Eötvös Loránd Tudományegyetem

Természettudományi Kar

Huong Dang Thu

Growth estimates of the solutions of linear systems in term of coefficients

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Témavezető:

Dr. Kovács Sándor

Numerikus Analízis Tanszék

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1 Systems of linear differential equation

Let I be the interval $\mathrm{I} \subset \mathbb{R}$, A be a $n \times n$ square matrix function defined on I with value in $\mathbb{C}^{n\times n}$ and $f : \mathrm{I} \to \mathbb{C}^n$ are function. Now we deal with the solution x which is an $n \times 1$ vector of functions of an underlying variable t in the following differential equation :

$$\dot{x}(t) = \mathrm{A}(t)x(t) + f(t) \quad (t \in \mathrm{I})\,. \tag{1.1}$$

1.1 Existence and uniqueness of solutions

In this section, we assumed that $f(t) = 0 \ \ (\forall t \in \mathrm{I})$ and the complex $n \times n$ matrix $\mathrm{A}(t)$ is continuous on an interval I. The linear system (1.1) becomes:

$$\dot{x}(t) = \mathrm{A}(t)x(t) \quad (t \in \mathrm{I})\,. \tag{1.2}$$

1.1. Definition. *A set of n linearly independent solutions: $x_1(t)$, $x_2(t)$,...., $x_n(t)$ of* (1.2) *is called a fundamental system of solution, and* $\mathrm{X}(t) = (x_1(t), x_2(t), ..., x_n(t))$ *whose columns are the vector of basis is called fundamental matrix.*

$$\Rightarrow \frac{d\mathrm{X(t)}}{dt} = \mathrm{A}(t)X(t). \tag{1.3}$$

1.1. Corollary. *Any nonsingular solution of equation* (1.3) *is a fundamental matrix of system* (1.2). *If one fundamental matrices* $\mathrm{X}_1(t)$ *is known, the complete set of fundamental matrices has the form*

$$\mathrm{X}(t) = \mathrm{X}_1(t)\mathrm{C}$$

where C *is a constant nonsingular* $n \times n$ *matrix.*

1.2. Definition. *A fundamental matrix* $\mathrm{X}(t)$ *is said to be normalized at a point* $t_0 \in \mathrm{I}$ *if* $\mathrm{X}(t_0) = \mathrm{E}$. *Then such a matrix is writen as* $\mathrm{X}(t) = \mathrm{X}(t, t_0)$.

If $\mathrm{X}(t)$ is a fundamental matrix of system (1.3) , a general solution of the linear homogeneous system (which means $f(t) \equiv 0$) is $x(t) = \mathrm{X}(t)\mathrm{C}$, where $\mathrm{C} \in \mathbb{C}^n$. and, all solution of non-homogeneous system is

$$x(t) = \mathrm{X}(t)\mathrm{C} + \mathrm{X}(t)\int \mathrm{X}^{-1}(\tau)f(\tau)d\tau \quad \forall t \in \mathrm{I}.$$

The solution of the Cauchy problem (1.2) with initial data $(t_0, x_0) \in \mathrm{I} \times \mathbb{C}^n$ of homogeneous system is written as

$$x(t) = \mathrm{X}(t)\mathrm{X}^{-1}(t_0)x_0 = \mathrm{X}(t, t_0)x_0$$

and, for non-homogeneous system as

$$\begin{aligned} x(t) &= \mathrm{X}(t)\mathrm{X}^{-1}(t_0)x_0 + \int_{t_0}^{t} \mathrm{X}(t)\mathrm{X}^{-1}(\tau)f(\tau)d\tau \\ &= \mathrm{X}(t, t_0)x_0 + \int_{t_0}^{t} \mathrm{X}(t, \tau)f(\tau)d\tau \end{aligned}$$

The matrix $\mathrm{X}(t, \tau)$ where $t, \tau \in \mathrm{I}$ is called the Cauchy matrix of system (1.2)

1.1.1 Existences of solution

Return the system (1.2) and (1.3) with the initial conditions $\mathrm{X}(t_0) = \mathrm{E}$. We shall look for a solution to homogeneous system by Picard's method of successive approximations:

$$\mathrm{X}_0(t) = \mathrm{E},$$

$$\mathrm{X}_k(t) = \mathrm{E} + \int_{t_0}^{t} \mathrm{A}(u)\mathrm{X}_{k-1}(u)du. \qquad (k = 1, 2.....) \tag{1.4}$$

or

$$\begin{aligned} \mathrm{X}_k(t) = \mathrm{E} &+ \int_{t_0}^{t} \mathrm{A}(u)du + \int_{t_0}^{t} \mathrm{A}(t_1)dt_1 \int_{t_0}^{t_1} \mathrm{A}(t_2)dt_2 + ... \\ &+ \int_{t_0}^{t} \mathrm{A}(t_1)dt_1 \int_{t_0}^{t_1} \mathrm{A}(t_2)dt_2 ... \int_{t_0}^{t_{k-1}} \mathrm{A}(t_k)dt_k. \end{aligned} \tag{1.5}$$

Let us show that the sequence (1.4) converges absolutely for $t \in \mathrm{I}$ and that this convergence is uniform in any closed interval. Indeed, the convergence of the sequence (1.4) is equivalent to the convergence of the series

$$\mathrm{X}_0(t) + [\mathrm{X}_1(t) - \mathrm{X}_0(t)] + [\mathrm{X}_2(t) - \mathrm{X}_1(t)] + ... \tag{1.6}$$

To prove this, we shall show that for any $t \in \mathrm{I}$

$$\|\mathrm{X}_k(t) - \mathrm{X}_{k-1}(t)\| \leq \frac{1}{k!} \left(\left| \int_{t_0}^{t} \|\mathrm{A}(u)\| du \right| \right)^k (k = 1, 2, ...) \tag{1.7}$$

For k=1,

$$\|\mathrm{X}_1(t) - \mathrm{X}_0(t)\| \leq \left|\int_{t_0}^{t} \|\mathrm{A}(u)\| du\right|.$$

so that (1.7) is true .

Assuming now that inequality (1.5) holds for all $k \leq m$. We need to prove it in case of $k = m + 1$

$$\begin{aligned}
\|\mathrm{X}_{m+1}(t) - \mathrm{X}_m(t)\| &= \left|\int_{t_0}^{t} \mathrm{A}(t_1)dt_1 \int_{t_0}^{t_1} \mathrm{A}(t_2)dt_2 ... \int_{t_0}^{t_m} \mathrm{A}(t_{m+1})dt_{m+1}\right| \\
&\leq \left|\int_{t_0}^{t} \|\mathrm{A}(t_1)\| dt_1 \frac{1}{m!} \left(\left|\int_{t_0}^{t_1} \|\mathrm{A}(u)\| du\right|\right)^m\right| \\
&= \frac{1}{(m+1)!}\left(\left|\int_{t_0}^{t} \|\mathrm{A}(u)\| du\right|\right)^{m+1}
\end{aligned}$$

Thus inequality (1.7) is true for any k by induction, and this proves our assertion. Indeed, this series (1.6) is majorized by the convergent series (see (1.7))

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\left|\int_{t_0}^{t} \|\mathrm{A}(u)\| du\right|\right)^k = \exp\left|\int_{t_0}^{t} \|\mathrm{A}(u)\| du\right|; \tag{1.8}$$

this implies its convergence . Denoting the sum of series (1.6) by $\Omega_{\mathrm{A}}(t, t_0)$, we obtain

$$\Omega_{\mathrm{A}}(t, t_0) = \mathrm{E} + \int_{t_0}^{t} \mathrm{A}(u)du + \sum_{k=2}^{\infty}\left(\int_{t_0}^{t} \mathrm{A}(t_1)dt_1 \int_{t_0}^{t_1} \mathrm{A}(t_2)dt_2 ... \int_{t_0}^{t_{k-1}} \mathrm{A}(t_k)dt_k\right). \tag{1.9}$$

Differentiating the series (1.9) term by term, we obtain

$$\frac{d\Omega_{\mathrm{A}}(t, t_0)}{dt} = \mathrm{A}(t)\Omega_{\mathrm{A}}(t, t_0).$$

The matrix $\Omega_{\mathrm{A}}(t, t_0)$,i.e, the solution of problem (1.3) is called the *matriciant* of system (1.2).This is the fundamental matrix normalized at $t = t_0$ and connected with any other fundamental matrix $\mathrm{X}(t)$ in the following way:

$$\Omega_{\mathrm{A}}(t, t_0) = \mathrm{X}(t)\mathrm{X}^{-1}(t_0) = \mathrm{X}(t, t_0).$$

Now we note some properties of the matriciant.

1. $\|\Omega_{\mathrm{A}}(t, t_0)\| \leq \exp|\int_{t_0}^{t} \|\mathrm{A}(\mathrm{u})\| du|$ (this follows from (1.8)).

2. In the case of the constant matrix A we have

$$\Omega_{\mathrm{A}}(t,t_0) = \exp \mathrm{A}(t-t_0),$$

which immediately follows from (1.9), since the series (1.9) is nothing else but the expansion of $\exp \mathrm{A}(t-t_0)$ [5, 6].

3. $\Omega_{\mathrm{A}}(t,t_0) = \Omega_{\mathrm{A}}(t,t_1)\ \Omega_{\mathrm{A}}(t_1,t_0),\ \ \forall t_0, t_1, t \in \mathrm{I}$

1.1.2 Uniqueness of solution

1.1. Lemma. (Bellman-Gronwall) *Let* $\mathrm{I} = [t_0, \infty)$*. Let funtion* $u(t) \geq 0$*,* $v(t) \geq 0$ *be real-valued continuous function for* $t \geq t_0$ *and*

$$u(t) \leq \lambda + \int_{t_0}^{t} u(\tau)v(\tau)d\tau,$$

where $\lambda \in \mathbb{R}_+$ *. Then for* $t \geq t_0$ *we have*

$$u(t) \leq \lambda e^{\int_{t_0}^{t} v(\tau)d\tau} \tag{1.10}$$

Proof. Denote $g(t) = \lambda + \int_{t_0}^{t} u(\tau)v(\tau)d\tau$ be real-valued differential function defined on I and from the hypothesis, we get $g(t) \geq u(t) \forall t \in \mathrm{I}$; hence

$$\dot{g}(t) = u(t)v(t) \leq g(t)v(t),$$

or $\dot{g}(t) - g(t)v(t) \leq 0$. Multiplying the last inequality by $e^{-\int_{t_0}^{t} v(\tau)d\tau}$, we have

$$\frac{d\left[g(t)e^{-\int_{t_0}^{t} v(\tau)d\tau}\right]}{dt} \leq 0.$$

Integrating the result t_0 to t , we get

$$g(t)e^{-\int_{t_0}^{t} v(\tau)d\tau} - g(t_0) \leq 0,$$

or

$$g(t) \leq \lambda e^{\int_{t_0}^{t} v(\tau)d\tau}.$$

Taking into account that $u(t) \leq g(t)$, we obtain (1.10)

□

We know that the solution of system (1.2) has the form:

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{A}(\tau)x(\tau)d\tau, \quad t \in \mathrm{I}.$$

Hence,

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^{t} |\mathrm{A}(\tau)||x(\tau)|d\tau.$$

By the lemma, we have

$$|x(t)| \leq |x(t_0)|e^{\int_{t_0}^{t} |\mathrm{A}(\tau)|d\tau}. \tag{1.11}$$

If $x_1(t)$ and $x_2(t)$ are solution satisfying the same initial conditions, then $x(t) = x_1(t) - x_2(t)$ is also a solution and $x(0) = 0$. It follows from (1.11) that $x(t) = 0$, proving the uniqueness of the solution.

