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Approximation in Function spaces

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Contents

Introduction

Many problems in mathematics can be answered easier by representing a function with a function sequence. The main aim of my thesis is to give a brief summary on certain approximation theorems, and give examples on solving problems with help of approximation.

In the first chapter we analyze the notion of uniform convergence on the Banach space of continuous functions. We are going to prove the first and second Weierstrass approximation theorems about uniform approximation of continuous functions by polynomials, resp. trigonometric polinomials. After that we prove that continuous functions are dense in certain Banach spaces as well, and as an application prove a theorem on the convergence of quadrature formulas.

The second chapter is mainly about Fourier series. After proving well known theorems, we will consider different convergence notions, and give a sufficient condition for f to have its Fourier series converge uniformly. We will prove a theorem on Fourier coefficients, namely the Riemann-Lebesgue lemma and Fejér's theorem as well.

In the third chapter we prove another sufficient condition for f to have a convergent Fourier series, and solve two partial differential equations, namely the wave and heat equation in one dimension with the help of Fourier series.

Chapter 1

Elementary Approximation

Let $I \subseteq \mathbb{R}$ be an arbitrary interval, and $f : I \to \mathbb{R}$ be an arbitrary function, and the following $(T_n)_{n,a}(x)$ linear operator the following: for every $f \in C^\infty$, and for every $n \in \mathbb{N}, \quad a \in I$:

$$T_{n,a}(f)(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a) \cdot x^j}{j!}.$$

According to the Taylor-theorem, if f is analytical in a, $\lim_{n\to\infty} T_{n,a}(f) = f$. During this chapter, we will prove, that if $f \in C(I)$, where I is a compact interval, it can be approximated, prove, that periodic functions can be approximated as well, and we will show, that approximation can be done in compact metric spaces.

1.1 Weierstrass Theorems

Definition 1.1.1. An $f : \mathbb{R} \longrightarrow \mathbb{R}$ function is affine, if there exists $\alpha, \beta \in \mathbb{R}$, such that $f(x) = \alpha x + \beta$.

Notation 1.1.2. Let I be a compact interval and $f : I \to \mathbb{R}$ be a bounded function

$$\|f\|_\infty := \sup_{t\in I} |f(t)|$$

Two reminders about uniform convergence:

Definition 1.1.3. Let $f_n,\ f$, be $E \to \mathbb{R}$ function. We say $f_n \to f$ uniformly, if

$$(\forall \varepsilon \in \mathbb{R})\ (\exists N \in \mathbb{N}^+)\ (\forall x \in E)\ (\forall n \geq N)\colon |f_n(x) - f(x)| \leq \varepsilon.$$

It is a well known fact, that uniform convergence is metrizable, namely,

$$f_n \to f \text{ uniformly} \iff \|f_n - f\|_\infty \to 0.$$

Theorem 1.1.4 (Weierstrass Approximation Theorem)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuos function. Then there exists a $(p_n)_{n\in\mathbb{N}}$ polynomial sequence, such that $p_n \longrightarrow f$ uniformly on $[a,b]$.*

Proof. Let $R = b - a$ and define

$$q(t) = \begin{cases} R^2 - t^2 & , |t| \leq R \\ 0 & , |t| > R \end{cases}$$

Lemma 1.1.5. *For every $\delta \in \mathbb{R}^+$:* $\lim\limits_{n\to\infty} \dfrac{\displaystyle\int_{|t|\geq\delta} q(t)^n dt}{\displaystyle\int_{\mathbb{R}} q(t)^n dt} = 0$

Proof. Let $\delta \geq R$. In this case $\displaystyle\int_{|t|\geq\delta} q(t)^n dt = 0$, since $q(t) = 0 \ \forall t \in \{|t| \geq R\}$. Let $\delta \leq R$. In this case q has the following properties:

1. is continuos
2. even
3. positive on the $(0, R)$ open intervall
4. strictly decreasing in $(0, R)$ and 0 in $\mathbb{R} \setminus (-R, R)$

All of these properties can be read from the definition of q.

$$\int_{|t|\geq\delta} q(t)^n dt \leq (2R - 2\delta) \cdot q(\delta)^n \leq 2Rq(\delta)^n \tag{1.1}$$

$$\int_{\mathbb{R}} q(t)^n dt \geq \int_{|t|\leq\frac{\delta}{2}} q(t)^n dt \geq \delta q\left(\frac{\delta}{2}\right)^n \tag{1.2}$$

Equation (1.1) is true, due to monotonicity, and the length of the interval. In equation (1.2) the integral is less because: $\{| \ t \ |\leq \frac{\delta}{2}\} \subset \mathbb{R}$. Combining these we obtain

$$0 \leq \frac{\displaystyle\int_{|t|\geq\delta} q(t)^n dt}{\displaystyle\int_{\mathbb{R}} q(t)^n dt} \leq \frac{2R}{\delta}\left(\frac{q(\delta)}{q(\frac{\delta}{2})}\right)^n$$

and since $0 \leq q(\delta) \leq q\left(\frac{\delta}{2}\right)$, we get:

$$\lim_{n\to\infty} \frac{2R}{\delta}\left(\frac{q(\delta)}{q(\frac{\delta}{2})}\right)^n = 0$$

□

Now back to proving the Weierstrass theorem. We shall assume, that:

$$f(a) = f(b) = 0, \tag{1.3}$$

and $f(t) = 0\ \forall t \in \mathbb{R} \setminus [a, b]$. If not, by adding affine functions we can achieve it. Since $[a, b]$ is a compact interval and f is continuous, we know that f is uniformly continuous, due to Heine's theorem. Let:

$$\omega(f, \delta) = \sup\{|f(x) - f(t)| : |x - t| \leq \delta\}. \tag{1.4}$$

Due to uniform continuity:

$$\lim_{\delta \to 0} \omega(f, \delta) = 0. \tag{1.5}$$

Let:

$$c_n = \int_{\mathbb{R}} q(t)^n dt \qquad \text{and} \qquad Q_n(t) = \frac{1}{c_n} \cdot q(t)^n \qquad \forall n \in \mathbb{N}^+, \tag{1.6}$$

which has the following properties:

i) $Q_n \geq 0$, since $q(t) \geq 0$ for every $t \in \mathbb{R}$.

ii) $Q_n(t) = 0\ \forall t \in \mathbb{R}$, if $|t| \geq R$

iii) $\displaystyle\int_{\mathbb{R}} Q_n(t)dt = \frac{1}{cn} \int_{\mathbb{R}} q(t)^n dt = \frac{c_n}{c_n} = 1$

iv) $\displaystyle\lim_{n \to \infty} \int_{|t| \geq \delta} Q_n(t)dt = 0$ according to Lemma 1.1.5.

Let $p_n(x) = \displaystyle\int_{\mathbb{R}} f(t)Q_n(x - t)dt\ \forall x \in \mathbb{R}$

$$\begin{aligned}
|f(x) - p_n(x)\ | &= \left| \int_{\mathbb{R}} f(x) - f(t)Q_n(x - t)dt \right| \\
&= \left| \int_{\mathbb{R}} f(x)Q_n(x - t) - f(t)Q_n(x - t)dt \right| \\
&= \left| \int_{\mathbb{R}} (f(x) - f(t))Q_n(x - t)dt \right| \\
&\leq \int_{|x-t| \leq \delta} |f(x) - f(t)|Q_n(x - t)dt + \int_{|x-t| \geq \delta} |f(x) - f(t)|Q_n(x - t)dt \\
&\leq \omega(f, \delta) + 2 \|f\|_\infty \int_{|s \geq \delta} Q_n(s)ds.
\end{aligned}$$

1. we should choose $\delta > 0$ such, that $\omega(f, \delta) \leq \frac{\varepsilon}{2}$.

2. And let N be such, that $\forall n \geq N$: $2 \|f\|_\infty \displaystyle\int_{|s| \geq \delta} Q_n(s)ds \leq \frac{\varepsilon}{2}$.

Then, $|f(x) - p_n(x)| \le \varepsilon \ \forall x \in [a, b] \ \forall n \ge N$, which means $p_n \to f$ uniformly.
The only thing left, is to show, that (p_n) is a polynomial for every $n \in \mathbb{N}$. Since $f(x) = 0$ $\forall x \in \mathbb{R} \setminus [a, b]$ the following is true:

$$p_n(x) = \int_{\mathbb{R}} f(t)Q_n(x-t)dt = \int_{x-R}^{x+R} f(t)\frac{1}{c_n}(R^2 - (x-t)^2)^n dt$$
$$= \int_a^b f(t)\frac{1}{c_n}(R^2 - (x-t)^2)^n dt.$$

We can see, that the integrand is a polynomial of x, and according to the Newton-Leibniz theorem: it's integral will also be a polynomial of x. □

Corollary 1.1.6. This also proves, that $C([a, b])$ is separable, since polynomials with rational coefficienses are:

1. dense in $C([a, b])$
2. countable.

Now we will show a similar achievement, by Weierstrass for periodic functions.

Definition 1.1.7. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. We say, that $f \in C_{2\pi}$, if f is a continuous, periodic function with a period length of 2π.

Definition 1.1.8. Let $p : \mathbb{R} \longrightarrow \mathbb{R}$. We say, that p is a trigonometrical polynomial, if $p(x) = \sum_{j=0}^{n} a_j \sin(jx) + b_j \cos(jx) \ \forall x \in \mathbb{R}$,where $a_j, \ b_j \in \mathbb{R}$.

Theorem 1.1.9 (Weierstrass second approximation theorem)**.** *For every $f \in C_{2\pi}$ there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of trigonometric polynomials, such that $p_n \to f$ uniformly.*

Proof. Let

$$q(t) = \begin{cases} 1 + \cos(t), & |t| \le \pi \\ 0, & |t| > \pi \end{cases}$$

Lemma 1.1.10. *For every $\delta > 0$,* $\lim_{n\to\infty} \frac{\int_{|t|\ge\delta} q(t)^n dt}{\int_{\mathbb{R}} q(t)^n dt} = 0$

Proof. It is the same, like in Lemma 1.1.5. Let $\delta \ge \pi$. In this case $\int_{|t|\ge\delta} q(t)^n dt = 0$, since $q(t) = 0 \ \forall t \in \{|t| \ge \pi\}$.
Let $0 < \delta \le \pi$. In this case q has the following properties:

1. is continuous

2. even
3. positive on the $(0, \pi)$ open interval
4. strictly decreasing in $(0, \pi)$ and 0 in $\mathbb{R} \setminus (-\pi, \pi)$

All of these properties can be read from the definition of q.

$$\int_{|t|\geq\delta} q(t)^n dt \leq (2\pi - 2\delta) \cdot q(\delta)^n \leq 2\pi q(\delta)^n \tag{1.7}$$

$$\int_{\mathbb{R}} q(t)^n dt \geq \int_{|t|\leq\frac{\delta}{2}} q(t)^n dt \geq \delta q\left(\frac{\delta}{2}\right)^n \tag{1.8}$$

