Eötvös Loránd University
Faculty of Science

András Csirik

Differential Forms and Applications

Bachelor's Thesis in Mathematics

Supervisor:

Dr. Sándor Kovács
Department of Numerical Analysis

Budapest, 2019

Acknowledgement

I would like to express my gratitude to my supervisor, Dr. Sándor Kovács for his contributions to my thesis. His guidance and expert advice have been invaluable throughout all stages of the work.

Contents

1 Introduction **3**

2 Differential forms **5**

2.1 The elements of multilinear algebra 5
2.2 Differential forms in $\mathbb{R}^n$ 15
2.2.1 Derivation 17
2.2.2 The induced form 24
2.2.3 Closed and exact forms 28
2.2.4 Integration 31
2.3 The Poincaré-Stokes theorem 34

3 Applications **41**

3.1 Maxwell's equations 41
3.1.1 The classical form of the Maxwell's equations 41
3.1.2 Minkowski spacetime 42
3.1.3 The Hodge-star operator 44
3.1.4 Maxwell's equations in terms of differential forms 49
3.2 Brouwer's Fixed Point Theorem 54

Chapter 1

Introduction

The theory of differential forms is a relatively new field in mathematics. It was first introduced at the beginning of the 20th century with the pioneer work of H. Poincare, E. Goursat and E. Cartan. Differential forms contributed to the evolution of many fields in mathematics, such as geometry or topology. Moreover, they provide an essential tool in modern physics, for example in the areas of classical mechanics, electrodynamics and general relativity.

One of the main advantages of using differential forms is that it does not require coordinates. Originally, the laws of classical physics were described by using vector calculus. In this case one first had to choose a coordinate system for calculations. But nature does not come "equipped" with a coordinate system. This is merely a human construction in order to make the computations less difficult. However, the laws of nature are general truths, independent from any chosen coordinate system. Therefore we can say that describing the laws of physics with the language of differential forms captures the real and essential properties of nature.

Another advantage of this formalism in comparison with tensors is that tensor fields "do not behave themselves" under mappings. This means that if Φ is a map from the X space to the Y space and there is a given tensor field on the X space then there is no naturally induced field on the Y space. However, with differential forms it can be defined quite naturally. So if we have a differential form in a space, we can automatically define it in other spaces using various maps.

From a theoretical point of view, differential forms can be considered as an elegant generalization of vector calculus. This general, more abstract theory involves a lot about what an undergraduate student comes across in their studies. Consider the main theorems of multi-variable calculus:, the Green-, Stokes- and the Gauss-theorems. All these theorems can be derived as a special case from the more general Poincaré-Stokes theorem.

The purpose of this thesis is to construct the theoretical background of differential forms and to present some elegant applications of this formalism. As a physical application we will rewrite Maxwell's equations in terms of differential forms and as a mathematical we will present a proof for Brouwer's fixed point theorem.

Chapter 2

Differential forms

2.1 The elements of multilinear algebra

Definition 2.1.1. *Let $k \in \mathbb{N}$ and let $X_1, ..., X_k, Y$ be vector spaces over the field K. We say that the*

$$f : X_1 \times ... \times X_k \to Y$$

*map is **k-linear** (in notation: $f \in \mathcal{L}_k(X_1, ...X_k; Y)$), if for all $i \in \{1, ..., k\}$ and for all $a_j \in X_j$ $(j \in \{1, ..., k\}, j \neq i)$ the map*

$$\phi_i : X_i \to Y, \qquad \phi_i(x) := f(a_1, ...a_{i-1}, x, a_{i+1}, ..., a_k)$$

is linear. In the special case

$$X := X_1 = ... = X_k, \quad Y := K.$$

*f is called a **k-form** and we use the notation $f \in \mathcal{L}_k(X^k, K)$. By convention $\mathcal{L}_0(X^0, K) := K$*

Examples.

1. If $k \in \mathbb{N}$, and $X_1, ..., X_k, Y$ are vector spaces then

$$f_0 : X_1 \times ... \times X_k \to Y, \qquad f_0(a_1, ..., a_k) := 0 \in Y$$

is obviously a k-linear map.

2. If $a \in \mathbb{R}^n$, then

$$f_a : \mathbb{R}^n \to \mathbb{R}, \quad f_a(x) := \langle a, x \rangle := \sum_{i=1}^{n} a_i x_i$$

is a 1-form. If $\{e_1, ..., e_n\}$ is the canonical basis of $\mathbb{R}^n$, then for all $i \in \{1, ..., n\}$

$$f_{e_i}(x) = \langle e_i, x \rangle = x_i \quad (x \in \mathbb{R}^n)$$

is the 1-form which returns the ith coordinate of a vector. So for an arbitrary $a = (a_1, ..., a_n) \in \mathbb{R}^n$ vector

$$f_a(x) = \langle a, x \rangle = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i f_{e_i}(x) \quad (x \in \mathbb{R}^n)$$

which means

$$f_a = \sum_{i=1}^{n} a_i f_{e_i}.$$

So the 1-forms $f_{e_1}, ..., f_{e_n}$ compose the basis of the dual space $\mathbb{R}^n)'$. The following notations are often used as well for the dual basis:

$$f_{e_i} =: e^i =: dx_i.$$

In this thesis we will mainly use dx_i.

3. If $k, n \in \mathbb{N}$, $N := \{1, ..., n\}$, $i = (i_1, ..., i_k) \in N^k$, then

$$\Delta_i^{n,k} : (\mathbb{R}^n)^k \to \mathbb{R}, \qquad \Delta_i^{n,k}(x_1, ...x_k) := \det \begin{bmatrix} x_{1i_1} & \dots & x_{ki_1} \\ \vdots & & \vdots \\ x_{1i_k} & \dots & x_{ki_k} \end{bmatrix} \quad (x_1, ..., x_k \in \mathbb{R}^n)$$

is a k-form.

These $\Delta_i^{n,k}$ forms can be constructed in more general spaces. In the euclidean space $\mathbb{R}^n$ there is a scalar product. For an $x \in \mathbb{R}^n$ element the ith coordinate of x can be expressed as $\langle x, e_i \rangle$, so the above formula can be written as:

$$\Delta_i^{n,k}(x_1, ...x_k) := \det \begin{bmatrix} \langle x_1, e_{i_1} \rangle & \dots & \langle x_k, e_{i_1} \rangle \\ \vdots & & \vdots \\ \langle x_1, e_{i_k} \rangle & \dots & \langle x_k, e_{i_k} \rangle \end{bmatrix} \quad (x_1, ..., x_k \in \mathbb{R}^n).$$

And it can be generalized even more. In a so-called pseudo-euclidean space V (which we will cover in chapter 3) there is no scalar product only a g non-degenerate symmetric bilinear map. In these spaces the definition is:

$$\Delta_i^{n,k}(x_1, ...x_k) := \det \begin{bmatrix} g(x_1, e_{i_1}) & \dots & g(x_k, e_{i_1}) \\ \vdots & & \vdots \\ g(x_1, e_{i_k}) & \dots & g(x_k, e_{i_k}) \end{bmatrix} \qquad (x_1, ..., x_k \in V).$$

Due to the properties of the determinant if i is not an injective multiindex:

$$\exists r, s \in \mathbb{N}, r \neq s : i_r = i_s,$$

then

$$\Delta_i^{n,k} = f_0 \ \ (\in \mathcal{L}_k(\mathbb{R}^n)^k), \mathbb{R}).$$

If $\tilde{i}$ is a permutation of i then

$$\Delta_i^{n,k} = \pm \Delta_{\tilde{i}}^{n,k}$$

Thus, we only need to focus on the **strictly increasing** multiindice:

$$N_*^k = \begin{cases} N := \{1, ..., n\} & (k = 1), \\ \{i \in N^k : i_1 < ... < i_k\} & (k > 1). \end{cases}$$

Theorem 2.1.1. *Let $\{e_1, ..., e_n\}$ be a basis in X. Then any $f \in \mathcal{L}_k(X^k, \mathbb{R})$ form is uniquely determined by its $f(e_{i_1}, ...e_{i_k})$ $(i \in \{1, ..., n\}^k)$ values on the k-tuples of basis vectors.*

Proof. Let $x_1, ..., x_k$ be arbitrary elements in X. Since $\{e_1, ..., e_n\}$ is a basis of X

these elements can be expressed as follows:

$$x_i = \sum_{j_1=1}^{n} a_{ij_i} e_{j_i} \quad (a_{ij_l} \in \mathbb{R}).$$

Using the k-linearity of f we get:

$$f(x_1, ..., x_k) = f\left(\sum_{j_1=1}^{n} a_{1j_1} e_{j_1}, ..., \sum_{j_k=1}^{n} a_{1j_k} e_{j_k}\right) = \sum_{j_1=1}^{n} a_{1j_1}, ..., \sum_{j_k=1}^{n} a_{1j_k} f(e_{j_1}, ..., e_{j_k}).$$

On the other hand if we are given the numbers $f(e_{i_1}, ...e_{i_k})$, the above expression determines a k-form. □

Definition 2.1.2. *Let $2 \leq n \in \mathbb{N}$, X be a vector space over the field K and $f \in \mathcal{L}_k(X^k, K)$. f is called **alternating** if it changes sing whenever two of its arguments are interchanged:*

$$f(a_1, ..., a_i, , ..., a_j, ..., a_k) = -f(a_1, ..., a_j, ..., a_i, ..., a_k)$$

for all $i, j \in N, i \neq j, a_i \in X$.

The set of all alternating k-forms

$$\mathcal{A}_k(X) := \begin{cases} K & (k = 0), \\ X' & (k = 1), \\ \{f \in \mathcal{L}_k(X^k, K) : f \text{ is alternating}\} & (k > 1). \end{cases}$$

builds a subspace in $\mathcal{L}_k(X^k, K)$. **Examples.**

1. The previously defined $\Delta_i^{n,k}$ forms are alternating: $\Delta_i^{n,k} \in \mathcal{A}_k(\mathbb{R}^n)$.

2. If $\phi, \psi \in \mathcal{A}_1(X)$ then their **wedge product** is $\phi \wedge \psi \in \mathcal{A}_2(X)$:

$$\phi \wedge \psi(x_1, x_2) := \phi(x_1)\psi(x_2) - \phi(x_2)\psi(x_1) \qquad (x_1, x_2 \in X).$$

3. If $x_1, ..., x_n \in \mathbb{R}^n$, then

$$f(x_1, ..., x_n) := \det(x_1, ...x_n)$$

is an alternating n-form: $f \in \mathcal{A}_n(\mathbb{R}^n)$.

Definition 2.1.3. *The* ***wedge product*** *of k 1-form is defined as the following. Let $\phi_1, ..., \phi_k \in \mathcal{A}_1(X)$, $x_1, ..., x_k \in X$. Then*

$$\phi_1 \wedge ... \wedge \phi_k \in \mathcal{A}_k(X), \qquad \phi_1 \wedge ... \wedge \phi_k(x_1, ..., x_k) := \det(\phi_i(x_j))_{i,j=1}^n.$$

This definition coincides with the one given before for two 1-forms. Moreover the following equation holds:

$$\Delta_i^{n,k} = dx_1 \wedge ... \wedge dx_k.$$

Theorem 2.1.2. *Let $\{e_1, ..., e_n\}$ be a basis in X. Then the set $\{\Delta_i^{n,k} : i \in N_*^k\}$ is a basis of $\mathcal{A}_k(X)$.*

Proof.

Step 1

First we show that these elements are linearly independent. Let's suppose that some linear combination of these forms is zero and show that all the coefficients must be zero.

$$\sum_{i \in N_*^k} a_i \Delta_i^{n,k} = f_0.$$

For an arbitrary $l \in N_*^k$ multiindex the following equation holds:

$$0 = f_0(e_{l_1}, ..., e_{l_k}) = \sum_{i \in N_*^k} a_i \Delta_i^{n,k}(e_{l_1}, ..., e_{l_k}) = a_l.$$

The last equality is true, because

$$\Delta_i^{n,k}(e_{l_1}, ..., e_{l_k}) = \begin{cases} 1 & (i = l), \\ 0 & (i \neq l). \end{cases}$$

(If $i = l$ then we get the determinant of the identity matrix and if $i \neq l$ there will be a full zero column in the determinant.)

Step 2

Now we prove that the $\Delta_i^{n,k}$ forms span the space, meaning that any arbitrary $f \in \mathcal{A}_k(\mathbb{R}^n)$ element can be expressed as their linear combination:

$$f = \sum_{i \in N_*^k} a_i \Delta_i^{n,k}$$

with some a_i coefficients. For this set

$$g = \sum_{i \in N_*^k} f(e_{i_1}, ..., e_{i_k}) \Delta_i^{n,k}.$$

This way $g \in \mathcal{A}_k(\mathbb{R}^n)$ and

$$g(e_{i_1}, ..., e_{i_k}) = f(e_{i_1}, ..., e_{i_k})$$

for all $i \in N_*^k$. Which means $f = g$, and if we choose the coefficients $a_i := f(e_{i_1}, ..., e_{i_k})$ we get the desired format for f.

□

We will almost always use the basis of strictly increasing multiindice except in the case $n = 3, k = 2$. Then we will use the basis $(\Delta_{(2,3)}^{3,2}, \Delta_{(3,1)}^{3,2}, \Delta_{(1,2)}^{3,2})$.

Definition 2.1.4. *Let $f \in \mathcal{A}_k(X)$ and $g \in \mathcal{A}_l(X)$. The* ***wedge product*** *or* ***exterior product*** *of f and g is the $f \wedge g \in \mathcal{A}_{k+l}(X)$ alternating form defined as follows:*

$$f \wedge g(x_1, ...x_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, ...x_{\sigma(k)}) g(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}).$$

This definition is made in a way that it coincides with the previous ones. Moreover the following equation holds:

$$\Delta_j^{n,k} \wedge \Delta_i^{n,l} = \Delta_{j,i}^{n,k+l} \qquad (j \in N_*^k, i \in N_*^l).$$

The next theorem summarizes the properties of the exterior product.