1.2 Linear systems with constant coefficients

Let $x : \mathrm{I} \to \mathbb{C}^n$ where $\mathrm{I} = [0, +\infty)$. Consider a system

$$\dot{x}(t) = \mathrm{A}x(t).$$

with a constant matrix A,i.e, an autonomous system. We showed that

$$\Omega_{\mathrm{A}}(t, 0) = \mathrm{X}(t, 0) = e^{\mathrm{A}t}.$$

Let us find the structure of this fundamental matrix . Let S be the matrix transforming A to its Jordan canonical form, i.e.,

$$\mathrm{B} = \mathrm{S}^{-1}\mathrm{AS} = \mathrm{diag}[\mathrm{J}_{\rho_1}(\lambda_1), ..., \mathrm{J}_{\rho_k}(\lambda_k)],$$

where $k \leq n$, $\sum_1^k \rho_i = n$, λ_i are the eigenvalues of the matrix A, and $\mathrm{J}_v(\lambda)$ is a $v \times v$ Jordan block.

$$\mathrm{J}_v(\lambda) = \begin{bmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \lambda & \cdots & \cdots & \vdots \\ 0 & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda \end{bmatrix}$$

Let us find out what the diagonal blocks of the matrix are,

$$e^{\mathrm{J}_v(\lambda)t} = e^{\mathrm{E}_v\lambda t + \mathrm{J}_v(0)t} = e^{\mathrm{E}_v\lambda t}e^{\mathrm{J}_v(0)t}$$
$$= e^{\lambda t}e^{\mathrm{J}_v(0)t} = e^{\lambda t}\sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{J}_v^k(0)t^k$$

where $\mathrm{J}_v(0)$ is a nilpotent matrix; thus, $\exp(\mathrm{J}_v(0)t)$ is a finite sum of v terms and can be easily calculated:

$$e^{\mathrm{J}_v(\lambda)t} = e^{\lambda t}.\begin{bmatrix} 1 & 0 & \cdots & 0 \\ t & 1 & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ \frac{t^v-1}{(v-1)!} & \cdots & t & 1 \end{bmatrix}$$

Now returning to our problems, we have

$$e^{\mathrm{A}t} = e^{\mathrm{SBS}^{-1}t} = \mathrm{S}e^{\mathrm{B}t}\mathrm{S}^{-1}.$$

and

$$e^{\mathrm{B}t} = \mathrm{diag}[e^{\mathrm{J}_{\rho_1}(\lambda_1)t}, ..., e^{\mathrm{J}_{\rho_k}(\lambda)t}].$$

If the matrix A is complex, then our problem is solved. If the matrix A is real, then $\exp(\mathrm{A}t)$ is real, which follows from the construction of the matriciant. If all the eigenvalues of the matrix A are real, then the matrices S and B are real and the expression for $e^{\mathrm{A}t} = \mathrm{S}e^{\mathrm{B}t}\mathrm{S}^{-1}$ does not need further explanations. On the other hand, when there are complex eigenvalues of the matrix A, then the matrices S and B are complex, and the expression for $\exp(\mathrm{A}t)$, the latter being real, does need further clarification. Let the matrix A be real and let the Jordan block $\mathrm{J}_{\rho_j}(\lambda_j)$ of order v correspond to its eigenvalue $\lambda_j = \alpha + i\beta$ in order to get

$$\mathrm{B} = \mathrm{S}^{-1}\mathrm{AS} = \mathrm{diag}[\mathrm{J}_{\rho_1}(\lambda_1), \ldots, \mathrm{J}_{\rho_k}(\lambda_k)].$$

Since A is real, there exists an eigenvalue $\lambda_{j+1} = \bar{\lambda}_j$ with the corresponding Jordan block

$$\mathrm{J}_{\rho_{j+1}}(\lambda_{j+1}) = \bar{\mathrm{J}}_{\rho_j}(\lambda_j).$$

Let us consider the diagonal element of the matrix B of dimension $2v \times 2v$ containing these blocks ($v = \rho_j = \rho_{j+1}$; the index of λ is omitted)

$$\mathrm{B}_{2v} = \begin{bmatrix} \mathrm{J}_\rho(\alpha + i\beta) & 0 \\ 0 & \mathrm{J}_\rho(\alpha - i\beta) \end{bmatrix} = \begin{bmatrix} \mathrm{J}_\rho(\alpha) + i\beta\mathrm{E}_v & 0 \\ 0 & \mathrm{J}_\rho(\alpha) - i\beta\mathrm{E}_v \end{bmatrix}$$

and carry out a special similarity transformation of the matrix B_{2v} reducing it to a real form. This standard transformation has the following form:

$$\mathrm{S}_{2v} = \begin{bmatrix} \mathrm{E}_v & i\mathrm{E}_v \\ \mathrm{E}_v & -i\mathrm{E}_v \end{bmatrix}, \quad \mathrm{S}_{2v}^{-1} = \frac{1}{2}\begin{bmatrix} \mathrm{E}_v & \mathrm{E}_v \\ -i\mathrm{E}_v & i\mathrm{E}_v \end{bmatrix}.$$

Thus, we have

$$\tilde{\mathrm{B}}_{2v}(\lambda) = \mathrm{S}_{2v}^{-1}\mathrm{B}_{2v}\mathrm{S}_{2v} = \begin{bmatrix} \mathrm{J}_\rho(\alpha) & -\beta\mathrm{E}_v \\ \beta\mathrm{E}_v & \mathrm{J}_\rho(\alpha) \end{bmatrix}.$$

This matrix $\tilde{\mathrm{B}}_{2v}$ is called the real canonical form of the matrix B_{2v}. By means of the similarity transformation with the matrix S, we reduce the matrix A to the diagonal form; then to each pair of complex-conjugate Jordan blocks (there are no other complex blocks) we apply the transformation of the form S_{2v} and finally obtain the real matrix

$$\tilde{\mathrm{B}} = \mathrm{diag}[\mathrm{J}_{\rho_1}(\alpha_1), \ldots, \mathrm{J}_{\rho_m}(\alpha_m), \tilde{\mathrm{B}}_{2v_{m+1}}(\lambda_{m+1}), \ldots, \tilde{\mathrm{B}}_{2v_{k-1}}(\lambda_{k-1})],$$

where the first m blocks correspond to real eigenvalues of the matrix A and the subsequent blocks correspond to the complex ones. The matrix $\tilde{\mathrm{B}}$ is similar to the matrix A and is obtained from it by means of a repeated complex transformation. Can we choose this transformation so that it is real?

1.2. Lemma. *If matrices* A *and* B *are real and there exists a complex matrix* S *such that*

$$\mathrm{B} = \mathrm{S}^{-1}\mathrm{A}\mathrm{S},$$

then there exists a real matrix S_0 *realizing this equality.*

Proof. Let $\mathrm{S} = \mathrm{S}_1 + \mathrm{S}_2$, where S_1 and S_2 are real. From AS = SB we have $\mathrm{A}(\mathrm{S}_1 + i\mathrm{S}_2) = (\mathrm{S}_1 + i\mathrm{S}_2)\mathrm{B}$ or $\mathrm{A}\mathrm{S}_1 = \mathrm{S}_1\mathrm{B}$, $\mathrm{A}\mathrm{S}_2 = \mathrm{S}_2\mathrm{B}$. The matrices S_i, $i = 1, 2$, are real but they ay be singular. We continue with the proof. We multiply the last equality by a number ρ and add it to the preceding one:

$$\mathrm{A}(\mathrm{S}_1 + \rho \mathrm{S}_2) = (\mathrm{S}_1 + \rho \mathrm{S}_2)\mathrm{B}.$$

Note that $\det(\mathrm{S}_1 + \rho \mathrm{S}_2)$ is a polynomial in ρ of degree at most n with real coefficients not equal simultaneously to zero, since $\det(\mathrm{S}_1 + i\mathrm{S}_2) \neq 0$. Hence, there exists a $\rho_0 \in \mathbb{R}$ such that the matrix $\mathrm{S} = \mathrm{S}_1 + \rho \mathrm{S}_2$ is nondegenerate.

□

It follows from the above Lemma that there exists a real matrix S_0 such that $\tilde{\mathrm{B}} = \mathrm{S}_0^{-1}\mathrm{A}\mathrm{S}_0$; therefore, we have

$$\begin{aligned} e^{\mathrm{A}t} &= \mathrm{S}_0 e^{\tilde{\mathrm{B}}t}\mathrm{S}_0^{-1} \\ &= \mathrm{S}_0 \operatorname{diag}[e^{\mathrm{J}_{\rho_1}(\alpha_1)t}, \ldots, e^{\mathrm{J}_{\rho_m}(\alpha_m)t}, e^{\tilde{\mathrm{B}}_{2v_{m+1}}(\lambda_{m+1})t}, \ldots, e^{\tilde{\mathrm{B}}_{2v_{k-1}}(\lambda_{k-1})t}]\mathrm{S}_0^{-1}. \end{aligned}$$

The form of the first m blocks on the diagonal is given previously. Now we consider the other ones:

$$\begin{aligned} e^{\tilde{\mathrm{B}}_{2v}(\lambda)t} &= \mathrm{S}_{2v}^{-1} e^{\mathrm{B}_{2v}(\lambda)t}\mathrm{S}_{2v} = \mathrm{S}_{2v}^{-1}\begin{bmatrix} e^{\mathrm{J}_v(\alpha)t}e^{i\beta t} & 0 \\ 0 & e^{\mathrm{J}_v(\alpha)t}e^{-i\beta t}\end{bmatrix}\mathrm{S}_{2v} \\ &= \operatorname{diag}\left[e^{\mathrm{J}_v(\alpha)t}, e^{\mathrm{J}_v(\alpha)t}\right]\begin{bmatrix} \mathrm{E}_v \cos\beta t & -\mathrm{E}_v \sin\beta t \\ \mathrm{E}_v \sin\beta t & \mathrm{E}_v\cos\beta t\end{bmatrix}. \end{aligned}$$

Thus we have finally found the form of the fundamental matrix $e^{\mathrm{A}t}$ of system $\dot{x} = \mathrm{A}x$ in relation to Jordan canonical form of the matrix A. Thus,

$$\|e^{\mathrm{A}t}\| \leq \|\mathrm{S}\|.\|\mathrm{S}^{-1}\|.\|e^{\mathrm{B}t}\| \leq \|\mathrm{S}\|.\|\mathrm{S}^{-1}\| \max_{1\leq j\leq k} \|e^{\mathrm{J}_{\rho_j}(\lambda_j)t}\|.$$

Let

$$\alpha = \max_j \mathrm{Re}\lambda_j, \forall j = 1, ..., k.$$

Then

$$\|e^{\mathrm{J}_{\rho_j}(\lambda_j)t}\| \leq e^{\alpha t}\|e^{\mathrm{J}_{\rho_j}(0)t}\|.$$

The matrix $e^{\mathrm{J}_{\rho_j}(0)t}$ contains powers of t and for $t \geq 0$ can be estimated in the following two ways :

a, for any $\epsilon > 0$ there exists a constant C_ϵ such that

$$\|e^{\mathrm{J}_{\rho_j}(0)t}\| \leq C_\epsilon e^{\epsilon t} \qquad j = 1 \ldots k.$$

Indeed,

$$\|e^{\mathrm{J}_{\rho_j}(0)t}\| = \|e^{\mathrm{J}_{\rho_j}(0)t}\| e^{-\epsilon t} e^{\epsilon t}.$$

Thus, choosing

$$C_\epsilon = \max_{\mathbb{R}_+} \|e^{\mathrm{J}_{\rho_j}(0)t}\| e^{-\epsilon t}$$

satisfies this previous inequality.

b,

$$\|e^{\mathrm{J}_{\rho_j}(0)t}\| \leq D(1+t)^{\rho_j - 1} \qquad j = 1 \ldots k.$$

Indeed, the power of t on the left-hand side of the last inequality does not exceed $\rho_j - 1$, the constant $D \geq 1$ depends on the choice of the norm, and on the right hand side of this inequality, there is the expression $(1+t)$, which guarantees the validity of the inequality for $t = 0$. Therefore, the estimates a, and b, generate the following estimates :

1.
$$\|e^{\mathrm{A}t}\| \leq C_\epsilon e^{(\alpha+\epsilon)t}, \qquad t \geq 0,$$

2.
$$\|e^{\mathrm{A}t}\| \leq K(1+t)^{n-1} e^{\alpha t}, \qquad t \geq 0.$$

In the last estimate the number n may be replaced with the maximal order of Jordan blocks. Note that in the case of simple eigenvalues of the matrix A the estimate acquires the following form:

$$\|e^{\mathrm{A}t}\| \leq M e^{\alpha t}, \qquad t \geq 0.$$

1.2.1 Example

1.1. Example.

$$\dot{x}_1 = x_1 + 3x_2$$

$$\dot{x}_2 = 3x_1 - 2x_2 - x_3$$

$$\dot{x}_3 = -x_2 + x_3$$

Here the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

The eigenvalues of the matrix A *are* $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -4$. *The matrix* S *consists of the eigenvectors of the matrix* A. *Hence*

$$S = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 2 & -5 \\ 3 & -1 & -1 \end{pmatrix}, \qquad S^{-1} = \frac{1}{70}\begin{pmatrix} 7 & 0 & 21 \\ 15 & 10 & -5 \\ 6 & -10 & -2 \end{pmatrix}.$$

Thus,

$$B = S^{-1}AS = \operatorname{diag}[1, 3, -4].$$

Therefore, we have

$$X(t,0) = e^{At} = \frac{1}{70}\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{-4t} \end{pmatrix}\begin{pmatrix} 7 & 0 & 21 \\ 15 & 10 & -5 \\ 6 & -10 & -2 \end{pmatrix}.$$

Finally, we have

$$\|e^{At}\|_I \leq \|S\|_I \|S^{-1}\|_I e^{Bt}\|_I = \frac{7 \cdot 30}{70} e^{3t} = 3e^{3t}.$$