Equation (1.7) is true, due to q's monotonicity, and the length of the interval. In equation (1.8) the integral is less because: $\{| \ t \ |\leq \frac{\delta}{2}\} \subset \mathbb{R}$. combining these:

$$0 \leq \frac{\int_{|t|\geq\delta} q(t)^n dt}{\int_{\mathbb{R}} q(t)^n dt} \leq \frac{2\pi}{\delta}\left(\frac{q(\delta)}{q(\frac{\delta}{2})}\right)^n$$

and since:$0 \leq q(\delta) \leq q\left(\frac{\delta}{2}\right)$ we get:

$$\lim_{n\to\infty} \frac{2\pi}{\delta}\left(\frac{q(\delta)}{q(\frac{\delta}{2})}\right)^n = 0$$

□

We shall assume, that:

$$f(-\pi) = f(\pi) = 0, \tag{1.9}$$

Since $[-\pi, \pi]$ is a compact interval, and f is continuos, we know that, f is uniformly continous, due to Heine's theorem. Let:

$$\omega(f, \delta) = \sup\{|f(x) - f(t)| : |x - t| \leq \delta\}. \tag{1.10}$$

Due to uniform continuity:

$$\lim_{\delta\to 0} \omega(f, \delta) = 0. \tag{1.11}$$

Let:

$$c_n = \int_{\mathbb{R}} q(t)^n dt \text{ and } Q_n(t) = \frac{1}{c_n} \cdot q(t)^n \quad \forall n \in \mathbb{N}^+, \tag{1.12}$$

which has the following properties:

i) $Q_n \geq 0$, since $q(t) \geq 0 \ \forall t \in \mathbb{R}$.

ii) $Q_n(t) = 0$ for every $t \in \mathbb{R}$, $|t| \geq R$

iii) $\int_{\mathbb{R}} Q_n(t)dt = \frac{1}{c_n} \int_{\mathbb{R}} q(t)^n dt = \frac{c_n}{c_n} = 1$

iv) $\lim_{n\to\infty} \int_{|t|\geq\delta} Q_n(t)dt = 0$ according to the Lemma.

Let $p_n(x) = \int_{\mathbb{R}} f(t)Q_n(x-t)dt \ \forall x \in \mathbb{R}$

$$\begin{aligned}
|f(x) - p_n(x)\ | &= \left| \int_{\mathbb{R}} f(x) - f(t)Q_n(x-t)dt \right| \\
&= \left| \int_{\mathbb{R}} f(x)Q_n(x-t) - f(t)Q_n(x-t)dt \right| \\
&= \left| \int_{\mathbb{R}} (f(x) - f(t))Q_n(x-t)dt \right| \\
&\leq \int_{|x-t|\leq\delta} |f(x) - f(t)|Q_n(x-t)dt + \int_{|x-t|\geq\delta} |f(x) - f(t)|Q_n(x-t)dt \\
&\leq \omega(f,\delta) + 2\,\|f\|_\infty \int_{|s\geq\delta} Q_n(s)ds.
\end{aligned}$$

1. we should choose $\delta \geq 0$ such, that $\omega(f,\delta) \leq \frac{\varepsilon}{2}$.

2. And let N be such, that $\forall n \geq N$: $2\,\|f\|_\infty \int_{|s|\geq\delta} Q_n(s)ds \leq \frac{\varepsilon}{2}$.

Therefore, $|f(x) - p_n(x)| \leq \varepsilon \ \forall x \in \mathbb{R} \ \forall n \geq N$. The only thing left is to show that $(p_n)_{n\in\mathbb{N}}$ is a trigonometrical polynomial.

Lemma 1.1.11. *If $p, q \in C_{2\pi}$ are trigonometrical polynomials, then $pq \in C_{2\pi}$ is also a trigonometrical polynomial.*

Proof. Since $p(x) = \sum_{j=0}^{n} a_j \sin(jx) + b_j \cos(jx)$, and $q(x) = \sum_{j=0}^{m} a_j \sin(jx) + b_j \cos(jx)$, it is enough to show, that

$$\cos(mt)\cos(nt), \sin(mt)\sin(nt), \cos(mt)\sin(mt)$$

are also trigonometric polynomials.

i) $\cos(nt)\cos(mt) = \frac{\cos((m-n)t)+\cos((n+m)t)}{2}$

ii) $\sin(nt)\sin(mt) = \frac{\cos((m-n)t)-\cos((n+m)t)}{2}$

iii) $\sin(nt)\cos(mt) = \frac{\cos((m-n)t)-\cos((n+m)t)}{2}$

due to the additional formulas. □

Exploiting this, and the definition of Q_n we obtain:

$$\begin{aligned} p_n(x) &= \int_{\mathbb{R}} f(t)Q_n(x-t)dt = \frac{1}{c_n}\int_{x-\pi}^{x+\pi} f(t)(1+cos(x-t))^n, \\ &= \frac{1}{c_n}\int_{-\pi}^{\pi} f(t)(1+cos(x-t))^n, \\ &= \frac{1}{c_n}\int_{-\pi}^{\pi} f(t)(1+cos(x)cos(t)+sin(x)sin(t))^n, \\ &= \sum_{j=0}^{n} a_k\cos(kx)+b_k\sin(kx). \end{aligned}$$

□

1.2 The Stone-Weierstrass Theorem

We have seen that every $f \in C([a,b])$ can be approximated, where $[a,b]$ is a compact interval. We proceed by proving density of continuous functions in certain metric spaces, with the help of our previous results.

Definition 1.2.1. Let K be a compact metric space. A linear subset $M \subset C(K)$ is a subalgebra, if for every $f, g \in M$, $fg \in M$ stands.

Definition 1.2.2. Let K be a compact metric space, and $M \subset C(K)$ a subspace . If for every $f, g \in M$, implies that $f \wedge g$, and $f \vee g \in M$, M is a vector lattice, where

$$f \wedge g(t) = \max(f(t), g(t)), \text{ and } f \vee g(t) = \min(f(t), g(t)).$$

Remark 1.2.3. Polynomials and trigonometric polynomials form a subalgebra in $C[a,b]$.

Proposition 1.2.4. Let K be a compact metric space, and $M \subset C(K)$ be a subspace. If for every $h \in M$, $|h| \in M$ stands, then M is a vector lattice.

Proof. We need to show, that $f \wedge g$ and $f \vee g$ is in M, which is true since

$$f \wedge g(t) = \max_K\{f(t), g(t)\} = \begin{cases} f(t), & f(t) \geq g(t) \\ g(t), & g(t) > f(t) \end{cases} = \begin{cases} f(t), & f(t)-g(t) \geq 0 \\ g(t), & g(t)-f(t) > 0, \end{cases}$$

and

$$f \vee g(t) = \min_K\{f(t), g(t)\} = \begin{cases} f(t), & f(t) \leq g(t) \\ g(t), & g(t) < f(t) \end{cases} = \begin{cases} f(t), & f(t)-g(t) \leq 0 \\ g(t), & g(t)-f(t) < 0, \end{cases}$$

from which we get, that $f \wedge g = \dfrac{f+|f-g|+g}{2}$, and $f \vee g = \dfrac{f-|f-g|+g}{2}$. □

Theorem 1.2.5 (Stone-Weierstrass Theorem)**.** *Let K be a compact, metric space. If $M \subset C(K)$ is a subalgebra, for which stands:*

i) every constant function is in M,

ii) M separates the points of K, which means:

$$\forall x, y \in K \; \exists h \in M : h(x) \neq h(y),$$

then M is dense in $C(K)$.

Proof. First we will show, that: $f_n \to f, g_n \to g \Longrightarrow f_n g_n \to fg$ uniformly, as $n \to \infty$.

$$\|fg - f_n g_n\|_\infty \leq \|f - f_n\|_\infty \|g\|_\infty + \|f_n\|_\infty \|g - g_n\|_\infty \to 0. \tag{1.1}$$

From this we know, that

1. M is a subalgebra
2. The closure: $\overline{M}$ is also a subalgebra.

Now we will show, that $\overline{M}$ is a vector lattice. Let $h \in M$ be aribtary, and $T \in \mathbb{R}$, such that $\|h\|_\infty \leq T$. Since M is a vector space we only have to show, that $|h| \in \overline{M}$ stands. Let $I := [-T, T]$, we wish to approximate h on I.
According to the first Weierstrass approximation theorem:

$$\exists (p_n)_{n \in \mathbb{N}} : \lim_{n \to \infty} p_n = |x|. \tag{1.2}$$

$p_n \circ h \in \overline{M}$ stands, since $p_n(h) = \sum_{j=0}^{n} a_j h^j$, and $\overline{M}$ is a closed subalgebra.
Let $f \in C(K), \varepsilon \in \mathbb{R}^+$ be arbitrary, and $h(t)$, such that $h(x) \neq h(y)$, for every $x, y \in K$, if $x \neq y$. We define the function f_{xy} for every $x, y \in K$:

$$f_{xy}(t) := f(x) \frac{h(t) - h(y)}{h(x) - h(y)} + f(y) \frac{h(t) - h(x)}{h(y) - h(x)}. \tag{1.3}$$

Therefore:$f_{xy}(x) = f(x)$ and $f_{xy}(y) = f(y)$. We continue by defining the following set:

$$U_y := \{z \in K : f_{xy}(z) > f(z) - \varepsilon\} \qquad \forall y \in K. \tag{1.4}$$

i) $K \subset \bigcup_{y \in K} U_y$

ii) U_y is an open set for every $y \in K$

iii) K is compact

Because of these, we can choose a finite semicover: $K \subset \bigcup_{i=1}^{n} U_{y_i}$, and let $f_x := \max_{i=1,\ldots,n} \{f_{xy_i}\}$.From this we know, that $f_x \geq f - \varepsilon$, and $f_x(x) = f(x)$.
Let

$$V_x := \{z \in K : f_x(z) < f(z) + \varepsilon\} \qquad \forall x \in K \tag{1.5}$$

This is an open set, K is compact, we can choose a finite semicover, V_{x_i} $(i = 1..n)$, and $g := \min_{i=1,\ldots,n} \{f_{x_i}\}$. Combining these:

$$\forall t \in K : f(t) - \varepsilon \leq g(t) \leq f(t) + \varepsilon, \tag{1.6}$$

since g is in both U and V. but this also means that:

$$\|f - g\|_\infty \leq \varepsilon \Longrightarrow \qquad g \to f \text{ uniformly.}$$

□

1.3 Korovkin's First Theorem

Definition 1.3.1. An operator $T \in L(C([0,1]), C([0,1]))$ is positive, if for every $f \in C(K)$, for which $f(t) \geq 0$ for every $t \in K$, then $Lf(t) \geq 0$ is also true.