Theorem 2.1.3. *Suppose that $f_1, f_2, f \in \mathcal{A}_k(X), g_1, g_2, g \in \mathcal{A}_l(X), h \in \mathcal{A}_m(X)$ and $a \in K$. Then*

1. $(f_1 + f_2) \wedge g = f_1 \wedge g + f_2 \wedge g,$
2. $f \wedge (g_1 + g_2) = f \wedge g_1 + f \wedge g_2,$
3. $(af) \wedge g = f \wedge (ag) = a(f \wedge g),$
4. $(f \wedge g) \wedge h = f \wedge (g \wedge h),$
5. $f \wedge g = (-1)^{kl} g \wedge f.$

Proof.

1. Let $x_1, ..., x_{k+l} \in X$ arbitrary elements. Then $(f_1 + f_2) \wedge g(x_1, ..., x_{k+l}) =$

$$
\begin{aligned}
=&\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(f_1 + f_2)(x_{\sigma(1)}, ...x_{\sigma(k)}) g(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) = \\
=&\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f_1(x_{\sigma(1)}, ...x_{\sigma(k)}) g(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) + \\
&+ \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f_2(x_{\sigma(1)}, ...x_{\sigma(k)}) g(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) = \\
=&f_1 \wedge g(x_1, ..., x_{k+l}) + f_2 \wedge g(x_1, ..., x_{k+l}).
\end{aligned}
$$

2. For $x_1, ..., x_{k+l} \in X$ we have $f \wedge (g_1 + g_2)(x_1, ..., x_{k+l}) =$

$$
\begin{aligned}
=&\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, ...x_{\sigma(k)})(g_1 + g_2)(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) = \\
=&\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, ...x_{\sigma(k)}) g_1(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) + \\
&+ \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, ...x_{\sigma(k)}) g_2(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) = \\
=&f \wedge g_1(x_1, ..., x_{k+l}) + f \wedge g_2(x_1, ..., x_{k+l}).
\end{aligned}
$$

3. If $x_1, ..., x_{k+l} \in X$, then $(af) \wedge g(x_1, ...x_{k+l}) =$

$$= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(af)(x_{\sigma(1)}, ...x_{\sigma(k)})g(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) =$$
$$= a\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)f(x_{\sigma(1)}, ...x_{\sigma(k)})g(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) =$$
$$= a(f \wedge g)(x_1, ..., x_{k+l}) =$$
$$= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)f(x_{\sigma(1)}, ...x_{\sigma(k)})(ag)(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}) =$$
$$= f \wedge (ag)(x_1, ..., x_{k+l}).$$

4. Set again any elements. $x_1, ..., x_{k+l+m} \in X$. From the definition we get $((f \wedge g) \wedge h)(x_1, ..., x_{k+l+m}) =$

$$= \frac{1}{(k+l)!m!} \sum_{\sigma \in S_{k+l+m}} \operatorname{sgn}(\sigma)(f \wedge g)(x_{\sigma(1)}, ...x_{\sigma(k+l)})h(x_{\sigma(k+l+1)}, ..., x_{\sigma(k+l+m)}).$$

   Now we decompose the permutation group S_{k+l+m} into so-called residual classes with respect to the subgroup S_{k+l}. This means grouping the permutations of S_{k+l+m} based on how they act on the last m element. This way each class contains $(k+l)!$ elements. Let C denote the set of these classes. Let R be an arbitrary class and σ_R a particular element in R. Each element $\sigma \in R$ can be decomposed as $\sigma = \sigma_R \circ \pi$ where $\pi \in S_{k+l}$. Then the above sum can be written as

$$\frac{1}{(k+l)!m!} \sum_{R \in C} \operatorname{sgn}(\sigma_R) \left(\sum_{\sigma \in R} \operatorname{sgn}(\pi)(f \wedge g)(x_{\sigma(1)}, ..., x_{\sigma(k+l)}) \right) \cdot$$
$$\cdot h(x_{\sigma_R(k+l+1)}, ..., x_{\sigma_R(k+l+m)}).$$

   Now note that all terms in the big parenthesis are equal, because they are all the permutation π from a fixed ordering given by σ_R. Since all classes in

C have $(k+l)!$ elements we can write the equation

$$\frac{1}{m!}\sum_{R\in C}\operatorname{sgn}(\sigma_R)(f\wedge g)(x_{\sigma_R(1)},...,x_{\sigma_R(k+l)})h(x_{\sigma_R(k+l+1)},...,x_{\sigma_R(k+l+m)}).$$

Now calculating the wedge product of f and g we obtain

$$\frac{1}{k!l!m!}\sum_{R\in C}\operatorname{sgn}(\sigma_R)\sum_{\tau\in S_{k+l}}\operatorname{sgn}(\tau)f(x_{\tau(\sigma_R(1))},...,x_{\tau(\sigma_R(k))})g(x_{\tau(\sigma_R(k+1))},...,x_{\tau(\sigma_R(k+l))})\cdot$$
$$\cdot h(x_{\sigma_R(k+l+1)},...,x_{\sigma_R(k+l+m)})$$

But again, all permutations $\sigma\in S_{k+l+m}$ can be decomposed as $\sigma=\tau\circ\sigma_R$ and, since τ acts as the identity on the last m indices, for the indices used in the argument of h $\sigma=\sigma_R$ is true. Thus we finally obtain:

$$((f\wedge g)\wedge h)(x_1,...,x_{k+l+m})=\frac{1}{k!l!m!}\sum_{\sigma\in S_{k+l+m}}\operatorname{sgn}(\sigma)f(x_{\sigma(1)},...,x_{\sigma(k)})\cdot$$
$$\cdot g(x_{\sigma(k+1)},...,x_{\sigma(k+l)})h(x_{\sigma(k+l+1)},...,x_{\sigma(k+l+m)}).$$

Because this result does not depend on the order we associate the operations the statement is proven.

5. Let's write f and g as the following:

$$f=:\sum_{i\in N_*^k}a_i dx_{i_1}\wedge...\wedge dx_{i_k}\quad(a_i\in\mathbb{R}),$$

$$g=:\sum_{j\in N_*^l}b_j dx_{j_1}\wedge...\wedge dx_{j_l}\quad(b_j\in\mathbb{R}).$$

Then

$$f \wedge g = \left(\sum_{i \in N_*^k} a_i dx_{i_1} \wedge ... \wedge dx_{i_k} \right) \wedge \left(\sum_{j \in N_*^l} b_j dx_{j_1} \wedge ... \wedge dx_{j_l} \right) =$$

$$= \sum_{i \in N_*^k} \sum_{j \in N_*^l} a_i b_j dx_{i_1} \wedge ... \wedge dx_{i_k} \wedge dx_{j_1} \wedge ... \wedge dx_{j_l}$$

Now we want to interchange the dx_i forms with the dx_j ones and use the fact that

$$dx_i \wedge dx_j = -dx_j \wedge dx_i.$$

All k dx_i-s have to "move through" l dx_j-s, thus the expression changes sign kl times:

$$f \wedge g = \sum_{i \in N_*^k} \sum_{j \in N_*^l} a_i b_j (-1)^k dx_{j_1} \wedge (dx_{i_1} \wedge ... \wedge dx_{i_k}) \wedge (dx_{j_2} \wedge ... \wedge dx_{j_l}) =$$

$$= \sum_{i \in N_*^k} \sum_{j \in N_*^l} a_i b_j (-1)^{kl} (dx_{j_1} \wedge ... \wedge dx_{j_l}) \wedge (dx_{i_1} \wedge ... \wedge dx_{i_k}) =$$

$$= (-1)^{kl} \sum_{i \in N_*^k} \sum_{j \in N_*^l} b_j a_i (dx_{j_1} \wedge ... \wedge dx_{j_l}) \wedge (dx_{i_1} \wedge ... \wedge dx_{i_k}) =$$

$$= (-1)^{kl} \left[\left(\sum_{j \in N_*^l} b_j dx_{j_1} \wedge ... \wedge dx_{j_l} \right) \wedge \left(\sum_{i \in N_*^k} a_i dx_{i_1} \wedge ... \wedge dx_{i_k} \right) \right] =$$

$$= (-1)^{kl} g \wedge f.$$

$\square$

Remarks.

1. The dimension of $\mathcal{A}_k(X)$ is:

$$\dim(\mathcal{A}_k(X)) = \begin{cases} \binom{n}{k} & (k \leq n), \\ 0 & (k > n). \end{cases}$$

2. The $\mathcal{A}(X)$ symbol denotes the direct sum of the $\mathcal{A}_k(X)$ vector spaces:

$$\mathcal{A}(X) := \mathcal{A}_0(X) \oplus ... \oplus \mathcal{A}_n(X) \oplus ... := \oplus_{m \in \mathbb{N}} \mathcal{A}_m(X).$$

It is clear that if $\dim(X) = n$, then:

$$\dim(\mathcal{A}(X)) = 1 + \binom{n}{1} + ... + \binom{n}{n} + 0 + ... = 2^n.$$

If $r, s \in \mathbb{N}_0$ and $\phi \in \mathcal{A}_r(X), \psi \in \mathcal{A}_s(X)$, then let

$$f := \sum_{r=0}^{\infty} \phi_r \in \mathcal{A}(X), \quad g := \sum_{s=0}^{\infty} \psi_s \in \mathcal{A}(X)$$

and

$$f \wedge g := \sum_{l=1}^{\infty} \sum_{r+s=l} \phi_r \wedge \psi_s \in \mathcal{A}(X).$$

Then the $\mathcal{A}$(X) vector space equipped with the exterior product constitutes the $(\mathcal{A}(X), \wedge)$ **exterior algebra**.

2.2 Differential forms in $\mathbb{R}^n$

Definition 2.2.1. *Let $n \in \mathbb{N}, k \in \mathbb{N}_0, \emptyset \neq V \subset \mathbb{R}^n$ be an open set. Then the map:*

$$\omega : V \to \mathcal{A}_k(\mathbb{R}^n)$$

*is called a **differential form of degree k** (over $\mathbb{R}^n$).*

Since $\Delta_i^{n,k} (i \in N_*^k)$ is a basis in $\mathcal{A}_k(\mathbb{R}^n)$ it is clear, that for every $x \in V$ exist the unique numbers

$$\omega_i(x) \in \mathbb{R} \ (i \in N_*^k)$$

such that

$$\omega(x) = \sum_{i \in N_*^k} \omega_i(x) \Delta_i^{n,k}.$$

The $\omega_i : V \to \mathbb{R}$ maps are called the **coordinate functions** of the differential form ω. The above discussed notations can be written shortly as:

$$\omega = \sum_{i \in N_*^k} \omega_i \Delta_i^{n,k}.$$

This is the so-called **canonical form** of ω.

Definition 2.2.2. *Let $r \in \mathbb{N}$. The space $\Lambda_k^r(V)$ denotes the differential forms of degree k, whose coordinate functions are r-times continuously differentiable:*

$$\Lambda_k^r(V) := \{\omega : V \to \mathcal{A}_k(\mathbb{R}^n) : \omega \in \mathcal{C}^r\}$$

The property $\omega \in \mathcal{C}^r$ is equivalent to $\omega_i \in \mathcal{C}^r \ \forall i \in N_*^k$.

The following operations are defined between differential forms:

Definition 2.2.3. *Let $f : V \to \mathbb{R}, \omega, w \in \Lambda_k^r(V), \sigma \in \Lambda_l^r(V)$. Then:*

1. $(f\omega)(x) := f(x)\omega(x)$,
2. $(\omega + w)(x) := \omega(x) + w(x)$,
3. $(\omega \wedge \sigma)(x) := \omega(x) \wedge \sigma(x)$.

Remarks.

1. In the special case $k = 0$ $\mathcal{A}_0(\mathbb{R}^n) = \mathbb{R}$, hence a differential 0-form is an $\omega : V \to \mathbb{R}$ n-variable function (scalar field).
2. If $k > n$ then $\mathcal{A}_k(\mathbb{R}^n) = \{f_0\}$.

3. It is clear that if $\alpha : V \to \mathbb{R}, \omega, w \in \Lambda_k^r(V)$ and $\sigma \in \Lambda_l^r(V)$, then

$$\alpha\omega = \sum_{i \in N_*^k} (a\omega_i)\Delta_i^{n,k} \in \Lambda_k^r(V), \ \ \omega + w = \sum_{i \in N_*^k} (\omega_i + w_i)\Delta_i^{n,k} \in \Lambda_k^r(V)$$

and

$$\omega \wedge \sigma = \sum_{i \in N_*^k} \sum_{j \in N_*^l} (\omega_i \sigma_i)\Delta_i^{n,k} \wedge \Delta_j^{n,l} \in \Lambda_{k+l}^r(V).$$

4. Let us introduce the following somewhat simplifying notation:

$$\omega(x; x_1, ..., x_k) := \omega(x)(x_1, ..., x_k) = \sum_{i \in N_*^k} \omega_i(x)\Delta_i^{n,k}(x_1, ..., x_k)$$

$$(x \in V, x_1, ..., x_k \in \mathbb{R}^n).$$

5. If $e_1, ..., e_n$ are the canonical basis vectors of $\mathbb{R}^n$, then for all $i \in N_*^k$ the coordinate functions of a differential k-form can be computed as follows:

$$\omega(x; e_{i_1}, ..., e_{i_k}) = \sum_{j \in N_*^k} \omega_j(x)\Delta_j^{n,k}(e_{i_1}, ..., e_{i_k}) = \omega_i(x) \qquad (x \in V).$$

2.2.1 Derivation

Definition 2.2.4. *The **exterior derivative** of a differential form:*

$$d : \Lambda_k^r(V) \to \Lambda_{k+1}^{r-1}(V), \qquad d\omega := d(\omega) := \begin{cases} \sum_{j \in N} \partial_j \omega \Delta_j^{n,1} \ (k = 0), \\ \\ \sum_{j \in N} \sum_{i \in N_*^k} \partial_j \omega_i \Delta_{(j,i)}^{n,k+1} \ (k > 0) \end{cases}$$

where $(j, i) := (j, i_1, ..., i_k) \in N \times N_*^k$.