1.2. Example.

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = -x_1 + 2x_2 + x_3$$

$$\dot{x}_3 = x_1 + x_3$$

Here the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

The eigenvalues of the matrix A *are* $\lambda_1 = 2$, $\lambda_{2,3} = 1 \pm i$. *The matrix* S *consists of the eigenvectors of the matrix* A *and has the following form:*

$$S = \begin{pmatrix} 1 & i & -i \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad S^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 2 \\ -2i & -1+i & 1+i \\ 2i & -1-i & 1-i \end{pmatrix}.$$

Thus,

$$\mathrm{B} = \mathrm{S}^{-1}\mathrm{A}\mathrm{S} = \mathrm{diag}\,[2, 1+i, 1-i]\,.$$

This matrix has two complex conjugate Jordan blocks of order one, i.e, $\rho = 1$, *and*

$$\mathrm{S}_{2\rho} = \mathrm{S}_2 = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \qquad \mathrm{S}_2^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

Thus we have

$$\tilde{\mathrm{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{-i}{2} & \frac{-i}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 1 & -i \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

By virtue of our notation, $\tilde{\mathrm{B}} = \mathrm{S}_0^{-1}\mathrm{A}\mathrm{S}_0$. *Then*

$$\mathrm{S}_0 = \mathrm{S}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 1 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -2 & 0 \\ 1 & 2 & 0 \end{pmatrix}.$$

$$\mathrm{S}_0^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 2 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}.$$

Therefore, we have

$$e^{\mathrm{A}t} = \mathrm{S}_0 e^{\tilde{\mathrm{B}}t}\mathrm{S}_0^{-1} = \mathrm{S}_0 \begin{pmatrix} e^2 t & 0 & 0 \\ 0 & e^t\cos t & -e^t \sin t \\ 0 & e^t \sin t & e^t \cos t \end{pmatrix} \mathrm{S}_0^{-1},$$

Finally, we have

$$\|e^{\mathrm{A}t}\|_I \leq \|\mathrm{S}_0\|_I \|\mathrm{S}_0^{-1}\|_I e^{\tilde{\mathrm{B}}t}\|_I = \frac{3}{4}(2+2\sqrt{2})e^{2t} = \frac{3}{2}(1+\sqrt{2})e^{2t}.$$

1.3 Homogeneous system with periodic coefficient

We shall consider systems (1.2) with the assumption:

$$\mathrm{A}(t+\omega) = \mathrm{A}(t) \quad \text{for} \quad \omega > 0, t \in \mathbb{R}. \tag{1.12}$$

1.3.1 Monodromy matrix

Note some properties of fundamental matrices of these systems.

1. If $\mathrm{X}(t)$ is a fundamental matrix, $\mathrm{X}(t+\omega)$ is also fundamental matrix. Because:

$$\frac{d\mathrm{X}(t)}{dt} = \mathrm{A}(t)\mathrm{X}(t), \quad t \in \mathbb{R},$$

   Hence,

$$\frac{d\mathrm{X}(t+\omega)}{dt} = \mathrm{A}(t+\omega)\mathrm{X}(t+\omega) = \mathrm{A}(t)\mathrm{X}(t+\omega).$$

2. There exists B non-singular matrix satifying:

$$\mathrm{X}(t+\omega) = \mathrm{X}(t)\mathrm{B}$$

   Since the matrices $\mathrm{X}(t)$, $\mathrm{X}(t+\omega)$ are fundamental matrix.

3.

$$\mathrm{X}(t+m\omega) = \mathrm{X}(t)\mathrm{B}^m$$

   Using the fundamental matrix normalized at $t = 0$. By the above property, $\Omega_{\mathrm{A}}(t+\omega, 0) = \Omega_{\mathrm{A}}(t, 0)$ B or

$$\Omega_{\mathrm{A}}(t+\omega, 0) = \mathrm{B} \qquad \text{for } t = 0$$

   This matrix B is call monodromy matrix.

1.3.2 Multiplier

1.3. Definition. *The eigenvalues of the monodromy matrix are multipliers of* (1.2) *and the characteristic equation is* $det[\mathrm{B} - \rho\mathrm{E}] = 0$.

1.1. Remark. *From Ostrograkri Jacobi-Lioville , we have* $\det\ \mathrm{B} = \exp\ \int_0^\omega Sp(\mathrm{A}(u))du$. *It follows that no multiplier is zero.*

1.1. Theorem. *A number ρ is a multiplier of system* (1.2) $\Longleftrightarrow$ *There exists a nontrivial solution $x(t)$ of this system such that*

$$x(t+\omega) = \rho x(t) \tag{1.13}$$

1.4. Definition. *A solution $x(t)$ satisfying* (1.13) *is called normal.*

Proof. *Necessity.* If ρ is a multiplier of (1.2), ρ is an eigenvalue of monodromy matrix $\mathrm{X}(\omega)$, where $\mathrm{X}(t) \equiv \Omega_{\mathrm{A}}(t, 0)$. The eigenvector v corresponding to this ρ satisfies the condition $\mathrm{X}(\omega)v = \rho v$. Let us show that the solution $x(t) = \mathrm{X}(t)v$ is required one. Indeed,

$$x(t+\omega) = \mathrm{X}(t+\omega)v = \mathrm{X}(t)\mathrm{X}(\omega)v = \mathrm{X}(t)\rho v = \rho x(t).$$

Sufficiency. Let us show that the number ρ from the equality (1.13) is a multiplier. For the solution $x(t)$ satisfying the equality (1.13) we have

$$x(\omega) = \rho x(0) \qquad \text{for } t = 0$$

Let us rewrite this solution as $x(t) = \mathrm{X}(t)x(0)$; this implies

$$x(\omega) = \mathrm{X}(\omega)x(0) \qquad \text{for } t = \omega$$

From these two equalities it follows that

$$\mathrm{X}(\omega)x(0) = \rho x(0).$$

Since $\|x(0)\| \neq 0$, we obtain that ρ is a root of characteristic equation of $\mathrm{X}(\omega)$ or ρ is a multiplier.

□

1.2. Corollary. *(on the structure of a normal solution) Let us set $\rho = \exp \lambda\omega$ and rewrite a normal solution in the form $x(t) = \exp(\lambda t)\,\varphi(t)$ where $t \in \mathbb{R}$. We show that $\varphi(t+\omega) = \varphi(t)$.Indeed,*

$$\varphi(t+\omega) = e^{-\lambda(t+\omega)}x(t+\omega) = e^{-\lambda(t+\omega)}\rho x(t) = e^{-\lambda(t+\omega)}e^{\lambda\omega}e^{\lambda t}\varphi(t) = \varphi(t), \qquad \forall t \in \mathbb{R}.$$

1.5. Definition. *If the equation*

$$e^{\mathrm{Y}} = \mathrm{X}$$

is valid for two square matrices X *and* Y , *then the matrix* Y *is called the logarithm of the matrix* X *and is denoted by* $\mathrm{Y} = Ln\ \mathrm{X}$

1.3.3 Example

Let us consider the example of the definition of the logarithm of a matrix.

1.3. Example.

$$\mathrm{A} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

the eigenvalues of A *are*

$$\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -4.$$

So we have the logarithms:

$$Ln\mathrm{A} = \mathrm{S}\,\mathrm{diag}\,[0, ln3, ln4 + i\pi]\,\mathrm{S}^{-1}.$$

where S *is consists of eigenvectors, which is determined in the section 1.2 .*

1.3.4 Floquet theorem

1.2. Theorem. *[Floquet]. The matriciant of* (1.2) *with condition in* (1.12) *can be represented in the form*

$$\Omega_{\mathrm{A}}(t,0) = \Phi(t)e^{\Lambda t} \text{ where } \Lambda = \frac{1}{\omega} Ln\ \Omega_{\mathrm{A}}(\omega,0), \Phi(t) = \Phi(t+\omega). \qquad (1.14)$$

Proof.

$$\Omega_{\mathrm{A}}(t,0) = \Omega_{\mathrm{A}}(t,0)e^{-\Lambda t}e^{\Lambda t} \equiv \Phi(t)e^{\Lambda t}.$$

Verifying that $\Phi(t) = \Phi(t+\omega)$. Indeed,

$$\Phi(t+\omega) = \Omega_{\mathrm{A}}(t+\omega,0)e^{-\Lambda(t+\omega)} = \Omega_{\mathrm{A}}(t,0)e^{-\Lambda t}\Omega_{\mathrm{A}}(\omega,0)e^{-\Lambda\omega} = \Phi(t).$$

□

1.2. Remark. *The matrix* $\Lambda = \frac{1}{\omega} Ln\ \Omega_{\mathrm{A}}(\omega,0)$ *and, therefore* $\Phi(t)$ *are, generally speaking, complex. In the case when* $\mathrm{A}(t)$ *is real, and the matrix* Λ *can be chosen real if and only if among the multipliers of the system there are no negative ones, or every elementary divisor corresponding to negative multipliers is of even multiplicity.*

1.3. Remark. *Let eigenvalues* λ_1,λ_2, ..., λ_n *of matrix* Λ *are connected with the multipliers* ρ_1, ρ_2...,ρ_n *in the following way:*

$$\lambda_k = \frac{1}{\omega} Ln\ \rho_k = \frac{1}{\omega}[\ln\ |\rho_k| + i(arg\rho_k + 2m\pi)],\ m = 0, \pm 1, \ldots,\ k = 1, 2, \ldots, n.$$

and the elementary divisors corresponding to λ_k and ρ_k coincide. This and the statements for a multiplier ρ of the system (1.2) the following is valid:

1. If $|\rho| > 1$, then all the corresponding solutions exponentially increase as $t \to \infty$,
2. If $|\rho| < 1$, then all the corresponding solutions exponentially decrease as $t \to \infty$,
3. If $|\rho| = 1$, then, in the case when only simple elementary divisors correspond to ρ, all the solutions are bounded, and there exist solutions growing as powers of t if among the elementary divisors there are multiple ones.

1.4 Definition and main properties of characteristic exponents

Let a complex-valued function $f(t)$ be defined on the interval $[t_0, \infty)$.

1.6. Definition. *The number (or the symbol* $\pm\infty$*) defined as*

$$\chi[f] = \overline{\lim_{t \to \infty}} \frac{1}{t}\ ln|f(t)|$$

is called the characteristic exponent of the function $f(t)$*.*

The characteristic exponent determines the growth of the absolute value of a function with respect to the scale of exponential functions $\exp(\alpha t)$. Obviously, the characteristic exponent for the latter is the number α. For an arbitraiy function $f(t)$ the identity

$$|f(t)| = \exp\left\{\left(\frac{1}{t}\ln|f(t)|\right) t\right\}.$$

is valid; this clarifies the definition given above.

1.4. Example.

$$\chi[t^m] = 0, \qquad \chi[\exp(\pm t \sin(t))] = 1,$$

$$\chi[c \neq 0] = 0, \qquad \chi[\exp(-t \exp\sin(t))] = e^{-1},$$

$$\chi[0] = -\infty, \qquad \chi[\exp(t \exp \sin(t))] = e,$$
$$\chi\left[\exp\left(t \cos \frac{1}{t}\right)\right] = 1, \qquad \chi[t^t] = \infty,$$
$$\chi\left[\exp\left(-t \cos \frac{1}{t}\right)\right] = -1, \qquad \chi[t^{-t}] = -\infty.$$

Now the obvious properties following straightforwardly from the above definition:

1. $\chi[f] = \chi[|f|]$,
2. $\chi[cf] = \chi[f]$, $\quad |c| \neq 0$,
3. if $|f(t)| \leq |\mathrm{F}(t)|$ for $t \geq a$, then $\chi[f] \leq \chi[\mathrm{F}]$.

Let consider the matrix

$$\mathrm{F}(t) = \{f_{ij}(t)\} \qquad i = 1, \dots, n, \quad j = 1, \dots, m, \quad m \leq n.$$

defined for $t \in [t_0, \infty)$.

1.7. Definition. *The number (or the symbol $\pm\infty$) defined as*

$$\chi[\mathrm{F}] = \max_{ij} \chi[f_{ij}]$$

is called the characteristic exponent of the matrix $\mathrm{F}(t)$.

Note that $\chi[\mathrm{F}] = \chi[\mathrm{F}^*]$, which is follows from the definition. In what follows we shall use one of the three norms:

$$\|x\|_I = \max_i |x_i| \;\Rightarrow\; \|\mathrm{A}\|_I = \max_i \sum_{j=1}^{n} |a_{ij}|,$$
$$\|x\|_{II} = \sum_{i=1}^{n} |x_i| \;\Rightarrow\; \|\mathrm{A}\|_I = \max_j \sum_{i=1}^{n} |a_{ij}|,$$
$$\|x\|_{III} = \sqrt{\langle x, x\rangle} \;\Rightarrow\; \|\mathrm{A}\|_{III} = (\text{the greatest eigenvalue of } \mathrm{A}^*\mathrm{A})^{\frac{1}{2}}.$$

Here x_i $(i = 1, \dots, n)$ are the element of the vector x, and a_{ij} $(i, j = 1, \dots, n)$, are the elements of the matrix A.