Theorem 1.3.2 (Korovkin's First Theorem)**.** *Let $(T_n)_{n\in\mathbb{N}} \in L(C(K), C(K))$ be a positive operator sequence, and set $x_i(t) := t^i (i = 0, 1, 2)$. If $T_n(x_i) \to x_i$ for $i = 0, 1, 2$ then $T_n(x) \to x$ uniformly for every $x \in C[0, 1]$.*

Proof. Let $x \in C([0, 1])$ be arbitrary. Since $[0, 1]$ is compact, and x is continous, it is also uniformly continuous. Which means:

$$(\forall \varepsilon \geq 0)\ (\exists \delta \geq 0) : |s - t| \leq \delta \Longrightarrow |x(s) - x(t)| \leq \varepsilon \tag{1.1}$$

We shall show, that: $|x(s) - x(t)| \leq \varepsilon + \alpha(t - s)^2$ for a suitable α.
Let $\alpha = \frac{2\|x\|_\infty}{\delta}$. If $|t - s| \leq \sqrt{\delta}$, than $|x(s) - x(t)| \leq \varepsilon$ is true, due to the uniform continuity of x. If $|t - s| \geq \sqrt{\delta}$:

$$|x(s) - x(t)| \leq |x(s)| + |x(t)| < \varepsilon + 2\|x\|_\infty = \varepsilon + \frac{2\|x\|_\infty \delta}{\delta} < \varepsilon + \alpha(t - s)^2. \tag{1.2}$$

By defining $y_t(s) := (t - s)^2$ (1.2) can be rewritten as:

$$-\varepsilon - \alpha y_t \leq x - x(t) \leq \varepsilon + \alpha y_t \ \forall t \in [0, 1] \tag{1.3}$$

Now we apply the T_n operator sequence, and use it's positivity:

$$-\varepsilon T_n(x_0) - \alpha T_n(y_t) \leq T_n(x) - x(t)T_n(x_0) \leq \varepsilon T_n(x_0) + \alpha T_n(y_t) \ \forall t \in [0, 1], \tag{1.4}$$

which shows:

$$|T_n(x) - x(t)T_n(x_0)| \leq |\varepsilon T_n(x_0) + \alpha T_n(y_t)| \ \forall t \in [0,1], \tag{1.5}$$

First we will prove pointwise convergence, therefore let t be an arbitrary point from $[0, 1]$.

$$|T_n(x)(t) - x(t)T_n(x_0)(t)| \leq |\varepsilon T_n(x_0)(t) + \alpha T_n(y_t)(t)| = \varepsilon,$$

since $T_n(x_i) \to x_i$ $(i = 0, 1, 2)$, and $y_t(t) = 0$, which delivers the pointwise convergence for every continous function in $[0, 1]$. The uniform converge is true as well, since:

$$\begin{aligned} y_t = \alpha x_0 + \beta x_1 + \gamma x_2 \qquad &\text{therefore:} \quad T_n y_t - y_t = T_n((t-s)^2) - (t-s)^2 \\ = T_n(\alpha x_0 + \beta x_1 + \gamma x_2) &- \alpha x_0 + \beta x_1 + \gamma x_2 \\ = \alpha(T_n(x_0) - x_0) &+ \beta(T_n(x_1) - x_1) + \gamma(T_n(x_2) - x_2), \end{aligned}$$

since x_0, x_1, x_2 span the quadratic polynomials. From this we can see that $T_n(y_t) \to y_t$ uniformly, therefore $\forall x \in C[0, 1]\colon T_n x \to x$ uniformly. □

We proceed by proving a theorem on quadrature formulas, for which we mention the well known Banach-Steinhaus theorem, without a proof, which can be read in [3].

Theorem 1.3.3. *Let X be a Banach space, Y be a normed space. An $(A_n)_{n\in\mathbb{N}} \in L(X, Y)$ operator sequence is pointwise bounded if and only if $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded.*

With the help of this, we can prove a theorem of Szegő, namely:

Theorem 1.3.4 (Szegő)**.** *Let $Q_n(x) \in L(C([0, 1]), \mathbb{R})$ be for every $x \in C([0, 1])$ the following: $Q_n(x) = \sum_{i=0}^{n} \alpha_i x(t_i)$, where $\alpha_i \in \mathbb{R}$, $t_i \in [0, 1]$, and $t_i \neq t_j$, if $i \neq j$. The following are equivalent:*

i) $Q_n(x) \to \int_0^1 x(t)dt$ *for every* $x \in C([0, 1])$,

ii) $Q_n(p) \to \int_0^1 p(t)dt$ *for every p polynomial, and* $\sup_n \sum_{i=0}^{n} |\alpha_i| < \infty$.

Proof. We start with showing: $\|Q_n\| = \sum_{i=0}^{n} |\alpha_i|$. Since $Q_n \in L(C([0, 1]), \mathbb{R})$,

$$\|Q_n\| = \inf\{c : |Q_n(x)| \leq c \|x\|_\infty \ \forall x \in C([0, 1])\},$$

the following is true:

$$|Q_n(x)| = \left| \sum_{i=0}^{n} x(t_i)\alpha_i \right| \leq \sum_{i=0}^{n} |x(t_i)||\alpha_i| \leq \|x\|_\infty \sum_{i=0}^{n} |\alpha_i|$$

and $|Q_n(1)| = \sum_{i=0}^{n} |\alpha_i|$, therefore $\|Q_n\| = \sum_{i=0}^{n} |\alpha_i|$. We proceed by showing $i) \Rightarrow ii)$. Since every p polynomial is continuous we only have to show, that $\sup_n \sum_{i=0}^{n} |\alpha_i| < \infty$, which stands since $C([0,1])$ is a Banach space, $\mathbb{R}$ is a normed space, and Q_n is pointwise bounded, thus according to the Banach-Steinhaus theorem Q_n is uniformly bounded, i.e. $\sup \|a_n\| < \infty$.

We continue by showing, implication $ii) \Rightarrow i)$. Namely for every polynomial p, and every continuous function x the following stands:

$$Q_n(p) \to \int_0^1 p(t)dt \qquad \sum_{i=0}^{n} |\alpha_i| < \infty \Rightarrow Q_n(x) \to \int_0^1 x(t)dt.$$

Due the the continuity of x Theorem 1.1.4 can be used: let $p_n : [0,1] \to \mathbb{R}$, such polynomial that $\|p_n - x\|_\infty \to 0$, as $n \to \infty$. Therefore:

$$|Q_n(p_n) - Q_n(x)| = \left| \sum_{i=0}^{n} \alpha_i (p_n(t_i) - x(t_i)) \right| \leq \|p_n - x\|_\infty \sum_{i=0}^{n} |\alpha_i| \leq \sum_{i=0}^{n} |\alpha_i| \varepsilon, \qquad (1.6)$$

thus $Q_n(p_n) \to Q_n(x)$, as $n \to \infty$. We proceed by showing:

$$\int_0^1 p_n(t)dt \to \int_0^1 x(t)dt, \text{ as } n \to \infty,$$

which stands due to:

$$\left| \int_0^1 p_n(t) - \int_0^1 x(t)dt \right| \leq \int_0^1 |p_n(t) - x(t)|dt \leq \varepsilon. \qquad (1.7)$$

From (1.6), and (1.7) we get $i)$. □

Here we mention Korovkin's second theorem without proving. It will be proved in Chapter 2.

Theorem 1.3.5 (Korovkin's Second Theorem)**.** *Let* $(T_n)_{n\in\mathbb{N}} \in L(C_{2\pi}, C_{2\pi})$ *be a positive operator sequence, and* $x_0(t) := 1$ $x_1(t) := \cos(t)$ $x_2(t) := \sin(t)$. *Then*

$$T_n(x_i) \to x_i (i = 0, 1, 2) \Longrightarrow T_n x \to x \qquad \forall x \in C_{2\pi}.$$

We have seen certain approximation methods already, but none of them could be used in practice. The next theorem will give a new, constructive proof for Theorem 1.1.4.

Definition 1.3.6. Let I be a closed, bounded interval in $\mathbb{R}$,and $f \in C(I)$.

$$(B_n f)(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \qquad \forall x \in I \ \forall n \in \mathbb{N}^+. \qquad (1.8)$$

$B_n f(x)$ is the so called n-th Bernstein polynomial.

Theorem 1.3.7 (Bernstein's Approximation Theorem)**.** *For every $f \in C(I)$, $B_n(f) \to f$ uniformly:*

Proof. We assume, that $I = [0, 1]$ for easier calculation. B_n is linear, since:

$$(B_n(cf))(x) := \sum_{k=0}^{n} \binom{n}{k} cf\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \tag{1.9}$$

$$= c\sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} = c(B_n f)c \in \mathbb{R} \tag{1.10}$$

$$(B_n(f+g))(x) := \sum_{k=0}^{n} \binom{n}{k} \left(f\left(\frac{k}{n}\right) + g\left(\frac{k}{n}\right)\right) x^k (1-x)^{n-k} \tag{1.11}$$

$$= \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} + \sum_{k=0}^{n} \binom{n}{k} g\left(\frac{k}{n}\right) x^k (1-x)^{n-k} = (B_n f) + (B_n g) \ g \in C(I). \tag{1.12}$$

B_n is positive, since if $f \geq 0$, $B_n f(x) \geq 0$ holds. Due to the binomial theorem:

$$(B_n 1)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + 1 - x)^n = 1$$

$$\begin{aligned}(B_n id)(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} = x(x+1-x)^{n-1} = x.\end{aligned}$$

And according to Korovkin's first theorem, by proving convergence for id^2, we will get the result, that B_n converges on $[0, 1]$. We start, by showing:

$$\begin{aligned}B_n\left(id^2 - \frac{id}{n}\right)(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k(k-1)}{n^2} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \frac{n!k(k-1)}{k!(n-k)!n^2} x^k (1-x)^{n-k} \\ &= \frac{n-1}{n} \sum_{k=2}^{n} \binom{n-2}{k-2} x^k (1-x)^{n-k} \\ &= \frac{n-1}{n} x^2\end{aligned}$$

From this, and the fact, that B_n is a linear operator we get:

$$B_n(id^2) = \frac{n-1}{n} id^2 + \frac{id}{n}.$$

and finally:

$$\left\|id^2 - B_n(id^2)\right\|_\infty = \frac{1}{n}\left\|id^2 - id\right\|_\infty \to 0,$$

therefore $B_n(f) \to f$ uniformly for every $f \in C([0,1])$. $\square$

Chapter 2

Fourier series

It is a well known fact from elementary analysis, that not every $f : \mathbb{R} \to \mathbb{R}$ integrable function's Fourier series converge uniformly. In this chapter we will inspect, under what conditions can uniform convergence be achieved, and inspect the general theory of Fourier series.

2.1 Basics

Definition 2.1.1. Let $f \in C_{2\pi}$ .The Fourier series of f is the following:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \text{ and } b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt.$$

Definition 2.1.2. a_i, b_j,where $i \in \mathbb{N}$, $j \in \mathbb{N}^+$, are called the corresponding Fourier coefficients.

Remark 2.1.3. In order to determine, whether we are talking about a series, or the sum of the series, we will write $\sum_{j \in \mathbb{N}} f_j$, when we mean the series, and write$\sum_{j=0}^{\infty} f_j$, when we mean the sum of the series.

Theorem 2.1.4. *Let* $a_0 + \sum_{k \in \mathbb{N}^+} a_k \cos(kt) + b_k \sin(kt)$ *be uniformly convergent on* $\mathbb{R}$. *if* $a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt) = f(t)$, *then* f *is continuous, and*

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt,$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \ , \ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt.$$

We will need two lemmas to prove this.

