Finite Hyper POSETs

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Contents

1 Acknowledgement

I would like to thank Gyula Katona for his enthusiastic guidance and supervising.

2 Abstract & motivation

It is a well known problem, that if given a series of real number, $a_1, ..., a_{mn+1}$, then it contains a monotone increasing subseries with $n+1$ elements or a monotone decreasing subseries with $m+1$ elements. Look at this problem in the following way: let $H = \{a_1, ..., a_{mn+1}\}$ and define the relation $<_1$ such as $a_i <_1 a_j$ if and only if $i < j$ and $a_i \leq a_j$ and define $<_2$ such that $a_i <_2 a_j$ if and only if $i < j$ and $a_i \geq a_j$. Then $(H, <_1)$ and $(H, <_2)$ are POSETs. Plus in the structure $(H, <_1, <_2)$ every to different elements can be compared due to $<_1$ or $<_2$, and every monotone increasing subseries is a $<_1$ chain and every monotone decreasing subseries is a $<_2$ chain. Thus it is enough to prove the following general problem: if $(H, <_1, <_2)$ is a structure with the properties above, then there exists a $<_1$ chain with $n+1$ elements or a $<_2$ chain with $m+1$ elements. This new view of the problem opens up opportunities for generalizations and a couple of new problems occur as well.

More preciously, in this article a generalization of POSETs are being studied, which I call Hyper POSET (HPOSET). It is a structure $(H, <_1, ..., <_n)$ where $<_1, ..., <_n$ are transitive relations such that every two different elements can be compared due to at least one of the relations. My goal is to study the chains and anti chains in these structures and to show some of its applications.

3 General Hyper POSETs

3.1 Introduction to Hyper POSETs

Definition 1 *$(H, <_1, <_2, ..., <_n)$ is called a Hyper POSET (HPOSET) if H is a set, $(H, <_k)$ is a POSET ($k = 1, ..., n$) and for any $x, y \in H$, $x \neq y$ there is a $1 \leq m \leq n$ such that $x <_m y$ or $y <_m x$ (there can be more than one such m's). For an r positive integer an $(H, <_1, ..., <_n)$ Hyper POSET is called an r-Hyper POSET, if for every $x, y \in H$, $x \neq y$ there are at least r different relations between x and y.*

The following statement is trivial and I let the reader to figure out its solution.

Statement 1 *Let $(H, <_1, ..., <_n)$ be a Hyper POSET and $1 \leq k \leq n$ be arbitrary. If $G \subset H$ is an antichain due to $<_k$, then $(G, <_1, ..., <_{k-1}, <_{k+1}, ..., <_n)$ is a Hyper POSET.*

There are some different ways, that the product of Hyper POSETs can be defined, but the following definition proved to be the most useful and obvious.

Definition 2 *Let n and k be positive integers and for $i = 1, ...k$ let $\mathfrak{H}_i = (H_i, <_1, ..., <_n)$ be Hyper POSETs. Let*

$$\mathfrak{H}_1 \star .... \star \mathfrak{H}_k = (H_1 \times ... \times H_k, <_1, ..., <_n)$$

be the ordered product of $\mathfrak{H}_1, ..., \mathfrak{H}_k$ where the relations are defined in the following way: let $(x_1, ..., x_k), (y_1, ..., y_k) \in H_1 \times ... \times H_k$ that $(x_1, ..., x_k) \neq (y_1, ..., y_k)$ and let r be the smallest index, that $x_r \neq y_r$. Then $(x_1, ..., x_k) <_m (y_1, ..., y_k)$ if and only if $x_r <_m y_r$.

Statement 2 *If $\mathfrak{H}_1, ..., \mathfrak{H}_k$ are r-Hyper POSETs with n relations, then $\mathfrak{H}_1 \star .... \star \mathfrak{H}_k$ is an r-Hyper POSET.*

Proof Let $x = (x_1, ..., x_k)$ and $y = (y_1, ..., y_k)$ be different elements of $H_1 \times ... \times H_k$, such that p is the smallest index, that $x_p \neq y_p$. Then because of $\mathfrak{H}_\mathfrak{p}$ is an r-Hyper POSET, there are at least r relations between x_p and y_p, and these relations will hold between x and y as well.

Now it has to be proved, that the relations $<_1, ..., <_m$ are transitive. Lets suppose, that $(x_1, ..., x_k) <_m (y_1, ..., y_k)$ and $(y_1, ..., y_k) <_m (z_1, ..., z_k)$ for some m. Let p be the smallest index, such that $x_p \neq y_p$ and q be the smallest index, that $y_q \neq z_q$. Then $x_p <_m y_p$ and $y_q <_m z_q$.
If $p = q$ then $x_1 = y_1 = z_1, ..., x_{p-1} = y_{p-1} = z_{p-1}$ and $x_p <_m y_p$ and $y_p <_m z_p$. But $\mathfrak{H}_p$ is a Hyper POSET, so $x_p <_m z_p$ and p is the smallest i index, that $x_i \neq z_i$, so $(x_1, ..., x_k) <_m (z_1, ..., z_k)$. If $p < q$ then $x_1 = y_1 = z_1, ..., x_{p-1} = y_{p-1} = z_{p-1}$ and $x_p <_m y_p = zp$, so p is the smallest i index, that $x_i \neq z_i$ and $x_p <_m z_q$ so $(x_1, ..., x_k) <_m (z_1, ..., z_k)$.
If $p > q$ then $x_1 = y_1 = z_1, ..., x_{q-1} = y_{q-1} = z_{q-1}$ and $x_q = y_q <_m zq$, so q is

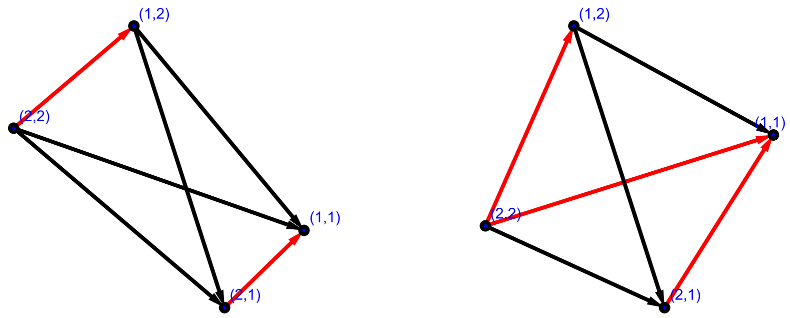
the smallest i index, that $x_i \neq z_i$ and $x_q <_m z_q$ so $(x_1, ..., x_k) <_m (z_1, ..., z_k)$. So $<_m$ is transitive for $m = 1, ..., k$ which means, that $\mathfrak{H}_1 \star .... \star \mathfrak{H}_k$ is an r-Hyper POSET.■

3.2 Longest chain

My goal is to study the chains in the HPOSETs. With the help of this a tool named *Erdős-Szekeres code*[3] (ESz code) will be used.

Definition 3 *Let $(H, <_1, ..., <_n)$ be a Hyper POSET. Let $f : H \to (\mathbb{Z}^+)^n$ be the function such that for any $x \in H$ the k'th $(k = 1, ..., n)$ coordinate of $f(x)$ is the length of the longest chain in $(H, <_k)$ with smallest point x. Then f is the Erdős-Szekeres code of the Hyper POSET.*

Unluckily the ESz code does not determine the HPOSET due to isomorphism. The following picture shows two HPOSET's with the same ESz code, not isomorphic to each other.



Obviously, they cannot be isomorph, because the one on the left has 2 $<_1$ and 4 $<_2$ relations and the one on the right has 3 of both of the relations. Nevertheless, the ESz code is a very useful tool in some problems.

Lemma 1 *Let $(H, <_1, ..., <_n)$ be an r-HPOSET and f its Erdős-Szekeres code. Let $x, y \in H$ such that $x \neq y$. Then $f(x)$ and $f(y)$ differ in at least r coordinates.*

Proof Let $x, y \in H$ be such that $x \neq y$. Then there exists $1 \leq m_1 \leq ... \leq m_r \leq n$ such that $x <_{m_s} y$ or $y <_{m_s} x$ $(s = 1, ..., r)$. It will be proved, that

$f(x)$ and $f(y)$ differ in the m_s'th coordinate, for $s = 1, ..., r$.
It can be assumed that $x <_{m_s} y$. Let $C \subset H$ one of the longest chains due to $<_{m_s}$ with smallest element y. Then the m_s'th coordinate of $f(y)$ is $|C|$. But $C \cup \{x\}$ is a longer chain with smallest element x, so by the definition of f the m_s'th coordinate of $f(x)$ is at least $|C| + 1$. By that the m_s'th coordinate of $f(x)$ and $f(y)$ differ.■

Now I prove a little Lemma, which will be very useful in some constructions.

Lemma 2 *Let $\mathfrak{H}_1, ..., \mathfrak{h}_k$ be Hyper POSETs and $\mathfrak{H} = \mathfrak{H}_1 \star ... \star \mathfrak{h}_k$. Let $c_{j,m}$ ($1 \leq j \leq k$, $1 \leq m \leq n$) be the size of the longest chain in $\mathfrak{H}_j$ due to $<_m$ and $a_{j,m}$ be the size of the biggest anti chain. Then the size of the biggest chain in $\mathfrak{H}$ due to $<_m$ is $c_{1,m}...c_{k,m}$ and the size of the biggest anti chain due to $<_m$ is $a_{1,m}...a_{k,m}$.*

Proof Firstly, lets prove it for the chains. Let $C_{j,m} \subset H_j$ be a chain such that $|C_{j,m}| = c_{j,m}$. Then if $C_m = C_{1,m} \times ... \times C_{k,m}$, then $C_m \subset H_1 \times ... \times H_k$ and $|C_m| = c_{1,m}...c_{k,m}$. Plus if $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$ are different elements of C_m and r is the smallest index, that $x_r \neq y_r$, then x_r and y_r are different elements of $C_{r,m}$, which is a $<_m$ chain, so $x_r <_m y_r$ or $y_r <_m x_r$. That means, that $(x_1, ..., x_k) <_m (y_1, ..., y_k)$ or $(y_1, ..., y_k) <_m (x_1, ..., x_k)$. So C_m is a $<_m$ chain. That proves, that the longest chain is at least $c_{1,m}...c_{k,m}$ long.
Now it will be proved with induction on k, that every $<_m$ chain has at most $c_{1,m}...c_{k,m}$ elements. For $k = 1$ it is obvious. Now let assume, that it is known for the ordered product of $k - 1$ Hyper POSETS and it will be proved for k. Let $C' \subset H_1 \times ... \times H_k$ be a $<_m$ chain and let

$$pr_1 \ C' = \{t \in H_1 \big| \exists (t, x_2, ..., x_k) \in C'\}.$$

Then $pr_1 \ C'$ is a $<_m$ chain, because if $t_1, t_2 \in pr_1 \ C'$ cannot be compared by $<_m$, then neither $(t_1, x_1, ..., x_k), (t_2, y_1, ..., y_k) \in C'$. So $|pr_1 \ C'| < c_{1,m}$. Now for every $t \in pr_1 \ C'$ let

$$C'_t = \{(x_2, ..., x_k \in H_2 \times ... \times H_k \big| (t, x_2, ..., x_k) \in C'\}.$$

Then C'_t is a $<_m$ chain in $\mathfrak{H}_2 \star ... \star \mathfrak{H}_k$ so by the assumption of the induction $|C'_t| \leq c_{2,m}...c_{k,m}$. But

$$\bigcup_{t \in pr_1 \ C'} \{t\} \times C'_t = C'$$

so

$$|C'| = \sum_{t \in pr_1 \ C'} |C'_t| \leq \sum_{t \in pr_1 \ C'} c_{2,m}...c_{k,m} \leq c_{1,m}...c_{k,m}$$

which is exactly what we wanted to prove.
For anti chains it can be proved by the same idea as for chains.■

The next theorem is a generalization of the well known theorem, that in every POSET with $t^2 + 1$ elements, there is a chain or an anti chain with at least $t + 1$ elements.