1.3. Lemma. *The characteristic exponent of a finite-dimensional matrix* $\mathrm{F}(t)$ *coincides with the characteristic exponent of its norm, i.e,*

$$\chi[\mathrm{F}] = \chi[\|\mathrm{F}\|]$$

Proof. The estimate

$$|f_{ij}(t)| \leq \|\mathrm{F}(t)\|, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad m \leq n, \quad t \in [t_0, \infty)$$

is valid for any of these norms. This is obvious for $\|\mathrm{F}(t)\|_I$ and $\|\mathrm{F}(t)\|_{II}$ and for the third norm there is the estimate

$$\max_i (\sum_{j=1}^{m} |f_{ij}(t)|^2)^{\frac{1}{2}} \leq \|\mathrm{F}(t)\|_{III}.$$

From these two inequalities, by the monotonicity of the characteristic exponet , we have

$$\chi[\mathrm{F}] \leq \chi[\|\mathrm{F}\|]. \tag{1.15}$$

At the same time

$$\|\mathrm{F}(t)\| \leq \sum_{i,j} |f_{ij}(t)|,$$

which is obvious for the first two norms, and for the third norm we use the inequality

$$\|\mathrm{F}(t)\|_{III} \leq (\sum_{ij} |f_{ij}|^2)^{\frac{1}{2}}.$$

It follows from the last two inequalities that

$$\chi[\mathrm{F}] \geq \chi[\|\mathrm{F}\|]. \tag{1.16}$$

Comparing (1.15) with (1.16), we obtain what was required.

□

1.3. Corollary. *Let* $x(t)$ *be a vector-function defined for* $t \in [t_0, \infty)$ *; then*

$$\chi[x] = \overline{\lim_{t\to\infty}} \frac{1}{t} ln\|x(t)\|.$$

1.5 Stability of linear differential systems

1.5.1 On the stability of linear homogeneous and nonhomogeneous system

Let a normal system

$$\dot{x} = f(t, x), \qquad x : [t_0, +\infty) \to \mathrm{C}^n$$

have a solution $x(t)$ with the initial condition $x(t_0) = x_0$. We will denote them by $x(t, t_0, x_0)$.

1.8. Definition. *The solution $x(t, t_0, x_0)$ is said to be Lyapunov stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that the condition*

$$\|x_0 - \eta\| < \delta$$

implies

$$\|x(t, t_0, x_0) - x(t, t_0, \eta)\| < \epsilon \quad \textit{for all } t \geq t_0.$$

1.9. Definition. *The solution $x(t, t_0, x_0)$ is said to be unstable if it is not stable, i.e., if there exists an $\epsilon > 0$ such that for any $\delta > 0$ there is an η_1 satisfying the condition*

$$\|x_0 - \eta_1\| < \delta$$

and a moment of time $t_1 > t_0$ such that

$$\|x(t, t_0, x_0) - x(t, t_0, \eta)\| \geq \epsilon.$$

1.10. Definition. *The solution $x(t, t_0, x_0)$ is said to be asymptotically stable if it is Lyapunov stable, and there exists an $\Delta > 0$ such that the condition*

$$\|x_0 - \eta\| < \Delta$$

implies

$$\lim_{t \to \infty} \|x(t_1, t_0, x_0) - x(t_1, t_0, \eta_1)\| = 0.$$

Note that the ball

$$\|x_0 - \eta\| < \Delta$$

is called the domain of attraction of the solution $x(t, t_0, x_0)$

1.11. Definition. *The solution $x(t, t_0, x_0)$ is said to be globally aymptotically stable if, in the previous definition, $\Delta = \infty$*

Let us consider some example to clarify the definitions:

1.5. Example.

$$\dot{x} = -x, \quad t \geq 0$$

The solutions are $x(t, t_0, x_0) = x_0 e^{-(t-t_0)}$. All the solutions tends to zero as $t \to \infty$, consequently, the equation is asymptotically stable.

1.6. Example.

$$\dot{x} = x, \quad t \geq 0$$

The solutions are $x(t, t_0, x_0) = x_0 e^{(t-t_0)}$. Obviously, the equation is unstable, because all the solutions increase without bound as t increases.

1.7. Example.

$$\dot{x} = 0, \quad t \geq 0$$

All the solutions are $x(t, t_0, x_0) = x_0$. The equation is obviously stable, but not asymptotically stable.

Let $x : [t_0, +\infty) \to \mathbb{C}^n$, $\mathrm{A} \in \mathrm{C}([t_0, +\infty)$, $f \in \mathrm{C}([t_0, +\infty)$, and $\sup_{t \geq t_0} \|A(t)\| < \infty$. We shall consider the corresponding homogeneous system

$$\dot{x} = \mathrm{A}(t)x$$

and nonhomogenous system

$$\dot{x} = \mathrm{A}(t)x + f(t).$$

We will list some basically following theorem, which is founded in [1].

1.3. Theorem. *A linear nonhomogeneous system is stable (asymptoticcally stable) if and only if the trivial solution of the linear homogeneous system is stable (asymptoticcally stable).*

1.4. Remark. *A linear nonhomogeneous system is stable (asymptoticcally stable) if and only if the corresponding homogeneous system is stable (asymptoticcally stable).*

1.4. Theorem. *A linear homogeneous system* (1.2)*, which* $\sup_{t\geq t_0}\|\mathrm{A}(t)\| < \infty$*, is stable if and only if each of its solutions* $x(t)$ *is bounded for* $t \geq t_0$*.*

Proof. We show that the stability of the system implies the boundedness of any of its solutions. Let, conversely, there exists a solution $z(t)$ of system (1.2) unbounded for $[t_0, \infty)$. Obviously,

$$\|z(t_0)\| \neq 0.$$

Fix $\epsilon > 0$; for it determine $\delta > 0$, whose existence is guaranteed by the stability of the trivial solution, and form the solution

$$x(t) = \frac{z(t)}{\|z(t_0)\|}\frac{\delta}{2}.$$

Since

$$\|x(t_0)\| = \frac{\delta}{2} < \delta,$$

by virtue of stability, we have

$$\|x(t)\| < \epsilon \quad \text{for} \quad t \geq t_0;$$

this contradicts our assumption that $z(t)$ is unbounded. Let us carry out the considerations in the opposite direction. Suppose $\mathrm{X}(t, t_0)$ is the fundamental matrix of (1.2), its columns are solutions of the system, and they are bounded; therefore, there exists a constant C such that

$$\|\mathrm{X}(t, t_0)\| \leq C \text{ for } t \geq t_0.$$

Any solution $x(t)$ is written in the form

$$x(t) = \mathrm{X}(t, t_0)x(t_0).$$

Hence,

$$\|x(t)\| = \|\mathrm{X}(t, t_0)\|\|x(t_0)\| \leq C\|x(t_0)\|.$$

Take $\epsilon > 0$ and determine $\delta = \frac{\epsilon}{C}$ according to it. Obviously, the inequality

$$\|x(t)\| < \epsilon \text{ for } t \geq t_0,$$

follows from the inequality

$$\|x(t_0)\| < \delta.$$

Thus, the trivial solution is stable and system (1.2) is also stable.

□

1.5. Remark. *All the solutions of a stable linear homogeneous system are simultaneously either bounded or unbounded for $t \geq t_0$.*

1.5. Theorem. *A linear homogeneous system is asymptotically stable if and only if all its solutions $x(t)$ tend to zero as $t \to \infty$,i.e.,*

$$\lim_{t\to\infty} \|x(t)\| = 0$$

Proof. Let the trivial solution be asymptotically stable, i.e., let there exist $\Delta > 0$ such that the inequality

$$\|x(t_0)\| < \Delta$$

implies

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

Take an arbitrary solution $x(t)$ and write it in the following way

$$\begin{aligned} x(t) &= \frac{x(t)}{\|x(t_0\|} \frac{\|x(t_0\|}{\Delta/2} \frac{\Delta}{2} \\ &= z(t)\frac{2\|x(t_0\|}{\Delta}. \end{aligned}$$

Obviously,

$$\|z(t_0)\| = \frac{\Delta}{2} < \Delta;$$

therefore,

$$\|z(t)\| \to 0 \text{ when } t \to \infty,$$

and, thus,

$$\|x(t)\| \to 0 \text{ when } t \to \infty.$$

conversely, let

$$\|x(t)\| \to 0 \text{ when} t \to \infty, \text{ for any solution } \; x(t).$$

For each solution there exists a moment of time $T > t_0$ such that

$$\|x(t)\| < 1 \text{ for } t \geq T.$$

On the interval $[t_0, T]$ the boundedness of $\|x(t)\|$ follows from the continuity. According to the previous theorem, the trivial solution is stable. But for any

$$\|x(t_0)\| < \infty,$$

according to our condition, we have

$$\|x(t)\| \to 0 \text{ when } t \to \infty,$$

i.e., $\Delta = \infty$ and the asymptotic stability if proved.

□

1.5.2 On stability of linear homogeneous systems whose coefficients are constant, periodic.

We will list a basically following theorem, which is founded in [1]

1.6. Theorem. *A linear homogeneous system* (1.2) *with constant coefficients is*

1. *stable if and only if all the eigenvalues of the coefficient matrix have nonpositive real parts, simple elemetary divisors correspond to the eigenvalues with zero real part.*

2. *asymptotically stable if and only if all the eigenvalues of the coefficient matrix have negative real parts.*

From the remark 2.3, we obtain:

1.7. Theorem. *A linear homogeneous system* (1.2) *with periodic coefficients is*

1. *stable if and only if all its multipliers belong to the closed disc, and simple elementary divisors correspond to the multipliers lying on the unit circle.*

2. *asymptotically stable if and only if all its multipliers belong to the iterior of the unit disc.*

2 Growth estimates of the solutions of linear systems in term of the coefficient

Let $x : \mathbb{R}_+ \to \mathbb{C}^n$, $\mathrm{A} \in \mathrm{C}(\mathbb{R}_+)$, $\sup_{t \geq 0} \|\mathrm{A}(t)\| < \infty$. Now we are considering a system

$$\dot{x} = \mathrm{A}(t)x \tag{2.17}$$

In this section we deal with six different coefficient criteria for the growth of solutions as $t \to \infty$. The first estimate is based on Lyapunov. As a consequence, it has been proved that the charateristic exponents of linear systems are finite in the case when the coefficients are bounded.

2.1 Lyapunov's estimate

2.1.1 Lyapunov's estimate theorem

2.8. Theorem. *[Lyapunov]. For any solution $x(t)$ of system (2.17), the following inequality is valid:*

$$\|x(t_0)\|e^{-\int_{t_0}^{t} \|\mathrm{A}(\tau)\|d\tau} \leq \|x(t)\| \leq \|x(t_0)\|e^{\int_{t_0}^{t} \|\mathrm{A}(\tau)\|d\tau}, \qquad t \geq t_0 \geq 0. \tag{2.18}$$

Proof. Assume that the solution of system (2.17) is $x(t)$ with the intitial solution $x(t_0) \in \mathbb{C}^n, t_0 \in \mathbb{R}_+$.This problem is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{A}(\tau)x(\tau)d\tau, \qquad t, t_0 \in \mathbb{R}_+,$$

Hence ,

$$\|x(t)\| \leq \|x(t_0)\| + \left| \int_{t_0}^{t} \|\mathrm{A}(\tau)\|\|x(\tau)\|d\tau \right|,$$

and by the Gronwall-Bellman lemma we obtain (2.18).

□

2.2 Bogdanov's estimate

2.2.1 Bogdanov's estimate theorem

2.9. Theorem. *[Bogdanov] [3]. For any solution $x(t)$ of system (2.17) with the real* *valued matrix* A *defined on* $t \in \mathbb{R}_+$, *the following estimate is valid:*

$$\|x(0)\| e^{-\frac{1}{2}\int_0^t \sum_{i,j=1}^n |a_{ij}(\tau)+a_{ji}(\tau)| d\tau} \leq x(t) \leq \|x(0)\| e^{\frac{1}{2}\int_0^t \sum_{i,j=1}^n |a_{ij}(\tau)+a_{ji}(\tau)| d\tau}, \tag{2.19}$$

where

$\|x(t)\|$ *is the Euclidean norm.*

Proof. Take the nontrivial solution

$$x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$$

of the system (2.17). We have:

$$\frac{dx_i}{dt} = \sum_{j=1}^n a_{ij}(t) x_j, \quad i = 1, ..., n.$$

or

$$x_i \dot{x}_i = \sum_{j=1}^n a_{ij}(t) x_i x_j$$

Summing up the last identities, we get

$$\frac{d}{dt}\sum_{i=1}^n x_i^2 = 2\sum_{i,j=1}^n a_{ij}(t) x_i x_j = \sum_{i,j=1}^n [a_{ij}(t) + a_{ji}(t)] x_i x_j.$$

Moreover, we have

$$\begin{aligned}\left|\frac{d}{dt}\|x\|^2\right| &\leq \sum_{i,j=1}^n |a_{ij}(t) + a_{ji}(t)| \frac{x_i^2 + x_j^2}{2} \\ &= \sum_{i=1}^n (\sum_{j=1}^n |a_{ij}(t) + a_{ji}(t)| x_i^2) \\ &\leq (\sum_{i=1}^n x_i^2) \sum_{l,s=1}^n |a_{ls}(t) + a_{sl}(t)|.\end{aligned}$$

Then,

$$-\sum_{l,s=1}^n |a_{ls}(t) + a_{sl}(t)| \|x\|^2 \leq \frac{d}{dt}\|x\|^2 \leq \sum_{l,s=1}^n |a_{ls}(t) + a_{sl}(t)| \|x\|^2.$$

Dividing the result by $\|x\|^2$ and integrating, we obtain

$$-\int_0^t \sum_{l,s=1}^n |a_{ls}(\tau) + a_{sl}(\tau)| d\tau \leq 2(ln\|x(t)\| - ln\|x(0)\|) \leq \int_0^t \sum_{l,s=1}^n |a_{ls}(\tau) + a_{sl}(\tau)| d\tau$$

this trivially implies (2.19).