Lemma 2.1.5. *If $f_n \to f$ uniformly on $H \subset \mathbb{R}$, and f_n is continuous in $a \in H \implies f$ is continuous in $a \in H$.*

Proof. We have to show, that:

$$(\forall \varepsilon \geq 0)\ (\exists \delta \geq 0)\ :\ |t-a| \leq \delta \Rightarrow |f(t) - f(a)| \leq \varepsilon.$$

To this aim fix $\varepsilon > 0$, then by exploiting uniform convergence there is $N \in \mathbb{N}$, so that $|f_n(t) - f(t)| \leq \frac{\varepsilon}{3}$ for each $n > N$ and $t \in H$.
By exploiting f_n's continousity, let $\delta > 0$, such that, $|f_N(t) - f_N(a)| \leq \dfrac{\varepsilon}{3}$,
if $|t - a| \leq \delta$. Combining these:

$$\begin{aligned}|f(t) - f(a)| =& |f(t) - f_n(t) + f_n(t) - f_n(a) + f_n(a) - f(a)| \\ \leq& |f(t) - f_n(t)| + |f_n(t) - f_n(a)| + |f_n(a) - f(a)| \\ \leq& \varepsilon.\end{aligned}$$

Which proves f's continuousity. □

Lemma 2.1.6. *Let $n \in \mathbb{N}^+$.*

i) $\displaystyle\int_{-\pi}^{\pi} \sin^2(nt)dt = \int_{-\pi}^{\pi} \cos^2(nt)dt = \pi$

ii) $n \neq m \displaystyle\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0$

iii) $n \neq m \displaystyle\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = \int_{\pi}^{\pi} \cos(nt)\cos(mt)dt = 0$

Proof.

$$\int_{-\pi}^{\pi} \sin^2(nt)dt = \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos(2nt)}{2} dt = \left[\frac{t}{2} - \frac{\sin(2nt)}{4n}\right]_0^{2\pi} = \pi$$

$$\int_{-\pi}^{\pi} \cos^2(nt)dt = \int_{-\pi}^{\pi} 1 - \sin^2(nt)dt = 2\pi - \pi,$$

thus (i) is proved.

$$\int_{-\pi}^{\pi} \cos(mt)\sin(nt)dt = \left[\frac{m\sin(mt)\sin(nt) + n\cos(mt)\cos(nt)}{m^2 - n^2}\right]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \left[\frac{m\sin(mt)\cos(nt) - n\cos(mt)\sin(nt)}{m^2 - n^2}\right]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \left[\frac{n\sin(mt)\cos(nt) - m\cos(mt)\sin(nt)}{m^2 - n^2}\right]_{-\pi}^{\pi} = 0,$$

since sin and cos $\in C_{2\pi}$. □

Now we proceed, by proving Theorem 2.1.4.

Proof. Since $a_0 + \sum_{k\in\mathbb{N}^+} a_k \cos(kt) + b_k \sin(kt)$ is continuous, by exploiting Lemma 2.1.5, f is continuous, and according to the theorem on integrating uniformly convergent sequences, both side of the equation: $a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt) = f(t)$ can be integrated. By doing so we get:

$$\int_{-\pi}^{\pi} a_0 + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt) dt = \int_{-\pi}^{\pi} f(t)dt, \tag{2.1}$$

which gives us: $a_0 = \int_{-\pi}^{\pi} \frac{f(t)}{2\pi} dt$. We proceed, by showing, that: for every $m \geq 0$

$$a_0 \cos(mt) + \sum_{k=1}^{\infty} a_k \cos(kt) \cos(mt) + b_k \sin(kt) \cos(mt) \tag{2.2}$$

converges uniformly on $\mathbb{R}$.
Let $\varepsilon \geq 0$, due to the uniform convergence of $a_0 + \sum_{k\in\mathbb{N}^+} a_k \cos(kt) + b_k \sin(kt)$, there is $N \in \mathbb{N}$, such that for every $n \geq N$

$$\left| f(t) - a_0 + \sum_{k=1}^{n} a_k \cos(kt) + b_k \sin(kt) dt \right| \leq \varepsilon.$$

Since $|\cos(mt)| \leq 1$:

$$\left| f(t) \cos(mt) - a_0 \cos(mt) + \sum_{k=1}^{n} (a_k \cos(kt) \cos(mt) + b_k \sin(kt) \cos(mt) \right| \leq \varepsilon,$$

for every $n \in \mathbb{N}$, and every $x \in \mathbb{R}$,which means, that,(2.2) converges uniformly, therefore it can be integrated. This gives us the following:

$$\int_{-\pi}^{\pi} f(t) \cos(mt) dt = \pi a_m.$$

The same can be done with $\sin(mx)$, which gives:

$$\int_{-\pi}^{\pi} f(x) \sin(mx) dx = \pi b_m,$$

which means the theorem has been proved. □

This theorem sounds a bit strange, since it states, that if the Fourier series converges uniformly, it converges to f, where the Fourier-coefficients can be obtained by integration. However, if we take an arbitrary $f \in C_{2\pi}$, we can calculate its Fourier coefficients, but will not know whether the Fourier series will converge uniformly to f or not. Emphasizing this problem, we will proceed, by showing that, there are $f \in C_{2\pi}$, such that, it's Fourier series will not even converge pointwise.

Remark 2.1.7. When calculating the Fourier coefficients, f doesn't have to be integrated on $[-\pi, \pi]$. Any 2π long interval will do,since if f is periodical, with a period length of 2π,and integrable on $[-\pi, \pi]$ then: :

$$\int_{a}^{a+2\pi} f(t)dt = \int_{-\pi}^{\pi} f(t) \qquad \forall a \in \mathbb{R}$$

Lemma 2.1.8. *Let $f \in C_{2\pi}$. Let $S_m f(x)$ be the m-th partial sum of f's Fourier series, i.e.,*

$$a_0 + \sum_{k=1}^{m} a_k \cos(kx) + b_k \sin(kx).$$

Then $S_m f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x-t)f(t)dt$, where $D_m \in C_{2\pi}$, and

$$D_m(2s) = \begin{cases} \dfrac{\sin((2m+1)s)}{\sin(s)} & if \quad s \neq 0 \\ 2m+1 & if \quad s = 0 \end{cases} \tag{2.3}$$

and $D_m(s)$ is the so called Dirichlet-kernel.

Remark 2.1.9. Continuity is granted, since $\lim_{s\to 0} \dfrac{\sin((2m+1)s)}{\sin(s)} = 2m+1$.

Proof.

$$\begin{aligned} S_m f(x) =& \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos(kx) + b_k \sin(kx) \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(1 + 2\sum_{k=1}^{m} \cos(kx)\cos(kt) + \sin(kx)\sin(kt)\Big) f(t)dt \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(1 + 2\sum_{k=1}^{m} \cos(k(x-t))\Big) f(t)dt. \end{aligned}$$

The only thing left is to show, that:

$$1 + 2\sum_{k=1}^{m} \cos(2ks) = \frac{\sin((2m+1)s)}{\sin(s)} \ \forall m \in \mathbb{N}.$$

We prove by induction. If $m = 0$ then $1 = \frac{\sin(s)}{\sin(s)} = 1$. Suppose the proposition stands for $m = m$: $1 + 2\sum_{j=1}^{m} \cos(2js) = \dfrac{\sin((2m+1)s)}{\sin(s)}$
Consider $m = m + 1$.

$$2\sin(s)\cos(2(m+1)s) = \sin((2m+3)s) - \sin((2m+1)s) \ \ \forall m \in \mathbb{N}.$$

After dividing by $\sin(s)$, and exploiting the inductional assumption, the proof has been completed. ∎

We proceed, by defining the following $\varphi : (C_{2\pi}(\mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R}$. functional. Let $f \in C_{2\pi}$ be arbitrary, then $\varphi_m(f) := S_m f(0)$.Since φ is a functional, the following stands:
$\|\varphi\| = \inf\{c \in \mathbb{R}^+ : |\varphi(f)| \leq c\,\|f\| \ : \ \forall f \in C_{2\pi}\}$.

Lemma 2.1.10. *φ_m is continuous, and $\|\varphi_m\| \to \infty$.*

Proof. Since $|a_j|,\ |b_i| \leq \|f\|_\infty \ \forall i \in \mathbb{N}^+,\ \forall j \in \mathbb{N}$, due to $S_m f$'s definition:

$$\|S_m f\|_\infty \leq \left(2m + \frac{1}{2}\right) 2\,\|f\|_\infty = (4m+1)\,\|f\|_\infty \Rightarrow \|\varphi_m\| \leq 4m + 1 < \infty$$

Let $f(2s) := (\mathrm{sign}(\sin(s))\sin((2m+1)s)$. It can be easily seen, that $f \in C_{2\pi}$, and $\|f\|_\infty = 1$.

$$\varphi_m(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(-t)f(t)dt = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_m(-2s)f(2s)ds \tag{2.4}$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2((2m+1)s)}{|\sin(s)|}ds = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{\sin^2((2m+1)s)}{\sin(s)}ds \tag{2.5}$$

$$\geq \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{\sin^2((2m+1)s)}{s}ds = \frac{2}{\pi}\int_{0}^{\frac{(2m+1)\pi}{2}} \frac{\sin^2(s)}{s}ds \tag{2.6}$$

$$\geq \frac{2}{\pi}\sum_{j=1}^{m}\int_{(j-1)\pi}^{j\pi} \frac{\sin^2(s)}{s}ds \geq \frac{2}{\pi}\int_{0}^{\pi}\sum_{j=1}^{m} \frac{\sin^2(s)}{j\pi}ds \tag{2.7}$$

$$= \frac{1}{\pi}\sum_{j=1}^{m}\frac{1}{j}. \tag{2.8}$$

(2.4) is true due to the definition of the Dirichlet-Kernel, and changing the integration variable, in (2.5) we used f's definition,and that $\sin^2$ is an even function. In (2.6) we exploited the periodicity of sin, in (2.7) we used the integral's additivity, and the fact, that the sum and integral can be interchanged. From these we get, that $\|\varphi_m\| \geq \varphi_m(f) > \frac{1}{\pi}\sum_{j=1}^{m}\frac{1}{j} \to \infty$. □

Combining these we get:

Theorem 2.1.11. *There are $f \in C_{2\pi}$, whose Fourier-series doesn't converge to f pointwise.*

Proof. We prove indirect.

$$(\forall f \in C_{2\pi}) : \varphi_m(f) \to f(0).$$

We will use the Banach-Steinhaus theorem, which states the following:
Let X be a Banach space, Y be a normed space. An $(A_n)_{n\in\mathbb{N}} \in L(X,Y)$ operator sequence is bounded pointwise if and only if $(A_n)_{n\in\mathbb{N}}$ is bounded uniformly.
Using this and the fact, that $C_{2\pi}$ is a Banach space, and $\mathbb{R}$ is a normed space we get, that $\|\varphi_m\| < \infty$, which contradicts the previous lemma. □

In the previous theorem we have seen, that not even pointwise convergence of Fourier series is guaranteed for arbitrary functions. There are two possible ways to procced

1. strengthening f's properties, in order to achieve uniform convergence

2. weakening the notion of convergence.

We continue by showing that every Fourier series converges in L^2. In order to do so we generalize Fourier series in arbitrary Hilbert spaces.

2.2 Abstract Fourier Series In Hilbert Spaces

In this chapter, H is always a Hilbert space, and $\langle \cdot, \cdot \rangle$, is the inner product of the Hilbert space.