Theorem 1 *Let $(H, <_1, ..., <_n)$ be an r-Hyper POSET such that $|H| \geq t^{n-r+1} + 1$. Then there is an $1 \leq m \leq n$ and a $C \subset H$ such that $|C| \geq t + 1$ and C is a $<_m$ chain.*

Proof Let the HPOSET's ESz code be f. Let assume indirectly that the length of every chain is at most t. That means that for every $x \in H$ every coordinate of $f(x)$ is at most t, so $f(x) \in \{1, ..., t\}^n$. But $|H| \geq t^{n-r+1} + 1$ and the first $n-r+1$ coordinates of the vectors in $f(H)$ can have maximum t^{n-r+1} different values, so by the pigeon hole theorem there are two vectors, $f(x)$ and $f(y)$, whose first $n-r+1$ coordinates is equal. But then $f(x)$ and $f(y)$ can only differ in the last $r-1$ coordinates. This contradicts with the previous *lemma*, which claims, that every two different vectors should have at least r different coordinates. ■

Remark This theorem will be used in the case $r = 1$ for which the statement is that in a Hyper POSET $(H, <_1, ..., <_n)$ there is a chain at least $\lceil \sqrt[n]{|H|} \rceil$ long.

Now it will be shown, that if given n and r then the previous theorem is strict for infinitely many t. More preciously there exists an r-Hyper POSET $(H, <_1, ..., <_n)$ that $|H| = t^{n-r+1}$ and the longest chain has t elements.

Theorem 2 *Let $r < n$ be positive integers. Then there exists infinitely many t positive integer, that there exists an r-Hyper POSET $(H_0, <_1, ..., <_n)$ with t^{n-r+1} elements such that every chain has a length at most t.*

Proof First a little lemma will be proved, which can be a useful tool in other constructions as well:

Lemma 3 *Let $r < n$ be positive integers and $G \subset (\mathbb{Z}^+)^n$ a finite subset which satisfies the following conditions:*

(i) *if $v, w \in G$ and $v \neq w$, then v and w differs in at least r coordinates*

(ii) *if $(x_1, ..., x_n) \in H$ then for $i = 1, ..., n$ and $s = 1, ..., x_i - 1$ there exists a vector $v_{i,s} \in G$, whose i'th coordinate is s*

Then there exists an r-Hyper POSET $(H, <_1, ..., <_n)$ that if f its Erdős-Szekeres code, then $f(H) = G$.

Proof of Lemma Let $H = G$ and for $(x_1, ..., x_n), (y_1, ..., y_n) \in G$ let $(x_1, ..., x_n) <_i (y_1, ..., y_n)$ if and only if $x_i > y_i$ (for $i = 1, ..., n$). Then $(H, <_1, ..., <_n)$ is an r-Hyper POSET, because it is easy to see, that $(H, <_i)$ is a POSET and by condition (i) every two vectors differ in at least r coordinates, so there are at least r different relations between any two different vectors in H.
Now it will be proved, that if f is the Erdős-Szekeres code, then for $x = (x_1, ..., x_n) \in G$ it is true, that $f(x) = x$ (this proves, that $f(H) = G$). Let $f(x) = (p_1, ..., p_n)$. By condition (ii), for $s = 1, ..., x_i - 1$ there is a vector $v_{i,s} \in H$ whose i'th coordinate is s. So by the determination of $<_i$ the

set $\{x, v_{i,x_i-1}, ..., v_{i,1}\}$ is a $<_i$ chain with x_i elements and starting point x, so $p_i \geq x_i$. Plus if $\{x, v_1, ..., v_l\}$ is a chain with starting point x, than the i'th coordinates of $v_1, ..., v_l$ are pairwise different positive integers smaller than x_i, so $p_i \leq x_i$, which means, that $p_i = x_i$ (for $i = 1, ..., n$). So it is proved, that $f(x) = x$ and the proof of the lemma is done. ■

Let p be a prime number greater than n. From now on all the calculations meant over the field $\mathbb{F}_p$. Let $M \in \mathbb{F}_p^{(r-1)\times n}$, whose (i,j)'th entry is $m_{i,j} = j^{i-1}$. For any $1 \leq a_1 < ... < a_{r-1} \leq n$ integers let $M_{a_1,...,a_{r-1}}$ be the $(r-1)\times(r-1)$ submatrix of M, whose j'th column is the a_j'th column of M. Then $M_{a_1,...,a_{r-1}}$ is a Vandermonde-matrix, so its determinant can be calculated in the following way:

$$\det M_{a_1,...,a_{r-1}} = \prod_{1\leq i<j\leq r-1} (a_i - a_j).$$

Because of $1 \leq a_1 < ... < a_{r-1} \leq n < p$ it is true, that $a_i - a_j \neq 0$ so

$$\det M_{a_1,...,a_{r-1}} \neq 0$$

which means, that $M_{a_1,...,a_{r-1}}$ is invertible (over $\mathbb{F}_p$). Now let

$$G = \{v \in \mathbb{F}_p^n | Mv = 0\}.$$

$rank(M) = r-1$ so G is a $n-r+1$ dimensional subspace of $\mathbb{F}_p^n$, which means that $|G| = p^{n-r+1}$.
It will be shown, that if $v, w \in G$ and $v \neq w$, then v and w has at least r different coordinates. Let $v - w = (x_1, ..., x_n)$ and let assume indirectly, that there are at least $n-r+1$ zeros among $x_1, ..., x_n$. Let $1 \leq a_1 < ... < a_{r-1} \leq n$ indices, that $x_j = 0$ if $j \notin \{a_1, ..., a_{r-1}\}$. $v - w \in G$ so $M(v-w) = 0$, which means that

$$M_{a_1,...,a_{r-1}} \begin{pmatrix} x_{a_1} \\ ... \\ x_{a_r} \end{pmatrix} = 0.$$

But $M_{a_1,...,a_{r-1}}$ is invertible so necessarily $(x_{a_1}, ..., x_{a_{r-1}}) = 0$, which means that $v - w = 0$, which is a contradiction.
Finally for every $m \in \mathbb{F}^p$ and $1 \leq j \leq n$ there is a vector in G, whose j'th coordinate is m. It is true because every $M_{a_1,...,a_{r-1}}$ is invertible, so if $1 \leq b_1 < ... < b_{n-r+1} \leq n$ and $s_1, ..., s_{n-r+1} \in \mathbb{F}^p$ are fixed, then one can find a vector $(x_1, ..., x_n) \in G$, that $x_{b_1} = s_1, ..., x_{b_{n-r+1}} = s_{n-r+1}$.
Let $\phi : \mathbb{F}_p \to \{1, ..., p\}$ be any bijection, than the set $\phi(G)$ satisfies the conditions (i) and (ii) in the previous lemma, so there exists an r-Hyper POSET $(H, <_1, ..., <_n)$, that for its f Erdős-Szekeres code, $f(H) = \phi(G)$. But every vector's every coordinate in $\phi(G)$ is at most p, so by the definition of f it is clear, that every chain in $(H, <_1, ..., <_n)$ has at most p elements. So if $t = p$, then $|H| = t^{n-r+1}$ and every chain has at most t elements.
It has been proved, that if $n < t = p$ is a prime, then there exists an r-Hyper POSET, which satisfies the conditions of the theorem.■

Remark It is true, that if every prime divisor of t is bigger than n, then there exists an r-Hyper POSET with t^{n-r+1} elements, that the biggest chain has at most t elements. Let $t = p_1...p_k$ the prime factorization of t and let $\mathfrak{H}_i$ be an r-Hyper POSET with p_i^{n-r+1} elements, that the longest chain has at most p_i elements $(i = 1, ..., k)$. Then if $\mathfrak{H} = \mathfrak{H}_1 \star ... \star \mathfrak{H}_k$, then $\mathfrak{H}$ has $p_1^{n-r+1}...p_k^{n-r+1} = t^{n-r+1}$ elements and due to *Lemma 2* the longest chain has at most $p_1...p_k = t$ elements.
My conjecture is, that the previous theorem is true for every t big enough respect to n. It cannot be true for small t. I will show that one can not pick t^{n-r+1} elements from $\{1, ..., t\}^n$, that every two differ in at least r coordinates, which is sufficient by the train of thoughts presented in the proof of *Theorem 1*.
If $v, w \in \{1, ..., t\}$ let $d(v, w)$ be the number of non-zero coordinates of $v - w$. Then $(\{1, ..., t\}^n, d)$ is a metric space, and v and w differ in r coordinates if and only if $d(v, w) \geq r$. Let assume, that $v_1, ..., v_s$ are vectors from $\{1, ..., t\}^n$ that $d(v_i, v_j) \geq r$ for every $1 \leq i < j \leq s$, then the spheres $B_{\lfloor \frac{r-1}{2} \rfloor}(v_i)$ are disjoint $(B_R(x) = \{y \in \{1, ..., t\}^n \big| d(x, y) \leq R\})$. But it can be calculated easily, that

$$|B_{\lfloor \frac{r-1}{2} \rfloor}(v_i)| = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} \binom{n}{j} (t-1)^j.$$

So because of $|\{1, ..., t\}^n| = t^n$ and that the spheres $B_{\lfloor \frac{r-1}{2} \rfloor}(v_i)$ are disjoint it is

$$s \leq \frac{t^n}{\sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} \binom{n}{j} (t-1)^j}.$$

If t is small (for example smaller than $\frac{1}{2} \binom{n}{\lfloor \frac{r-1}{2} \rfloor}^{\frac{1}{\lceil \frac{r-1}{2} \rceil}}$) than the right side is smaller than t^{n-r+1}, so the theorem cannot be strict.