□

2.2.2 Collolary

2.4. Corollary. *Both these results have a serious shortcoming , i.e., the upper bound for the characteristic exponents is positive . Indeed from the inequalities* (2.18) *we get*

$$-\underline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t}\|\mathrm{A}(\tau)\|d\tau \leq \chi[x] \leq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t}\|\mathrm{A}(\tau)\|d\tau \tag{2.20}$$

From the Bogdanov's estimate (2.19) *we have*

$$-\underline{\lim_{t\to\infty}}\frac{1}{2t}\int_{0}^{t}\sum_{l,s=1}^{n}|a_{ls}(\tau)+a_{sl}(\tau)|\,d\tau \leq \chi[x] \leq \overline{\lim_{t\to\infty}}\frac{1}{2t}\int_{0}^{t}\sum_{l,s=1}^{n}|a_{ls}(\tau)+a_{sl}(\tau)|\,d\tau \tag{2.21}$$

Thus, inequalities (2.20) *and* (2.21) *show that, in the case of abritrary matrix* A, *the estimates, generally speaking, are useless. Exceptions are the cases when* $\mathrm{A}\equiv 0$ *and when* $\mathrm{A}(t)$ *is skew-symmetric; both estimates in the first case and* (2.19) *in the second case reflect the real state of affairs:*

$$\|x(t)\| = const, \qquad t\in\mathbb{R}_+$$

.

Now let us consider more exact estimates.

2.3 Vazhevskii's estimate

This result does not require the assumption that $\|\mathrm{A}(t)\|$ in the system (2.17) be bounded.

2.3.1 Vazhevskii's estimate theorem

2.10. Theorem. *[Vazhevskii][6]. Now we consider the system* (2.17) *and let* $\lambda(t)$ *and* $\wedge(t)$ *are the smallest and the greatest eigenvalues of the matrix*

$$\mathrm{A}^H(t) = \frac{[\mathrm{A}(t)+\mathrm{A}^*(t)]}{2}$$

. *For any solution* $x(t)$ *of system the inequality*

$$\|x(0)\|e^{\int_0^t \lambda(u)du} \leq \|x(t)\| \leq \|x(0)\|e^{\int_0^t \wedge(u)du} \tag{2.22}$$

is valid, where $\|x(t)\|$ *is the Euclidean norm.*

Proof. Let us take a nontrivial solution $x(t)$ of system (2.17); for it $\|x\|^2 = x^*x$. Therefore we have

$$\frac{d\|x\|^2}{dt} = x^*\frac{dx}{dt} + x\frac{dx^*}{dt} = x^*\mathrm{A}(t)x + x^*\mathrm{A}^*(t)x = 2x^*\mathrm{A}^H(t)x.$$

Because A^H is Hermitian; hence, it is unitarily similar to the diagonal matrix

$$\mathrm{D} = \mathrm{diag}[\lambda_1(t), ..., \lambda_n(t)]$$

Let $\mathrm{U}(t)$ be such that

$$\mathrm{U}^*(t)\mathrm{U}(t) = \mathrm{E}$$

and

$$\mathrm{A}^H(t) = \mathrm{U}^*(t)\mathrm{D}(t)\mathrm{U}(t);$$

hence

$$x^*\mathrm{A}^H(t)x = x^*\mathrm{U}^*(t)\mathrm{D}(t)\mathrm{U}(t)x = y^*\mathrm{D}(t)y = \sum_{j=1}^{n} \lambda_j y_j \overline{y}_j, \tag{2.23}$$

where $y = \mathrm{U}(t)x$; therefore, $\|x\| = \|y\|$. Let

$$\lambda(t) = \min_i \lambda_i(t), \quad \wedge(t) = \max_i \lambda_i(t)$$

.

From (2.23) we have

$$\lambda(t)\|x\|^2 \leq x^*\mathrm{A}^H(t)x \leq \wedge(t)\|x\|^2,$$

or

$$2\lambda(t)\|x\|^2 \leq \frac{d\|x\|^2}{dt} \leq 2\wedge(t)\|x\|^2.$$

Diving the result by $\|x\|^2$ and integrating the last inequality from 0 to t, we get

$$\int_0^t \lambda(u)du \leq ln\|x(t)\| - ln\|x(0)\| \leq \int_0^t \wedge(u)du.$$

This is desired.

□

2.3.2 Collolary

2.6. Remark. *From Vazhevskii's theorem we have*

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t \lambda(u)du \leq \chi[x] \leq \overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t \wedge(\tau)d\tau. \tag{2.24}$$

The right hand side of the inequality (2.24) *may be negative.*

2.3.3 Example

2.8. Example. *Consider system* (2.17) *with the matrix*

$$\mathrm{A} = \begin{pmatrix} -1 & t \\ -t & -4 \end{pmatrix}$$

where

$$\mathrm{A}^H(t) = \frac{[\mathrm{A}(t) + \mathrm{A}^*(t)]}{2} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}.$$

Therefore, $\lambda = -4$ *and* $\wedge = -1$ *and the solution of the system*

$$\dot{x_1} = -x_1 + tx_2,$$

$$\dot{x_2} = -tx_1 - 4x_2.$$

have the estimate

$$e^{-4t}\|x(0)\| \leq \|x(t)\| \leq e^{-t}\|x(0)\|;$$

we note that the coefficients of the system are unbounded .

2.4 Lozinskii's estimate.

In this subsection we deal with the Lozinskii's logarithmic norm and its application to estimate for solutions of linear systems.

2.4.1 Logarithmic norm

2.12. Definition. *[8] The number* $\gamma(\mathrm{A})$ *defined as*

$$\gamma(\mathrm{A}) = \lim_{h\to 0+} \frac{\|\mathrm{E} + h\mathrm{A}\| - 1}{h}. \tag{2.25}$$

is called the logarithmic norm of the matrix A.

Note that $\gamma(\mathrm{A})$ depends of the choice of matrix norm and is defined for any norm such that $\|\mathrm{E}\| = 1$. The term is conditional because this norm does not have the properties of an ordinary norm; for example, it may be negative.

2.4.2 Example

2.9. Example. *For the matrix*

$$\mathrm{A} = \begin{pmatrix} -5 & 1 \\ 2 & -1 \end{pmatrix} \textit{ we have } \quad \gamma(\mathrm{A})_I = -1.$$

2.4.3 Properties of logarithmic norm

2.1. Properties. *For any $n \times n$ matrix A and B, we will present some properties of logarithmic norm from below:*

1. $\gamma(\mathrm{A} + \mathrm{B}) \leq \gamma(\mathrm{A}) + \gamma(\mathrm{B})$. *Because:*

$$\|\mathrm{E} + h(\mathrm{A} + \mathrm{B}\| \leq \|\frac{1}{2}\mathrm{E} + h\mathrm{A}\| + \|\frac{1}{2}\mathrm{E} + h\mathrm{A}\| \leq \frac{1}{2}\left[\|\mathrm{E} + 2h\mathrm{A}\| + \|\mathrm{E} + 2h\mathrm{A}\|\right].$$

Substract 1 from both sides and divide by h , then pass to the limit.

2. $\gamma(\alpha \mathrm{A}) = \alpha\gamma(\mathrm{A}), \quad \alpha \in \mathrm{R}_+$.

3. $\gamma(\mathrm{A}) \leq \|\mathrm{A}\|$. *Indeed,*

$$\|\mathrm{E} + h\mathrm{A}\| - 1 = \|\mathrm{E} + h\mathrm{A}\| - \|\mathrm{E}\| \leq \|\mathrm{E}\| + \|h\mathrm{A}\| - \|\mathrm{E}\| = h\|\mathrm{A}\|,$$

it remains to divide by h and to pass the limit.

4. $\gamma(\mathrm{A}) - \gamma(\mathrm{B}) \leq \|\mathrm{A} - \mathrm{B}\|$*; this follows from*

$$\gamma(\mathrm{A}) \leq \gamma(\mathrm{B}) + \gamma(\mathrm{A} - \mathrm{B}) \leq \gamma(\mathrm{B}) + \|\mathrm{A} - \mathrm{B}\|.$$

We give the values of the logarithmic norms for the three matrix norms indicated:

$$\gamma_I(\mathrm{A}) = \max_{\mu}\left\{ Re\ a_{\mu\mu} + \sum_{\eta \neq \mu} |a_{\mu\eta}| \right\},$$

$$\gamma_{II}(\mathrm{A}) = \max_{\eta}\left\{ Re\ a_{\eta\eta} + \sum_{\mu \neq \eta} |a_{\mu\eta}| \right\},$$

$$\gamma_{III}(\mathrm{A}) = \textit{the greatest eigenvalue of } \frac{(\mathrm{A}+\mathrm{A}^*)}{2}.$$

The caculations of these norms are carried out straightforwardly. For example,

$$\begin{aligned}|\mathrm{E}+h\mathrm{A}|_I &= \max_{\mu} \sum_{\eta=1}^{n} |(\mathrm{E}+h\mathrm{A})_{\mu\eta}| \\ &= \max_{\mu} \Big\{ |1+ha_{\mu\mu}| + \sum_{\eta\neq\mu} h|a_{\mu\eta}| \Big\} \\ &= \max_{\mu} \Big\{ 1 + hRe\ a_{\mu\mu} + O(h^2) + \sum_{\eta\neq\mu} h|a_{\mu\eta}| \Big\}\end{aligned}$$

Substracting 1 and dividing by h, we pass to the limit and obtain $\gamma_I(\mathrm{A})$

2.4.4 Lozinskii's estimate theorem

Now we turn to system(2.17) . Let $\mathrm{X}(t,s)$ be its Cauchy matrix.

2.11. Theorem. *[Lozinskii]. For any matrix norm the following estimate is valid:*

$$\|\mathrm{X}(t,s)\| \leq e^{\int_s^t \gamma(\mathrm{A}(\tau))d\tau}, \qquad t \geq s \geq 0. \tag{2.26}$$

Proof. Let $t_k = s + (k/N)(t-s)$; however, the division of the interval $[s,t]$ does not necessarily have to be uniform, only the maximal length of the subintervals must not exceed h . We have

$$\mathrm{X}(t,s) = \mathrm{X}(t_N)\mathrm{X}^{-1}(t_{N-1})\mathrm{X}(t_{N-1})\mathrm{X}^{-1}(t_{N-2})\ldots\mathrm{X}(t_2)\mathrm{X}^{-1}(t_1)\mathrm{X}(t_1)\mathrm{X}^{-1}(s),$$

or

$$\mathrm{X}(t,s) = \prod_{k=N}^{1} \mathrm{X}(t_k, t_{k-1}).$$

Recall that

$$\mathrm{X}(t_k,t_{k-1}) \equiv \Omega_{\mathrm{A}}(t_k,t_{k-1})$$

and turn to the expansion (1.5) , we have

$$\begin{aligned}\mathrm{X}(t_k,t_{k-1}) &= \mathrm{E} + \int_{t_{k-1}}^{t_k} \mathrm{A}(u)du + \int_{t_{k-1}}^{t_k} \mathrm{A}(u_1)du_1 \int_{t_{k-1}}^{u_1} \mathrm{A}(u_2)du_2 + ... \\ &= \mathrm{E} + \int_{t_{k-1}}^{t_k} \mathrm{A}(u)du + O(h^2) \\ &= \mathrm{E} + h\mathrm{A}(t_{k-1}) + O(h^2).\end{aligned}$$

We pass to the estimate

$$
\begin{aligned}
\|\mathrm{X}(t,s)\| &\le \prod_{k=1}^{N} \|\mathrm{E} + h\mathrm{A}(t_{k-1}) + O(h^2)\| \\
&\le \prod_{k=1}^{N} \left(\|\mathrm{E} + h\mathrm{A}(t_{k-1})\| + O(h^2)\right) \\
&= \prod_{k=1}^{N} \left[1 + h\gamma\left(\mathrm{A}(t_{k-1})\right) + O(h)\right] \quad \text{(by definition of } \gamma \text{ logarithmic norm)} \\
&= \prod_{k=1}^{N} e^{ln[1+h\gamma(\mathrm{A}(t_{k-1}))+O(h)]} \\
&= \prod_{k=1}^{N} e^{ln[1+h\gamma(\mathrm{A}(t_{k-1}))+O(h)]} \\
&= e^{\sum_{k=1}^{N}[h\gamma(\mathrm{A}(t_{k-1}))+O(h)]} \qquad (ln(1+t) = t \text{ when } t \text{ is small enough.}) \\
&= e^{\sum_{k=1}^{N} h\gamma(\mathrm{A}(t_{k-1}))+NO(h)} \\
&= e^{\sum_{k=1}^{N} h\gamma(\mathrm{A}(t_{k-1}))+\left(\frac{(t-s)}{h}\right)O(h)}.
\end{aligned}
$$

Passing to the limit as $h \to 0+$, we obtain the estimate (2.26).