Remarks.

1. If $k = 0$ and $\omega \in \mathcal{C}^1(V, \mathbb{R})$, then for all $x \in V$, and $\xi \in \mathbb{R}^n$ we have

$$d\omega(x)(\xi) = \sum_{j=1}^{n} \partial_j \omega(x)\Delta_j^{n,1}(\xi) = \langle \operatorname{grad} \omega(x), \xi \rangle.$$

With the commonly used notation

$$\partial_j \omega = \frac{\partial \omega}{\partial x_j} \qquad \text{and} \qquad \Delta_j^{n,1} = dx_j$$

we get

$$d\omega = \sum_{j=1}^{n} \frac{\partial \omega}{\partial x_j} dx_j,$$

which is the so-called **differential** of the ω scalar field at the point x.

2. If $k \in N$, then using the equality $\Delta_{(j,i)}^{n,k+1} = \Delta_j^{n,1} \wedge \Delta_i^{n,k}$ we have

$$d\omega = \sum_{j \in N} \sum_{i \in N_*^k} \partial_j \omega_i \Delta_j^{n,1} \wedge \Delta_i^{n,k} = \sum_{i \in N_*^k} \sum_{j \in N} \partial_j \omega_i \Delta_j^{n,1} \wedge \Delta_i^{n,k} = \sum_{i \in N_*^k} d\omega_i \wedge \Delta_i^{n,k}.$$

Theorem 2.2.1. *The exterior derivative has the following properties:*

1. *The d operator is linear, meaning if $\omega, w, \in \Lambda_k^r(V)$ and $\alpha \in \mathbb{R}$, then*

$$d(\omega + w) = d\omega + dw \quad \text{and} \quad d(\alpha\omega) = \alpha d\omega.$$

2. *If $\omega \in \Lambda_k^r(V)$ and $\sigma \in \Lambda_l^r(V)$, then*

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge (d\sigma).$$

3. *If $\omega \in \Lambda_k^2(V)$, then*

$$d(d\omega) = 0 \qquad (\in \Lambda_{k+2}^0(V)).$$

Proof.

1. (a) First we prove the additivity.

- If $k = 0$ then

$$d(\omega + w) = \sum_{j\in N} \partial_j(\omega + w)\Delta_j^{n,1} = \sum_{j\in N}(\partial_j\omega + \partial_j w)\Delta_j^{n,1} =$$

$$= \sum_{j\in N} \partial_j\omega\Delta_j^{n,1} + \sum_{j\in N} \partial_j w\Delta_j^{n,1} = d\omega + dw.$$

- If $k > 0$ then

$$d(\omega + w) = \sum_{j\in N}\sum_{i\in N_*^k} \partial_j(\omega_i + w_i)\Delta_{(j,i)}^{n,k+1} =$$

$$= \sum_{j\in N}\sum_{i\in N_*^k} (\partial_j\omega_i + \partial_j w_i)\Delta_{(j,i)}^{n,k+1} =$$

$$= \sum_{j\in N}\sum_{i\in N_*^k} \partial_j\omega_i\Delta_{(j,i)}^{n,k+1} + \sum_{j\in N}\sum_{i\in N_*^k} \partial_j w_i\Delta_{(j,i)}^{n,k+1} =$$

$$= d\omega + dw.$$

(b) Now we prove the homogeneity, which goes analogously.

- If $k = 0$ then

$$d(\alpha\omega) = \sum_{j\in N} \partial_j(\alpha\omega)\Delta_j^{n,1} = \alpha\sum_{j\in N} \partial_j\omega\Delta_j^{n,1} = \alpha d\omega.$$

- If $k > 0$ then

$$d(\alpha\omega) = \sum_{j\in N}\sum_{i\in N_*^k} \partial_j(\alpha\omega_i)\Delta_{(j,i)}^{n,k+1} = \alpha\sum_{j\in N}\sum_{i\in N_*^k} \partial_j\omega_i\Delta_{(j,i)}^{n,k+1} = \alpha d\omega.$$

2. Using the linearity of d:

$$d(\omega \wedge \sigma) = d\left(\sum_{i \in N_*^k} \sum_{j \in N_*^l} (\omega_i \sigma_j) \Delta_i^{n,k} \wedge \Delta_j^{n,l}\right) = \sum_{i \in N_*^k} \sum_{j \in N_*^l} d(\omega_i \sigma_j) \wedge \Delta_i^{n,k} \wedge \Delta_j^{n,l} =$$

$$= \sum_{i \in N_*^k} \sum_{j \in N_*^l} (\sigma_j d\omega_i + \omega_i d\sigma_j) \wedge \Delta_i^{n,k} \wedge \Delta_j^{n,l} =$$

$$= \sum_{i \in N_*^k} \sum_{j \in N_*^l} (\sigma_j d\omega_i) \wedge \Delta_i^{n,k} \wedge \Delta_j^{n,l} + \sum_{i \in N_*^k} \sum_{j \in N_*^l} (\omega_i d\sigma_j) \wedge \Delta_i^{n,k} \wedge \Delta_j^{n,l} =$$

$$= \left(\sum_{i \in N_*^k} d\omega_i \wedge \Delta_i^{n,k}\right) \wedge \left(\sum_{j \in N_*^l} \sigma_j \Delta_j^{n,l}\right) +$$

$$+ \sum_{i \in N_*^k} \sum_{j \in N_*^l} \omega_i (d\sigma_j \wedge \Delta_i^{n,k}) \wedge \Delta_j^{n,l} =$$

$$= (d\omega \wedge \sigma) + \sum_{i \in N_*^k} \sum_{j \in N_*^l} \omega_i ((-1)^k \Delta_i^{n,k} \wedge d\sigma_j) \wedge \Delta_j^{n,l} =$$

$$= (d\omega \wedge \sigma) + (-1)^k \left(\sum_{i \in N_*^k} \omega_i \Delta_i^{n,k}\right) \wedge \left(\sum_{j \in N_*^l} d\sigma_j \wedge \Delta_j^{n,l}\right) =$$

$$= d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma.$$

3. • If $k = 0$ then by using the linearity of d and Young's theorem

$$d(d\omega) = d\left(\sum_{j\in N} \partial_j \omega \Delta_j^{n,1}\right) = \sum_{j\in N} d(\partial_j\omega) \wedge \Delta_j^{n,1} =$$

$$= \sum_{j\in N}\left(\sum_{i\in N} \partial_i(\partial_j\omega)\Delta_i^{n,1}\right) \wedge \Delta_j^{n,1} = \sum_{i,j\in N} \partial_{ij}\omega\Delta_i^{n,1} \wedge \Delta_j^{n,1} =$$

$$= \sum_{i<j} (\partial_{ij}\omega - \partial_{ij}\omega)\Delta_i^{n,1} \wedge \Delta_j^{n,1} = 0 \in \Lambda_2^0(V).$$

- If $k > 0$ then using the linearity, the rule for differentiating an exterior product and the $k = 0$ case we have

$$d(d\omega) = d\left(\sum_{i\in N_*^k} d\omega_i \wedge \Delta_i^{n,k}\right) = \sum_{i\in N_*^k} d(d\omega_i \wedge \Delta_i^{n,k}) =$$
$$= \sum_{i\in N_*^k} [d(d\omega_i) \wedge \Delta_i^{n,k} - (d\Delta_i^{n,k}) \wedge (d\omega_i)] = 0 \in \Lambda_{k+2}^0(V).$$

The last equation holds, since for all $i \in N_*^k$: $\omega_i \in \Lambda_0^2(V)$, so because of the $k = 0$ case we have

$$d(d\omega_i) = 0 \in \Lambda_2^0(V)$$

and

$$d\Delta_i^{n,k} = d(1\Delta_i^{n,k}) = \sum_{j\in N} (\partial_j 1)\Delta_{(j,i)}^{n,k+1} = \sum_{j\in N} 0\Delta_{(j,i)}^{n,k+1} = 0 \in \Lambda_{k+1}^1 \quad (i \in N_*^k).$$

□

Definition 2.2.5. *Let* $f_i : V \to \mathbb{R}$ $(i \in N_*^k)$ *and* $f := (f_i, i \in N_*^k) : V \to \mathbb{R}^{\binom{n}{k}}$, *then the* ***differential form generated by f*** *is:*

$$\omega_f := \sum_{i \in N_*^k} f_i \Delta_i^{n,k}$$

meaning that:

$$\omega_f(x; x_1, ..., x_k) := \sum_{i \in N_*^k} f_i(x) \Delta_i^{n,k}(x_1, ..., x_k) \ (x \in V, x_1, ..., x_k \in \mathbb{R}^n).$$

Special cases.

1. *If* $k = 0$, *then* $f : V \to \mathbb{R}$, *so* $\omega_f = f$, *thus:*

$$d\omega_f = \sum_{j \in N} \partial_j f \Delta_j^{n,1} = \omega_{\mathrm{grad}(f)}.$$

2. *If* $n = 2, k = 1$, *then* $f : V \to \mathbb{R}^2$, *so* $\omega_f = f_1 \Delta_1^{2,1} + f_2 \Delta_2^{2,1}$, *thus:*

$$d\omega_f = \sum_{j=1}^{2} \sum_{i=1}^{2} \partial_j f_i \Delta_{(j,i)}^{2,2} = \sum_{j=1}^{2} \left(\partial_j f_1 \Delta_{(j,1)}^{2,2} + \partial_j f_2 \Delta_{(j,2)}^{2,2} \right) =$$

$$= \partial_1 f_1 \Delta_{(1,1)}^{2,2} + \partial_1 f_2 \Delta_{(1,2)}^{2,2} + \partial_2 f_1 \Delta_{(2,1)}^{2,2} + \partial_2 f_2 \Delta_{(2,2)}^{2,2} =$$

$$= 0 + \partial_1 f_2 \Delta_{(1,2)}^{2,2} - \partial_2 f_1 \Delta_{(1,2)}^{2,2} + 0 = \omega_{\partial_1 f_2 - \partial_2 f_1}.$$

3. *If* $n = 3, k = 1$, *then* $f : V \to \mathbb{R}^3$, *so* $\omega_f = f_1 \Delta_1^{3,1} + f_2 \Delta_2^{3,1} + f_3 \Delta_3^{3,1}$, *thus*

$$d\omega_f = \sum_{j=1}^{3}\sum_{i=1}^{3} \partial_j f_i \Delta^{3,2}_{(j,i)} = \sum_{j=1}^{3} \left(\partial_j f_1 \Delta^{3,2}_{(j,i)} + \partial_j f_2 \Delta^{3,2}_{(j,2)} + \partial_j f_3 \Delta^{3,2}_{(j,3)} \right) =$$

$$= \partial_1 f_1 \Delta^{3,2}_{(1,1)} + \partial_1 f_2 \Delta^{3,2}_{(1,2)} + \partial_1 f_3 \Delta^{3,2}_{(1,3)} +$$

$$+ \partial_2 f_1 \Delta^{3,2}_{(2,1)} + \partial_2 f_2 \Delta^{3,2}_{(2,2)} + \partial_2 f_3 \Delta^{3,2}_{(2,3)} +$$

$$+ \partial_3 f_1 \Delta^{3,2}_{(3,1)} + \partial_3 f_2 \Delta^{3,2}_{(3,2)} + \partial_3 f_3 \Delta^{3,2}_{(3,3)} =$$

$$= (\partial_2 f_3 - \partial_3 f_2) \Delta^{3,2}_{(2,3)} + (\partial_3 f_1 - \partial_1 f_3) \Delta^{3,2}_{(3,1)} + (\partial_1 f_2 - \partial_2 f_1) \Delta^{3,2}_{(1,2)} =$$

$$= \omega_{\mathrm{rot}(f)}.$$

4. *If* $n = 3, k = 2$, *then* $f : V \to \mathbb{R}^3$, *so*

$$\omega_f = f_1 \Delta^{3,2}_{(2,3)} + f_2 \Delta^{3,2}_{(3,1)} + f_3 \Delta^{3,2}_{(1,2)},$$

thus

$$d\omega_f = \sum_{j=1}^{3}\sum_{i=1}^{3} \partial_j f_i \Delta^{3,3}_{(j,i)} = \sum_{j=1}^{3} \left(\partial_j f_1 \Delta^{3,3}_{(j,2,3)} + \partial_j f_2 \Delta^{3,3}_{(j,3,1)} + \partial_j f_3 \Delta^{3,3}_{(j,1,2)} \right) =$$

$$= (\partial_1 f_1 \Delta^{3,3}_{(1,2,3)} + 0 + 0) + (0 + \partial_2 f_2 \Delta^{3,3}_{(2,3,1)} + 0) + (0 + 0 + \partial_3 f_3 \Delta^{3,3}_{(3,1,2)}) =$$

$$= (\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3) \Delta^{3,3}_{(1,2,3)} = \omega_{\mathrm{div}(f)}.$$

2.2.2 The induced form

One of the most important features of differential forms is the way they behave under differentiable maps.

Definition 2.2.6. *Let $r \in \mathbb{N}_0$, $n, m \in \mathbb{N}$, $k \in \{0, 1, ..., n\}$, $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets, $\omega \in \Lambda_k^r(V)$ and $\Phi \in \mathcal{C}^{r+1}(U, V)$. Let us define the **induced form** or the **pullback** of ω as follows:*

$$\omega * \Phi \in \Lambda_k^r(U).$$

If $k = 0$, then

$$\omega * \Phi := \omega \circ \Phi.$$

If $k > 0$, then

$$\omega * \Phi(x; x_1, ...x_k) := \omega(\Phi(x); \Phi'(x)x_1, ..., \Phi'(x)x_k) \qquad (x \in U, x_1, ...x_k \in \mathbb{R}^m).$$

This definition makes sense since $\Phi(x) \in V$, $\Phi'(x) \in \mathbb{R}^{n \times m}$, thus $\Phi'(x)x_i \in \mathbb{R}^n$.