Definition 2.2.1. An $(e_n)_{n\in\mathbb{N}} \subset H$ vector sequence is a complete system, if

$$(\forall x \in H) : \langle x, e_n \rangle = 0 \ \ \forall n \in \mathbb{N} \Rightarrow x = 0$$

Definition 2.2.2. An $(e_n)_{n\in\mathbb{N}}$ is a total system, if

$$\overline{\operatorname{span}((e_n)_{n\in\mathbb{N}})} = H$$

From the elementary theory of Hilbert spaces the following theorem is well known: Let M be a subset in H, the following are equivalent

1. M is a complete set, i.e. $M^{\perp} = 0$

2. M is a total system.

Definition 2.2.3. An $(e_n)_{n\in\mathbb{N}} \subset H$ vector sequence is

1. orthogonal, if $\langle e_i, e_j \rangle = 0 \ \forall i \neq j \in \mathbb{N}$, and $\langle e_i, e_i \rangle = c \ c \in \mathbb{R}^+$

2. orthonormal, if $(e_n)_{n\in\mathbb{N}}$ is orthogonal, and $\langle e_i, e_i \rangle = 1 \ \forall i \in \mathbb{N}$

Proposition 2.2.4. Let $(e_n)_{n\in\mathbb{N}} \subset H$, be an orthogonal sequence, then

$$\sum_{j=0}^{\infty} e_j \text{ converge } \iff \sum_{j=0}^{\infty} \|e_j\|^2 \text{ converge}$$

Proof. Let $s_n := \sum_{j=0}^{n} e_j$, and $\sigma_n := \sum_{j=0}^{n} \|e_j\|^2$, and $n, m \in \mathbb{N}, n \geq m$. By exploiting orthogonality of the vector sequences, we get:

$$\|s_n - s_m\|^2 = \left\| \sum_{j=m+1}^{n} e_j \right\|^2 = \sum_{j=m+1}^{n} \|e_j\|^2 = |\sigma_n - \sigma_m|. \tag{2.1}$$

Since both H, and $\mathbb{R}$ are complete, and on the two sides of (2.1) we have Cauchy-sequences, they converge under the same conditions. □

Definition 2.2.5. Let $(e_n)_{n\in\mathbb{N}}$ be a complete orthonormal system, and $x \in H$ be arbitrary. $\sum\limits_{j\in\mathbb{N}} \langle x, e_j\rangle e_j$ is called x's Fourier series, and $\langle x, e_j\rangle$ are x's Fourier coefficients.

Theorem 2.2.6. *Let H be a Hilbert-space, $(e_n) \subset H$ complete orthonormal system. for every $x \in H$ x's Fourier series converge, and their sum equals x.*

Proof. Let $x_i := \langle x, e_i\rangle e_i \ \forall i \in \mathbb{N}$, which is a complete orthonormal system. Using 2.2.4:

$$\sum_{i=1}^{\infty} x_i \text{ converge } \iff \sum_{i=1}^{\infty} \|x_i\|^2 \text{ converge}$$

Let $s_n := \sum\limits_{i=1}^{n} x_i$, and $H_n := \operatorname{span}\{e_1, \dots e_n\}$ $n \in \mathbb{N}^+$. In this case $s_n \subset H_n$, and $\|s_n\|^2 \leq \|x\|^2$, since

$$0 \leq \|x - s_n\|^2 = \|x\|^2 + \|s_n\|^2 - 2\mathrm{Re}\langle x, sn\rangle = \|x\|^2 - \|s_n\|^2,$$

since

$$\langle x, s_n\rangle = \sum_{j=1}^{n} \overline{\langle x, e_j\rangle}\langle x, e_j\rangle = \sum_{j=1}^{n} |\langle x, e_j\rangle|^2 = \left\|\sum_{j=1}^{n}\langle x, e_j\rangle e_j\right\|^2 = \|s_n\|^2.$$

From this, we get the convergence:

$$\sum_{i=0}^{n} \|x_i\|^2 = \|s_n\|^2 \leq \|x\|^2 \Rightarrow \sum_{i=0}^{\infty} \|x_i\|^2 < \infty \Rightarrow \sum_{i=1}^{\infty} x_i \text{ converge.}$$

We have to prove, that the sum is x. Let $s := \sum\limits_{i=1}^{\infty} x_i$, we will show, that $s = x$. Due to the fact, that $(e_i)_{i\in\mathbb{N}}$, is complete, it is enough to show, that $\langle s - x, e_j\rangle = 0$ $\forall j \in \mathbb{N}$. Using $\langle\cdot,\cdot\rangle$'s continuity:

$$\langle s, e_j\rangle = \left\langle \sum_{i=1}^{\infty} x_i, e_j \right\rangle = \left\langle \sum_{i=1}^{\infty}\langle x, e_i\rangle e_i, e_j \right\rangle = \sum_{i=1}^{\infty}\langle x, e_i\rangle\langle e_i, e_j\rangle = \langle x, e_j\rangle,$$

therefore $\langle x - s, e_j\rangle = 0 \ \forall j \in \mathbb{N}^+$. □

Remark 2.2.7. This theorem doesn't bring us closer to giving sufficient conditions for Fourier series to converge uniformly, since $\|\cdot\|_\infty$ is not generated by an inner product.

Remark 2.2.8. Using this theorem, generalized Fourier series can be inspected, using arbitrary orthogonal systems. A few examples for it:

1. $H := L^2(0,\pi)$ $\ e_n(t) = \sqrt{\frac{2}{\pi}}\sin(nt)\ \forall n \in \mathbb{N}$
2. $H := L^2(0,\pi)$ $\ e_0(t) = \frac{1}{\sqrt{\pi}}$ $\ e_n(t) = \sqrt{\frac{2}{\pi}}\cos(nt)\ \forall n \in \mathbb{N}^+$

3. $H := L^2(0, 2\pi)$ $e_n(t) = \sqrt{\frac{1}{2\pi}} e^{inx}$ $\forall n \in \mathbb{N}$

4. Fourier series, defined in **Definition** 2.1.1

Remark 2.2.9. In order to avoid misunderstandings, from now on, if we write Fourier series, we are reffering to the traditional ones, defined in Section 2.1. When we need the generalized notion of Fourier series, we will refer to it as generalized Fourier series.

2.3 Uniform Convergence Of Fourier Series

Definition 2.3.1. Let $f : E \to \mathbb{R}$, if the range of f is finite, then we say that f is simple.

Theorem 2.3.2. *The set of continuous functions on $[a, b]$ is a dense set in $L^2([a, b])$.*

Proof. Let $f \in L^2([a, b])$, and we need $g \in C([a, b])$, such that $\|f - g\|_2 \leq \varepsilon$. Let $A \subseteq [a, b]$ be a closed subset, and χ_A the characteristic function of A, namely:

$$\chi_A(t) = \begin{cases} 0, & t \notin A \\ 1, & t \in A. \end{cases}$$

Let $h(t) := \inf_{t \in [a,b]} |t - y|$, for every $y \in A$, and $g_n(t) := \dfrac{1}{1 + nh(t)}$ for every $n \in \mathbb{N}$. For g_n the following hold:

i) $g_n(t)$ is continuous for every $t \in [a, b]$,

ii) $g_n(t) = 1$, if $t \in A$, and $g_n(t) \to 0$ for every $t \notin A$ as $n \to \infty$.

Exploiting this we get that $\|g_n - \chi_A\|_2 = \left(\int_{[a,b] \setminus A} g_n^2 \right)^{\frac{1}{2}} \to 0$ as $n \to \infty$, which means that characteristic functions of closed sets can be approximated with continuous functions. From this we obtain, that every simple function can be approximated with continuous functions.
Let $f \in L^2([a, b])$, $f \geq 0$, and $(s_n)_{n \in \mathbb{N}}$ non negative, simple,monotone increasing set of functions, such that $s_n(t) \to f(t)$. Since $|f - s_n|^2 \leq f^2$, according to the dominated convergence theorem $\|f - s_n\|_2 \to 0$, as $n \to \infty$. To prove for arbitrary f we have to decompose f into f^+ and f^-. □

Theorem 2.3.3 (Completeness of Fourier series)**.** *Let $f \in C_{2\pi}$ be arbitrary. If $a_i, b_i = 0$ $\forall i \in \mathbb{N}^+$, and $a_0 = 0$ then $f = 0$, where a_i, b_i are the corresponding Fourier coefficients.*

Proof. We have to show, that the trigonometrical polynomials, are a total system, in $L^2([-\pi, \pi])$, which is equivalent, to being a complete orthonormal system, so we proceed, by showing, that the trigonometrical polynomials, are a complete orthonormal system. Let $f \in L^2[-\pi, \pi], \varepsilon > 0$, we will need a p trigonometrical polinom , such that, $\|f - p\|_2 \leq \varepsilon$ We know from Theorem 2.3.1, that $C[-\pi, \pi]$ is dense in $L^2[-\pi, \pi]$.Let $g \in C(-\pi, \pi)$ such that, $\|f - g\|_2 \leq \varepsilon$.

For this g, we can construct a $\bar{g}$, such that $\bar{g}(-\pi) = \bar{g}(\pi) = 0$, and $\|g - \bar{g}\|_2 \leq \varepsilon$. by noticing the fact: $\bar{g}$ is periodical, with a period length of 2π, according to the second Weierstrass theorem, it can be approximated with a p trigonometrical polynom, such that $\|\bar{g} - p\|_\infty \leq \varepsilon^2$. From this:

$$\int_{-\pi}^{\pi} |\bar{g} - p|^2 \leq 2\pi \|\bar{g} - p\|_\infty^2 ,$$

which implies

$$\|\bar{g} - p\|_2 \leq \sqrt{2\pi} \|\bar{g} - p\|_\infty \leq \sqrt{2\pi}\varepsilon,$$

therefore

$$\|f - p\|_2 \leq \|f - g\| + \|g - \bar{g}\| + \|\bar{g} - p\| \leq \varepsilon(2 + \sqrt{2\pi}).$$

Which means, that the trigonometric polynomials form a complete orthonormal set in $L^2([-\pi, \pi])$. Using this, let $f \in C_{2\pi}$, be arbitrary, whose Fourier coefficients are 0. Since f's every Fourier coefficients are 0, f is orthogonal to a complete orthonormal system in L^2, which means, $f = 0$ a.e. But since f is continuous, it also means, that $f = 0$ everywhere. □

Proposition 2.3.4. Let $f \in C_{2\pi}$, be arbitrary. If f's Fourier series converge uniformly on $\mathbb{R}$, then $a_0 + \sum_{j=1}^{\infty} a_j \cos(jt) + b_j \sin(jt) = f(t)\ \forall t \in \mathbb{R}$