Theorem 1 can be generalized in the following way:

Theorem 3 *Let $s \leq r < n$ be positive integers and let $(H, <_1, ..., <_n)$ be an r-Hyper POSET. Let*

$$\alpha = \min \left\{ (n-r+1)^s, 2^{s-1} \binom{n}{s} - \binom{r}{s} + 1 \right\}$$

and let assume, that $|H| \geq t^\alpha + 1$. Than there exists a $C \subset H$ and $1 \leq i_1 \leq ... \leq i_s \leq n$ that $|C| \geq t + 1$ and C is a chain due to $<_{i_1}, ..., <_{i_s}$.

Proof First it will be proved for the case $\alpha = (n-r+1)^s$. It will be proved by induction on s. For $s = 1$ it is equivalent with *Theorem 1*, so it is done. If $s > 1$ let assume, that the statement is proved for $s - 1$. If $|H| > t^{(n-r+1)^s} + 1$ than by *Theorem 1* there exists a $B \subset H$ that $|B| \geq t^{(n-r+1)^{s-1}} + 1$ and B is a chain

due to one of the relations. It can be assumed without the loss of generality, that B is a $<_n$ chain. Now $(B, <_1, ..., <_{n-1})$ is a an $(r-1)$-Hyper POSET with $n-1$ relations and $|B| \geq t^{((n-1)-(r-1)+1)^{s-1}} + 1$ so using the assumption of the induction there exists a $C \subset B$ and $1 \leq i_1 \leq ... \leq i_{s-1} \leq n-1$ that C is a chain due to $<_{i_1}, ..., <_{i_{s-1}}$. But then C is a chain in $(H, <_1, ..., <_n)$ due to $<_{i_1}, ..., <_{i_{s-1}}, <_n$, so it is done.
Now we prove the case where $\alpha = 2^{s-1}\binom{n}{s} - \binom{r}{s} + 1$. Let

$$J = \{(j_1, \epsilon_2 j_2, ..., \epsilon_s j_s) \big| \epsilon_2, ..., \epsilon_n \in \{-1, 1\}; 1 \leq j_1 < ... < j_s \leq n\}.$$

Then $|I| = 2^{s-1}\binom{n}{s}$. Define the Hyper POSET $(H, (\prec_j)_{j \in J})$ as follows: if $j \in J$ and $j = (j_1, \epsilon_2 j_2, ..., \epsilon_s j_s)$ where $\epsilon_2, ..., \epsilon_n \in \{-1, 1\}$ and $1 \leq j_1 < ... < j_s \leq n$ than $x \prec_j y$ if and only if $x <_{j_1} y$ and for $k = 2, ..., s$ if $\epsilon_k = 1$ then $x <_{j_k} y$ and if $\epsilon_k = -1$ then $y <_{j_k} x$. It is clear that the relation $\prec_j$ is transitive. Because $(H, <_1, ..., <_n)$ is an r-Hyper POSET, there are at least r relations between any two different elements of H, and every s-tuples of this r relations determine clearly an $\prec_j$ relation between these two elements. So $(H, (\prec_j)_{j \in J})$ is an $\binom{r}{s}$-Hyper POSET. Apply *Theorem 1* to the $(H, (\prec_j)_{j \in J})$ $r_0 = \binom{r}{s}$-Hyper POSET with $n_0 = 2^{s-1}\binom{n}{s}$ relations. Because $|H| \geq t^{2^{s-1}\binom{n}{s} - \binom{r}{s} + 1} + 1 = t^{n_0 - r_0 + 1} + 1$ there exists a $C \subset H$ that C is a $\prec_j$ chain for some $j \in J$. If $j = (j_1, \epsilon_2 j_2, ..., \epsilon_s j_s)$ then it is clear, that C is a chain in $(H, <_1, ..., <_n)$ due to $<_{j_1}, ..., <_{j_s}$, so the proof is complete.$\blacksquare$

Remark If n is big respect to $r > s > 2$ then $(n - r + 1)^s \sim n^s$ and $2^{s-1}\binom{n}{s} - \binom{r}{s} + 1 \sim \frac{2^{s-1}}{s!} n^s$ so then $\alpha = 2^{s-1}\binom{n}{s} - \binom{r}{s} + 1$. But if r is big enough, then $\alpha = (n - r + 1)^s$.

Now we may ask, that what can be said about the biggest anti chain in a HPOSET. We have to assume, that there is only one relation between any two elements or else it can be that every anti chain has only one element (if H is totally ordered due to every relation). If $n = 2$, then an $<_1$ anti chain is an $<_2$ chain and a $<_1$ anti chain is a $<_2$ chain, so the answer is the same as it was for chains, so if $|H| \geq t^2 + 1$ then there is an anti chain with $t + 1$ elements. Now let's examine the case $n > 2$. If $|H| \geq t^2 + 1$ then there is a $<_1$ chain or an $<_1$ anti chain with $t + 1$ elements. But a $<_1$ chain is a $<_2$ anti chain, so there is an anti chain in $(H, <_1, ..., <_n)$ with $t + 1$ elements. My conjecture is that if t is big enough respect to n then this is strict.
For the proof of this one have to construct a HPOSET $(H, <_1, ..., <_n)$ with t^2 elements such that for every $1 \leq k \leq n$ the set H is the union of t pieces of disjoint $<_k$ chains with length t (this is the only way the construction can look like, if we don't want an anti chain with $t + 1$ elements). The construction doesn't seem to be hard rather need a lot of work and case separation. Instead of that I show a construction if $n = q + 1$ where q is the power of an arbitrary prime number and $t = q^m$ $(m \in \mathbb{N})$.

Definition 4 *A Hyper POSET is called Strong Hyper POSET or SHPOSET if there is only one relation between any two elements.*

Theorem 4 *Let q be the power of a prime number, m an integer, $n = q + 1$ and $t = q^m$. Then there exists a Strong Hyper POSET $(H, <_1, ..., <_n)$ such that $|H| = t^2$ and every anti chain due to any of the relations has at most t elements.*

Proof First a Strong Hyper POSET $\mathfrak{G} = (G, <_1, ..., <_n)$ will be constructed with q^2 elements such that the biggest anti chain has q elements. Let the elements of G be $x_1, ..., x_{q^2}$. If q is a power of a prime number, then there exists a field with q elements, $\mathbb{F}_q$. Let $\mathfrak{A}$ be the affine plane over $\mathbb{F}_q$, then $\mathfrak{A}$ has q^2 elements, so there exist bijections between the elements of G and $\mathfrak{A}$, let one of them be $\varphi : G \to \mathfrak{A}$. The lines in $\mathfrak{A}$ has exactly $q + 1$ different directions, let them be $\mathbf{v}_1, ..., \mathbf{v}_{q+1}$. Define the relation $<_k$ $(k = 1, ..., n)$ in G as follows: $x_i <_k x_j$ if and only if $i < j$ and the direction of the line lying on $\varphi(x_i), \varphi(x_j)$ is $\mathbf{v}_k$. Then there is only one relation between any two elements of G and $<_k$ is transitive, because if $x_i <_k x_j$ and $x_j <_k x_l$ then $i < j < l$ and the direction of the line lying on $\varphi(x_i), \varphi(x_j)$ is the same as the direction of the line $\varphi(x_j), \varphi(x_l)$, which means that $\varphi(x_i), \varphi(x_j), \varphi(x_l)$ is collinear, so the direction of the line $\varphi(x_i), \varphi(x_l)$ is $\mathbf{v}_k$ too.
Now it will be shown that every anti chain in $(G, <_1, ..., <_n)$ has at most q elements. Let assume, that $A \subset G$ and $|A| \geq q + 1$. For any $1 \leq k \leq n$ there is exactly q lines in $\mathfrak{A}$ with direction $\mathbf{v}_k$ and their union contains every element of $\mathfrak{A}$. So because of $|\varphi(A)| = q + 1$ there is two elements $a, b \in \varphi(A)$ that lies on a line with direction $\mathbf{v}_k$ and so $\varphi^{-1}(a) <_k \varphi^{-1}(b)$ or $\varphi^{-1}(b) <_k \varphi^{-1}(a)$ and that means that A cannot be a $<_k$ anti chain for any k.
Now let $\mathfrak{H} = \mathfrak{G} \star ... \star \mathfrak{G}$ where $\mathfrak{G}$ is multiplied m times. Then $\mathfrak{H}$ is a Strong Hyper POSET with q^{2m} elements and by *Lemma 2* every anti chain has at most q^m elements. The construction is complete.■

3.3 Chain and anti chain decomposition

In this section my goal is to give an upper bound to the minimal number of chains needed to decompose a Hyper POSET with n relations an t elements. The following theorem is a generalization of Theorem 5 in my previous work [2].

Theorem 5 *Let $(H, <_1, ..., <_n)$ be Hyper POSET such that $|H| = t$. Then there is an $1 \leq m \leq n$, such that H is the union of at most*

$$\left\lceil \frac{n-1}{n} t \right\rceil$$

$<_m$-chains.

Proof It will be proved by induction on n. Firstly, let's prove for $n = 2$. Let $G \subset H$ be the biggest anti chain in the POSET due to $<_2$. By the Dilworth's[1] theorem H is the union of $|G|$ pieces of $<_2$ chains so if $|G| \leq \lceil \frac{t}{2} \rceil$ then it is done.

Now let assume that $|G| > \lceil \frac{t}{2} \rceil$. For every two elements $x, y \in G$ the relation $x <_2 y$ and $y <_2 x$ cannot hold, so it must be $x <_1 y$ or $y <_1 x$. That means that G is a $<_1$ chain. And the elements of $H \setminus G$ are individually $<_1$ chains, so H is the union of at most

$$1 + |H \setminus G| = 1 + t - |G| \leq 1 + t - \left(\left\lceil \frac{t}{2} \right\rceil + 1 \right) = t - \left\lceil \frac{t}{2} \right\rceil \leq \left\lceil \frac{t}{2} \right\rceil$$

$<_1$ chains.