Theorem 2.2.2. *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open subsets, $\Phi \in \mathcal{C}^{r+1}(\mathbb{R}^m, \mathbb{R}^n)$ $k \in \{0, 1, ..., n\}$, $\omega, \sigma \in \Lambda_k^r(\mathbb{R}^n)$ and $g \in \Lambda_0^r(\mathbb{R}^n)$. Then:*

1. *$(\omega + \sigma) * \Phi = \omega * \Phi + \sigma * \Phi$,*
2. *$(g\omega) * \Phi = g \circ \Phi \cdot \omega * \Phi$,*
3. *if $\omega_1, ..., \omega_k \in \Lambda_1^r(\mathbb{R}^n)$, then $(\omega_1 \wedge ... \wedge \omega_k) * \Phi = \omega_1 * \Phi \wedge ... \wedge \omega_k * \Phi$.*

Proof.

1. If $x \in U$ and $x_1, ..., x_k \in \mathbb{R}^m$ then

$$((\omega + \sigma) * \Phi)(x; x_1, ..., x_k) = (\omega + \sigma)(\Phi(x); \Phi'(x)x_1, ..., \Phi'(x)x_k) =$$

$$= \omega(\Phi(x); \Phi'(x)x_1, ..., \Phi'(x)x_k) + \sigma(\Phi(x); \Phi'(x)x_1, ..., \Phi'(x)x_k) =$$

$$= \omega * \Phi(x; x_1, ..., x_k) + \sigma * \Phi(x; x_1, ..., x_k).$$

2. For $x \in U$ and $x_1, ..., x_k \in \mathbb{R}^m$ we have

$$((g\omega) * \Phi)(x; x_1, ..., x_k) = (g\omega)(\Phi(x); \Phi'(x)x_1, ...\Phi'(x)x_k) =$$

$$= g(\Phi(x))\omega(\Phi(x); \Phi'(x)x_1, ...\Phi'(x)x_k) = g \circ \Phi(x) \cdot \omega * \Phi(x_1, ..., x_k).$$

3. Let $x \in U$ and $x_1, ..., x_k \in \mathbb{R}^m$. Then

$$((\omega_1 \wedge ... \wedge \omega_k) * \Phi)(x; x_1, ..., x_k) = (\omega_1 \wedge ... \wedge \omega_k)(\Phi(x); \Phi'(x)x_1, ..., \Phi'(x)x_k) =$$

$$= \det(\omega_i(\Phi(x; \Phi'(x)x_j))) = \det(\omega_i * \Phi(x; x_j)) =$$

$$= (\omega_1 * \Phi \wedge ... \wedge \omega_k * \Phi)(x; x_1, ..., x_k).$$

□

Now we can interpret the meaning of the induced form. Let $(x_1, ..., x_n)$ be coordinates in $\mathbb{R}^n$ and $(y_1, ..., y_m)$ be coordinates in $\mathbb{R}^m$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function which "substitutes" the coordinates:

$$y_1 = \Phi_1(x_1, ..., x_n), ..., y_m = \Phi_m(x_1, ..., x_n).$$

Let $\omega = \sum_{i \in N_*^k} \omega_i dy_{i_1} \wedge ... \wedge dy_{i_k}$ be a k-form in $\mathbb{R}^m$. Then using the above properties of the induced form we obtain

$$\omega * \Phi = \left(\sum_{i \in N_*^k} \omega_i dy_{i_1} \wedge ... \wedge dy_{i_k} \right) * \Phi = \sum_{i \in N_*^k} (\omega_i dy_{i_1} \wedge ... \wedge dy_{i_k}) * \Phi =$$

$$= \sum_{i \in N_*^k} (\omega_i \circ \Phi) \cdot (dy_{i_1} \wedge ... \wedge dy_{i_k}) * \Phi = \sum_{i \in N_*^k} (\omega_i \circ \Phi) \cdot dy_{i_1} * \Phi \wedge ... \wedge dy_{i_k} * \Phi.$$

Since for all $x \in U$ and for all $v \in \mathbb{R}^m$

$$(dy_j * \Phi)(x; v) = dy_j(\Phi(x); \Phi'(x)v) = dy_j(\Phi'(x)v) = (\Phi'(x)v)_j =$$

$$= \partial_1 \Phi_j(x) v_1 + ... + \partial_n \Phi_j(x) v_n = \langle \Phi_j'(x), v \rangle = d\Phi_j(x)(v),$$

we have

$$\omega * \Phi = \sum_{i \in N_*^k} (\omega_i \circ \Phi) d\Phi_{i_1} \wedge ... \wedge d\Phi_{i_k}.$$

Thus the pullback of ω is the same as substituting the x_i variables and their dx_i differentials by the functions of x_k and dx_k.

Theorem 2.2.3. *Let $\Phi \in \mathcal{C}^{r+1}(\mathbb{R}^m, \mathbb{R}^n)$, $\omega, \sigma \in \Lambda_k^r(\mathbb{R}^n)$ and $\Psi \in \mathcal{C}^{r+1}(\mathbb{R}^p, \mathbb{R}^m)$. Then*

1. $(\omega \wedge \sigma) * \Phi = \omega * \Phi \wedge \sigma * \Phi$,
2. $\omega * (\Phi \circ \Psi) = (\omega * \Phi) * \Psi$,
3. $d\omega * \Phi = d(\omega * \Phi)$.

Proof.
Let $(y_1, ..., y_n) = (\Phi_1(x_1, ..., x_m), ..., \Phi_n(x_1, ..., x_m)) \in \mathbb{R}^n$, $(x_1, ..., x_m) \in \mathbb{R}^m$, and

$$\omega = \sum_{i \in N_*^k} \omega_i dy_{i_1} \wedge ... \wedge dy_{i_k}, \qquad \sigma = \sum_{j \in N_*^k} \sigma_i dy_{j_1} \wedge ... \wedge dy_{j_k}.$$

1. Then $(\omega \wedge \sigma) * \Phi =$

$$= \left(\sum_{i \in N_*^k} \sum_{j \in N_*^k} \omega_i \sigma_j (dy_{i_1} \wedge ... \wedge dy_{i_k}) \wedge (dy_{j_1} \wedge ... \wedge dy_{j_k}) \right) * \Phi =$$

$$= \sum_{i \in N_*^k} \sum_{j \in N_*^k} \omega_i \circ \Phi \cdot \sigma_j \circ \Phi \cdot (d\Phi_{i_1} \wedge ... \wedge d\Phi_{i_k}) \wedge (d\Phi_{j_1} \wedge ... \wedge d\Phi_{j_k}) =$$

$$= \left(\sum_{i \in N_*^k} \omega_i \circ \Phi \cdot d\Phi_{i_1} \wedge ... \wedge d\Phi_{i_k} \right) \wedge \left(\sum_{j \in N_*^k} \sigma_j \circ \Phi \cdot d\Phi_{j_1} \wedge ... \wedge d\Phi_{j_k} \right) =$$

$$= \omega * \Phi \wedge \sigma * \Phi.$$

2. Let $x, x_1, ..., x_k \in \mathbb{R}^p$. Then

$$(\omega * (\Phi \circ \Psi))(x; x_1, ..., x_k) = \omega(\Phi \circ \Psi(x); (\Phi \circ \Psi)'(x)x_1, ..., (\Phi \circ \Psi)'(x)x_k) =$$

$$= \omega(\Phi(\Psi(x)); \Phi'(\Psi(x))\Psi'(x)x_1, ..., \Phi'(\Psi(x))\Psi'(x)x_k) =$$

$$= \omega * \Phi(\Psi(x); \Psi'(x)x_1, ..., \Psi'(x)x_k) =$$

$$= (\omega * \Phi) * \Psi(x; x_1, ..., x_k).$$

3. **Step 1**

   We will first prove the statement for 0-forms. This case $\omega : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function.

$$d\omega * \Phi = \left(\sum_{i=1}^{n} \frac{\partial \omega_i}{\partial y_i} dy_i \right) * \Phi = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial \omega_i}{\partial y_i} \circ \Phi \cdot \frac{\partial \Phi_i}{\partial x_j} dx_j = \sum_{j=1}^{m} \frac{\partial (\omega \circ \Phi)}{\partial x_j} dx_j =$$

$$= d(\omega \circ \Phi) = d(\omega * \Phi).$$

   Step 2 Now we consider the $k > 0$ case.

$$d(\omega * \Phi) = d \left(\sum_{i \in N_*^k} \omega_i \circ \Phi \cdot (dy_{i_1} * \Phi) \wedge ... \wedge (dy_{i_k} * \Phi) \right) =$$

$$= \sum_{i \in N_*^k} d(\omega_i \circ \Phi \cdot (d\Phi_{i_1} \wedge ... \wedge d\Phi_{i_k})) =$$

$$= \sum_{i \in N_*^k} d(\omega_i \circ \Phi) \wedge (d\Phi_{i_1} \wedge ... \wedge d\Phi_{i_k}) =$$

$$= \sum_{i \in N_*^k} (d\omega_i * \Phi) \wedge (dy_{i_1} * \Phi) \wedge ... \wedge (dy_{i_k} * \Phi) =$$

$$= \left(\sum_{i \in N_*^k} d\omega_i \wedge dy_{i_1} \wedge ... \wedge dy_{i_k} \right) * \Phi = d\omega * \Phi.$$

□

2.2.3 Closed and exact forms

From the theory of the Riemann-integral it is a well-known fact, that every continuous function on $\mathbb{R}$ has a primitive function. In the language of differential forms it means that if $\omega \in \Lambda_1^0(I)$ and has the form: $\omega(x) = f(x)dx$ with some continuous function f, there exists some function F such that $dF = \omega$.

Now we want to ask the analogous question of whether any differential form has a "primitive form". We would like to give a necessary and sufficient condition for the existence of this form.

Definition 2.2.7. *The $\omega \in \Lambda_k^r(V)$ differential form is said to be*

1. ***closed*** *if $d\omega = 0$,*
2. ***exact*** *if there is an $\eta \in \Lambda_{k-1}^{r+1}$ such that $d\eta = \omega$.*

Since for all $\omega \in \Lambda_k^2(V)$: $dd\omega = 0$ it is clear that if a form is exact, then it is closed. The following "lemma" presents a necessary condition for closed forms to be exact.

Theorem 2.2.4 (Poincaré-lemma). *Let $k \in \mathbb{N}$, $2 \le r \in \mathbb{N} \cup \{\infty\}$ and V be a star-shaped*[1] *domain. In this case $\omega \in \Lambda_k^r(V)$ is closed if and only if it is exact.*

Proof. It only remains to show that if ω is closed, then it is exact. Let us suppose that the star-point of V is 0[2], and the closed form ω is the folowing:

$$\omega = \sum_{i \in N_*^k} \omega_i \Delta_i^{n,k}.$$

We will define an operation $P : \Lambda_k^r(V) \to \Lambda_{k-1}^{r+1}(V)$ for which $dP(\omega) = \omega$, thus proves, that ω is exact. Let $x \in V$ and P be the following:

$$P(\omega)(x) := \sum_{i \in N_*^k} \sum_{l=1}^{k} (-1)^{l-1} \left(\int_0^1 t^{k-1} \omega_i(tx) dt \right) x_{i_l} dx_{i_1} \wedge \ldots \wedge dx_{i_{l-1}} \wedge dx_{i_{l+1}} \wedge \ldots \wedge dx_{i_k}.$$

Now if we use the rule for the derivative of a wedge product for the coordinate functions we get: $dP(\omega)(x) =$

$$\begin{aligned}
=& \sum_{i \in N_*^k} \sum_{l=1}^{k} \left(\int_0^1 t^{k-1} \omega_i(tx) dt \right) \Delta_i^{n,k} + \\
&+ \sum_{j=1}^{n} \sum_{i \in N_*^k} \sum_{l=1}^{k} (-1)^{l-1} \left(\int_0^1 t t^{k-1} \partial_j \omega_i(tx) dt \right) x_{i_l} dx_{i_1} \wedge \ldots \wedge dx_{i_{l-1}} \wedge dx_{i_{l+1}} \wedge \ldots \wedge dx_{i_k} = \\
=& k \sum_{i \in N_*^k} \left(\int_0^1 t^{k-1} \omega_i(tx) dt \right) \Delta_i^{n,k} + \\
&+ \sum_{j=1}^{n} \sum_{i \in N_*^k} \sum_{l=1}^{k} (-1)^{l-1} \left(\int_0^1 t t^{k-1} \partial_j \omega_i(tx) dt \right) x_{i_l} dx_{i_1} \wedge \ldots \wedge dx_{i_{l-1}} \wedge dx_{i_{l+1}} \wedge \ldots \wedge dx_{i_k}.
\end{aligned}$$

---

[1] It means that there exists an $a \in V$ (so-colled **star-point**) such that for all $x \in V$

$$[a, x] := \{a + t(x - a) \in \mathbb{R}^n : t \in [0, 1]\} \subset V.$$

[2] If $0 \ne a$ is a star-point of V (and 0 is not) we can apply the transformation $y := x - a$.