Proof. $a_0 + \sum_{j \in \mathbb{N}^+} a_j \cos(jt) + b_j \sin(jt)$ is continuous, and every Fourier coefficients of $a_0 + \sum_{j \in \mathbb{N}^+} a_j \cos(jt) + b_j \sin(jt)$, are identical to f's Fourier coefficients, therefore
$f(t) - a_0 + \sum_{j \in \mathbb{N}^+} a_j \cos(jt) + b_j \sin(jt) \in C_{2\pi}$, and the Fourier coefficients of
$f(t) - a_0 + \sum_{j \in \mathbb{N}^+} a_j \cos(jt) + b_j \sin(jt)$ are 0, therefore $f(t) - a_0 + \sum_{j \in \mathbb{N}^+} a_j \cos(jt) + b_j \sin(jt) = 0$
which implies $f(t) = a_0 + \sum_{j \in \mathbb{N}^+} a_j \cos(jt) + b_j \sin(jt)$ for every $t \in \mathbb{R}$. □

Notation 2.3.5. $f \in C_{2\pi}^k$, if $f \in C_{2\pi} \cap C^k(\mathbb{R})$

Lemma 2.3.6. *If $f \in C_{2\pi}^k$, then exists such $M \in \mathbb{R}^+$:*

$$(\forall n \in \mathbb{N}^+) : |a_n| \leq \frac{M}{n^k}, |b_n| \leq \frac{M}{n^k}.$$

Proof. We proceed using induction. If $k = 0$ $f \in C_{2\pi}$, therefore f is bounded. Let $K := \sup_{t\in[-\pi,\pi]} \{|f(t)|\}$. From the defintion of the Fourier coefficients, it can be seen, that: $|a_i|, |b_j| \leq 2K \quad \forall i \in \mathbb{N}\ \forall j \in \mathbb{N}^+$. Thus, the property holds if $k = 0$.
Assume, it is true for $k = k$, and consider $k = k+1$: $f \in C_{2\pi}^{k+1} \Rightarrow f' \in C_{2\pi}^{k}$. Which means, that the property holds, for the Fourier coefficients of f'. Let $n \in \mathbb{N}^+$.

$$\begin{aligned} a_n &= \int_{-\pi}^{\pi} f(t)\cos(nt)dt = \left[f(x)\frac{\sin(nt)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{f'(t)\sin(nt)}{n}dt \\ &= 0 - \frac{1}{n}\int_{-\pi}^{\pi} f'(t)\sin(nt)dt \Longrightarrow \left|\int_{-\pi}^{\pi} f(t)\cos(nt)dt\right| \leq \frac{M}{n^{k+1}}. \end{aligned}$$

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} f(t)\sin(nt)dt = \left[f(x)\frac{\cos(nt)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{f'(t)\cos(nt)}{n}dt \\ &= 0 - \frac{1}{n}\int_{-\pi}^{\pi} f'(t)\cos(nt)dt \Longrightarrow \left|\int_{-\pi}^{\pi} f(t)\sin(nt)dt\right| \leq \frac{M}{n^{k+1}}. \end{aligned}$$

□

Theorem 2.3.7. *Let $f \in C_{2\pi}^2$, then f's Fourier series converges uniformly to f.*

Proof. According to Lemma 2.3.6,

$$|a_j| \leq \frac{M}{j^2}, \qquad |b_j| \leq \frac{M}{j^2}, \qquad \forall j \in \mathbb{N},$$

therefore:

$$\begin{aligned} \left|a_0 + \sum_{j\in\mathbb{N}^+} a_k\cos(jt) + b_k\sin(jt)\right| &\leq |a_0| + \sum_{j=1}^{\infty} \frac{M(|\cos(jt)| + |\sin(jt)|)}{j^2} \\ &\leq |a_0| + \sum_{j=1}^{\infty} \frac{2M}{j^2}. \end{aligned}$$

This means, that $a_0 + \sum_{j\in\mathbb{N}^+} a_j\cos(jt) + b_j\sin(jt)$ fulfills the Weierstrass-criteria, therefore $a_0 + \sum_{j\in\mathbb{N}^+} a_j\cos(jt) + b_j\sin(jt) \to f$ uniformly. □

Our next theorem is on the order of the Fourier coefficients.

Theorem 2.3.8 (Riemann-Lebesgue Lemma)**.** *If $f \in C_{2\pi}$, then $\lim_{i\to\infty} a_i, b_i = 0$, where a_i, b_i are the corresponding Fourier coefficients.*

Proof. We want to show, that

i) $b_k = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(t)\sin(kt)dt = 0,$

ii) $a_k = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(t)\cos(kt)dt = 0,$

We start by proving i).

Let $\varepsilon > 0$ be arbitrary. Since f is continuous and periodical with a period length of 2π, we can use the Second Weierstrass Approximation theorem, therefore let $N \in \mathbb{N}$, such that: $\|f - p_N\|_\infty \le \varepsilon$, where p_N is a trigonometrical polynomial. Let $k \in \mathbb{N}$, and $k > N$, then

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt) - p_N(t)\sin(kt)dt,$$

according to Lemma 2.1.6. Exploiting this we get:

$$\begin{aligned}|b_k| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt\right| &\le \frac{1}{\pi}\int_{-\pi}^{\pi} |(f(t) - p_N(t))\sin(kt)|dt \\ &\le \|f - p_N\|_\infty \frac{1}{\pi}2\pi \le 2\varepsilon,\end{aligned}$$

thus $b_k \to 0$, as $k \to \infty$. Proving ii) starts the same as for i), the only difference arise during the calculation of the a_k coefficients, namely:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt) - p_N(t)\cos(kt)dt,$$

according to Lemma 2.1.6. Exploiting this we get:

$$\begin{aligned}|a_k| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt\right| &\le \frac{1}{\pi}\int_{-\pi}^{\pi} |(f(t) - p_N(t))\cos(kt)|dt \\ &\le \|f - p_N\|_\infty \frac{1}{\pi}2\pi \le 2\varepsilon.\end{aligned}$$

□

Definition 2.3.9. Let $\alpha \in (0, 1]$, $f : I \to \mathbb{R}$. We say that f is Hölder continuous, if for every $x, y \in I$ exists such $C \in \mathbb{R}$, that $|f(x) - f(y)| \le C|x - y|^\alpha$. The set of Hölder continuous functions with $\alpha \in \mathbb{R}$ constant is referred as $\mathrm{Lip}^\alpha(I)$.

Remark 2.3.10. If $\alpha = 1$, f is Lipschitz continuous.

Without proving we mention the following:

Remark 2.3.11. Let $\alpha \in (0, 1]$, and $f \in Lip^\alpha([-\pi, \pi])$, where $f(-\pi) = f(\pi)$. Then $a_0 + \sum_{j\in\mathbb{N}^+} a_j \cos(jt) + b_j \sin(jt) \to f$ uniformly.

2.4 Fejér-summation

Up to this point we have seen, that for every $f \in C^2_{2\pi}$ f's Fourier series will converge uniformly to f. We have seen that convergence can be achieved in arbitrary Hilbert spaces, and exploited the fact, that $L^2([-\pi,\pi])$ is a Hilbert space.We proceed by modifying Fourier series, in order to show that for every continuous function f its Fourier series converge to f uniformly. The motivation for this, is the well known fact, from elementary analysis, that:

$$a_n \to a \Rightarrow \frac{1}{n}\sum_{i=1}^{n} a_i \to a.$$

Definition 2.4.1. Let $\sigma_n(f) := \dfrac{1}{n+1}\displaystyle\sum_{m=0}^{n} S_m f$, where σ_n is the so called n-th Fejér-sum.

Lemma 2.4.2. *$\sigma_n f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} F_n(x-t)f(t)dt$, where $F_n(2s) = \dfrac{1}{n+1}\dfrac{\sin^2((n+1)s)}{\sin^2(s)}$. F_n is called the Fejér-kernel.*

Proof. Since

$$\sigma_n(f) = \frac{1}{n+1}\sum_{m=0}^{n} S_m f = \frac{1}{n+1}\sum_{m=0}^{n} S_m f = \sum_{m=0}^{n} \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x-t)f(t)dt}{n+1}$$
$$= \frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi}\sum_{m=0}^{n} D_m(x-t)f(t)dt.$$

Which means: $F_n = \dfrac{\sum_{j=0}^{n} D_j}{n+1}$, therefore it is enough to show:

$$\frac{\sin^2((n+1)s)}{\sin^2(s)} = \sum_{m=0}^{n}\frac{\sin((2m+1)s)}{\sin(s)}.$$

By direct calculation:

$$\sum_{m=0}^{n}\sin(s)\sin((2m+1)s) = \frac{1}{2}\sum_{m=0}^{n}\cos(2ms) - \cos((2m+2)s)$$
$$= \frac{1-\cos((2n+2)s)}{2} = \sin^2((n+1)s).$$

After dividing by $\sin^2(s)$, we get:

$$\frac{\sin^2((n+1)s)}{\sin^2(s)} = \sum_{m=0}^{n}\frac{\sin((2m+1)s)}{\sin(s)}.$$

□

Proposition 2.4.3. σ_n is a positive operator.

Proof. Let $f \geq 0$ arbitrary. Since $F_n f \geq 0 \Rightarrow \displaystyle\int_{-\pi}^{\pi} F_n(t)f(t)dt \geq 0$. □

Theorem 2.4.4 (Freud). *Let K be a compact metric space, $h_1, h_2, \dots h_m \in C(K)$, such functions, that separates K's points, and $h_0 := 1$. If*

$$\|h_i - L_n h_i\|_\infty \to 0 \qquad \forall h = (0, 1, 2, \dots m), \ \text{and} \ \left\| \sum_{i=1}^m h_i^2 - L_n\left(\sum_{i=1}^m h_i^2 \right) \right\|_\infty \to 0, \quad (2.1)$$

then:

$$\|f - L_n f\|_\infty \to 0 \qquad \forall f \in C(K).$$

Proof. Let $f \in C(K)$ be arbitrary, and $\varepsilon \geq 0$, $M := \{h_i : 1 \leq i \leq m\}$. Since M separates K's points, for every $x, y \in K$, we can choose $N_{x,y}$, such that:

$$|f(x) - f(y)| \leq \varepsilon + N_{x,y} \sum_{j=1}^m |h_j(x) - h_j(y)|^2. \qquad (2.2)$$

Let $\mathcal{G}_{x,y}$ be the following:

$$\mathcal{G}_{x,y} := \{(x', y') \in K \times K : |f(x') - f(y')| \leq \varepsilon + N_{x,y} \sum_{j=1}^m |h_j(x) - h_j(y)|^2\}.$$

By (2.2) we have $K \subset \bigcup_{x,y \in K} \mathcal{G}_{x,y}$, and due to the fact, that K is a compact set, we can choose a finite subcover. Let $\bigcup_{i \in I} \mathcal{G}'_{x_i, y_i} \subset \bigcup_{x,y} \mathcal{G}_{x,y}$ be a finite subcover, and N be the maximum of N_{x_i, y_i}'s. Therefore:

$$|f(x) - f(y)| \leq \varepsilon + N \sum_{j=1}^m |h_j(x) - h_j(y)|^2\} \qquad \forall x, y \in K.$$

Using this, for a fixed $x \in K$, and arbitrary $y \in K$:

$$|f(x)(L_n 1)(y) - (L_n f)(y)|$$

$$\leq \varepsilon (L_n 1)(y) + N \sum_{j=1}^m h_j^2(x)(L_n 1)(y) - 2N \sum_{j=1}^m h_j(x)(L_n h_j)(y) + N L_n\left(\sum_{j=1}^m h_j^2 \right)(y).$$

From this, by choosing $y = x$, we get:

$$|f - L_n f| = |f - L_n f + f L_n 1 - f L_n 1| \leq |f||1 - L_n 1| + |L_n f - f L_n 1| \leq$$
$$\leq |f||1 - L_n 1| + N \sum_{j=1}^m h_j^2 (L_n 1) - 2N \sum_{j=1}^m h_j (L_n h_j) + N L_n \left(\sum_{j=1}^m h_j^2 \right) \to 0,$$

because of (2.1). $\square$

We give another proof to Korovkin's first theorem.