Let assume, that it is true for $n - 1$ ($n \geq 3$) and it will be proved for n. Let $G \subseteq H$ be one of the biggest anti chains in the POSET defined by $<_n$. By the Dilworth's theorem H is the union of $|G|$ piece of $<_n$ chain so if

$$|G| \leq \left\lceil \frac{n-1}{n} t \right\rceil$$

then it is done. Now let assume that $|G| > \lceil \frac{n-1}{n} t \rceil$. G is an $<_n$ anti chain, so at least one of the relations $<_1, ..., <_{n-1}$ holds between any two elements of G. That means, that the assumption of the induction can be applied on G. Using that for an $1 \leq m \leq n-1$ the set G is the union of

$$\left\lceil \frac{n-2}{n-1} |G| \right\rceil$$

$<_m$ chains. The points of $H \setminus G$ are individually $<_m$ chains, so H is the union of at most

$$\left\lceil \frac{n-2}{n-1} |G| \right\rceil + t - |G|$$

$<_m$ chains. If $h(x) = \left\lceil \frac{n-2}{n-1} x \right\rceil + t - x$ then h is clearly monotone decreasing in the set of integers so if $|G| > \lceil \frac{n-1}{n} t \rceil$ then

$$h(|G|) \leq h \left(\left\lceil \frac{n-1}{n} t \right\rceil + 1 \right)$$

Let $s = \lceil \frac{n-1}{n} t \rceil - \frac{n-1}{n} t$ then $0 \leq s < 1$ and

$$h \left(\left\lceil \frac{n-1}{n} t + 1 \right\rceil \right) = h \left(\frac{n-1}{n} t + s + 1 \right) =$$

$$= \left\lceil \frac{n-2}{n-1} \left(\frac{n-1}{n} n + s + 1 \right) \right\rceil + t - \left(\frac{n-1}{n} t + s + 1 \right) =$$

$$= \left\lceil \frac{n-2}{n} t + (s+1) \frac{n-2}{n-1} \right\rceil + \frac{1}{n} t - 1 - s < \frac{n-2}{n} t + (s+1) \frac{n-2}{n-1} + 1 + \frac{1}{n} t - 1 - s < \frac{n-1}{n} t + 1.$$

So $h(|G|)$ is smaller than $\frac{n-1}{n} t + 1$ and it is an integer, so $h(|G|) \leq \lceil \frac{n-1}{n} t \rceil$. The theorem is proven.■

Remark If t is given it is not hard to find a $(H_0, <_1, ..., <_n)$ Hyper POSET, that $|H| = t$ and for which the above theorem is strict, so H cannot be decomposed into less than $\lceil \frac{n-1}{n} t \rceil$ pieces of $<_m$ chains $(m = 1, ..., n)$.

Theorem 6 *Let $(H, <_1, ..., <_n)$ be a Strong Hyper POSET such that $|H| = t$. Then there exists an $1 \leq m \leq n$ that H is the union of $\lfloor \frac{t}{n} + \frac{n-1}{2} \rfloor$ pieces of $<_m$ anti chains.*

Proof For $i = 1, ..., n$ let C_i be one of the biggest chains due to $<_i$. If $|C_i| \leq \lfloor \frac{t}{n} + \frac{n-1}{2} \rfloor$ for some i, then due to the Dilworth theorem H is the union of $\lfloor \frac{t}{n} + \frac{n-1}{2} \rfloor$ pieces of $<_i$ anti chains and it is done. So it can be assumed, that $|C_i| \geq \lfloor \frac{t}{n} + \frac{n-1}{2} \rfloor + 1$ for $i = 1, ..., n$. But $(H, <_1, ..., <_n)$ is a Strong Hyper POSET, so two different types of chains can intersect at maximum one point. That means, that

$$\left| \bigcup_{i=1}^{n} C_i \right| \geq \sum_{i=1}^{n} |C_i| - \binom{n}{2} \geq n \left(\left\lfloor \frac{t}{n} + \frac{n-1}{2} \right\rfloor + 1 \right) - \binom{n}{2}.$$

But $\bigcup_{i=1}^{n} C_i \subset H$ so $\left| \bigcup_{i=1}^{n} C_i \right| \leq t$ which means $n(\lfloor \frac{t}{n} + \frac{n-1}{2} \rfloor + 1) - \binom{n}{2} \leq t$. It is equivalent to $\lfloor \frac{t}{n} + \frac{n-1}{2} \rfloor + 1 < \frac{t}{n} + \frac{n-1}{2}$ which is a contradiction, so the proof is complete.

Theorem 7 *Let $n \geq 2$ be a positive integer and $(H, <_1, ..., <_n)$ a Strong Hyper POSET such that $|H| \leq \frac{t(t+1)}{2} - 1$. Than there exists $A_1, ..., A_{t-1}$ that A_i is an anti chain due to one of the relations $(i = 1, ..., t-1)$ and $\bigcup_{i=1}^{t-1} A_i = H$.*

Proof It will be proved by induction on t. For $t = 2$ it is $|H| \leq 2$. If $|H| = 1$ then it is trivial. If $H = \{x, y\}$ then it can be assumed without the loss of generality, that $x <_1 y$ and then H is a $<_2$ anti chain, so it is the union of 1 anti chain.
Now let assume, that the statement is true for $t = u - 1$ and now it will be proved for $t = u$. Let $|H| \leq \frac{u(u+1)}{2} - 1$ and let $C \subset H$ be one of the biggest $<_1$ chain. If $|C| \leq u - 1$ then by the Dilworth theorem H is the union of $u - 1$ pieces of $<_1$ anti chains, so the proof is done. If $|C| \geq u$ then let $H_0 = H \setminus C$. In that case

$$|H_0| \leq \frac{u(u+1)}{2} - 1 - u = \frac{u(u-1)}{2} - 1$$

so by the assumption of the induction there exists $A_1, ..., A_{u-2}$ anti chains, that $\bigcup_{i=1}^{u-2} A_i = H_0$. But C is a $<_1$ chain and H is a Strong Hyper POSET, so C is a

$<_2$ anti chain. With the choice of $A_{u-1} = C$ it is $\bigcup_{i=1}^{u-1} A_i = H$ and $A_1, ..., A_{u-1}$ are all anti chains due to one of the relations, so the proof is complete.

4 Lexicographic Hyper POSETs

Definition 5 *Let $H \subset \mathbb{N}^n$ and $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{y} = (y_1, ..., y_n)$ are different elements of H. Define the relations $<_k$ $(k = 1, ..., n)$ as the following: $\boldsymbol{x} <_k \boldsymbol{y}$ if and only if $x_1 = y_1, x_2 = y_2, ..., x_{k-1} = y_{k-1}$ and $x_k < y_k$. Then $(H, <_1, ..., <_n)$ is a Hyper POSET and it will be called as the Lexicographic Hyper POSET (LHPOSET) defined from H.*

In this section we will study the Lexicographic Hyper POSETs. These special Hyper POSETs come up naturally in some constructions because it is easy to characterize their chains and anti chains.

Statement 3 *Let $H \subset \mathbb{N}^n$ and $\mathcal{H} = (H, <_1, ..., <_n)$ be a Lexicographic Hyper POSET defined from H. Let f be it's Erdős-Szekeres code and $G = Imf$. Let $\mathcal{G} = (G, \prec_1, ..., \prec_n)$ be the Lexicographic Hyper POSET defined from G. Then $f : \mathcal{H} \to \mathcal{G}$ is an order changing bijection, so for every $x, y \in H$ it holds that $x <_k y \Leftrightarrow f(y) \prec_k f(x)$ for $k = 1, ..., n$.*

Proof Let assume that $x <_k y$ for some $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$ $(x, y \in H)$. That means that $x_1 = y_1, x_2 = y_2, ..., x_{k-1} = y_{k-1}, x_k < y_k$. Due to the definition of LHPOSET the $<_m$ relation only determined by the first m coordinates so if $1 \leq l \leq k-1$ then $x <_l z \Leftrightarrow y <_l z$ and $z <_l x \Leftrightarrow z <_l x$ for any $z \in H$. Which means, that due to $<_l$ relation x and y behaves the same, so the l'th coordinate of $f(x)$ and $f(y)$ are the same. The k'th coordinate of $f(x)$ is larger, than the k'th coordinate of $f(y)$, because if $C \subset H$ is a longest chain due to $<_k$ with smallest point y, then $C \cup \{x\}$ is a longer chain with smallest point x. So we got, that $f(y) \prec_k f(x)$. Now let assume, that for some $x, y \in H$ it is $f(x) \prec_k f(y)$. In a LHPOSET there is exactly one relation between two elements so because of upper written it must be $y <_k x$. So summarized: $x <_k y \Leftrightarrow f(y) \prec_k f(x)$.

Lemma 4 *Let $H \subset \mathbb{Z}^n$ be a finite set and let*

$$pr_i\ H = \{y | \exists (x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n) \in H\}$$

for $i = 1, ..., n$.Let $(H, <_1, ..., <_n)$ be the Lexicographic Hyper POSET defined from H. Then the longest chain due to the relation $<_i$ has maximum $|pr_i\ H|$ elements.

Proof Let $x_1, ..., x_k \in H$ such that $x_1 <_i ... <_i x_k$. It is enough to prove, that $k \leq |pr_i H|$. For $j = 1, ..., k$ let the coordinates of x_j be $(x_{j,1}, ..., x_{j,n})$. Then by the definition of $<_i$ it is $x_{1,r} = x_{2,r} = ... = x_{k,r}$ if $1 \leq r < i$ integer and $x_{1,i} < ... < x_{k,i}$. That means, that $x_{1,i}, ..., x_{k,i}$ are pairwise different. But $x_{1,i}, ..., x_{k,i} \in pr_i H$, so $k \leq |pr_i H|$. The lemma is proven. ■

Theorem 8 *Let t be a positive integer. Then there exists an $H_0 \subset \mathbb{Z}^n$ such that $|H| = t^n$ and if $(H_0, <_1, ..., <_n)$ is the Lexicographic Hyper POSET defined from H then the longest chain due to any of the relations has maximum t elements.*

Proof Let

$$H_0 = \{1, ..., t\}^n,$$

then $|H_0| = t^n$. Furthermore $|pr_i\ H_0| = t$ for $i = 1, ..., n$, so by the previous lemma it is obvious, that the longest chain has t elements due to every relations. ■

Theorem 9 *Let $H \subset \mathbb{Z}^n$ and $(H, <_1, ..., <_n)$ the Lexicographic Hyper POSET defined from H. If $t = |H|$ then there exists $1 \leq m \leq n$ and $A \subset H$ such that A is an anti chain due to $<_m$ and $|A| \geq t^{\frac{n-1}{n}}$.*

Proof A little stronger statement will be proved: let a_k be the size of the biggest anti chain due to $<_k$ $(k = 1, ..., n)$, then

$$\sum_{k=1}^{n} a_k \geq n t^{\frac{n-1}{n}}.$$

This will be proved by induction on n, but first a little analytical lemma needed.