Now let's calculate $P(d\omega)(x)$. Since

$$d\omega = \sum_{j=1}^{n} \sum_{i \in N_*^k} \partial_j \omega_i \Delta_{(j,i)}^{n,k}$$

we have $P(d\omega)(x) =$

$$= \sum_{j=1}^{n} \sum_{i \in N_*^k} \left(\int_0^1 t^{k-1} \omega_i(tx) dt \right) x_j \Delta_i^{n,k} -$$

$$- \sum_{j=1}^{n} \sum_{i \in N_*^k} \sum_{l=1}^{k} (-1)^{l-1} \left(\int_0^1 t t^{k-1} \partial_j \omega_i(tx) dt \right) x_{i_l} dx_{i_1} \wedge ... \wedge dx_{i_{l-1}} \wedge dx_{i_{l+1}} \wedge ... \wedge dx_{i_k}.$$

If we add the two formulas there are a lot of therms which cancel each other out, thus we get:

$$(P(d\omega) + dP(\omega))(x) = k \sum_{i \in N_*^k} \left(\int_0^1 t^{k-1} \omega_i(tx) dt \right) \Delta_i^{n,k} +$$

$$+ \sum_{j=1}^{n} \sum_{i \in N_*^k} \left(\int_0^1 t^{k-1} \omega_i(tx) dt \right) x_j \Delta_i^{n,k} =$$

$$= \sum_{i \in N_*^k} \left(\int_0^1 \frac{d}{dt} [t^k \omega_i(tx)] dt \right) \Delta_i^{n,k} =$$

$$= \sum_{i \in N_*^k} [1 \cdot \omega_i(x) - 0 \cdot \omega_i(0)] \, \Delta_i^{n,k} = \omega.$$

So we have

$$P(d\omega) + dP(\omega) = \omega.$$

Using our assumption that ω is closed this simplifies to $dP(\omega) = \omega$, which means that $P(\omega)$ is the form we were looking for. $\square$

2.2.4 Integration

Let us introduce the following notation:

$$\mathbb{I}^k := [0,1]^k \quad (k \in \mathbb{N})$$

the closed unit cube in $\mathbb{R}^k$. By convention

$$\mathbb{I} := \mathbb{I}^1 \qquad \text{and} \qquad \mathbb{I}^0 := \{0\}.$$

Definition 2.2.8. *Let $k \in \mathbb{N}_0, 2 \leq n \in \mathbb{N}, r \in \mathbb{N} \cup \{\infty\}$ and $\emptyset \neq V \subset \mathbb{R}^n$ be an open subset. Consider a map*

$$\Phi : \mathbb{I}^k \to V.$$

*Φ is called a **k-cube**, if it is continuous. If Φ is also r-times continuously differentiable, then it is called an **(r-times) smooth k-cube**.*

Definition 2.2.9. *Let $k, n \in \mathbb{N}, \emptyset \neq V \in \mathbb{R}^n$ be an open subset and Φ be a k-cube. If $j \in \{1, ..., k\}$ and $s \in \{0,1\}$, then Φ_{js} denotes the following $k-1$-cube:*

$$\Phi_{js} : \mathbb{I}^{k-1} \to V, \quad \Phi_{js}(x_1, ..., x_{k-1}) := \begin{cases} \Phi(s) \ (k=1), \\ \Phi(x_1, ...x_{j-1}, s, x_{j+1}, ..., x_{k-1}) \ (k>1). \end{cases}$$

The set

$$\partial\Phi := \{\Phi_{js} : j \in \{1, ..., k\}, s \in \{0,1\}\}$$

*is called the **boundary** of the k-cube Φ.*

Since $\dim(\mathcal{A}_k(\mathbb{R}^k)) = \binom{k}{k} = 1$, $\mathcal{A}_k(\mathbb{R}^k)$ has only one basis element: $\Delta^{k,k}_{(1,...,k)}$. So if $k \in \{1, ..., n\}$ and $\Phi \in \mathcal{C}^{r+1}(\mathbb{I}^k, V)$, then $\omega \star \Phi \in \Lambda^r_k(\mathbb{I}^k)$, which means the induced form has the following form:

$$\omega \star \Phi = \Omega \Delta^{k,k}_{(1,...,k)}$$

with some function $\Omega \in \mathcal{C}^{r+1}(\mathbb{I}^k, \mathbb{R})$.

Definition 2.2.10. *Let $k \in \{0, ..., n\}$, $\omega \in \Lambda^r_k(V)$ and $\Phi \in \mathcal{C}^{r+1}(\mathbb{I}^k, V)$. We define the **integral of the k-form** ω **over the k-cube** Φ as follows:*

If $k = 0$, then

$$\int_\Phi \omega := \omega(\Phi(0)),$$

If $k > 0$, then

$$\int_\Phi \omega := \int_{\mathbb{I}^k} \Omega.$$

Special cases.

- The case $k = 1$.

  In this case ω and Φ are the following:

  $$\omega = \omega_f = \sum_{i=1}^{n} f_i \Delta_i^{n,1}, \qquad \Phi \in \mathcal{C}^{r+1}([0,1], V),$$

  where $f = (f_1, ..., f_n) \in \mathcal{C}^r(V, \mathbb{R}^n)$. Let's compute the $\omega_f * \Phi \in \Lambda_1^r([0,1])$ induced form. Let $x \in [0,1]$ and $y \in \mathbb{R}$.

  $$(\omega_f * \Phi)(x)(y) = \sum_{i=1}^{n} f_i(\Phi(x)) \Delta_i^{n,1}(\Phi'(x)y) = \sum_{i=1}^{n} f_i(\Phi(x))(\Phi'(x)y)_i =$$

  $$= \sum_{i=1}^{n} f_i(\Phi(x))\Phi_i'(x)y = \sum_{i=1}^{n} f_i(\Phi(x))\Phi_i'(x)\Delta_1^{1,1}(y) =$$

  $$= \langle f \circ \Phi, \Phi' \rangle (x) \Delta_1^{1,1}(y) = \left(\langle f \circ \Phi, \Phi' \rangle \Delta_1^{1,1} \right)(x)(y)$$

  So in this case $\Omega = \langle f \circ \Phi, \Phi' \rangle$, thus

  $$\int_\Phi \omega_f = \int_{[0,1]} \langle f \circ \Phi, \Phi' \rangle = \int_\Phi f \qquad \textbf{(line integral)}.$$

- The case $k = n$.

  In this case ω and Φ are the following:

  $$\omega = \omega_f = f \Delta_{(1,...,n)}^{n,n}, \qquad \Phi \in \mathcal{C}^{r+1}(\mathbb{I}^n, V),$$

  where $f \in \mathcal{C}^r(V, \mathbb{R})$. Let $x \in \mathbb{I}^n$, $x_1, ..., x_n \in \mathbb{R}^n$ and suppose that Φ is regular.[3]

$$(\omega_f * \Phi)(x)(x_1, ..., x_n) = f(\Phi(x))\Delta^{n,n}_{(1,...,n)}(\Phi'(x)x_1, ..., \Phi'(x)x_n) =$$

$$= f(\Phi(x)) \det(\Phi'(x)x_1, ..., \Phi'(x)x_n) = f(\Phi(x)) \det(\Phi'(x)) \det(x_1, ..., x_n) =$$

$$= f(\Phi(x)) \det(\Phi'(x))\Delta^{n,n}_{(1,...,n)}(x_1, ..., x_n) = ((f \circ \Phi) \det(\Phi'))(x)\Delta^{n,n}_{(1,...,n)}(x_1, ..., x_n) =$$

$$= (((f \circ \Phi) \det(\Phi'))\Delta^{n,n}_{(1,...,n)})(x)(x_1, ..., x_n).$$

So in this case $\Omega = (f \circ \Phi) \det(\Phi')$. Thus

$$\int_\Phi \omega_f = \int_{\mathbb{I}^n} (f \circ \Phi) \det(\Phi') = \int_{\mathbb{I}^n} (f \circ \Phi) |\det(\Phi')| = \int_{R_\Phi} f \qquad \textbf{(multiple integral)}$$

- The case $n = 3$, $k = 2$.
  In this case ω and Φ are the following:

$$\omega = \omega_f = f_1\Delta^{3,2}_{(2,3)} + f_2\Delta^{3,2}_{(3,1)} + f_3\Delta^{3,2}_{(1,2)}, \qquad \Phi \in \mathcal{C}^{r+1}(\mathbb{I}^2, V),$$

  where $f = (f_1, f_2, f_3) \in \mathcal{C}^r(V, \mathbb{R}^3)$. Here we take a different approach to compute Ω. We use the formula for computing the coordinate functions of a differential form. Since $\omega_f * \Phi = \Omega\Delta^{2,2}_{(1,2)}$, we have:

$$\begin{aligned}\Omega(x) =&(\omega_f * \Phi)(x; e_1, e_2) = \omega_f(\Phi(x); \Phi'(x)e_1, \Phi'(x)e_2) = \omega_f(\Phi(x); \partial_1\Phi(x), \partial_2\Phi(x))\\ =&f_1(\Phi(x))\Delta^{3,2}_{(2,3)}(\partial_1\Phi(x), \partial_2\Phi(x)) + f_2(\Phi(x))\Delta^{3,2}_{(3,1)}(\partial_1\Phi(x), \partial_2\Phi(x))+\\ &+ f_3(\Phi(x))\Delta^{3,2}_{(1,2)}(\partial_1\Phi(x), \partial_2\Phi(x)) = f_1(\Phi(x))(\partial_1\Phi_2\partial_2\Phi_3 - \partial_2\Phi_2\partial_1\Phi_3)+\\ &+ f_2(\Phi(x))(\partial_1\Phi_1\partial_2\Phi_3 - \partial_2\Phi_1\partial_1\Phi_3) + f_3(\Phi(x))(\partial_1\Phi_1\partial_2\Phi_2 - \partial_2\Phi_1\partial_1\Phi_2) =\\ =&(f \circ \Phi)_1(x)(\partial_1\Phi \times \partial_2\Phi)_1(x) + (f \circ \Phi)_2(x)(\partial_1\Phi \times \partial_2\Phi)_2(x)+\\ &+ (f \circ \Phi)_3(x)(\partial_1\Phi \times \partial_2\Phi)_3(x) = \langle f \circ \Phi, \partial_1\Phi \times \partial_2\Phi\rangle(x) \quad (x \in \mathbb{I}^2).\end{aligned}$$

[3]It means that $\det(\Phi'(x)) > 0$ for all $x \in V$.

Thus

$$\int_{\Phi} \omega_f = \int_{\mathbb{I}^2} \langle f \circ \Phi, \partial_1 \Phi \times \partial_2 \Phi \rangle = \int_{\Phi} f \qquad \textbf{(surface integral)}.$$

Definition 2.2.11. *Let $k \in \{0, ..., n\}$, $\omega \in \Lambda_k^r(V)$ and $\Phi \in \mathcal{C}^{r+1}(\mathbb{I}^{k+1}, V)$. We define the* ***integral of the k-form ω over the boundary of the k+1-cube Φ*** *as follows:*

$$\int_{\partial \Phi} \omega := \sum_{j=1}^{k+1} \sum_{s=0}^{1} (-1)^{j+s} \int_{\Phi_{js}} \omega.$$

2.3 The Poincaré-Stokes theorem

Theorem 2.3.1. *Let $\emptyset \neq V \subset \mathbb{R}^n$ be an open set and $\omega \in \Lambda_k^r(V)$ $(k = 0, ..., n-1)$. Let $\Phi \in \mathcal{C}^{r+1}(\mathbb{I}^{k+1}, V)$ be a $k+1$-cell. Then*

$$\int_{\Phi} d\omega = \int_{\partial \Phi} \omega.$$

Proof.

Step 1 First we prove the $k = 0$ case.

In this case $\omega : V \to \mathbb{R}$ and $\Phi : [0, 1] \to V$. Using the Newton-Leibniz-rule for line integrals we have:

$$\int_{\partial \Phi} \omega = \int_{\Phi_{11}} \omega - \int_{\Phi_{10}} \omega = \omega(\Phi(1)) - \omega(\Phi(0)) = \int_{\Phi} \operatorname{grad} \omega = \int_{\Phi} d\omega.$$

Step 2 Now we prove the $k > 0$ case where Φ is the identity map.

$$V = \mathbb{I}^{k+1}, \qquad \Phi : \mathbb{I}^{k+1} \to \mathbb{I}^{k+1} \qquad \Phi(x) = x \qquad (x \in \mathbb{I}^{k+1}).$$

Let us denote the the multiindex $i_* := (1, ..., i-1, i+1, ..., k+1)$. This way

$\omega \in \Lambda_k^r(\mathbb{I}^{k+1})$ can be expressed as:

$$\omega = \sum_{i=1}^{k+1} \omega_i \Delta_{i_*}^{k+1,k}.$$

The derivative of ω is then

$$d\omega = \sum_{i=1}^{k+1} \sum_{j \in (K+1)_*^k} \partial_j \omega_i \Delta_{(j,i_*)}^{k+1,k+1}.$$

The only non-zero terms are when $j = i$ in which case the number i needs to be interchanged with $i-1$ other to reach the identity permutation. So we have

$$d\omega = \left(\sum_{i=1}^{k+1} (-1)^{i-1} \partial_i \omega_i \right) \Delta_{(1,\dots,k+1)}^{k+1,k+1}.$$

Hence, by the definition of the integral

$$\int_\Phi d\omega = \sum_{i=1}^{k+1} (-1)^{i-1} \int_{\mathbb{I}^{k+1}} \partial_i \omega_i.$$

Let's compute the other side of the equation. The boundary of Φ by definition is

$$\partial\Phi = \{\Phi_{js} : j \in \{1, ..., k+1\}; s \in \{0,1\}\},$$

where

$$\Phi_{js} : \mathbb{I}^k \to \mathbb{I}^{k+1} \qquad \Phi_{js}(x_1, ..., x_k) = \Phi(x_1, ..., x_{j-1}, s, x_{j+1}, ..., x_k)$$

$$(x_1, ..., x_k \in [0,1]).$$

Since

$$\omega * \Phi_{js} \in \Lambda_k^r(\mathbb{I}^k)$$

it can be written as

$$\omega * \Phi_{js} = \Omega_{js} \Delta_{(1,\dots,k)}^{k,k} \qquad (\Omega_{js} : \mathbb{I}^k \to \mathbb{R})$$

Let $x = (x_1, ..., x_k) \in \mathbb{I}^k$ and let $\{e_1, ..., e_k\}$ be the canonical basis of $\mathbb{R}^k$. Using the formula for the coordinate functions we have

$$\Omega_{js}(x) = (\omega * \Phi_{js})(x; e_1, ..., e_k) = \omega(\Phi_{js}(x); \Phi_j s'(x)e_1, ..., \Phi_j s'(x)e_k) =$$
$$= \sum_{i=1}^{k+1} \omega_i(\Phi_{js}(x))\Delta_{i_*}^{k+1,k}(\partial_1\Phi_{js}(x), ..., \partial_k\Phi_{js}(x))$$