□

2.4.5 Collolary

2.7. Remark. *It is possible to obtain analogous estimates for the Cauchy matrix from below :*

$$
\|\mathrm{X}(t,s)\| \ge \exp\left[\int_s^t \lim_{h\to 0-} \frac{\|\mathrm{E} + h\mathrm{A}(\tau)\| - 1}{h} d\tau\right] \qquad (t \ge s \ge 0)\,. \tag{2.27}
$$

Combining the estimate (2.26) *and* (2.27)*, we write them for the norms I, II, III, respectively:*

1.

$$
\exp\left(\int_s^t \min_{\mu}\left(Re\ a_{\mu\mu} - \sum_{\eta\neq\mu}|a_{\mu\eta}|\right)d\tau\right) \le \|\mathrm{X}(t,s)\| \le \exp\left(\int_s^t \max_{\mu}\left(Re\ a_{\mu\mu} + \sum_{\eta\neq\mu}|a_{\mu\eta}|\right)d\tau\right). \tag{2.28}
$$

2.

$$\exp\left(\int_s^t \min_\eta \left(Re\ a_{\eta\eta} - \sum_{\mu\neq\eta} |a_{\mu\eta}|\right) d\tau\right) \leq \|\mathrm{X}(t,s)\| \leq \exp\left(\int_s^t \max_\eta \left(Re\ a_{\eta\eta} + \sum_{\mu\neq\eta} |a_{\mu\eta}|\right) d\tau\right). \tag{2.29}$$

3. for the third norm we obtain Vazhevskii's estimate (2.22).

Let us illustrate Lozinskii's result by examples.

2.4.6 Example 1.

2.10. Example.

$$\dot{x_1} = -\frac{x_1}{t} + \frac{x_2}{t}, \qquad \dot{x_2} = \frac{x_1}{t} - x_2$$

where $t \in [1, +\infty)$*,* x_1, x_2 *are complex-valued differential function defined on* $[1, +\infty)$*.*

The first two methods given in this chapter will give too crude estimates from above; Vazhevskii's method will give the result after some calculations, while here (by Lozinskii theorem) we straightforwardly have

$$\gamma_I(\mathrm{A}(t)) = 0,$$

hence, the system is stable.

2.4.7 Example 2.

2.11. Example. *Consider the system*

$$\dot{x_1} = -\mu(1 - 2\sin(t))x_1 + \mu x_2,$$

$$\dot{x_2} = \mu x_1 - x_2.$$

where $0 \leq \mu$*,* $t \in \mathbb{R}_+$*,* x_1, x_2 *are complex-valued differential function defined on* $\mathbb{R}_+$*.*

Let us show when this system is asymptotically stable by using the method of logarithmic norms . The condition

$$\int_0^t \gamma(\mathrm{A}(u))du \to -\infty \text{ as } t \to \infty$$

guarantees the asymptotic stability. Obviuously, the logarithmic norms

$$\gamma_I(\mathrm{A}) = \gamma_{II}(\mathrm{A}) = \max\ \{2\mu \sin t, \mu - 1\}.$$

are not satisfactory. Let us turn to the third norm

$$\gamma_{III}(\mathrm{A}) = \frac{1}{2}\{-1 - \mu + 2\mu \sin t + ([1 - \mu(1 - 2\sin t)]^2 + 4\mu^2)^{\frac{1}{2}}\}$$

and, since this function is 2π periodic, let us consider the integral over the period,

$$\begin{aligned}\varphi(\mu) &= \int_0^{2\pi} \gamma_{III}(\mathrm{A}(\tau))d\tau \\ &= \frac{1}{2}\left\{-2\pi - 2\pi\mu + \int_0^{2\pi} ([1 - \mu(1 - 2\sin t)]^2 + 4\mu^2)^{\frac{1}{2}} d\tau\right\}.\end{aligned}$$

Note that for $0 \leq \mu < \infty$ the function $\varphi(\mu)$ is convex downwards since $\varphi''(\mu) > 0$; moreover,

$$\varphi(0) = 0, \quad \varphi'(0) < 0.$$

This implies that if

$$\varphi(\overline{\mu}) < 0,$$

then

$$\varphi(\mu) < 0 \quad \textit{for} \ \ 0 < \mu < \overline{\mu}.$$

Straightforward calculations show that

$$\varphi(0.67) < 0,\ \ \varphi(0.68) > 0.$$

Thus, the asymptotic stability of Malkin's system is guaranteed for $0 < \mu < \mu^*$, where

$$0.67 < \mu^* < 0.68.$$

2.5 The method of freezing

Let us turn the system (2.17). Ther naturally arises the question: is the behavior of solutions of this system connected with the character of the eigenvalues $\{\gamma_1(t), ..., \gamma_n(t)\}$ of the matrix $\mathrm{A}(t)$? For example, if

$$\gamma = \sup_{t \geq 0} \max\{\gamma_1(t), ..., \gamma_n(t)\}, \tag{2.30}$$

then can we, say, make a conclusion on the asymptotic stability from the fact that $\gamma < 0$? Alas, in the general case, we cannot. And let see an example of contradiction.

2.5.1 Example

2.12. Example. *Consider the system* $\dot{x} = \mathrm{A}(t)x$*, where*

$$\mathrm{A}(t) = \begin{pmatrix} -(1+2\cos 4t) & 2(1+\sin 4t) \\ 2(\sin 4t - 1) & -1+2\cos 4t \end{pmatrix}$$

where $t \in \mathbb{R}_+$*,* x *is complex-valued differential function defined on* $\mathbb{R}_+$*. Here* $\gamma_1 = \gamma_2 = -1$*, and the system has a solution*

$$x(t) = e^t(\sin 2t, \cos 2t)^T, \qquad \chi[x] = 1.$$

Thus, the eigenvalues of $\mathrm{A}(t)$ are not directly connected with the character of the behavior of solutions of the system in the autonomous case. However, if the coefficients of the system are functions of small variation, then such a connection can be obtained. The idea of the method of freezing is that, fixing $t_1 \in \mathbb{R}_+$, we reduce our system (2.17) to an almost constant one,

$$\dot{x} = [\mathrm{A}(t_1) + (\mathrm{A}(t) - \mathrm{A}(t_1))]\, x, \tag{2.31}$$

where

$$t \in \mathbb{R}_+,\ \mathrm{A} \in \mathrm{C}(\mathbb{R}_+),\ \sup_{\mathbb{R}_+} \|\mathrm{A}\| \leq \infty,\ x : \mathbb{R}_+ \to \mathbb{C}^n \ \text{ differential function.}$$

The result given below is due to [4]. We write a solution $x(t)$ of system (2.31) as follows:

$$x(t) = e^{\mathrm{A}(t_1)t}\ x(0) + \int_0^t e^{\mathrm{A}(t_1)(t-\tau)}[\mathrm{A}(\tau) - \mathrm{A}(t_1)]x(\tau)d\tau.$$

Therefore,

$$\|x(t)\| = \|e^{\mathrm{A}(t_1)t}\|.\|x(0)\| + \int_0^t \|e^{\mathrm{A}(t_1)(t-\tau)}\|.\|[\mathrm{A}(\tau) - \mathrm{A}(t_1)]\|.\|x(\tau)\|d\tau.$$

This inequality holds for all t, including $t = t_1$. Set i$t = t_1$ and then omit the subscript, i.e., denote t_1 by t:

$$\|x(t)\| = \|e^{\mathrm{A}(t)t}\|.\|x(0)\| + \int_0^t \|e^{\mathrm{A}(t)(t-\tau)}\|.\|[\mathrm{A}(\tau) - \mathrm{A}(t)]\|.\|x(\tau)\|d\tau.$$

We introduce a restriction for the rate of change of $\mathrm{A}(t)$. Let

$$\delta = \sup_{t\geq 0} \|\dot{\mathrm{A}}(t)\| \Rightarrow \|\mathrm{A}(\tau) - \mathrm{A}(t)\| \leq \delta(t-\tau) \quad (0 \leq \tau \leq t)\,. \tag{2.32}$$

To estimate $e^{\mathrm{A}(t)t}$, we use the inequality in section 2.2 , whence ,

$$\|e^{\mathrm{A}(t)t}\| \leq \mathrm{D}(1+t)^{n-1}e^{\gamma t}. \tag{2.33}$$

where γ is defined by formula (2.30). Note that

$$\begin{aligned} \delta(1+t-\tau)^{n-1}(t-\tau) &\leq \delta^{\frac{1}{n+1}}\delta^{\frac{n}{n+1}}(1+t-\tau)^n \\ &= \delta^{\frac{1}{n+1}}e^{nln[\delta^{\frac{1}{n+1}}(1+t-\tau)]} \\ &\leq \delta^{\frac{1}{n+1}}e^{n\delta^{\frac{1}{n+1}}(1+t-\tau)}. \end{aligned} \tag{2.34}$$

By the virtue of the conditions (2.32) and (2.33), we write

$$\|x(t)\| \leq \mathrm{D}(1+t)^{n-1}e^{\gamma t}\|x(0)\| + \int_0^t \mathrm{D}(1+t-\tau)^{n-1}e^{\gamma(t-\tau)}\delta(t-\tau)\|x(\tau)\|d\tau,$$

and, taking into account (2.34), we have

$$\|x(t)\| \leq \mathrm{D}(1+t)^{n-1}e^{\gamma t}\|x(0)\| + \int_0^t \mathrm{D}e^{\gamma(t-\tau)}\delta^{\frac{1}{n+1}}e^{n\delta^{\frac{1}{n+1}}(1+t-\tau)}\|x(\tau)\|d\tau.$$

Divide the last inequality by

$$\varphi(t) = (1+t)^{n-1}e^{(\gamma+n\delta^{\frac{1}{n+1}})t} > 0. \tag{2.35}$$

Then

$$\frac{\|x(t)\|}{\varphi(t)} \leq \mathrm{D}e^{-n\delta^{\frac{1}{(n+1)}}t}\|x(0)\| + \int_0^t \frac{\mathrm{D}\delta^{\frac{1}{n+1}}}{(1+t)^{n-1}}e^{-(\gamma+n\delta^{\frac{1}{(n+1)}})\tau}e^{n\delta^{\frac{1}{n+1}}}\|x(\tau)\|d\tau;$$

therefore,

$$\frac{\|x(t)\|}{\varphi(t)} \leq \mathrm{D}\|x(0)\| + \int_0^t \mathrm{D}e^{n\delta^{\frac{1}{n+1}}}\delta^{\frac{1}{n+1}}\frac{\|x(\tau)\|}{\varphi(\tau)}d\tau.$$

By using the Gronwall - Bellman formula, we have

$$\frac{\|x(t)\|}{\varphi(t)} \leq \mathrm{D}\|x(0)\|e^{\int_0^t \mathrm{D}_1\delta^{\frac{1}{n+1}}d\tau}, \qquad \mathrm{D}_1 = \mathrm{D}e^{n\delta^{\frac{1}{n+1}}},$$

or taking into account (2.35), we obtain

$$\|x(t)\| \leq \mathrm{D}\|x(0)\|(1+t)^{n-1}e^{(\gamma+n\delta^{\frac{1}{n+1}})t}e^{\mathrm{D}_1\delta^{\frac{1}{n+1}}t} \quad \forall t \in \mathbb{R}_+. \tag{2.36}$$

Thus, we have proven the following statement.