Lemma 5 *Let $x_1, ..., x_r$ and y be nonnegative real numbers such that $x_1 + ... + x_r = t$ and $x_i \leq y$ $(i = 1, 2, ..., r)$. Let $0 < \alpha < 1$ then*

$$x_1^\alpha + ... + x_r^\alpha \geq t y^{\alpha - 1}.$$

Proof of lemma Let

$$C = \{(x_1, ..., x_r) \big| 0 \leq x_i \leq y; x_1 + ... + x_r = t\}$$

and $f : C \to \mathbb{R}$

$$f((x_1, ..., x_r)) = x_1^\alpha + ... + x_r^\alpha.$$

Then C is compact and f is continuous so f has a minimum and it takes it on some element $\mathbf{z} = (z_1, ..., z_r) \in C$. Now it will be shown that except at most one $1 \leq i \leq r$ it is $z_i = y$ or $z_i = 0$. Let assume that there exists a $1 \leq i < j \leq n$ such that $0 < z_i, z_j < y$. Let's check two cases: if $z_i + z_j \leq y$ then $z_i^\alpha + z_j^\alpha > (z_i + z_j)^\alpha$ (using that $0 < \alpha < 1$) and with $z_i' = z_i + z_j$, $z_j' = 0$ it is $\mathbf{z'} = (z_1, ..., z_i', ..., z_j', ..., z_r) \in C$ and $f(\mathbf{z'}) < f(\mathbf{z})$ which is a contradiction. If $z_i + z_j > y$ then using that id^α is concave, it is $z_i^\alpha + z_j^\alpha > y^\alpha + (z_i + z_j - y)^\alpha$ so with $z_i' = y$ and $z_j' = x_i + x_j - y$ it's $\mathbf{z'} \in C$ and $f(\mathbf{z'}) < f(\mathbf{z})$ which is again a contradiction.
The zeros from $z_1, ..., z_r$ can be left, so it can be assumed, that every $z_i = y$

except for at most one. Then because their sum is t, there must be $\left\lfloor \frac{t}{y} \right\rfloor$ pieces of y and one $t - y\left\lfloor \frac{t}{y} \right\rfloor = y\left\{\frac{t}{y}\right\}$. So the minimum of f on C is

$$\left\lfloor \frac{t}{y} \right\rfloor y^\alpha + \left(y\left\{\frac{t}{y}\right\}\right)^\alpha = \left(\frac{t}{y} - \left\{\frac{t}{y}\right\}\right) y^\alpha + \left(y\left\{\frac{t}{y}\right\}\right)^\alpha =$$

$$= ty^{\alpha-1} + y^\alpha\left(\left\{\frac{t}{y}\right\}^\alpha - \left\{\frac{t}{y}\right\}\right) \geq ty^{\alpha-1}$$

where the last inequality holds because $0 < \alpha < 1$ and $0 \leq \left\{\frac{t}{y}\right\} < 1$.■

Let's get back to the proof of the statement. If $n = 1$ then the statement claims, that $a_1 \geq 1$, which is obvious, because every element as a set is an anti chain. Now let assume that for any LHPOSET with $n-1$ relations the above statement is true and now it will be proved for any LHPOSET $(H, <_1, ..., <_n)$ that

$$\sum_{k=1}^{n} a_k \geq nt^{\frac{n-1}{n}}.$$

Let assume that the set of the first coordinates of the elements of H is $\{w_1, ..., w_r\}$ and let $A_{w_i} \subset H$ be the set of vectors, whose first coordinate is w_i $(i = 1, ..., r)$. Then A_{w_i} is an anti chain due to $<_1$. Let $y = a_1$, then y is the size of the biggest anti chain due to $<_1$, so $|A_{w_i}| \leq y$. Let $x_i = |A_{w_i}|$ and $\mathcal{H}_i = (A_{w_i}, <_2, ..., <_n)$, then this an Lexicographic Hyper POSET with $n-1$ relations, so if $B_{i,k}$ is the biggest anti chain due to $<_k$ in it $(k = 2, ..., n)$ and $|B_{i,k}| = b_{i,k}$, then by the induction

$$\sum_{k=1}^{n} b_{i,k} \geq (n-1)x_i^{\frac{n-2}{n-1}}.$$

Let's notice that for $k = 2, ..., n$

$$C_k = \bigcup_{i=1}^{r} B_{i,k}$$

is an anti chain due to $<_k$, because if $x \in B_{i,k}$ and $y \in B_{j,k}$ where $i \neq j$ then $x <_1 y$ or $y <_1 x$. So $a_k \geq |C_k|$ and now some calculations can be done:

$$\sum_{k=1}^{n} a_k \geq y + \sum_{k=2}^{n} |C_k| = y + \sum_{k=2}^{n} \left| \bigcup_{i=1}^{r} B_{i,k} \right| =$$

$$= y + \sum_{k=2}^{n}\sum_{i=1}^{r} b_{i,k} = y + \sum_{i=1}^{r}\sum_{k=2}^{n} b_{i,k} \geq y + \sum_{i=1}^{r}(n-1)x_i^{\frac{n-2}{n-1}}.$$

Now using the lemma with $0 \leq x_i \leq y$, $x_1 + ... + x_n = t$ and $\alpha = \frac{n-2}{n-1}$ the inequality

$$y + (n-1)\sum_{i=1}^{r} x_i^{\frac{n-2}{n-1}} \geq y + (n-1)ty^{-\frac{1}{n-1}}$$

holds. Apply the A-G inequality for the numbers y and $n-1$ pieces of $ty^{-\frac{1}{n-1}}$. It claims, that

$$y + (n-1)ty^{-\frac{1}{n-1}} \geq n\sqrt[n]{y(ty^{-\frac{1}{n-1}})^{n-1}} = nt^{\frac{n-1}{n}}$$

which is exactly that needed to be proven.
So it is proved that

$$\sum_{k=1}^{n} a_k \geq nt^{\frac{n-1}{n}}$$

and from that it easily follows that there exists an $1 \leq m \leq n$ such that $a_m \geq t^{\frac{n-1}{n}}$ which means that the biggest $<_m$ anti chain has size at least $t^{\frac{n-1}{n}}$.■

Theorem 10 *There exists an $H_0 \subset \mathbb{N}^n$ such that $|H_0| = t^n$ and the Lexicographic Hyper POSET $(H_0, <_1, ..., <_n)$ defined from H_0 has the property, that for $m = 1, ..., n$ the size of the biggest anti chain due to $<_m$ is at most t^{n-1}.*

Proof The construction is the same as in the theorem with the longest chain. Let $H_0 = \{(x_1, ..., x_n \big| x_i = 1, ..., t; i = 1, ..., n\}$ be the set. Let

$$C_{u_1,...,u_{m-1},u_{m+1},...,u_n} = \{(u_1, ..., u_{m-1}, s, u_{m+1}, ..., u_n \big| s = 1, ..., t\},$$

then $C_{u_1,...,u_{m-1},u_{m+1},...,u_n}$ is a $<_m$ chain and

$$\bigcup_{u_1=1}^{t} ... \bigcup_{u_{m-1}=1}^{t} \bigcup_{u_{m+1}=1}^{t} ... \bigcup_{u_n=1}^{t} C_{u_1,...,u_{m-1},u_{m+1},...,u_n} = H_0,$$

so H_0 is the union of t^{n-1} pieces of $<_m$ chains, which means that the biggest $<_m$ anti chain has a size at most t^{n-1}.■

Theorem 11 *Let $H \subset \mathbb{N}^n$ be a set such that $|H| < \binom{t+n-1}{n}$. Let $(H, <_1, ..., <_n)$ be the Lexicographic Hyper POSET defined from H. Then there exist $A_1, ..., A_{t-1} \subset H$, that for $i = 1, ..., t-1$ the set A_i is an anti chain due to one of the relations $<_1, ..., <_n$ and $H = \bigcup_{i=1}^{t-1} A_i$.*

Proof It will be proved by induction on n. If $n = 1$ the statement claims that if $|H| < t$ then there exists $<_1$ anti chains $A_1, ..., A_{t-1}$ that $H = \bigcup_{i=1}^{t-1} A_i$. But this is trivial, because $|H| \leq t-1$ and all points of H as a set are anti chains. Now let assume, that the statement is true for $1, ..., n-1$ ($n \geq 2$), it will be shown for n.
Now an induction on t will be used ($t \geq 2$). If $t = 2$ then it claims that if $|H| < n+1$ then H is an anti chain itself due to one of the relations $<_1, ..., <_n$. If H is not an $<_n$ anti chain, then there is $x, y \in H$ such that $x <_n y$, so x and y differ in only the last coordinate. Let $H' \subset \mathbb{N}^n$ be the set, whose elements are

the elements of H without the last coordinate. Then because x and y is the same without the last coordinate, $|H'| < n$ and $(H', <_1, ..., <_{n-1})$ is a Lexicographic Hyper POSET, so by the induction on n it comes that H' is an anti chain due one of the relations $<_1, ..., <_{n-1}$ and H is an anti chain due to the same relation. Let assume that the statement is true for $t-1$ ($t \geq 3$), it will be proved for t. Let M be the size of the biggest anti chain due to $<_n$. Then let's examine two cases: first case when $M \geq \binom{t+n-1}{n-1}$. Let A_1 be one of the biggest $<_n$ anti chains and $H' = H \setminus A_1$. Then

$$|H'| < \binom{t+n-1}{n} - \binom{t+n-2}{n-1} = \binom{t+n-2}{n}$$

and $(H', <_1, ..., <_n)$ is an LHPOSET so by the induction on t it is clear, that there exists $A_2, ..., A_{t-1}$ anti chains that $H' = \bigcup_{i=2}^{t-1} A_i$, so it is $H = \bigcup_{i=1}^{t-1} A_i$ and $A_1, ..., A_{t-1}$ are anti chains, so that case is proven.
Second case, when $M < \binom{t+n-2}{n-1}$. By the definition of LHPOSET's if $x, y \in H$ and $x \neq y$, then ($x <_n y$ or $y <_n x$)$\Leftrightarrow$(the vectors x and y differ in the first $n-1$ coordinate). So a $B \subset H$ is an $<_n$ anti chain if and only if there are no two vectors in B, that they are the same in the first $n-1$ coordinate. Let

$$H'' = \{(x_1, ..., x_{n-1}) \big| \exists y, (x_1, ..., x_{n-1}, y) \in H\}$$

then $|H''| = M$ because of the previous ideas. The relations $<_1, ..., <_{n-1}$ only depends on the first $n-1$ coordinates, so it can be said that $(H'', <_1, ..., <_{n-1})$ is an LHPOSET with $n-1$ relations. It is $|H''| = M < \binom{t+n-2}{n-1}$, so because of the induction on n it is clear, that H'' is de union of $t-1$ subsets $B_1, ..., B_{t-1}$, such that B_i is an anti chain due to one of the relations $<_1, ..., <_{n-1}$. Now for $i = 1, ..., t-1$ let

$$A_i = \{(x_1, ..., x_n) \big| (x_1, ..., x_{n-1}) \in B_i; (x_1, ..., x_n) \in H\}.$$

Then it's easy to check that if B_i was an anti chain due to $<_m$ ($1 \leq m \leq n-1$) then A_i is also an anti chain in $(H, <_1, ..., <_n)$ due to $<_m$, and because of $H'' = \bigcup_{i=1}^{t-1} B_i$ it is $H = \bigcup_{i=1}^{t-1} A_i$, so $A_1, ..., A_{t-1}$ satisfies the conditions. The proof is complete.■

Theorem 12 *There exists an $H_0 \subset \mathbb{N}^n$ that $|H_0| = \binom{t+n-1}{n}$ and if $(H_0, <_1, ..., <_n)$ is the Lexicographic Hyper POSET defined from H_0, and $A_1, ..., A_r$ are subsets of H_0 such that $\bigcup_{i=1}^{r} A_i = H_0$ and A_i is an anti chain due to one of the relations $<_1, ..., <_n$ then $r \geq t$.*