Here the partial derivatives are

$$\partial_i\Phi_{js}(x_1, ..., x_k) = \partial_i\Phi(x_1, ..., x_{j-1}, s, x_{j+1}, ..., x_k) = (0, ..., 1, ..., 0) \quad (i \in \{1, ..., k\}),$$

where the 1 is in the ith position if $i < j$ and in the $(i + 1)$th position if $i \geq j$. It means that the argument of $\Delta_{i_*}^{k+1,k}$ is a matrix with $k + 1$ row and k column, which is almost the "identity" matrix, but the jth row is filled with 0-s. It follows that the only non-zero term in the summation is where $i = j$ for which

$$\Delta_{j_*}^{k+1,k}(\partial_1\Phi_{js}(x), ..., \partial_k\Phi_{js}(x)) = 1,$$

the determinant of the identity matrix. It follows that

$$\Omega_{js}(x) = \omega_j(\Phi_{js}(x)) = \omega_j(\Phi_{js}(x_1, ..., x_k)) = \omega_j(\Phi(x_1, ..., x_{j-1}, s, x_{j+1}, ..., x_k)).$$

And since Φ is the identity map we get

$$\Omega_{js}(x) = \omega_j(x_1, ..., x_{j-1}, s, x_{j+1}, ..., x_k),$$

so

$$(\omega * \Phi_{js})(x) = \omega_j(x_1, ..., x_{j-1}, s, x_{j+1}, ..., x_k)\Delta_{(1,...,k)}^{k,k}.$$

Now we can compute the integral over the boundary:

$$\int_{\partial\Phi}\omega = \sum_{j=1}^{k+1}\sum_{s=0}^{1}(-1)^{j+s}\int_{\Phi_{js}}\omega = \sum_{j=1}^{k+1}(-1)^{j+1}\left(\int_{\Phi_{j1}}\omega - \int_{\Phi_{j0}}\omega\right) =$$

$$= \sum_{j=1}^{k+1}(-1)^{j-1}\left(\int_{\mathbb{I}^k}\omega_j(x_1,...,x_{j-1},1,x_{j+1},...,x_k) - \right.$$

$$\left. - \int_{\mathbb{I}^k}\omega_j(x_1,...,x_{j-1},0,x_{j+1},...,x_k)\right) =$$

$$= \sum_{j=1}^{k+1}(-1)^{j-1}\int_{\mathbb{I}^k}(\omega_j(x_1,...,x_{j-1},1,x_{j+1},...,x_k) -$$

$$- \omega_j(x_1,...,x_{j-1},0,x_{j+1},...,x_k)) =$$

$$= \sum_{j=1}^{k+1}(-1)^{j-1}\int_{\mathbb{I}^k}\left(\int_0^1 \partial_j\omega_j(x_1,...,x_{j-1},t,x_{j+1},...,x_k)dt\right) =$$

$$= \sum_{j=1}^{k+1}(-1)^{j-1}\int_{\mathbb{I}^{k+1}}\partial_j\omega_j$$

which is the same result we got earlier.

Step 3 Now we prove the theorem for arbitrary $\Phi : \mathbb{I}^{k+1} \to V$ $k+1$-cubes. We use the fact that the exterior derivative commutes with the pullback of forms.

$$\int_{\Phi} d\omega = \int_{\mathbb{I}^{k+1}} d\omega * \Phi = \int_{\mathbb{I}^{k+1}} d(\omega * \Phi)$$

Since $d(\omega * \Phi) \in \Lambda_{k+1}^{r-1}(\mathbb{I}^{k+1})$ we can use the result from step 2:

$$\int_{\mathbb{I}^{k+1}} d(\omega * \Phi) = \int_{\partial \mathbb{I}^{k+1}} \omega * \Phi = \sum_{j=1}^{k+1} \sum_{s=0}^{1} \int_{\mathbb{I}_{js}^{k+1}} \omega * \Phi = \sum_{j=1}^{k+1} \sum_{s=0}^{1} \int_{\Phi \circ \mathbb{I}_{js}^{k+1}} \omega = \int_{\partial \Phi} \omega.$$

□

Special cases.

- The case $k = 0$.
  In this case
  $$\omega \in \Lambda_0^r = \mathcal{C}^r(V, \mathbb{R}) \qquad \Phi \in \mathcal{C}^{r+1}([0,1], V).$$
  We have already shown that if
  $$f \in \mathcal{C}^r(V, \mathbb{R}) \qquad \omega := \omega_f$$
  then
  $$d\omega = \omega_{\text{grad}(f)}.$$
  Thus
  $$\int_{\Phi} \omega_{\text{grad}(f)} = \int_{\Phi} d\omega = \int_{\partial \Phi} \omega = \sum_{s=0}^{1} (-1)^{1+s} \int_{\Phi_{1s}} \omega = \sum_{s=0}^{1} (-1)^{1+s} \omega(\Phi_{1s}(0)) =$$
  $$= \sum_{s=0}^{1} (-1)^{1+s} \omega(\Phi(s)) = \omega(\Phi(1)) - \omega(\Phi(0)).$$
  This is the same as
  $$\int_{\Phi} \text{grad}(f) = f(\Phi(1)) - f(\Phi(0))$$ **(Newton-Leibniz formula for line integrals)**.

- The case $n = 2, k = 2$.
  In this case
  $$\omega \in \Lambda_1^r(V), \qquad \Phi \in \mathcal{C}^{r+1}(\mathbb{I}^2, V).$$

We have already shown that if

$$f = (f_1, f_2) \in \mathcal{C}^r(V, \mathbb{R}^2), \quad \omega := \omega_f = f_1 \Delta_1^{2,1} + f_2 \Delta_2^{2,1}$$

then

$$d\omega = \omega_{\partial_1 f_2 - \partial_2 f_1}.$$

Thus

$$\int_{R_\Phi} \partial_1 f_2 - \partial_2 f_1 = \int_\Phi \omega_{\partial_1 f_2 - \partial_2 f_1} = \int_\Phi d\omega = \int_{\partial\Phi} \omega = \int_{\partial\Phi} f$$

This is the **Green-theorem**.

- The case $n = 3, k = 2$.
  In this case

$$\omega \in \Lambda_1^r(V), \qquad \Phi \in \mathcal{C}^{r+1}(\mathbb{I}^2, V).$$

We have already shown that if

$$f = (f_1, f_2, f_3) \in \mathcal{C}^r(V, \mathbb{R}^3), \quad \omega := \omega_f = f_1 \Delta_1^{3,1} + f_2 \Delta_2^{3,1} + f_3 \Delta_3^{3,1}$$

then

$$d\omega = \omega_{\mathrm{rot}(f)}.$$

Thus if Φ is regular

$$\int_\Phi \mathrm{rot}(f) = \int_\Phi \omega_{\mathrm{rot}(f)} = \int_\Phi d\omega = \int_{\partial\Phi} \omega = \int_{\partial\Phi} f.$$

This is the **Stokes-theorem**.

- The case $n = 3, k = 3$.
  In this case

$$\omega \in \Lambda_2^r(V), \qquad \Phi \in \mathcal{C}^{r+1}(\mathbb{I}^3, V).$$

We have already shown that if

$$f = (f_1, f_2, f_3) \in \mathcal{C}^r(V, \mathbb{R}^3), \quad \omega := \omega_f = f_1 \Delta_{(2,3)}^{3,2} + f_2 \Delta_{(3,1)}^{3,2} + f_3 \Delta_{(1,2)}^{3,2}$$

then

$$d\omega = \omega_{\mathrm{div}(f)}$$

thus if Φ is regular

$$\int_{R_\Phi} \mathrm{div}(f) = \int_\Phi \omega_{\mathrm{div}(f)} = \int_\Phi d\omega = \int_{\partial\Phi} \omega = \int_{\partial\Phi} f.$$

This is the **Gauss-theorem**.

Chapter 3

Applications

3.1 Maxwell's equations

3.1.1 The classical form of the Maxwell's equations

In his work published in 1865 Maxwell worked out a unified theory which connected the seemingly different phenomena of electricity and magnetism. His four equations describe the impact of an electromagnetic field on a distribution of electrical charges in space as well as the interaction between the electric field and the magnetic field.

Physical quantities can be modeled by different mathematical objects. In this section first we introduce the model most commonly used in classical electrodynamics. Then we present another approach: a model using differential forms.

Definition 3.1.1. *Let $\Omega \subset \mathbb{R}^3$ be a domain. The* ***electric field*** *and the* ***magnetic field*** *are described by the time dependent differentiable vector fields E and B defined on the domain Ω:*

$$E : \Omega \times \mathbb{R} \to \mathbb{R}^3, \qquad B : \Omega \times \mathbb{R} \to \mathbb{R}^3.$$

The ***electric charge density*** *ρ is described by a time dependent scalar field and* *the* ***current density*** j *by a time dependent vector field on the domain* Ω*:*

$$\rho : \Omega \times \mathbb{R} \to \mathbb{R}, \qquad j : \Omega \times \mathbb{R} \to \mathbb{R}^3.$$

The electric and magnetic field has an impact on the electric charge and the current density. Also electric charges and currents generate the electric and magnetic field. Maxwell's equations describe how the interaction between these quantities work. Having chosen units in which $\mu_0 = \epsilon_0 = c = 1$ the equations take the following form:

$$\operatorname{div} E = \rho, \qquad \operatorname{rot} E = -\frac{\partial B}{\partial t},$$
$$\operatorname{div} B = 0, \qquad \operatorname{rot} B = j + \frac{\partial E}{\partial t}.$$

3.1.2 Minkowski spacetime

One of the most important principles in physics is that every law describing nature needs to be expressed with equations which are independent from the location of the observer. Physical phenomena do not depend on the coordinate system in which we describe it. Thus the equations must be invariant under changes of coordinate systems.

This principle led to the birth of the theory of special relativity and the concept of spacetime, since Maxwell's equations are not invariant. However if we don't distinguish time from the spacial coordinates we can overcome this trouble. To read more about the topic see [8].

Now we construct the mathematical background for the Minkowski spacetime.

Definition 3.1.2. *Let* V *be a finite dimensional vector space.* V *is called a* ***pseudo-euclidean space*** *if on* V *there is symmetric bilinear non-degenerate map*

$$g : V \times V \to \mathbb{R}$$

such that for all $u, v, w \in V$ and $\lambda \in \mathbb{R}$ the following properties are satisfied:

1. $g(\lambda u + v, w) = \lambda g(u, w) + g(v, w)$,

2. $g(u, \lambda v + w) = \lambda g(u, w) + g(u, w)$ *(bilinear),*

3. $g(u, v) = g(v, u)$ *(symmetric),*

4. $g(u, v) = 0 \quad \forall u \in V \iff v = 0$ *(non-degenerate).*

For a given basis $\{e_1, ...e_n\}$ in V let's define the matrix:

$$M(g) := (g(e_i, e_j))_{i,j=1}^n := (g_{ij})_{i,j=1}^n.$$

We will denote this pseudo-euclidean space (V,g).

A pseudo-euclidean space is "less" than an euclidean space, since the g map is not positive-definite. If it were, it would be a scalar product. However, in physics' literature g is often called (mistakenly) a scalar product, since their algebraic properties are quite similar.

Definition 3.1.3. *Let (V, g) be a pseudo-euclidean space and $\{e_1, ...e_n\}$ a basis in V. We say that $\{e_1, ...e_n\}$ is an* ***orthonormal basis*** *if*

$$g(e_i, e_j) = \begin{cases} \pm 1 & (i = j), \\ 0 & (i \neq j). \end{cases}.$$

Theorem 3.1.1 (Sylvester)**.** *Let (V, g) be a pseudo-euclidean space. Then there exists a basis $\{e_1, ...e_n\}$ in V such that the matrix $M(g)$ is diagonal and only contains $+1$'s and -1's, i.e.*

$$M(g) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

The numbers of +1's (p) and −1's (q) are independent of the particular basis. The pair (p,q) is called the ***signature*** *of g.*

We will not prove this theorem here, for a proof see for example [9].

Since the matrix $M(g)$ is diagonal, and has no 0 elements it follows that it is invertable. We will denote the entries of the inverse matrix g^{ij}.

Definition 3.1.4. *The pseudo-euclidean space* $(\mathbb{R}^4, g)$ *is called* ***Minkowski-spacetime*** *if g has a signature (3,1).*

A point in Minkowski-spacetime can be given by four coordinates:

$$x = (x_1, x_2, x_3, ct) \in \mathbb{R}^4,$$

where the first three coordinates are the usual "spatial" coordinates, and the fourth one is the time coordinate, c being the speed of light in vacuum. In theoretical works one frequently chooses a unit system where $c = 1$ (we will also do it). Another commonly used notation is $x_0 := t$, so in computations the 0 index always refers to the time coordinate.

3.1.3 The Hodge-star operator

In this section we introduce the Hodge-star operator, which is an isomorphism between the $\mathcal{A}_k(V)$ and $\mathcal{A}_{n-k}(V)$ spaces. It will be an essential tool for modeling the electric and magnetic fields with differential forms. It will turn out that these quantities are strongly depend on each other and it is better to model them as one object.

First we extend the g map on a pseudo-euclidean space (V, g) to the $\mathcal{A}_k(V)$. Adding this new structure will make them pseudo-euclidean spaces as well. This extension will be done relative to an orthonormal basis.