2.5.2 Theorem

2.12. Theorem. *Let system* (2.17) *be such that*

$$\|\mathrm{A}(t_1) - \mathrm{A}(t_2)\| \leq \delta|t_1 - t_2|, \ \forall t_1, t_2 \in \mathrm{R}_+.$$

where δ *is sufficiently small. Then*

1. for any solution $x(t)$ *the estimate* (2.36) *is valid.*

2. we have

$$\chi[x] \leq \gamma + d\delta^{\frac{1}{n+1}}, \tag{2.37}$$

where γ *is defined by formula* (2.30) *and* $d = n + \mathrm{D}e^{n\delta^{\frac{1}{n+1}}}$*. Here* n *is the order of the system and* D *is the constant from the estimate that is in the section 2.2.*

2.5.3 Remark

2.8. Remark. *The equality of* (2.37) *is attainable (in [7]), which means that there exist systems whose greatest characteristic exponent is not less than*

$$\gamma + c_0\delta^{\frac{1}{n+1}},$$

where c_0 *is a constant independent of* δ*.*

2.6 Yakubovich's estimate for the characteristic exponents of systems with periodic coefficients

Let the matrix $\mathrm{A}(t)$ in system (2.17) be such that

$$\mathrm{A}(t) = \mathrm{A}(t + \omega), \qquad t \in \mathbb{R}.$$

We introduce the set of matrices $\mathrm{G}(t)$ satisfying the following conditions for $t \in \mathbb{R}$

1. $\mathrm{G}^*(t) = \mathrm{G}(t)$,

2. $\mathrm{G}(t + \omega) = \mathrm{G}(t)$,

3. $\mathrm{G} \in \mathrm{C}^1(\mathbb{R})$,

4. $\langle \mathrm{G}(t)a, a\rangle \ > 0$, where $a \in \mathbb{C}^n$ is arbitrary, $\|a\| \neq 0, t \in \mathbb{R}$

For example, for any ω-periodic and continuously differentiable matrix $\mathrm{F}(t)$ $(t \in \mathbb{R})$, the matrix

$$\mathrm{G}(t) = \mathrm{F}^*(t)\mathrm{F}(t)$$

satisfies the conditions given above. Let ρ be a multiplier of system (2.17). By Theorem Floquet, the solution $x(t) = e^{\lambda t}\varphi(t)$, where

$$\varphi(t+\omega) = \varphi(t) \text{ and } \lambda = \frac{1}{\omega}\mathrm{Ln}\rho,$$

corresponds to it. Consider the form

$$\xi(t) = \langle \mathrm{G}(t)x, x\rangle. \tag{2.38}$$

We take our normal solution as $x(t)$; then

$$\xi(t) = e^{(\lambda+\overline{\lambda})t}\langle \mathrm{G}\varphi, \varphi\rangle = e^{2t \ \mathrm{Re} \ \lambda}\langle \mathrm{G}\varphi, \varphi\rangle$$

whence

$$ln \ \xi(t) = 2t \ \mathrm{Re} \ \lambda + ln \ \langle \mathrm{G}\varphi, \varphi\rangle. \tag{2.39}$$

Note that

$$ln \ \langle \mathrm{G}(t)\varphi(t), \varphi(t)\rangle$$

is a periodic and continuous, consequently, a bounded function; therefore, dividing (2.39) by $2t$, in the limit we have

$$\mathrm{Re} \ \lambda = \frac{1}{2}\lim_{t\to\infty}\frac{1}{t}ln \ \xi(t) \tag{2.40}$$

Starting with this, we obtain estimates for Re λ. Differentiate the form (2.38) along the trajectories of system (2.17):

$$\dot{\xi} = \langle \dot{\mathrm{G}}x, x\rangle + \langle \mathrm{GA}x, x\rangle + \langle \mathrm{G}x, \mathrm{A}x\rangle = \langle \mathrm{Q}x, x\rangle,$$

where

$$\mathrm{Q} = \dot{\mathrm{G}} + \mathrm{GA} + \mathrm{A}^*\mathrm{G} \qquad (\mathrm{Q}^* = \mathrm{Q}). \tag{2.41}$$

consider the equation

$$\det(\mathrm{Q} - q\mathrm{G}) = 0 \tag{2.42}$$

Let $q_1(t)$ and $q_2(t)$ be its smallest and greatest roots; note that they are real and ω-periodic. Indeed, introduce the matrix $\mathrm{G}^{\frac{1}{2}}$. The matrix G is Hermitian; therefore, there exists a unitary matrix U such that $\mathrm{G} = \mathrm{U}^*\mathrm{D}\mathrm{U}$, where D is real diagonal matrix [5, 9]. We write

$$\mathrm{G} = \mathrm{U}^*\mathrm{D}^{\frac{1}{2}}\mathrm{U}\mathrm{U}^*\mathrm{D}^{\frac{1}{2}}\mathrm{U} \equiv \mathrm{G}^{\frac{1}{2}}\mathrm{G}^{\frac{1}{2}}.$$

Note that $\mathrm{G}^{\frac{1}{2}}$ is Hermitian; moreover,

$$\det[(\mathrm{G}^{-\frac{1}{2}})^*[\mathrm{Q} - q\mathrm{G}]\mathrm{G}^{-\frac{1}{2}}] = \det[(\mathrm{G}^{-\frac{1}{2}})^*\mathrm{Q}\mathrm{G}^{-\frac{1}{2}} - q\mathrm{E}],$$

whence we obtain the equivalence of (2.42) and

$$\det[(\mathrm{G}^{-\frac{1}{2}})^*\mathrm{Q}\mathrm{G}^{-\frac{1}{2}} - q\mathrm{E}] = 0. \tag{2.43}$$

The matrix

$$\mathrm{H} = (\mathrm{G}^{-\frac{1}{2}})^*\mathrm{Q}\mathrm{G}^{-\frac{1}{2}}$$

is Hermitian; hence we get the inequality

$$h_{\min}\langle y, y\rangle \ \leq \langle \mathrm{H}y, y\rangle \ \leq h_{\max}\langle y, y\rangle$$

holds for any vector $y \in \mathbb{C}^n$, where

$$h_{\min} = \min\{h_1(t), ..., h_n(t)\}, \qquad \text{and } h_{\max} = \max\{h_1(t), ..., h_n(t)\},$$

and $\{h_1(t), ..., h_n(t)\}$ are the eigenvalues of $\mathrm{H}(t)$. By virtue of this, (2.43) implies

$$q_1(t)\langle y, y\rangle \ \leq \ \langle \mathrm{Q}\mathrm{G}^{-\frac{1}{2}}y, \mathrm{G}^{-\frac{1}{2}}y\rangle \ \leq \ q_2(t)\langle y, y\rangle,$$

and, setting $y = \mathrm{G}^{\frac{1}{2}}x$, we obtain

$$q_1(t)\langle \mathrm{G}^{\frac{1}{2}}x, \mathrm{G}^{\frac{1}{2}}x\rangle \leq \langle \mathrm{Q}x, x\rangle \leq q_2(t)\langle \mathrm{G}^{\frac{1}{2}}x, \mathrm{G}^{\frac{1}{2}}x\rangle,$$

or

$$q_1(t)\langle \mathrm{G}x, x\rangle \leq \langle \mathrm{Q}x, x\rangle \leq q_2(t)\langle \mathrm{G}x, x\rangle,$$

or

$$q_1(t)\xi(t) \leq \dot{\xi}(t) \leq q_2(t)\xi(t). \tag{2.44}$$

The inequalities (2.44) give a two-sided estimate of the function $\xi(t)$:

$$\int_0^t q_1(\tau)d\tau \leq \ln\,\xi(t) - \ln\,\xi(0) \leq \int_0^t q_2(\tau)d\tau. \tag{2.45}$$

For any ω-periodic function $q(t)$ we have

$$\int_0^t q(\tau)d\tau = \frac{t}{\omega}\int_0^\omega q(\tau)d\tau + r(t),$$

where $r(t)$ is ω-periodic. Thus,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t q(\tau)d\tau = \frac{1}{\omega}\int_0^\omega q(\tau)d\tau.$$

Dividing the inequality (2.45) by $2t$, we pass to the limit , and by the virture of (2.40), we obtain

$$\frac{1}{2\omega}\int_0^\omega q_1(\tau)d\tau \leq \text{ Re }\lambda \leq \frac{1}{2\omega}\int_0^\omega q_2(\tau)d\tau \tag{2.46}$$

or equivalently,

$$\frac{1}{2}\,\overline{\lim}\,\frac{1}{t}\int_0^t q_1(\tau)d\tau \leq \text{Re }\alpha \leq \frac{1}{2}\,\underline{\lim}\,\frac{1}{t}\int_0^t q_2(\tau)d\tau$$

2.6.1 Theorem

2.13. Theorem. *[Yakubovich]. Let consider the system* (2.17) *with the matrix* $\mathrm{A}(t)$ *ω-pediodic. If an $n \times n$ matrix-function* $\mathrm{G}(t)$ *satifies the following conditions for* $t \in \mathbb{R}$

1. $\mathrm{G}^*(t) = \mathrm{G}(t)$,
2. $\mathrm{G}(t+\omega) = \mathrm{G}(t)$,
3. $\mathrm{G} \in \mathrm{C}^1(\mathbb{R})$,
4. $\langle \mathrm{G}(t)a, a\rangle > 0$ *, where $a \in \mathbb{C}^n$ is arbitrary,* $\|a\| \neq 0, t \in \mathbb{R}$

and let $q_1(t)$ *,* $q_2(t)$ *denote the smallest and largest roots of equation*

$$det(\mathrm{Q} - q\mathrm{G}) = 0, \qquad \textit{where } \mathrm{Q} = \dot{\mathrm{G}} + \mathrm{GA} + \mathrm{A}^*\mathrm{G} \qquad (\mathrm{Q}^* = \mathrm{Q}).$$

Then, this inequality

$$\frac{1}{2\omega}\int_0^\omega q_1(\tau)d\tau \leq \textit{ Re }\lambda \leq \frac{1}{2\omega}\int_0^\omega q_2(\tau)d\tau$$

is valid.

2.6.2 Remark

2.9. Remark. *The expressons* $\frac{1}{2\omega}\int_0^\omega q_i(\omega)d\omega$ *(i = 1, 2) in the above theorem clearly depend on the choice of* $\mathrm{G}(t)$

The next theorem shows that by suitable choice of $\mathrm{G}(t)$ the estimate (2.46) may be made arbitraily sharp.

2.6.3 Yakubovich's estimate theorem

2.14. Theorem. *[Yakubovich]. Let* α_1 *and* α_n *be characteristic exponents of system* (2.17) *with the minimum and maximum real parts , respectively. For any* $\epsilon \geq 0$*, there is a matrix-function* $\mathrm{G}(t)$ *satisfying conditions formulated above such that*

$$\frac{1}{2\omega}\int_0^\omega q_1(\tau)d\tau \leq \ Re\ \alpha_1 < \frac{1}{2\omega}\int_0^\omega q_1(\tau)d\tau + \epsilon, \frac{1}{2\omega}\int_0^\omega q_2(\tau)d\tau - \epsilon \leq \frac{1}{2\omega}\int_0^\omega q_2(\tau)d\tau \tag{2.47}$$

If the coefficient matrix $\mathrm{A}(t)$ *is real, then* $\mathrm{G}(t)$ *may also be assumed real.*

Proof. Perform the substitution

$$x = \mathrm{P}(t)y. \tag{2.48}$$

in the equation (2.17), where $\mathrm{P}(t)$ is a nonsingular differentiable matrix-function.

The result is

$$\frac{dy}{dt} = \mathrm{B}(t)y. \tag{2.49}$$

where

$$\mathrm{B}(t) = \mathrm{P}^{-1}(t)\mathrm{A}(t)\mathrm{P}(t) - \mathrm{P}^{-1}(t)\frac{d\mathrm{P}}{dt}. \tag{2.50}$$

If $\mathrm{P}(t+T) \equiv \mathrm{P}(t)$, then clearly also $\mathrm{B}(t+T) \equiv \mathrm{B}(t)$. For any solution $x(t)$ of type $x = u(t)e^{\lambda t}$, where $u(t+T) \equiv u(t)$, we have a solution $y(t) = v(t)e^{\lambda t}$, where $v(t+T) \equiv v(t)$, and converserly, This equation (2.17) and (2.48) have the same characteristic exponents. If may happen that $\mathrm{B}(t+T) \equiv \mathrm{B}(t)$ although $\mathrm{P}(t)$ is not T-periodic (This is the case, for example, when $\mathrm{A}(t)$ is a real matrix, $\mathrm{B}(t) = K$ is a constant real matrix, and some of the multipliers of equation (2.17) are real and negative). However, if $\mathrm{P}(t)$

and $P^{-1}(t)$ are bounded, then, repeating the simple reasoning given above, we see that the real parts of the characteristic exponents of equations (2.17) and (2.49) concide.