Proof Call the set H_0 (n, t)-ordered if

$$H_0 = \{(x_1, ..., x_n) \big| t \geq x_1 \geq ... \geq x_n \geq 1\}.$$

It will be proved by induction on n that the (n,t)-ordered set satisfies the conditions. If $n = 1$ then $H_0 = \{1, 2, ..., t\}$ so $|H_0| = t$ and H_0 is a totally ordered set, so every anti chain has maximum one elements. Because of that at least t anti chains needed to cover it. Now let assume, that the statement is true for $(n-1,u)$-ordered sets where $u = 1, 2...$, now it will be proved for n. Firstly, $|H_0| = \binom{n+t-1}{n}$, because $\binom{n+t-1}{n}$ is the number of n-tuples $(y_1, ..., y_n)$ such that $n + t - 1 > y_1 > ... > y_n \geq 1$ and the function $\varphi((y_1, ..., y_n)) = (y_1 - (n-1), y_2 - (n-2), ..., y_n)$ is a bijection between these n-tuples and H_0. Let $A_1, ..., A_r$ be anti chains such that $\bigcup_{i=1}^{r} A_i = H_0$ and let assume that exactly k of them is an $<_1$ anti chain. It can be assumed, that these are $A_1, ..., A_k$. For $y = 1, ..., t$ let

$$G_y = \{(y, x_2, ..., x_n) \big| y \geq x_2 \geq ... \geq x_n \geq 1\}$$

and

$$G'_y = \{(x_2, ..., x_n) \big| y \geq x_2 \geq ... \geq x_n \geq 1\}.$$

Then every $<_1$ anti chain is a subset of one of the $G_1, ..., G_t$, so if $A_1, ..., A_k$ are all of the $<_1$ anti chains, then there is at least $t-k$ indexes $i_1 < ... < i_{t-k}$ such that none of the elements of G_{i_j} $(j = 1, ..., t-k)$ are covered by any of $A_1, ..., A_k$. And then it must be $i_{t-k} \geq t-k$. So every element of $G_{i_{t-k}}$ is covered with one anti chain from $A_{k+1}, ..., A_r$. Now start examine $G'_{i_{t-k}}$. $(G'_{i_{t-k}}, <_2, ..., <_n)$ is an LHPOSET with $n-1$ relations which is $(n-1, i_{t-k})$-ordered. If

$$A'_s = \{(x_2, ..., x_n) \big| (i_{t-k}, x_2, ..., x_n) \in A_s\}$$

$(s = k+1, ..., r)$, then A'_s is anti chain in $G'_{i_{t-k}}$ with the same relation and $\bigcup_{i=k+1}^{r} A'_i = G'_{i_{t-k}}$ so by the induction on n it comes, that $r - k \geq i_{t-k} \geq t-k$, so $r \geq t$ and the proof is complete.■

Remark This construction shows, that if $n = 2$, then *Theorem 7* is strict for every $t \geq 2$ positive integer.

5 Geometric Hyper POSETs

Definition 6 *Let n, d be positive integers and H be a finite subset of $\mathbb{R}^d$. Let $D_1, ..., D_n$ be convex cones in $\mathbb{R}^d$, such that*

$$\left(\bigcup_{i=1}^{n} D_i\right) \cup \left(\bigcup_{i=1}^{n} -D_i\right) = \mathbb{R}^d.$$

Define the Hyper POSET $\mathfrak{H} = \mathfrak{H}(H, D_1, ..., D_n) = (H, <_1, ..., <_n)$ as follows: for $x, y \in H$, $x \neq y$ the relation $x <_i y$ holds $(i = 1, ..., n)$ if and only if $y - x \in D_i$. Lets call these type of Hyper POSETs Geometric Hyper POSET (GHPOSET), and lets call d the dimension of the Geometric Hyper POSET.

Statement 4 *The above definition is correct, so let $D_1, ..., D_n \subset \mathbb{R}^d$ be convex cones, such that*

$$\left(\bigcup_{i=1}^{n} D_i\right) \cup \left(\bigcup_{i=1}^{n} -D_i\right) = \mathbb{R}^d.$$

Define the structure $\mathfrak{H} = (H, <_1, ..., <_n)$, such that for $x, y \in H$, $x \neq y$ the relation $x <_i y$ holds $(i = 1, ..., n)$ if and only if $y - x \in D_i$. Then $\mathfrak{H}$ is a Hyper POSET.

Proof Firstly, it will be shown, that $<_i$ $(i = 1, ..., n)$ is transitive. Let assume, that for $x, y, z \in H$ it is $x <_i y$ and $y <_i z$. Then $y - x \in D_i$ and $z - y \in D_i$. But D_i is a convex cone, so it is closed for summation, so $D_i \ni (y - x) + (z - y) = z - x$, which means $x <_i z$.
Secondly, it will be shown, that if $x \neq y \in H$, then there exists $1 \leq i \leq n$, that $x <_i y$ or $y <_i x$. Because of the criteria

$$\left(\bigcup_{i=1}^{n} D_i\right) \cup \left(\bigcup_{i=1}^{n} -D_i\right) = \mathbb{R}^d$$

there exists at least one i, that $y - x \in D_i$ or $y - x \in -D_i$. If $y - x \in D_i$, then $x <_i y$ and if $y - x \in -D_i$, then $x - y \in D_i$, so $y <_i x$. This proves, that $\mathfrak{H}$ is a Hyper POSET.■

Definition 7 *Lets call the finite system $D_1, ..., D_n$ of convex cones covering, if $\left(\bigcup_{i=1}^{n} D_i\right) \cup \left(\bigcup_{i=1}^{n} -D_i\right) = \mathbb{R}^d$ and the intersection of any two different cones from $D_1, ..., D_n, -D_1, ..., -D_n$ is the origin. If $D_1, ..., D_n$ is a covering system, then lets call the Geometric Hyper POSET $\mathfrak{H}(H, D_1, ..., D_n)$ Strong Geometric Hyper POSET (SGHPOSET), .*

Remark Any Strong Geometric Hyper POSET is obviously a Strong Hyper POSET, but there are GHPOSETs, which are SHPOSETs, but not isomorph to a SGHPOSET.
For example let $0 = (0,0,0), x = (1,0,0), y = (0,1,0), z = (0,0,1)$ and let $D_1 = \{(a, -b, b) | a, b \geq 0\}$, $D_2 = \{(-a, a, b) | a, b \geq 0\}$, $D_3 = \{(-a, b, a) | a, b \geq 0\}$ and let $H = \{0, x, y, z\}$.
$\mathbb{R}^3 \setminus (D_1 \cup D_2 \cup D_3 \cup -D_1 \cup -D_2 \cup -D_3)$ is the union of a finite number of convex cones, let them be $E_1, ..., E_r$. Then

$$\mathfrak{H}(H, D_1, D_2, D_3, E_1, ..., E_r) = (H, <_1, <_2, <_3, \prec_1, ..., \prec_r)$$

is a GHPOSET and $0 <_1 x, y <_1 z, 0 <_2 z, y <_2 x, 0 <_3 y, x <_3 z$ are all the relations, so it is a Strong Hyper POSET.
But it is not isomorphic to a SGHPOSET, because if there is an isomorphism

$$f : \mathfrak{H}(H, D_1, D_2, D_3, E_1, ..., E_r) \to \mathfrak{H}(G, D_1', D_2', D_3', E_1', ..., E_r')$$

then $f(x) - f(0) \in D'_1, f(z) - f(y) \in D'_1$ so because D'_1 is a convex cone $f(x) - f(0) + f(z) - f(y) \in D'_1$. Similarly $f(z) - f(0) \in D'_2, f(x) - f(y) \in D'_2$, so $f(z) - f(0) + f(x) - f(y) \in D'_2$. So $f(x) + f(z) - f(0) - f(y) \in D'_1 \cap D'_2$, which is impossible, because it is not hard to show, that $f(x) + f(z) - f(0) - f(y) \neq 0$.

Statement 5 *Let $H_1, ..., H_k \subset \mathbb{R}^d$ and let $D_1, ..., D_n \subset \mathbb{R}^d$ be convex cones, such that the interior of $D_1, ..., D_n$ is not empty and $\mathfrak{H}_i = \mathfrak{H}(H_i, D_1, ..., D_n)$ is a Strong Geometric Hyper POSET ($i = 1, ..., k$). Lets suppose, that if $x, y \in H_i$ and $x <_j y$ ($1 \leq j \leq n$) then $y - x \in int\ D_j$. Then there exists an $H \subset \mathbb{R}^d$, such that*

$$\mathfrak{H}_1 \star ... \star \mathfrak{H}_k \simeq \mathfrak{H}(H, D_1, ..., D_n).$$

Proof The sets $H_1, ..., H_k$ are finite, so they are bounded, lets suppose that their union can be covered with a circle of radius R. For $i = 1, ..., k$ and $x, y \in D_i$ where $x \neq y$, if $x <_j y$ then $y - x \in int\ D_j$ which means that there exists a $0 < r_{i,x,y}$, that $B_{r_{i,x,y}}(y - x) \in D_j$ (where $B_r(x)$ is the open circle with center x and radius r). Let $r = \min_{i=1,...,k} (\min_{x \neq y \in D_i} r_{i,x,y})$. Finally let $t = \frac{R}{2kr}$.
Define $\phi : \mathbb{R}^2 \times ... \times \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbb{R}^2$ is multiplied k times, as follows:

$$\phi((x_1, ..., x_k)) = \sum_{i=1}^{k} x_i t^{k-i}.$$

It will be shown, that if $H = \phi(H_1 \times ... \times H_k)$ then ϕ extracts to a $\mathfrak{H}_1 \star ... \star \mathfrak{H}_k \to \mathfrak{H}(H, D_1, ..., D_n)$ isomorphism. Let $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$ be different elements of $H_1 \times ... \times H_k$ and let q be the smallest index, that $x_q \neq y_q$ and let suppose, that $x_q <_j y_q$. Then $(x_1, ..., x_k) <_j (y_1, ..., y_k)$. Now it has to be proved, that $\phi((x_1, ..., x_k)) <_j \phi(y_1, ..., y_k)$ which is equivalent to $\phi((y_1, ..., y_k)) - \phi((x_1, ..., x_k)) \in D_j$.

$$\phi((y_1, ..., y_k)) - \phi((x_1, ..., x_k)) \in D_j = \sum_{i=1}^{k} y_i t^{k-i} - \sum_{i=1}^{k} x_i t^{k-i} =$$

$$= \sum_{i=q}^{k} (y_i - x_i) t^{k-i} = t^{k-q} \left(y_q - x_q + \sum_{i=q+1}^{k} (y_i - x_i) t^{q-i} \right)$$