Theorem 2.4.5 (Korovkin's first theorem)**.** *If $L_n \in L(C(I), C(I))$ is a positive operator, and*

$$L_n f \to f \ uniformly, \quad if\ f = 1, t, t^2,$$

then $L_n f \to f$ uniformly, $\forall f \in C(I)$.

Proof. We can apply Theorem 2.4.4, with $K = I$, and $m = 1$. □

Theorem 2.4.6 (Korovkin's second theorem)**.** *If $L_n \in L(C_{2\pi}, C_{2\pi})$ is a positive operator, and*

$$L_n f \to f \ uniformly, \quad if\ f = 1, \cos, \sin,$$

then $L_n f \to f$ uniformly, $\forall f \in C_{2\pi}$.

Proof. We will prove, exploiting Theorem 2.4.4. Let $K \subset \mathbb{R}^m$ be a compact subset, and for every $x \in \mathbb{R}^m$, x_i is the i-th coordinate of x, and $h_j(x) := x_j$ for every $1 \leq j \leq m$. The h_i functions separate K's points, thus it fulfills h_i's demanded properties.
In the following let $m = 2$, $h_0 = 1,\ h_1 = x_1,\ h_2 = x_2$, and $K = \partial\overline{B_0(1)}$. Since $x_1^2 + x_2^2 = 1$, if $L_n f \to\ f$ uniformly, for $f = 1,\ x_1,\ x_2$, then $L_n f \to f$ for every $f \in C(K)$, according to 2.4.4.
Let $T(s) = (\cos(s), \sin(s))$. Using the fact, that $f \mapsto f \circ T$, is an isometric isomorphism between $C(K)$, and $C_{2\pi}$, and if $f \geq 0 \Rightarrow T(f) \geq 0$. Since $h_i \in C(K)$, $i = 0, 1, 2$, and $h(T(s)) = 1, \cos(s), \sin(s)$, we get, that the results for $C(K)$, is equivalent to the results for $C_{2\pi}$, thus the proof has ended. □

Theorem 2.4.7 (Fejér)**.** *Let $f \in C_{2\pi}$ be arbitrary. The sequence of the Fejér-sums converges uniformly to f:*

$$\forall f \in C_{2\pi} \Rightarrow \sigma_n(f) \to f \ uniformly$$

where $\sigma_n(f) = \dfrac{1}{n+1} \displaystyle\sum_{m=0}^{n} S_m f,\ \forall n \in \mathbb{N}.$

Proof. Using the definition, we get the following:

$$\sigma_n 1 = \frac{1}{n+1} S_m 1 = 1,$$

$$\sigma_n \cos = \frac{1}{n+1} S_m \cos = \frac{n}{n+1} \cos,$$

$$\sigma_n \sin = \frac{1}{n+1} S_m \sin = \frac{n}{n+1} \sin,$$

and since σ_n is a positive operator, by using Theorem 2.4.6, we get the theorem. □

Chapter 3

Solving Partial Differential Equations With Fourier Series

In this chapter we wish to proceed with our inspections to a rather more applied field. Our goal is to solve certain partial differential equations using Fourier series, such as the wave and heat equation.

Theorem 3.0.1 (Dirichlet)**.** *Let $f \in C^1_{2\pi}$. Then the Fourier series $a_0 + \sum_{k\in\mathbb{N}^+} a_k \cos(kx) + b_k \sin(kx)$ converges uniformly, to f.*

Proof. We will use the properties of the Dirichlet-kernel, namely:
$S_m f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x-t)f(t)dt$, where $D_m \in C_{2\pi}$, and

$$D_m(2s) = \begin{cases} \dfrac{\sin((2m+1)s)}{\sin(s)} & if \quad s \neq 0 \\ 2m+1 & if \quad s = 0, \end{cases} \tag{3.1}$$

where $S_m fx = a_0 + \sum_{j=1}^{m} a_j \cos(jx) + b_j \sin(jx)$. Let $0 \leq h \leq \pi$:

$$\begin{aligned} |S_m f(t) - t| &= \left| \frac{1}{\pi}\int_{-\pi}^{\pi} (f(s+t) - f(t))D_m(s)ds \right| \\ &\leq \frac{1}{\pi}\left(\left| \int_{-\pi}^{-h} (f(s+t) - f(t))D_m(s)ds \right| + \left| \int_{-h}^{h} (f(s+t) - f(t))D_m(s)ds \right| \right. \\ &\left. + \left| \int_{h}^{\pi} (f(s+t) - f(t))D_m(s)ds \right| \right) =: \frac{1}{\pi}(A+B+C). \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary, and $0 < h < \dfrac{\varepsilon}{\pi \|f'\|_\infty}$. Then according to the mean value theorem of integration, $f(s+t)-f(t)=f'(u)s$ for some $u\in[-h,h]$, hence

$$\left|\int_{-h}^{h}(f(s+t)-f(t))D_m(s)ds\right|\leq\int_{-h}^{h}\frac{f(s+t)-f(t)}{2\sin(\frac{|s|}{2})|}\sin\left(\left(m+\frac{1}{2}\right)s\right)ds$$
$$\leq\int_{-h}^{h}\frac{\|f'\|_\infty|s|}{2\sin(\frac{|s|}{2})}ds\leq 2h\|f'\|_\infty\frac{\pi}{2},$$

since $\dfrac{2u}{\pi}\leq\sin u$ for every $0\leq u\leq\dfrac{\pi}{2}$. From this $B\leq\varepsilon$.

We continue, by estimating C. Let $f_t(s)=\dfrac{f(s+t)-f(t)}{2\sin(\frac{s}{2})}$, and $n=m+\frac{1}{2}$. It can be easily seen, that $f_t\in C^1[h,\pi]$, and $\sup_{t\in\mathbb{R}}\|f_t'\|_\infty<\infty$ and $\sup_{t\in\mathbb{R}}\|f_t\|_\infty<\infty$. By partial integration we get:

$$\left|\int_h^\pi f_t(s)\sin(ns)ds\right|=\left|f_t(s)\frac{-1}{n}\right|_{s=h}^{s=\pi}-\left|\int_h^\pi f_t'(s)\frac{-1}{n}ds\right|\leq\frac{c}{n},$$

for a suitable constant c, therefore there is $N\in\mathbb{N}$, such that for every $n>N$ $C\leq\varepsilon$. We get $A\leq\varepsilon$ in the same way. Combining these: we get $S_mf(t)\to f(t)$ for every $t\in[-\pi,\pi]$. □

Theorem 3.0.2 (Term by Term Differentiation)**.** *Let f be a $C^2_{2\pi}$ function. If f' is in $C^1_{2\pi}$, then the Fourier series $a_0+\sum_{k\in\mathbb{N}^+}a_k\cos(kx)+b_k\sin(kx)=f(x)$ can be differentiated term by term, and the series so obtained converges uniformly to f'.*

Proof. Since f' is in $C^1_{2\pi}$ we can use the Theorem 3.0.1.

Lemma 3.0.3. *Let $f\in C_{2\pi}$ be an odd function, assuming, that the corresponding Fourier series converge. Then the corresponding Fourier series is $\sum_{k\in\mathbb{N}^+}b_k\sin(kt)$.*

Proof. Since the Fourier series converge, we only have to determine the Fourier coefficients, and show that $a_i=0$ for every $n\in\mathbb{N}$. Which is true, since f is an odd function $a_0=0$, and $a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(t)\cos(kt)dt=0$, since $f(t)\cos(kt)$ is also an odd function. □

Lemma 3.0.4. *Let $f\in C_{2\pi}$ be an even function, assuming, that the corresponding Fourier series converge. Then the corresponding Fourier series is $a_0+\sum_{k\in\mathbb{N}^+}a_k\cos(kt)$.*

Proof. Since the Fourier series converge, we only have to determine the Fourier coefficients, and show that $b_i=0$ for every $n\in\mathbb{N}^+$. Which is true, since $b_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(t)\sin(kt)dt=0$, since $f(t)\sin(kt)$ is an odd function. □

Definition 3.0.5. Let $f\in C_{2\pi}$ the Fourier series of f is a sine (resp., cosine) series, if it only consists of sine (resp., cosine) terms.

With the help of this, we can create a sine (resp., cosine) series of any continuous $f\in C([0,\pi])$, by expanding it onto $[-\pi,\pi]$, such that $f(-x)=-f(x)$ (resp., $f(-x)=f(x)$), if the expanded function's Fourier series converges.

□

3.1 The Wave Equation

Consider a string, bound at both endpoints, with a length of L. We wish to inspect the reaction of the string to physical interference. First we need to construct a mathematical model. Let $u(t,x) : \mathbb{R}^+ \times [0,L] \to \mathbb{R}$, $u(t,x) \in C^2$ be a function, which shows, where position x of the string after t time is. We also assume, that $u(t,0) = u(t,L) = 0$, since the string is bound at both end points, and is unable to move. We also consider the location, and the velocity of the string to be known, at the beggining of the experiment. Which means $u(0,x) = f(x)$, $\partial_t u(t,x) = g(x)$. Since $u \in C^2$ both $f, g \in C^2$ stands. We rely on the physical fact, that the partial differential equation, modeling the string is the following: $\partial_t^2 u(t,x) = c^2 \partial_x^2 u(t,x)$. Combining these, and assuming that $\pi = L$ (if not, we transform the system) we get, the following problem:

$$\partial_t^2 u(t,x) = c^2 \partial_x^2 u(t,x) \qquad \forall t \in \mathbb{R}^+ \ \forall x \in [0,\pi] \tag{3.1}$$

$$u(0,x) = f(x) \qquad \forall x \in [0,\pi] \tag{3.2}$$

$$\partial_t u(t,x) = g(x) \qquad \forall x \in [0,\pi] \tag{3.3}$$

$$u(t,0) = u(t,\pi) = 0 \qquad \forall t \in \mathbb{R}^+, \tag{3.4}$$

where c is a non zero constant, defined by properties, such the thickness, or the tension of the string.

We look for the solution in a form, such that $u(t,x) = T(t)X(x)$. This reduces our system to the following: $T''(t)X(x) = c^2 T(t)X''(x)$, and after dividing both sides, we get:$\dfrac{T''(t)}{c^2 T(t)} = \dfrac{X''(x)}{X(x)}$. It can be easily seen, that the right side of the equation only depends upon t, while the right side only upon x. From this we know, that both sides have to be a constant. Let $\dfrac{T''(t)}{c^2 T(t)} = \dfrac{X''(x)}{X(x)} = \eta$, from which we get two separable ordinary differential equations:

$$T''(t) - c^2 \eta T(x) = 0 \tag{3.5}$$

$$X''(x) - \eta X(x) = 0. \tag{3.6}$$

This has a solution for every η arbitrary constant.