The form $\xi = \langle Gx, x \rangle$, $\dot{\xi} = \langle Qx, x \rangle$ that is given in (2.41) become $\xi = \langle G_1 y, y \rangle$, and $\dot{\xi} = \langle Q_1 y, y \rangle$, where

$$G_1(t) = P^*(t)G(t)P(t), \text{and } Q_1(t) = P^*(t)Q(t)P(t). \tag{2.51}$$

Since

$$\det(Q_1 - qG_1) = |\det P|^2 det(Q - qG).$$

it follows that the equation

$$det(Q_1 - qG_1) = 0$$

has the same roots as equation (2.42) . Thus the integrals on the left and right of inequalities (2.46) for equation (2.17) and (2.49) are the same , provided that the corresponding matrices G and G_1 are related as in (2.51) where $P(t)$ is the matrix in (2.48). The fact that these integrals are the same for equation (2.17) and (2.49) and the real parts of the characteristic exponents coincide implies the following conclusion. Suppose we have been able to choose a matrix $G_1(t)$ for equation (2.49) so as to satify (2.47). Then there is a matrix $G(t)$ for equation (2.17) (defined by the first half of (2.51)) such that (2.47) is valid for equation (2.17) as well.

Suppose now that A and $G(t)$ are complex matrices. By Floquet theorem , there exists a T- periodic matrix $P(t)$ such that $B(t) = K$ ($= \Lambda$ in Floquet theorem) is a constant matrix. It follows that we need only prove the theorem for equations with constant coefficients. Thus we consider the equation

$$\frac{dy}{dt} = Ky. \tag{2.52}$$

Assume first that K may be reduced to diagonal form :

$$K = S^{-1}\,\mathrm{diag}[\alpha_1, \alpha_2, \ldots, \alpha_n]S. \tag{2.53}$$

where S is a nonsingular (complex) matrix, and the eigenvalues $\alpha_1, \ldots, \alpha_n$ are so numbered that $\mathrm{Re}\alpha_1 \leq \ldots \leq \mathrm{Re}\alpha_n$. Let

$$G_1 = S^*S \tag{2.54}$$

It is clear that G_1 is positive definite. By (2.41),

$$Q_1 = G_1 K + K^* G_1 = S^* \operatorname{diag}[2\mathrm{Re}\alpha_1, \ldots, 2\mathrm{Re}\alpha_n] S. \tag{2.55}$$

Since

$$\det(Q_1 - qG_1) = |\det S|^2 \prod_{j=1}^{n} (2\mathrm{Re}\alpha_j - q),$$

it follows that $q_1 = 2\mathrm{Re}\alpha_1$, $q_2 = 2\mathrm{Re}\alpha_n$. Thus we have (2.47) for equation (2.49) with $\epsilon = 0$. It follows from the foregoing that for equation (2.17) and $G = (P(t)^*)^{-1} S^* S P(t)^{-1}$ we again have $q_1 = 2\mathrm{Re}\alpha_1$, $q_2 = 2\mathrm{Re}\alpha_n$, so that (2.47) holds with $\epsilon = 0$. for equation (2.17) too.

Now suppose that K is not reducible to diagonal form. The proof follows the same general lines, but is rather more complicated. For any $\delta > 0$, there is a nonsingular matrix S_δ such that

$$K = S_\delta^{-1} \operatorname{diag}[Q_\delta(\alpha_{(1)}), \ldots Q_\delta(\alpha_{(s)})] S_\delta,$$

where $\alpha_{(j)}$ are the eigenvalues of K and $Q_\delta(\alpha)$ are the elementary Jordan matrices

$$Q_\delta(\alpha) = \begin{pmatrix} \alpha & \delta & 0 & \cdots & 0 & 0 \\ 0 & \alpha & \delta & \cdots & 0 & 0 \\ 0 & 0 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \alpha & \delta \\ 0 & 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}$$

We may assume that

$$\mathrm{Re}\alpha_{(1)} \leq \ldots \leq \mathrm{Re}\alpha_{(s)}.$$

Let

$$G_1 = S_\delta^* S_\delta. \tag{2.56}$$

Then

$$Q_1 = G_1 K + K^* G_1 = S_\delta^* \operatorname{diag}[2\mathrm{Re}Q_\delta(\alpha_{(1)}), \ldots, 2\mathrm{Re}Q_\delta(\alpha_{(s)})] S_\delta,$$

where ReA denotes the matrix

$$\mathrm{Re}A = \frac{A + A^*}{2}.$$

The roots of the equation

$$\det(Q_1 - qG_1) = |\det S|^2 \prod_{j=1}^{s} \det(2\mathrm{Re}Q_\delta(\alpha_{(j)}) - qI_j) = 0 \tag{2.57}$$

(where I_j are identity matrices) are the eigenvalues of all the matrices $2\mathrm{Re}Q_\delta(\alpha_{(j)})$. For small δ , the eigenvalues of the matrix

$$2\mathrm{Re}Q_\delta(\alpha) = \begin{pmatrix} 2\mathrm{Re}\alpha & \delta & 0 & \cdots \\ \delta & 2\mathrm{Re}\alpha & \delta & \cdots \\ 0 & \delta & 2\mathrm{Re}\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 2\mathrm{Re}\alpha \end{pmatrix}$$

are arbitrarily close to $2\mathrm{Re}\alpha$. Hence the roots of equation (2.57) are arbitrarily close to the numbers $2\mathrm{Re}\alpha_{(j)}$ for sufficiently small δ. Consequently, for any $\epsilon > 0$, we can find $\delta > 0$ such that the choice of (2.56) gives condition (2.47) for equation (2.49). As before, we arrive at the same conclusion for equation (2.17) by setting

$$G = [P(t)^*]^{-1} S_\delta^* S_\delta P(t)^{-1}.$$

Now suppose that $A(t)$ is a real matrix . It remains to prove that we can then choose $G(t)$ as a real and, as before, T-pediodic matrix. And according to Jordan form of real matrix, there exists a real nonsingular matrix $P(t)$ such that $B(t) = K$ is a constant real matrix in (2.49) and

$$P(t + T) = P(t)R, \tag{2.58}$$

where $R^2 = I$.

We consider the case when K has the simple elementary divisors. To fix ideas, let $\alpha_1$be real , and $\alpha_n = \beta + i\gamma$ complex eigenvalues. Then there exist a real nonsingular matrix S (Jordan form) such that $K = S^{-1}QS$, where

$$Q = \mathrm{diag}\left[\alpha_1, \cdots, \begin{pmatrix} \beta & \gamma \\ -\gamma & \beta \end{pmatrix}\right].$$

As before, we define G_1 by (2.54) . Then if

$$Q_1 = G_1K + K^*G_1 = S^*(Q + Q^*)S$$

we again get (2.55). As before, we see that $q_1 = \mathrm{Re}\alpha_1$, $q_2 = \mathrm{Re}\alpha_n$ for equation (2.49) , hence also for (2.17) and (2.47) is satisfied. Moreover

$$\mathrm{G} = [\mathrm{P}(t)^*]^{-1}\mathrm{S}^*\mathrm{S}\mathrm{P}(t)^{-1}$$

is a real coefficient matrix. We claim that it is also T-periodic. Using (2.58), we see that it will suffice to prove the equality

$$[\mathrm{R}^*]^{-1}\mathrm{S}^*\mathrm{S}\mathrm{R}^{-1} = \mathrm{S}^*\mathrm{S}. \tag{2.59}$$

Note that the matrix R may be defined as $+1$ or -1 on each of the root subspaces of K. This means that

$$\mathrm{R} = \mathrm{S}^{-1}\mathrm{Q}_0\mathrm{S},$$

where Q_0 is a diagonal matrix with diagonal elements ± 1 . Thus $\mathrm{Q}_0^2 = \mathrm{I}$. The desired relation (2.59) now follows from the expression for R , and so $\mathrm{G}(t+T) = \mathrm{G}(t)$.

If K has multiple elementary divisors, the reasoning is similar to that in the complex case with multiple elementary divisors. We then use the real Jordan form so that $\mathrm{G}(t)$ is T- periodic is proved as above.

□

2.6.4 Application for a second-oder system

Let us consider the case of a second-order system (2.17) with real coefficients, the matrix $\mathrm{G}(t)$ has the form:

$$\mathrm{G}(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}$$

Because $\mathrm{G}(t)$ is symmetrix, we can assume that $\mathrm{G}(t)$ is a real matrix-function. From the above conditions of $\mathrm{G}(t)$, $a(t)$, $b(t)$ and $c(t)$ are differentiable and ω-perodic functions, with

$$a(t) > 0, \quad c(t) > 0, \quad a(t)c(t) - b^2(t) > 0 \qquad (0 \le t \le \omega).$$

The matrix $\mathrm{Q}(t)$ is defined by (2.41), and the equation (2.42) can be written as

$$\det(\mathrm{Q}(t)\mathrm{G}^{-1}(t) - q\mathrm{E}) = 0,$$

or

$$q^2 - \mathrm{Sp}(\mathrm{QG}^{-1})q + \det(\mathrm{QG}^{-1}) = 0, \tag{2.60}$$

$$\mathrm{Sp}(\mathrm{QG}^{-1}) = \mathrm{Sp}(\dot{\mathrm{G}}\mathrm{G}^{-1} + \mathrm{GAG}^{-1} + \mathrm{A}^*) = \mathrm{Sp}(\dot{\mathrm{G}}\mathrm{G}^{-1}) + 2\mathrm{Sp}(\mathrm{A}).$$

We shall show that

$$\mathrm{Sp}(\dot{\mathrm{G}}\mathrm{G}^{-1}) = \frac{d}{dt}\ln\ \det(\mathrm{G}(t)).$$

Indeed, denoting $\mathrm{B} = \dot{\mathrm{G}}\mathrm{G}^{-1}$ and using the Ostrogradskii- Liouville formula, we have

$$\det(\mathrm{G}(t)) = \det(\mathrm{G}(0))e^{\int_0^t \mathrm{Sp}(\mathrm{B}(\tau))d\tau}.$$

whence follows the wanted equality. The solutions of equation (2.60) has the form

$$q_{1,2} = \frac{1}{2}(\mathrm{Sp}(\dot{\mathrm{G}}\mathrm{G}^{-1}) + 2\mathrm{Sp}(\mathrm{A})) \pm \left(\frac{1}{4}[\mathrm{Sp}(\dot{\mathrm{G}}\mathrm{G}^{-1}) + 2\mathrm{Sp}(\mathrm{A})]^2 - \det(\mathrm{Q}(t)\mathrm{G}^{-1}(t))\right)^{\frac{1}{2}}.$$

Hence, by the **Yakubovich theorem** , we obtain

$$\mathrm{Re}\,\alpha_{1,2} = \frac{1}{2\omega}\left[\int_0^\omega \mathrm{Sp}(\mathrm{A}(\tau))d\tau \pm \inf_G \int_0^\omega \left(\frac{1}{4}[\mathrm{Sp}(\dot{\mathrm{G}}(\tau)\mathrm{G}^{-1}(\tau)) + 2\mathrm{Sp}(\mathrm{A}(\tau))]^2 - \det(\mathrm{Q}(\tau)\mathrm{G}^{-1}(\tau))\right)^{\frac{1}{2}} d\tau\right]; \tag{2.61}$$

2.6.5 Example

2.13. Example. (Malkin.) *Consider the system*

$$\dot{x_1} = -\mu(1 - 2\sin(t))x_1 + \mu x_2,$$

$$\dot{x_2} = \mu x_1 - x_2.$$

the example in section 4 shows that the asymptotic stability of Malkin's system is guaranteed for $0 < \mu < \mu^*$ *, where*

$$0.67 < \mu^* < 0.68.$$

Now we shall show that the solutions of Malkin's system are asymptotically stable for $0 < \mu < \frac{2}{3} = 0.6666666$*. Define* $\mathrm{G} = \mathrm{E}_2$ *. Then ,*

$$\mathrm{Q} = \mathrm{A} + \mathrm{A}^* = 2\begin{pmatrix} \mu(-1 + 2\sin t) & \mu \\ \mu & -1 \end{pmatrix}$$

$$\det \mathrm{Q} = -4[\mu(-1 + 2\sin t) + \mu^2.$$

Formula (2.61) *gives*

$$Re\ a_{1,2} \leq \frac{1}{2}\left[-1 - \mu + \frac{1}{2\pi}\int_0^{2\pi} \sqrt{(1 + \mu(-1 + s\sin t))^2 + 4\mu^2 dt}\right].$$

For any function $\psi \geq 0$, *we have , by the Cauchy-Bunyakovskii inequality,*

$$\frac{1}{\omega}\int_0^{\omega} \sqrt{\psi(t)}dt \leq \left(\frac{1}{\omega}\int_0^{\omega} \psi(t)dt\right)^{\frac{1}{2}}.$$

Therefore,

$$Re\ \alpha_{1,2} \leq \frac{1}{2}(-1 - \mu + \sqrt{1 - 2\mu + 7\mu^2}).$$

The right-hand side of this inequality is negative for $0 < \mu < \frac{2}{3}$. *Consequently, the trivial solution of Malkin's system is asymptotically stable for* $0 < \mu < \frac{2}{3}$.

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