Here

$$\left\| \sum_{i=q+1}^{k} (y_i - x_i) t^{q-i} \right\| \leq \sum_{i=q+1}^{k} \|y_i - x_i\| t^{q-i} \leq \frac{1}{t} \sum_{i=q+1}^{k} \|y_i - x_i\|$$

and $\|y_i - x_i\| < 2R$ because $H_1, ..., H_k$ can be covered with circle with radius R, so

$$\frac{1}{t} \sum_{i=q+1}^{k} \|y_i - x_i\| < \frac{1}{t} k 2R = r.$$

So

$$\left\| \sum_{i=q+1}^{k} (y_i - x_i) t^{q-i} \right\| < r$$

which means that

$$\phi((y_1, ..., y_k)) - \phi((x_1, ..., x_k)) \in t^{k-q} B_r(y_q - x_q).$$

But $y_q - x_q \in D_j$ and by the definition of r it is true, that $B_r(y_q - x_q) \subset D_j$, so $t^{k-q} B_r(y_q - x_q) \subset D_j$ and thus $\phi((y_1, ..., y_k)) - \phi((x_1, ..., x_k)) \in D_j$. So $\phi((x_1, ..., x_k)) <_j \phi(y_1, ..., y_k)$ which proves, that ϕ can be extracted to an isomorphism. The proof is complete. ■

Theorem 13 *Let $\mathfrak{L}$ be a Lexicographic Hyper POSET with n relations and $D_1, ..., D_n$ be a covering system in the plane, that the interior of D_i is not empty ($i = 1, ..., n$). Then there exists an $H \subset \mathbb{R}^2$ that for the Strong Geometric Hyper POSET $\mathfrak{G} = \mathfrak{H}(H, D_1, ..., D_n)$ the Hyper POSETs $\mathfrak{L}$ and $\mathfrak{G}$ are isomorph.*

Proof For $i = 1, ..., n$ let $v_i \in int(D_i)$ any vector. Let $\mathfrak{L} = (L, <_1, ..., <_n)$, where $L \subset \mathbb{Z}^n$ and for $x \in L$ let pr_i be the i'th coordinate of x. Let $H_i = \{pr_i(x) v_i | x \in L\}$ and $\mathfrak{H}_i = (H_i, D_1, ..., D_n)$. Than $\mathfrak{H}_i$ is a $<_i$ chain and it satisfies, that if $x, y \in H_i$ and $x <_i y$ then $y - x \in int(D_i)$. Let $\phi : L \to H_1 \times ... \times H_n$ be the injection, that $\phi((x_1, ..., x_n)) = (x_1 v_1, ..., x_n v_n)$ and let $Im(\phi) = G^*$. Than $\phi : \mathfrak{L} \to (G^*, D_1, ..., D_n)$ is an isomorphism.
Let $\mathfrak{H}^* = \mathfrak{H}_1 \star ... \star \mathfrak{H}_n$. Then by the previous statement, there exists $\mathfrak{H} = \mathfrak{H}(H, D_1, ..., D_n)$ that $\mathfrak{H} \simeq \mathfrak{H}^*$. Let $\psi : \mathfrak{H}^* \to \mathfrak{H}$ be an isomorphism and $G = \psi(G^*)$, $\mathfrak{G} = \mathfrak{H}(G, D_1, ..., D_n)$. Then

$$(\psi |_{G^*}) \circ \phi$$

is an isomorphism between $\mathfrak{L}$ and $\mathfrak{G}$. ■

Theorem 14 *Let $D_1, ..., D_n$ be convex cones in $\mathbb{R}^2$, such that the interior of D_i is not empty ($i = 1, ..., n$), $\left(\bigcup_{i=1}^{n} D_i \right) \cup \left(\bigcup_{i=1}^{n} -D_i \right) = \mathbb{R}^2$ and $D_i \cap D_j = D_i \cap D_j' = \{0\}$ for any $1 \leq i, j \leq n$, $i \neq j$. Then there exists a constant C, that for infinitely many t positive integers there exists a set $H \subset \mathbb{R}^2$, that $|H| = t$ and the biggest anti chain in the SGPOSET $\mathfrak{H}(H, D_1, ..., D_n) = (H, <_1, ..., <_n)$ due to any of the relations is smaller than $C\sqrt{t}$.*

Proof For any s positive integer let $H_s = \{(a, b) | a = 1, ..., s; b = 1, ..., s\}$. Then it will be shown, that there exists a constant C (dependant on $D_1, ..., D_n$), that in $\mathfrak{H}(H_s, D_1, ..., D_n)$ the biggest anti chain due to any of the relations is smaller than Cs. Because of $|H_s| = s^2$, it proves the theorem for $t = s^2$.
For $i = 1, ..., n$ the interior of D_i in not empty, so there exists a vector in D_i, whose both coordinates are rational numbers, let it be $(\frac{a_i}{b_i}, \frac{c_i}{d_i})$, where a_i, c_i are integers and b_i, d_i are positive integers. But D_i is a cone, so $b_i d_i (\frac{a_i}{b_i}, \frac{c_i}{d_i}) =$

$(a_i d_i, b_i c_i) \in D_i$. Let $(p_i, q_i) = (a_i d_i, b_i d_i)$, then p_i, q_i are integers and $(p_i, q_i) \in D_i)$. It will be shown, that

$$C = \max_{1 \le i \le n} |p_i| + |q_i|$$

satisfies the conditions.
It will be proved, that the biggest anti chain due to the relation $<_i$ is smaller then $s(|p_i| + |q_i|)$. It can be assumed, that $p_i > 0$ and $q_i > 0$, the other four cases can be handled the same way. For any u, v integers let

$$A(u, v) = \{(u + ap_i, v + aq_i) \big| a \in \mathbb{Z}\}$$

then $A(u, v)$ is a $<_i$ chain, because if $x, y \in A(u, v)$, then $y - x$ is a multiple of $(p_i, q_i) \in D_i$. Plus

$$\left(\bigcup_{j=1}^{s} \bigcup_{k=1}^{q_i} A_{j,k} \right) \cup \left(\bigcup_{j'=1}^{p_i} \bigcup_{k'=1}^{s} A_{j',k'} \right) \supset H_s.$$

It is true, because if $(x, y) \in H_s$, then let j be the biggest integer, that $(x - jp_i, y - jq_i) \in H_s$. Then $x - jp_i \le p_i$ or $y - jq_i \le q_i$. In the first case $(x, y) \in \bigcup_{j'=1}^{p_i} \bigcup_{k'=1}^{s} A_{j',k'}$ and in the second case $(x, y) \in \bigcup_{j=1}^{s} \bigcup_{k=1}^{q_i} A_{j,k}$.
So H_s is the union of $s(p_i + q_i)$ pieces of $<_i$ chains, so the biggest anti chain due to $<_i$ is smaller than $s(p_i + q_i)$. The proof is complete.■

Remark If given the integer n, the most natural case is when

$$D_i = \left\{ (r \cos \alpha, r \sin \alpha) \big| r \ge 0; \frac{(i-1)\pi}{n} \le \alpha < \frac{i\pi}{n} \right\}.$$

Let C_n be the inf of the constants, which satisfy the conditions of the upper theorem for these $D_1, ..., D_n$. It might be a hard question to determine C_n. In every SHPOSET there is an antichain with $\sqrt{t}$ elements, if the basis set has t elements, so $C_n \ge 1$ for $n = 2, 3....$ For $n = 2$ it is $C_2 = 1$ and the $s \times s$ square lattice is a good construction.

For $n > 2$ it is $C_n \le n$. If we follow the proof of the upper theorem, it is enough to find $(p_i, q_i) \in D_i$, that p_i, q_i are integers, and $|p_i| + |q_i| \le n$. Let $z_j = (j, n-j)$, $z_{j+n} = (n-j, -j)$, $z_{j+2n} = (-j, j-n)$ and $z_{j+3n} = (j-n, j)$ for $j = 0, ..., n-1$. Then $z_0, ..., z_{4n-1}$ are all the vectors with integer coordinates, where the sum of the absolute value of the coordinates is n. Let β_j be the angle of the vectors z_j and z_{j+1} (β_{4n-1} is the angle of z_{4n-1} and z_0). Then $\beta_j = \beta_{2n-j} = \beta_{j+2n} = \beta_{4n-j}$ and if $0 < j < n-1$ then it can be calculated easily that

$$\sin \beta_j = \frac{n}{\sqrt{j^2 + (n-j)^2}\sqrt{(j+1)^2 + (n-j-1)^2}} \le \frac{n}{\frac{n}{\sqrt{2}} \frac{n}{\sqrt{2}}} = \frac{2}{n}.$$

If $n \geq 3$ then $\sin \frac{\pi}{n} > \frac{2}{n}$, which means that $\sin \beta_j < \sin \frac{\pi}{n}$, so $\beta_j \leq \frac{\pi}{n}$. That means, that for $j = 0, ..., 4n-1$ it is $\beta_j < \frac{\pi}{n}$, so for any given i one of the vectors of $z_0, ..., z_{4n-1}$ is an element of D_i, so there is a vector in D_i, where the sum of the absolute values of the coordinates is exactly n. This proofs the statement.

For $n = 3$ I will prove strict result for the problem above:

Theorem 15 *For* $i = 1, 2, 3$ *let* $D_i = \left\{(r \cos \alpha, r \sin \alpha) \middle| r \geq 0; \frac{(i-1)\pi}{3} \leq \alpha < \frac{i\pi}{3}\right\}$ *and let* $\mathfrak{H}(H, D_1, D_2, D_3) = (H, <_1, <_2, <_3)$*, where* $H \subset \mathbb{R}^2$ *and* $|H| > 3s^2 - 3s + 1$*, where* s *is a positive integer. Then there exists an* $A \subset H$ *such that* $|A| > 2s - 1$ *and* A *is an anti chain due to one of the relations* $<_1, <_2, <_3$*.*

Proof Firstly, I will define how a chain can be extracted to a broken line in $\mathbb{R}^2$ and some of its features will be studied. Let $D \subset \mathbb{R}^2$ be a convex cone and v the direction of the bisector of D (D is always an angle, so v is well definied). For $x, y \in \mathbb{R}^2$ let $x \prec y$ if $y - x \in D$. Let $C = \{x_1, ..., x_m\}$ be a chain due to $\prec$ such that $x_1 \prec ... \prec x_m$. Define the broken line $L(C)$ an follows: connect x_j and x_{j+1} with a segment if $j = 1, ..., m-1$ and draw a half line from x_1 to the direction of $-v$ and a half line from x_m to the direction v (L is dependent on D, but for simplicity, it will not be marked, and it will not cause any confusion). Then it is easy to see, that $L(C)$ is a $<_i$ chain too. Let $x, y \in L(C)$. If x and y are on the same segment, then $x - y$ is parallel to one of the vectors v or $x_{j+1} - x_j$, which are all in $D \cup (-D)$ so $x \prec y$ or $y \prec x$. If x and y are in different segments, let suppose, that $x \in [x_j, x_{j+1}]$, $y \in [x_l, x_{l+1}]$. If $j < l$ then $x \prec x_{j+1} \prec x_l \prec y$ and if $l > j$, then it is $y \prec x$. The cases where at least one of x or y is on an infinite segment can be proved similarly.