Before we proceed, we shall inspect the possible values of η. In the case, where $\eta > 0$, we get the solutions in the form $e^{\sqrt{\eta} t}$, which means, as time goes by, the oscillation of the string grows, which contradicts to the physical experiments. Which gives us, that $\eta < 0$. We rewrite (3.5),(3.6), using $\hat{\eta} = -\eta$:

$$T''(t) + c^2 \hat{\eta} T(x) = 0 \tag{3.7}$$

$$X''(x) + \hat{\eta} X(x) = 0. \tag{3.8}$$

Solving this will give us:

$$\begin{aligned} T(t) =& a_1 \cos(\sqrt{\hat{\eta}} tc) + a_2 \sin(\sqrt{\hat{\eta}} tc), \\ X(x) =& b_1 \cos(\sqrt{\hat{\eta}} x) + b_2 \sin(\sqrt{\hat{\eta}} x). \end{aligned}$$

Since $u(t,x) = T(t)X(x)$, (3.1) stands. We proceed with checking the other conditions. We start with (3.4). Since $u(t,0) = u(t,\pi)$ for every $t \in \mathbb{R}^+$, we get, we only need the sine terms from $X(x)$.

Only (3.2), and (3.3) is left to be shown. However apart from particular f and g functions, these conditions can not be satisfied by such $T(t)X(x)$ functions. Therefore we exploit the fact, that if we have two solutions, their sum will also be a solution to (3.1).

Let $\hat{\eta} = k^2$, and $u_k(t,x) = \sin(kx)(a_1(k)\cos(ckt) + a_2(k)(\sin(ckt))$ for every k, where $a_1(k), a_2(k)$ are suitable constants. Let

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t,x)$$

Nevertheless, with this definition a number of problems arise:

i) We do not know the constants $a_1(k),\ a_2(k)$.

ii) We do not know, whether $\sum_{k \in \mathbb{N}^+} u_k(t,x)$ converges or not.

iii) Even if $\sum_{k \in \mathbb{N}^+} u_k(t,x)$ converge, we do not know anything about the convergence of its derivatives.

We start by determining constants $a_1(k),\ a_2(k)$, with the help of the initial conditions.

$$\sum_{k=1}^{\infty} u_k(0,x) = \sum_{k=1}^{\infty} \sin(kx)(a_1(k)\cos(0) + a_2 \sin(0)) = \sum_{k=1}^{\infty} \sin(kx) a_1(k) = f(x),$$

and since $f \in C^2[0,\pi]$, f can be represented with sine expansion, the Fourier sine series of f will converge, and $a_1(k)$ are the corresponding Fourier coefficients.

$$\partial_t \sum_{k=1}^{\infty} u_k(0,x) = \sum_{k=1}^{\infty} \sin(kx)(a_1(k)(-ck)\sin(0) + a_2(ck)\cos(0)) = \sum_{k=1}^{\infty} ck \sin(kx) a_2(k) = g(x),$$

and since $g \in C^2[0,\pi]$ and g can be written in a sine expansion, the Fourier sine series of g will converge, and $cka_2(k)$ are the corresponding Fourier sine coefficients.

We proceed by showing, that $\sum_{k=1}^{\infty} u_k(t,x)$ converges uniformly.

$$\begin{aligned}\left|\sum_{k=1}^{\infty} u_k(t,x)\right| &= \left|\sum_{k=1}^{n} \sin(kx)(a_1(k)\cos(ckt) + a_2(k)(\sin(ckt))\right| \\ &\leq \sum_{k=1}^{\infty} |a_1(k)\cos(ckt) + a_2(k)\sin(ckt)| \\ &\leq \sum_{k=1}^{\infty} |a_1(k)| + |a_2(k)| \leq \sum_{k=1}^{\infty} \frac{M}{k^2} + \frac{\overline{M}}{k^2} < \infty,\end{aligned}$$

with suitable M, $\overline{M}$ constants. Thus $\sum_{k=1}^{\infty} u_k(t,x) = u(t,x)$ stands for every $t \in \mathbb{R}^+$, and $x \in [0,\pi]$. We only have to show, that the derivatives of $u(t,x)$ also converge, namely:

$$\partial_t \sum_{k=1}^{\infty} u_k(t,x),\ \partial_x \sum_{k=1}^{\infty} u_k(t,x),\ \partial_t^2 \sum_{k=1}^{\infty} u_k(t,x),\ \partial_x^2 \sum_{k=1}^{\infty} u_k(t,x).$$

However the conditions for Theorem 3.0.2 are fulfilled, thus we can differentiate term by term.

3.2 The Heat Equation

We wish to inspect the model of heat condensation, in a long thin rod. We assume, both ends of the rod is kept at steady temperature, the length of the rod is L and the temperature of the rod, in a positon x after t time, is $u(t,x)$ $t \in \mathbb{R}^+,\ x \in [0,L]$, and u is twice differentiable in x, and once in t. The physical model is the following:

$$\partial_t u(t,x) = c\partial_x^2 u(t,x)$$

where c is a constant, representing the conductivity of the rod. We also assume, that $u(t,0) = T_1$ and $u(t,L) = T_2$, which expresses, that the temperature does not change at the ends of the rod. We consider the initial temperature to be known: $u(0,x) = f(x)$, and $f \in C^2([0,L])$, since $u(t,x)$ is twice differentiable in x. This gives us the following system:

$$\partial_t u(t,x) = c\partial_x^2 u(t,x) \qquad \forall t \in \mathbb{R}^+\ \forall x \in [0,L] \tag{3.1}$$

$$u(t,0) = T_1 \qquad \forall t \in \mathbb{R}^+ \tag{3.2}$$

$$u(t,L) = T_2 \qquad \forall t \in \mathbb{R}^+ \tag{3.3}$$

$$u(0,x) = f(x) \qquad \forall x \in [0,L]. \tag{3.4}$$

We start with a simple observation. Let $s(x) = \frac{T_2 - T_1}{L}x + T_1$, which would be a solution, if we dismissed (3.4). This solution, does not change through time, thus it is called the steady state solution. Later on we will see, that every system, without external stimulation tends to this state, as time passes by. We solve this, by considering an alternate system, such that:

$$\partial_t u(t,x) = c\partial_x^2 u(t,x) \qquad \forall t \in \mathbb{R}^+\ \forall x \in [0,L] \tag{3.5}$$
$$u(t,0) = 0 \qquad \forall t \in \mathbb{R}^+ \tag{3.6}$$
$$u(t,L) = 0 \qquad \forall t \in \mathbb{R}^+ \tag{3.7}$$
$$u(0,x) = f(x) - s(x) = h(x) \qquad \forall x \in [0,L]. \tag{3.8}$$

We proceed just like in the case of the wave equation. We look for $u(t,x)$ in such form: $u(t,x) = T(t)X(x)$. According this, we get (3.5) to be the following: $T'(t)X(x) = cT(t)X''(x)$, and ofter a division, we get: $\frac{T'(t)}{cT(t)} = \frac{X''(x)}{X(x)}$. Since the right side depends only upon t, and the right side only upon x, we get that both side is a constant: $\frac{T'(t)}{cT(t)} = \frac{X''(x)}{X(x)} = \eta$. From this, we get the following system:

$$T'(t) - c\eta T(t) = 0 \tag{3.9}$$
$$X''(x) - \eta X(x) = 0. \tag{3.10}$$

We proceed by inspecting the possible values of η. A solution of (3.9) is $T(t) = a\exp(c\eta t)$, where $a \in \mathbb{R}$ is an arbitrary constant. If $\eta > 0$, $\lim_{t\to\infty} T(t) = \infty$, thus we assume $\eta \leq 0$. In the case $\eta = 0$, $T(t)$ is constant, therefore we consider $\eta < 0$.Exploiting this, we change our system to the the following:

$$T'(t) + c\eta_1 T(t) = 0 \tag{3.11}$$
$$X''(x) + \eta_1 X(x) = 0, \tag{3.12}$$

where $\eta_1 = -\eta$. A solution to (3.11) is $T(t) = \exp(-c\eta_1 t)$, and to (3.12), is $X(x) = a_1 \sin(\sqrt{\eta_1}t) + a_2\cos(\sqrt{\eta_2}t)$. Combining this, we get the following:

$$u(t,x) = T(t)X(x) = \exp(-c\eta_1 t)(a_1\sin(\sqrt{\eta_1}x) + a_2\cos(\sqrt{\eta_2}x)).$$

Up to this point, we have a solution to (3.5), for every $\eta_1 > 0$. We want to have a rich enough set of solutions, among which the initial conditions can be satisfied with a series expansion. Therefore let $\sqrt{\eta_1} = \frac{k\pi}{L}$. According to this, we have set of solutions in the following form:

$$u_k(t,x) = T_k(t)X_k(x) = \exp\left(-c\frac{k^2\pi^2 t}{L^2}\right)\left(a_1\sin\left(\frac{k\pi x}{L}\right) + a_2\cos\left(\frac{k\pi x}{L}\right)\right).$$

Assuming, the rod satisfies conditions (3.6),(3.7),(3.8), we have $f(0) = T_1,\ f(L) = T_2$, which is equivalent to $h(0) = h(L) = 0$. We wish to create a solution $u_{init}(t,x)$, which fulfilles condition (3.8). To do so, we look for the solution represented by its sine expansion:

$$u_{init}(t,x) = \sum_{k=1}^{\infty} a(k) \exp\left(-c\frac{k^2\pi^2 t}{L^2}\right) \sin\left(\frac{k\pi x}{L}\right). \tag{3.13}$$

The problems are the same, as during the wave equation.

i) We do not know constants $a(k)$.

ii) We do not know, whether $\sum_{k=1}^{\infty} u_k(t,x)$ converges or not.

iii) Even if $\sum_{k=1}^{\infty} u_k(t,x)$ converge, we do not know anything about the convergence of its derivatives.

We start by determining constants $a(k)$, with the help of the sine expansion: $a_k = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{k\pi x}{L}\right)$, since $h \in C^2([-\pi,\pi])$, the Fourier series converge uniformly.

The convergence of $\sum_{k=1}^{\infty} u_k(t,x)$ now stands, since:

$$\left|\sum_{k=1}^{\infty} u_k(t,x)\right| = \left|\sum_{k=1}^{\infty} a(k) \exp\left(-c\frac{k^2\pi^2 t}{L^2}\right) \sin\left(\frac{k\pi x}{L}\right)\right| \le \sum_{k=1}^{\infty} a_k \le \frac{M}{k^2} < \infty,$$

since $h \in C^2([0,\pi])$.

The only thing left to show, is that the derivatives converge, namely:

$$\partial_t \sum_{k=1}^{\infty} u_k(t,x),\ \partial_x \sum_{k=1}^{\infty} u_k(t,x),\ \partial_x^2 \sum_{k=1}^{\infty} u_k(t,x).$$

However the conditions of Theorem 3.0.2 are fulfilled, thus we can derivate term by term. To get a solution to the original system, let $u(t,x) = s(x) + u_{init}(t,x)$, which is a solution indeed.

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