Definition 3.1.5. *Let* (V, g) *be a pseudo-euclidean space and* $\{e_1, ...e_n\}$ *an or-*

thonormal basis in it. Let $f_1, f_2 \in \mathcal{A}_k(V)$*. Then*

$$g_k : \mathcal{A}_k(V) \times \mathcal{A}_k(V) \to \mathbb{R},$$

$$g_k(f_1, f_2) := \sum_{i \in N_*^k} g^{i_1 i_2} ... g^{i_k i_k} f_1(e_{i_1}, ..., e_{i_k}) f_2(e_{i_1}, ..., e_{i_k}).$$

Theorem 3.1.2. *The above defined* g_k *map is non-degenerate, symmetric and bilinear. Moreover if* $\{e_1, ..., e_n\}$ *is an orthonormal basis in* V *(with respect to* g*), then* $\Delta_i^{n,k}$ $(i \in N_*^k)$ *is an orthonormal basis in* $\mathcal{A}_k(V)$*.*

Proof. The bilinearity and symmetry of g is clear, so we focus on the non-degenerativity. Let $h \in \mathcal{A}_k(V)$ and suppose that

$$g_k(f, h) = \sum_{i \in N_*^k} g^{i_1 i_2} ... g^{i_k i_k} f(e_{i_1}, ..., e_{i_k}) h(e_{i_1}, ..., e_{i_k}) = 0 \in \mathbb{R} \quad \forall f \in \mathcal{A}_k(V).$$

We need to show that

$$h = 0 \in \mathcal{A}_k(V).$$

Since a k-form is uniquely determined by its value on the k-tuples of vectors with strictly increasing multiindex let's choose f as the following:

$$f(e_{i_1}, ..., e_{i_k}) := g^{i_1 i_2} ... g^{i_k i_k} h(e_{i_1}, ..., e_{i_k}).$$

This way

$$g_k(f, h) = \sum_{i \in N_*^k} \left(g^{i_1 i_2} ... g^{i_k i_k} \right)^2 h(e_{i_1}, ..., e_{i_k})^2 = 0.$$

This is only possible if all the terms in the summation are 0. Since the numbers $g^{i_j i_j}$ are not 0 it follows that all $h(e_{i_1}, ...e_{i_k})$ must be 0. Which means that $h = 0 \in \mathcal{A}_k(V)$, thus g is non-degenerate.
To prove the orthonormality set two multiindex $i, j \in N_*^k$.

$$g_k(\Delta_i^{n,k}, \Delta_j^{n,k}) = \sum_{l \in N_*^k} g^{l_1 l_1} ... g^{k_l k_l} \Delta_i^{n,k}(e_{l_1}, ..., e_{l_k}) \Delta_j^{n,k}(e_{j_1}, ..., e_{j_k})$$

This expression does not equal 0 if and only if $i = j = l$, in which case it is ± 1, thus the basis $\{\Delta_i^{n,k}\ (i \in N_*^k)\}$ is in fact orthonormal. $\square$

Apart from the g map we fix an orientation on V. For this consider the set $\mathcal{B}(V)$ of all ordered bases $\mathcal{B} = (v_1, ..., v_n)$. For two ordered bases $\mathcal{B} = (v_1, ..., v_n)$ and $\mathcal{B}' = (v_1', ..., v_n')$ there exists a linear transformation $A(\mathcal{B}, \mathcal{B}') = (a_{ij})_{i,j=1}^n$ which transforms one into the other:

$$v_i = \sum_{j=1}^{n} a_{ij} v_j'.$$

We can define an equivalence relation $\sim$ on the set $\mathcal{B}(V)$ by requiring that

$$\mathcal{B} \sim \mathcal{B}' \quad \Longleftrightarrow \quad \det(A(\mathcal{B}, \mathcal{B}')) > 0.$$

This way we have two equivalence classes.

Definition 3.1.6. *The* ***orientation*** *of V is a choice of one of the two equivalence classes in the set $\mathcal{B}(V)$.*

Definition 3.1.7. *Let (V, g) be an oriented pseudo-euclidean space. Let $\{e_1, ...e_n\}$ be a basis in V such that the matrix $M(g)$ has the diagonal format of Sylvester's theorem, and $(e_1, ..., e_n)$ is positively oriented. Now we define the* ***volume-form*** *$dV \in \mathcal{A}_n(V)$ as follows:*

$$dV(v_1, ..., v_n) := \det(g(v_i, e_j))_{i,j=1}^n = \Delta_{(1,...,n)}^{n,n}(v_1, ..., v_n) \quad (v_1, ..., v_n \in V).$$

Theorem 3.1.3. *Let (V, g) be a pseudo-euclidean space and $f : V \to \mathbb{R}$ be a linear function. Then for all $\alpha \in V$ there exists a unique $\beta \in V$ such that $f(\alpha) = g(\alpha, \beta)$.*

Proof.

Step 1 (uniqueness)

Suppose that $f(\alpha) = g(\alpha, \beta) = g(\alpha, \gamma)$. We have

$$g(\alpha, \beta) - g(\alpha, \gamma) = 0.$$

Using the bilinearity of g we get

$$g(\alpha, \beta - \gamma) = 0.$$

Since g is non-degenerate it follows that

$$\beta - \gamma = 0 \qquad \Longrightarrow \qquad \beta = \gamma.$$

Step 2 (existence)

Let $\{e_1, ...e_n\}$ be an orthonormal basis in V, expand α in terms of the basis elements:

$$\alpha = \sum_{i=1}^{n} \alpha_i e_i$$

and let β be the following:

$$\beta := \sum_{j=1}^{n} g(e_j, e_j) f(e_j) e_j.$$

Using the bilinearity g we get:

$$g(\alpha, \beta) = g\left(\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} g(e_j, e_j) f(e_j) e_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i g(e_j, e_j) f(e_j) g(e_i, e_j).$$

Because the basis is orthonormal this simplifies to:

$$g(\alpha, \beta) = \sum_{i=1}^{n} \alpha_i g(e_i, e_i)^2 f(e_i).$$

Since $g(e_i, e_i)^2 = 1$, and again using the bilinearity

$$g(\alpha, \beta) = \sum_{i=1}^{n} \alpha_i f(e_i) = f\left(\sum_{i=1}^{n} \alpha_i e_i\right) = f(\alpha).$$

□

Now let's use this result for the vector space $\mathcal{A}_{n-k}(V)$. Let $\lambda \in \mathcal{A}_k(V)$ a fixed

k-form. Then for any $\theta \in \mathcal{A}_{n-k}(V)$ we have $\lambda \wedge \theta \in \mathcal{A}_n(V)$. Since $\mathcal{A}_n(V)$ is one dimensional, and the basis element is the volume-form there exists a unique $a \in \mathbb{R}$ such that $\lambda \wedge \theta = a \cdot dV$. Using this equality we can define the following function:

$$f_\lambda : \mathcal{A}_{n-k}(V) \to \mathbb{R}, \qquad f_\lambda(\theta) := a.$$

With this definition f_λ is a linear function, thus we can apply the previous theorem, which tells us that there exists a unique element $\phi \in \mathcal{A}_{n-k}(V)$ such that

$$f_\lambda(\theta) = g(\theta, \phi) \quad (\theta \in \mathcal{A}_{n-k}(V)).$$

We can finally define the Hodge-dual $\star\lambda$ of λ to be

$$\star\lambda := (-1)^q \phi \in \mathcal{A}_{n-k}(V).$$

Definition 3.1.8. *The* ***Hodge-star operator*** *is a map between k-forms and $n-k$ forms*

$$\star : \mathcal{A}_k(V) \to \mathcal{A}_{n-k}(V)$$

defined by the following equation:

$$\lambda \wedge \theta = (-1)^q g(\theta, \star\lambda) dV \quad (\theta \in \mathcal{A}_{n-k}(V)).$$

This quite abstract definition is not sufficient to actually compute the Hodge-dual of some form. In order to do this we use a different approach. Since the Hodge-star operator is linear it suffices to compute the duals of the basis elements, then expend the results in a linear and alternating way.
So we wish to compute $\star\Delta_i^{n,k}$ $(i \in N_*^k)$. Let $j \in N_*^{n-k}$. Then from the definition of $\star$:

$$\Delta_i^{n,k} \wedge \Delta_j^{n,n-k} = (-1)^q g(\Delta_j^{n,n-k}, \star\Delta_i^{n,k}) \Delta_{(1,\dots,n)}^{n,n}.$$

Note that the left side of the equation differs from 0 only if j is the complementary index of i. Since the $\Delta_j^{n,n-k}$ $(j \in N_*^{n-k})$ basis elements are orthonormal the right side tells us that $\star\Delta_i^{n,k}$ has the following form:

$$\star\Delta_i^{n,k} = c\Delta_j^{n,n-k}$$

where j is the complementary index and $c \in \mathbb{R}$. Putting this back into the original equation yields:

$$\begin{aligned} \operatorname{sgn}(\tau)\Delta^{n,n}_{(1,...,n)} &= \Delta_i^{n,k} \wedge \Delta_j^{n,n-k} = (-1)^q g(\Delta_j^{n,n-k}, c\Delta_j^{n,n-k})\Delta^{n,n}_{(1,...,n)} = \\ &= (-1)^q c g^{j_1 j_1} ... g^{j_{n-k}, j_{n-k}} \Delta^{n,n}_{(1,...,n)}. \end{aligned}$$

Where τ is the number of inversions in the permutation which takes the sequence $(i_1, ..., i_k, j_1, ..., j_{n-k})$ to $(1, ..., n)$. Using the fact that $g^{j_r j_r} = \frac{1}{g_{j_r j_r}}$ we obtain

$$c = (-1)^q \operatorname{sgn}(\tau) g_{j_1 j_1} ... g_{j_{n-k} j_{n-k}}.$$

Hence

$$\star\Delta_i^{n,k} = (-1)^q \operatorname{sgn}(\tau) g_{j_1 j_1} ... g_{j_{n-k} j_{n-k}} \Delta_j^{n,n-k}.$$

3.1.4 Maxwell's equations in terms of differential forms

First we consider the homogeneous Maxwell equations:

$$\operatorname{div} B = 0, \qquad \operatorname{rot} E + \frac{\partial B}{\partial t} = 0.$$

We have shown that divergence of a vector field can be thought of as the exterior derivative of a differential 2-form, and the rotation of a vector field as the exterior derivative of a 1-form. Therefore it seems logical to represent the magnetic field as a 2-form and the electric field as a 1-form:

$$B := B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2,$$

$$E := E_1 dx_1 + E_2 dx_2 + E_3 dx_3.$$

Now, these forms are the inhabitants of spacetime, and we can combine them into one differential form.

Definition 3.1.9. *The **electromagnetic field F** is a differential 2-form on spacetime defined by the following equation:*

$$F \in \Lambda_2^1(\mathbb{R}^4) \qquad F := B + E \wedge dx_0$$

where dx_0 denotes the time coordinate of spacetime.

The full form of F is the following:

$$\begin{aligned} F =& E_1 dx_1 \wedge dx_0 + E_2 dx_2 \wedge dx_0 + E_3 dx_3 \wedge dx_0 + \\ &+ B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2. \end{aligned}$$

Now let's take the exterior derivative of F:

$$\begin{aligned} dF =& \partial_2 E_1 dx_2 \wedge dx_1 \wedge dx_0 + \partial_3 E_1 dx_3 \wedge dx_1 \wedge dx_0 + \partial_1 E_2 dx_1 \wedge dx_2 \wedge dx_0 + \\ &+ \partial_3 E_2 dx_3 \wedge dx_2 \wedge dx_0 + \partial_1 E_3 dx_1 \wedge dx_3 \wedge dx_0 + \partial_2 E_3 dx_2 \wedge dx_3 \wedge dx_0 + \\ &+ \partial_1 B_1 dx_1 \wedge dx_2 \wedge dx_3 + \partial_0 B_1 dx_0 \wedge dx_2 \wedge dx_3 + \partial_2 B_2 dx_2 \wedge dx_3 \wedge dx_1 + \\ &+ \partial_0 B_2 dx_0 \wedge dx_3 \wedge dx_1 + \partial_3 B_3 dx_3 \wedge dx_1 \wedge dx_2 + \partial_0 B_3 dx_0 \wedge dx_1 \wedge dx_2. \end{aligned}$$

Now collecting the terms and using the anti-symmetry of the wedge product we get:

$$\begin{aligned} dF =& (\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3) dx_1 \wedge dx_2 \wedge dx_3 + \\ &+ (\partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1) dx_0 \wedge dx_2 \wedge dx_3 + \\ &+ (\partial_3 E_1 - \partial_1 E_3 + \partial_0 B_2) dx_0 \wedge dx_3 \wedge dx_1 + \\ &+ (\partial_1 E_2 - \partial_2 E_1 + \partial_0 B_3) dx_0 \wedge dx_1 \wedge dx_2. \end{aligned}$$

Note that $dF = 0$ is the same as

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0$$

and

$$\begin{cases} \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0, \\ \\ \partial_3 E_1 - \partial_1 E_3 + \partial_0 B_2 = 0, \\ \\ \partial_1 E_2 - \partial_2 E_1 + \partial_0 B_3 = 0. \end{cases}$$

