survival probabilities
rl62← rep(0,38)
rl62[1]← 1
for(i in 2:38) rl62[i]← prod(rp62[1:i-1])
```

probability of reaching age i assuming to be alive at 62
```
ERL62← sum(rl62[])
```
expected remaining lifetime at 62

7.1.3 Monte Carlo simulation

```
qxs← matrix(0, nrow=101, ncol=1000)
```
initial matrix to fill up with simulated mortality rates
```
for(i in 1:63) for(j in 1:1000) qxs[i,j]← qx[i,58]
```
$q_n^r = q_n^{2006}$, when $n \leq 62$
```
z← seq(1,57)

ma← rep(0,57)
for(i in 1:57) ma[i]← mean(a[1:(i+1)])
mb← rep(0,57)
for(i in 1:57) mb[i]← mean(b[1:(i+1)])
```
expected value
```
s11← rep(0,57)
for(i in 1:57) s11[i]← cov(a[1: (i+1)], a[1: (i+1)])
s12<- rep(0,57)
for(i in 1:57) s12[i]← cov(a[1: (i+1)], b[1: (i+1)])
s21<- rep(0,57)
for(i in 1:57) s21[i]← cov(b[1: (i+1)], a[1: (i+1)])
s22<- rep(0,57)
for(i in 1:57) s22[i]← cov(b[1: (i+1)], b[1: (i+1)])
```
kovariance matrix
```
k← 1
while (k<1001)
{
y1← rnorm(57,0,1)
y2← rnorm(57,0,1)
as← rep(0,57)
for(i in 1:57) as[i]← s11[i]*y1[i]+s12[i]*y2[i]+ma[i]
bs<- rep(0,57)
for(i in 1:57) bs[i]← s21[i]*y1[i]+s22[i]*y2[i]+mb[i]
```

```
As← cov(z,as)/var(z,z)
Bs← mean(as)-A*mean(z)
Cs← cov(z,bs)/var(z,z)
Ds← mean(bs)-C*mean(z)
for(i in 64:101) if (exp((As*(i+2)+Bs)*x[i]+(Cs*(i+2) +Ds))>1) qxs[i,k]=1
+ else qxs[i,k]=exp((As*(i+2) +Bs)*x[i]+(Cs*(i+2) +Ds))
k← k+1
}
```

7.1.4 Sensitivity analysis

```
Dr ← read.csv2("C:/discrates.csv")
    # read discount rates
z← seq(2006,2125)
    # z=vector of calendar years
Dx← Dr[1:38,1]
s1← c(0.7,0.9,1,1.1,1.3)
    # factors that change mortality rates
s2← c(0.97,0.99,1,1.01,1.03)
    # factors that change discount rates
V← matrix(0,nrow=5,ncol=5)
    # matrix for proof of theorem
P← matrix(0,nrow=5,ncol=5)
    # option price in different cases

for (n in 1:5) {
qxsn←s1[n]*qxs
qxn←s1[n]*qx

for (l in 1:5) {
Dxl←s2[l]*Dx
pxkn← matrix(0, nrow=38, ncol=1000)
    # initial matrix to fill up with p_32(30,n) values
for(i in 1:38) for(j in 1:1000)
pxkn[i,j]←prod(1-qxsn[63:(i+61),j])
    # p_32(30,n) = ∏_{i=30}^{n-1} p_{32+i}, n = 31...68
```

```
lambda←matrix(0, nrow=38, ncol=1000)
```

initial matrix to fill up with $\Lambda_{32}(30,n)$ values

```
for(i in 1:38) for(j in 1:1000) lambda[i,j]←Dxl[i]*pxkn[i,j]
```

#$\Lambda_{32}(30,n) = D(30,n)\,p_{32}(30,n)\,n = 31...68$

```
Lambda←prod(1-qxn[33:62,58])
```

strike price $\Lambda = \prod_{i=32}^{61} p_i^{58}$

```
maxl←matrix(0, nrow=38, ncol=1000)
for(i in 1:38) for(j in 1:1000)
maxl[i,j]<-max(lambda[i,j]-Lambda,0)
```

#$max(\Lambda_{32}(30,n) - \Lambda, 0)\,n = 31...68$

```
Cx← matrix(0, nrow=38, ncol=1000)
for(i in 1:38) for(j in 1:1000) Cx[i,j]←Lambda*maxl[i,j]
```

#$C_{32}(30,n|\Lambda) = \prod_{i=1}^{30} \Lambda(i-1,i)\,max(\Lambda_{32}(30,n) - \Lambda, 0),\ n = 31...68$

```
C←rep(0,1000)
for(j in 1:1000) C[j]←sum(Cx[,j])
V1←lambda-Cx
```

proof of theorem

```
for(i in 1:38) for(j in 1:1000) if(V1[i,j]<0) V1[i,j]←1 else V1[i,j]←0
V[n,l]←sum(V1[])
P[n,l]←mean(C[])
}
}
P
```

#option price in 25 different cases