The first equality corresponds with $\operatorname{div} B = 0$ and next three with

$$\operatorname{rot} E + \frac{\partial B}{\partial t} = 0.$$

Thus the homogeneous equations can be written in the compact form of:

$$dF = 0$$

which means that the electromagnetic field 2-form is closed.
For the inhomogeneous equations let's compute Hodge-dual of F:

$$\begin{aligned} \star(dx_1 \wedge dx_0) &= - \star (dx_0 \wedge dx_1) = - \left((-1)^q \operatorname{sgn}(\tau) g_{22} g_{33} dx_2 \wedge dx_3\right) \\ &= - \left((-1)^1 \cdot 1 \cdot 1 \cdot 1\right) dx_2 \wedge dx_3 = dx_2 \wedge dx_3, \end{aligned}$$

$$\begin{aligned} \star(dx_2 \wedge dx_0) &= - \star (dx_0 \wedge dx_2) = - \left((-1)^q \operatorname{sgn}(\tau) g_{11} g_{33} dx_1 \wedge dx_3\right) \\ &= - \left((-1)^1 \cdot (-1) \cdot 1 \cdot 1\right) dx_2 \wedge dx_3 = dx_3 \wedge dx_2, \end{aligned}$$

$$\begin{aligned} \star(dx_3 \wedge dx_0) &= - \star (dx_0 \wedge dx_3) = - \left((-1)^q \operatorname{sgn}(\tau) g_{11} g_{22} dx_1 \wedge dx_2\right) \\ &= - \left((-1)^1 \cdot 1 \cdot 1 \cdot 1\right) dx_2 \wedge dx_3 = dx_1 \wedge dx_2, \end{aligned}$$

$$\begin{aligned} \star(dx_2 \wedge dx_3) &= (-1)^q \operatorname{sgn}(\tau) g_{00} g_{11} dx_0 \wedge dx_1 = \\ &= (-1)^1 \cdot 1 \cdot (-1) \cdot 1 \cdot dx_0 \wedge dx_1 = -dx_1 \wedge dx_0, \end{aligned}$$

$$\star(dx_3 \wedge dx_1) = -\star(dx_1 \wedge dx_3) = -\left((-1)^q \operatorname{sgn}(\tau) g_{00} g_{22} dx_0 \wedge dx_2\right) =$$
$$= -\Big((-1)^1 \cdot (-1) \cdot (-1) \cdot 1\Big)\, dx_0 \wedge dx_2 = -dx_2 \wedge dx_0,$$

$$\star(dx_1 \wedge dx_2) = (-1)^q \operatorname{sgn}(\tau) g_{00} g_{33} dx_0 \wedge dx_3$$
$$= (-1)^1 \cdot 1 \cdot (-1) \cdot 1 \cdot dx_0 \wedge dx_3 = -dx_3 \wedge dx_0.$$

So the Hodge-dual of F is:

$$dF = -B_1 dx_1 \wedge dx_0 - B_2 dx_2 \wedge dx_0 - B_3 dx_3 \wedge dx_0 +$$

$$+ E_1 dx_2 \wedge dx_3 + E_2 dx_3 \wedge dx_1 + E_3 dx_1 \wedge dx_2.$$

We can also unify the electric charge and current density into a 1-form:

$$J := j_1 dx_1 + j_2 dx_2 + j_3 dx_3 - \rho dx_0.$$

Let's compute the exterior derivative of $\star F$:

$$d \star F = -\partial_2 B_1 dx_2 \wedge dx_1 \wedge dx_0 - \partial_3 B_1 dx_3 \wedge dx_1 \wedge dx_0 - \partial_1 B_2 dx_1 \wedge dx_2 \wedge dx_0 -$$

$$-\partial_3 B_2 dx_3 \wedge dx_2 \wedge dx_0 - \partial_2 B_3 dx_2 \wedge dx_3 \wedge dx_0 - \partial_1 B_3 dx_1 \wedge dx_3 \wedge dx_0 +$$

$$+\partial_0 E_1 dx_0 \wedge dx_2 \wedge dx_3 + \partial_1 E_1 dx_1 \wedge dx_2 \wedge dx_3 + \partial_0 E_2 dx_0 \wedge dx_3 \wedge dx_1 +$$

$$+\partial_2 E_2 dx_2 \wedge dx_3 \wedge dx_1 + \partial_0 E_3 dx_0 \wedge dx_1 \wedge dx_2 + \partial_3 E_3 dx_3 \wedge dx_1 \wedge dx_2.$$

Collecting the terms we obtain:

$$d \star F = (\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3) dx_1 \wedge dx_2 \wedge dx_3 +$$

$$+ (\partial_3 B_2 - \partial_2 B_3 + \partial_0 E_1) dx_0 \wedge dx_2 \wedge dx_3 +$$

$$+ (\partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2) dx_0 \wedge dx_1 \wedge dx_3 +$$

$$+ (\partial_2 B_1 - \partial_1 B_2 + \partial_0 E_3) dx_0 \wedge dx_1 \wedge dx_2.$$

Now let's take the dual of $d \star F$

$$\star d \star F = - (\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3) dx_0 + (\partial_3 B_2 - \partial_2 B_3 + \partial_0 E_1) dx_1 +$$

$$+ (\partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2) dx_2 + (\partial_2 B_1 - \partial_1 B_2 + \partial_0 E_3) dx_3.$$

Now $\star d \star F = J$ corresponds to

$$\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \rho$$

and

$$\begin{cases} \partial_3 B_2 - \partial_2 B_3 + \partial_0 E_1 = j_1, \\ \partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2 = j_2, \\ \partial_2 B_1 - \partial_1 B_2 + \partial_0 E_3 = j_3. \end{cases}$$

The first equation is the same as $divE = \rho$ and the next three is the same as $rotB - \frac{\partial E}{\partial t} = j$. Thus the inhomogeneous equations take the form:

$$\star d \star F = J.$$

3.2 Brouwer's Fixed Point Theorem

Theorem 3.2.1. *In the Euclidean space* $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ *every continous map*

$$f : \overline{B_1(0)} \to \overline{B_1(0)}$$

from the closed unit ball to itself has a fixed point.

Proof. **Step 1** First we prove the theorem for $f \in \mathcal{C}^1$ functions. Indirectly let's suppose that

$$f : \overline{B_1(0)} \to \overline{B_1(0)} \qquad (f \in \mathcal{C}^1)$$

is a function with no fixed points:

$$f(x) \neq x \quad (x \in \overline{B_1(0)}).$$

Now we construct the function which assigns to each $x \in \overline{B_1(0)}$ point the point of intersection from $f(x)$ through x with the sphere $\partial\overline{B_1(0)}$. The line from x through $f(x)$ can be parameterized as follows:

$$L = \{x + t(x - f(x)) : t \in \mathbb{R}\}.$$

This line intersects with $\partial\overline{B_1(0)}$ if for some $\lambda \in \mathbb{R}$ the following condition is satisfied:

$$\|x + \lambda(x - f(x))\|^2 = 1.$$

Let's compute the norm squared of this expression:

$$\begin{aligned}\|x + \lambda(x - f(x))\|^2 &= \langle x + \lambda(x - f(x)), x + \lambda(x - f(x))\rangle = \\ &= \langle x, x\rangle + 2\lambda\langle x, x - f(x)\rangle + \lambda^2\langle x - f(x), x - f(x)\rangle = \\ &= \|x\|^2 + 2\lambda\langle x, x - f(x)\rangle + \lambda^2\|x - f(x)\|^2.\end{aligned}$$

So we need to solve the following quadratic equation for λ:

$$\|x - f(x)\|^2\lambda^2 + 2\langle x, x - f(x)\rangle\lambda + \|x\|^2 - 1 = 0.$$

The solution is

$$\lambda_{\pm}(x) = \frac{\langle x, f(x) - x\rangle \pm \sqrt{\langle x, f(x) - x\rangle^2 + (1 - \|x\|^2)\|f(x) - x\|^2}}{\|f(x) - x\|^2}.$$

Now we can construct our function:

$$F = (F_1, ..., F_n) : \overline{B_1(0)} \to \partial\overline{B_1(0)}$$

$$F(x) := x + \frac{\langle x, f(x) - x\rangle \pm \sqrt{\langle x, f(x) - x\rangle^2 + (1 - \|x\|^2)\|f(x) - x\|^2}}{\|f(x) - x\|^2}(x - f(x)).$$

From the formula we can see that F is a $\mathcal{C}^1$ function. Moreover F acts on the boundary of the unit ball as the identity map:

$$F(x) = x \quad (x \in \partial\overline{B_1(0)}).$$

Since the image of any x is on the boundary of the unit ball the following relation holds for all $x \in \overline{B_1(0)}$:

$$\|F(x)\|^2 = \sum_{i=1}^{n} F_i(x)^2 = 1.$$

Differentiating the above formula yields:

$$2\sum_{i=1}^{n} F_i(x)F_i'(x) = 2\sum_{i,j=1}^{n} F_i(x)\partial_j F_i(x)dx_j = 0$$

and it follows that for each index j

$$\sum_{i=1}^{n} F_i(x)\partial_j F_i(x) = 0.$$

This equality shows that the system of equations

$$\sum_{i=1}^{n} \alpha_i \partial_j F_i(x) = 0.$$

has a non-trivial solution. $(\alpha_1, ...\alpha_n) = (F_1(x), ..., F_n(x)) \neq (0, ..., 0)$. Hence the determinant of the following matrix vanishes.

$$\det(\partial_j F_i(x)) = 0$$

Now we define an $\omega \in \Lambda^r_{n-1}$ differential form and using the observation made above conclude that its derivative vanishes. If

$$\omega := F_1 \cdot dF_2 \wedge ... \wedge dF_n$$

then

$$d\omega = dF_1 \wedge dF_2 \wedge ... \wedge dF_n = \det(\partial_j F_i(x))\, dx_1 \wedge ... \wedge dx_n = 0.$$

Now we use Stokes' theorem and the fact that F acts on the boundary of the unit ball as the identity map and arrive at a contradiction.

$$0 = \int_{\overline{B_1(0)}} d\omega = \int_{\overline{\partial B_1(0)}} \omega = \int_{\overline{\partial B_1(0)}} x_1 dx_2 \wedge ... \wedge dx_n = \int_{\overline{B_1(0)}} dx_1 \wedge ... \wedge dx_n =$$
$$= \mu(\overline{B_1(0)}).$$

Which would mean that the volume of the unit ball is zero, thus this is a contradiction.

Step 2 Now we consider the general case, where f is continuous. This case will be reduced to the previous one by using the Stone-Weierstrass approximation theorem. This theorem states (see [3]) that if $\Omega \subset \mathbb{R}^n$ is compact, $f : \Omega \to \mathbb{R}$ is continuous, then for all $\epsilon > 0$ there exists a $p : \Omega \to \mathbb{R}$ polynomial such that $|f(x) - p(x)| < \epsilon$ for all $x \in \Omega$. In our case $\overline{B_1(0)}$ is compact, so by applying this theorem to the $f_1, ..., f_n$ coordinate functions of f we can conclude that for each $\epsilon > 0$ there exists a

$$p : \overline{B_1(0)} \to \mathbb{R}^n \quad (p \in \mathcal{C}^1)$$

polynomial such that

$$\|f(x) - p(x)\| < \epsilon.$$

Now consider the normalized function:

$$\tilde{p}(x) := \frac{p(x)}{1+\epsilon}.$$

Because of

$$\|p(x)\| - \|f(x)\| \leq \|p(x) - f(x)\| \leq \epsilon \quad \text{and} \quad \|f(x)\| \leq 1$$

we have $\|p(x)\| \leq 1 + \epsilon$, thus $\|\tilde{p}(x)\| \leq 1$. So $\tilde{p}$ is a map from the unit ball to itself:

$$\tilde{p} : \overline{B_1(0)} \to \overline{B_1(0)}.$$

Moreover $\tilde{p}$ can be estimated against f:

$$\|f(x) - p(x)\| \leq \|f(x) - p(x)\| + \|p(x) - \tilde{p}(x)\| \leq \epsilon + \|p(x)\| \left|1 - \frac{1}{1+\epsilon}\right| \leq$$
$$\leq \epsilon + (1+\epsilon)\frac{\epsilon}{1+\epsilon} \leq 2\epsilon.$$

To sum up we have proved so far that for each $\epsilon > 0$ there exists a map $\tilde{p} : \overline{B_1(0)} \to \overline{B_1(0)}$ ($\tilde{p} \in \mathcal{C}^1$) such that for all $x \in \overline{B_1(0)}$ the following estimate holds:

$$\|f(x) - \tilde{p}(x)\| \leq 2\epsilon.$$

Now let's suppose that the continuous map $f : \overline{B_1(0)} \to \overline{B_1(0)}$ has no fixed points. This means that

$$0 < \mu := \inf_{x \in \overline{B_1(0)}} \|f(x) - x\|$$

Now for $\epsilon = \mu/2$ we choose a smooth map $\tilde{p}$ with the above discussed properties. Since $\tilde{p} \in \mathcal{C}^1$ we can apply the first part of this proof to $\tilde{p}$ and conclude that it has a fixed point $x_0 \in \overline{B_1(0)}$. But this means

$$\|f(x_0) - \tilde{p}(x_0)\| = \|f(x_0) - x_0\| < \mu$$

which contradicts the definition of μ. So our initial assumption (that f has no fixed points) is false.

$\square$

Bibliography

[1] Agricola I. and Friedrich T.: *Global Analysis: Differential Forms in Analysis, Geometry and Physics*, Graduate Studies in Mathematics, vol. 52 (2002.)

[2] Manfredo P. Do Carmo: *Differential Forms and Applications*, Univeritext 1994.

[3] Kovács, S.: *Funkcionálanalízis feladatokban, egyetemi jegyzet*, Budapest, 2013.
ISBN: 978-963-284-445-9
(http://numanal.inf.elte.hu/~alex/hu/anyag/PROGINF/FunkAnal/FunkAnalKS.pdf)

[4] Kovács, S.: *Alkalmazott analízis gyakorlat, egyetemi jegyzet*, Budapest, 2018.
ISBN 978-963-489-032-4
(http://numanal.inf.elte.hu/~alex/AlkAnalGyak/AlkAnalGyakKS.pdf)

[5] Simon, P.: *Válogatott fejezetek a matematikából, egyetemi jegyzet*, Budapest, 2019. ELTE Eötvös kiadó
(http://numanal.inf.elte.hu/~simon/ujfolyt.mod.latex.pdf)

[6] Tevian Dray: *The Hodge Dual Operator, university lecture note* 1999.
(http://people.oregonstate.edu/~drayt/Courses/MTH434/2009/dual.pdf)

[7] Solomon Akaraka Owelle: *Maxwell's Equations in Terms of Differential Forms, postgraduate thesis* 2010.

(https://bbs.pku.edu.cn/attach/13/c8/13c819b28e8fb43c/maxwell_hodge.pdf)

[8] Gregory L. Naber: *The Geometry of Minkowski Spacetime: An Introduction to the Mathematics of the Special Theory of Relativity (Applied Mathematical Sciences)* 2012.

[9] Igor R. Shafarevich, Alexey O. Remizov: *Linear Algebra and Geometry*, Springer 2012.