Lemma 6 *Let* $C_1, ..., C_k$ *be finite* $\prec$ *chains. Then there exists* $C'_1, ..., C'_k$ *chains, that* $\bigcup_{j=1}^{k} C_j = \bigcup_{j=1}^{k} C'_j$ *and the broken lines* $L(C_1), ..., L(C_k)$ *are pairwise disjoint.*

Proof of lemma It can be assumed, that $C_1, ..., C_k$ are pairwise disjoint, else some points can be left out from each C_j without changing the union. Let S be a square with two sides parallel to v and which covers $\bigcup_{j=1}^{k} C_j$ and define $L_0(C) = L(C) \cap S$ for any C chains. Then $L_0(C)$ have a finite length, let $l(C)$ be the length of $L_0(C)$. Plus $L(C_1) \setminus L_0(C_1), ..., L(C_k) \setminus L_0(C_k)$ are unions of parallel half lines, so every intersection of $L(C_1), ..., L(C_k)$ are inside of S.
Lets suppose that for some a and b the broken lines $L_0(C_a)$ and $L_0(C_b)$ have intersection. Let $x_1 \prec ... \prec x_m$ be the points of C_a and $y_1, ..., y_n$ be the points of C_b. If $[x_j, x_{j+1}]$ and $[y_l, y_{l+1}]$ intersect, then let $\{z\} = [x_j, x_{j+1}] \cap [y_l, y_{l+1}]$ and let $C_a^* = \{x_1, ..., x_j, y_{l+1}, ..., y_n\}$ and $C_b^* = \{y_1, ..., y_l, x_{j+1}, ..., x_m\}$. Then C_a^* and C_b^* are $\prec$ chains, because $y_l \prec z$ and $z \prec x_{j+1}$ so $y_l \prec x_{j+1}$ and for the

same reason $x_j \prec y_{l+1}$. Plus $l(C_a) + l(C_b) > l(C_a^*) + l(C_b^*)$ because

$$l(C_a)+l(C_b)-(l(C_a^*)+l(C_b^*)) = |x_{j+1}-x_j|+|y_{l+1}-y_l|-|x_j-y_{l+1}|-|y_l-x_{j+1}| =$$

$$= |x_{j+1} - z| + |z - x_j| + |y_{l+1} - z| + |z - y_l| - |x_j - y_{l+1}| - |y_l - x_{j+1}| =$$

$$= (|x_{j+1} - z| + |z - y_l| - |y_l - x_{j+1}|) + (|z - x_j| + |y_{l+1} - z| - |x_j - y_{l+1}|) > 0$$

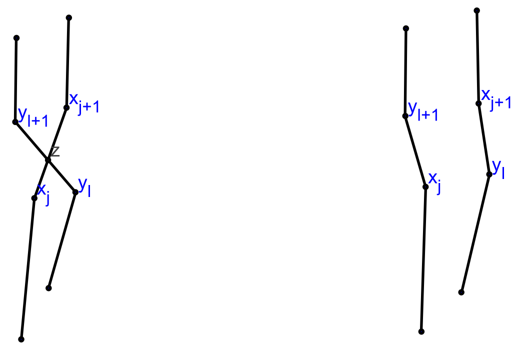
where the last inequality holds because of the triangle inequality. If $[x_j, x_{j+1}]$ intersects with the half line from y_1, then let there intersection be z and let $C_a^* = \{x_1, ..., x_j, y_1, ..., y_n\}$ and $C_b^* = \{x_{j+1}, ..., x_m\}$. Then C_a^* and C_b^* are also $<_i$ chains and $l(C_a) + l(C_b) > l(C_a^*) + l(C_b^*)$. It is true, because let d_1 be the distance from the side of S, which intersects with the half line from y_1. Let d_2 be the distance from the same side of S and d_3 be the distance of z from that side. Then

$$l(C_a) + l(C_b) - l(C_a^*) - l(C_b^*) =$$

$$= |x_j - x_{j+1}| + d_1 - d_2 - |x_j - y_1| = |x_j - z| + |z - x_{j+1}| + d_3 + |y_1 - z| - d_2 - |x_j - y_1| =$$

$$= (|x_j - z| + |z - y_1| - |x_j - y_1|) + |z - x_{j+1}| + d_3 - d_2 > |z - x_{j+1}| + d_3 - d_2 > 0$$

where the last inequality holds because $d_3 - d_2$ is the length of the perpendicular projection of $z - x_{j+1}$ to the vector v. If $L_0(C_a)$ and $L_0(C_b)$ intersects in other way, it can be handled as this last case.

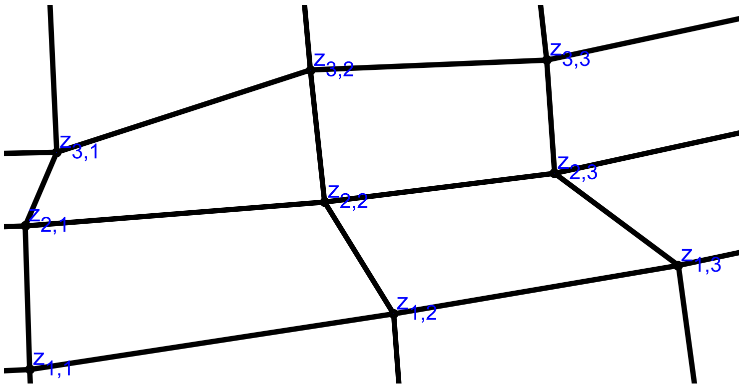


So if $L_(C_a)$ and $L_(C_b)$ intersects, then C_a and C_b can be replaced with C_a^* and C_b^*, such that $C_a \cup C_b = C_a^* \cup C_b^*$ and $l(C_a) + l(C_b) > l(C_a^*) + l(C_b^*)$. Repeating this procedure we will arrive to a state $C_1', ..., C_k'$, that $L_0(C_1'), ..., L_0(C_k')$ are disjoint, because there are only finite ways to select k chains from the finite set $\bigcup_{j=1}^{k} C_j$, and in every step, the sum $\sum_{j=1}^{k} l(C_k')$ is strictly decreasing. So the procedure have to stop after N steps, where N is the number of possible ways, to select k chains from $\bigcup_{j=1}^{k} C_j$.■

Let assume indirectly, that for $i = 1, 2, 3$ the biggest anti chain due to $<_i$ is at most $2s - 1$. Then by the Dilworth theorem H is the union of $2s - 1$

pieces of $<_i$ chains. Let $T_1, ..., T_{2s-1}$ be $<_1$ chains, such that $\bigcup_{j=1}^{2s-1} T_j = H$ and let $U_1, ..., U_{2s-1}$ be $<_2$ chains, that $\bigcup_{j=1}^{2s-1} U_j = H$. Then by the previous lemma it can be assumed, that $L(T_1), ..., L(T_{2s-1})$ are pairwise disjoint and $L(U_1), ..., L(U_{2s-1})$ are pairwise disjoint.

Let v_1, v_2 be the bisector of D_1 and D_2. For $j = 1, ..., n$ the broken line $L(T_j)$ divides the plane into two parts, let them be $L_+(T_j)$ and $L_-(T_j)$ determined by the following: let $x \in \mathbb{R}^2 \setminus L(T_j)$ and $X = \{x\} - L(T_j)$. Then $X - X = L(T_j) - L(T_j) \subset D_1 \cup -D_1$. If there exists $y \in D_2 \cup D_3$ and $z \in -D_2 \cup D_3$, then $y - z \in D_2 \cup D_3$, because $D_2 \cup D_3$ is also a convex cone. But $y - z \in X - X \subset D_1 \cup -D_1$, which is disjoint from $D_2 \cup D_3$, so it is impossible. That means, that $X \cap D_2 \cup D_3$ or $X \cap -D_2 \cup -D_3$ is empty, if the first one is empty, then $x \in L_-(T_j)$, else $x \in L_+(T_j)$. The broken lines $L(T_1), ..., L(T_n)$ are pairwise disjoint, so if $j \neq l$ then $L_+(T_j) \subset L_+(T_l)$ or $L_+(T_l) \subset L_+(T_j)$. Without the loss of generality it can be assumed, that $L_+(T_1) \subset ... \subset L_+(T_{2s-1})$. Similarly, it can be assumed, that $L_+(U_1) \subset ... \subset L_+(U_{2s-1})$ where $L_+(U_j)$ and $L_-(U_j)$ is defined by $-D_3$ and D_1 instead of D_2 and D_3.



For every $1 \leq j, l \leq 2s - 1$ the broken lines $L(T_j)$ and $L(U_l)$ intersect, let their intersection be $z_{j,l}$, It is obvious, that $H \subset \{(z_{j,l} | j, l = 1, ..., 2s - 1\}$. Then it will be proved, that if $1 \leq j < l \leq 2s - 1$ and $1 \leq k \leq 2s - 1$, then $z_{j,k} <_2 z_{l,k}$. It is true, because $z_{j,k}, z_{l,k} \in L(U_k)$, so $z_{j,k} <_2 z_{l,k}$ or $z_{l,k} <_2 z_{j,k}$, but $z_{l,k} \in L(T_l) \subset L_+(T_j)$, so $z_{l,k} - z_{j,k} \in D_1 \cup -D_1 \cup D_2 \cup D_3$, which means, that $z_{j,k} <_2 z_{l,k}$. Similarly, it is true, that $z_{k,j} <_1 z_{k,l}$.

If for some $1 \leq a, b, c, d \leq 2s - 1$ it is $z_{a,b} <_3 z_{c,d}$, then $a < c$, because if $a = c$ then $z_{a,b} <_1 z_{c,d}$ or $z_{c,d} <_1 z_{a,b}$ and if $a > c$, then $z_{a,b} \in L(T_a) \in L_+(T_c)$, so $z_{a,b} - z_{c,d} \in D_1 \cup -D_1 \cup D_2 \cup D_3$, which means, that $z_{c,d} - z_{a,b} \notin D_3$. Similarly, it can be proved, that $d < b$, because if $b = d$ then $z_{a,b} <_2 z_{c,d}$ or $z_{c,d} <_2 z_{a,b}$ and if $b < d$, then $z_{c,d} \in L(U_d) \in L_+(U_b)$, so $z_{c,d} - z_{a,b} \in D_2 \cup -D_2 \cup D_1 \cup -D_3$, which means, that $z_{c,d} - z_{a,b} \notin D_3$.

Lemma 7 *Let n be a positive integer and $S = \{(a,b)\big|a,b = 1,...,n\}$. Define the relation $\prec$ on S such that $(a,b) \prec (c,d)$ if $a < c$ and $d < b$. Then clearly $(S,\prec)$ is a POSET. Let $C_1,...,C_k$ be $\prec$ chains ($k \in \mathbb{N}$). Then*

$$\left|\bigcup_{i=1}^{k} C_i\right| \leq kn - \frac{k^2-1}{4}$$

if k is odd, and if k is even, then

$$\left|\bigcup_{i=1}^{k} C_i\right| \leq kn - \frac{k^2}{4}.$$

First proof of lemma Let $D = \{(a,b) \in mathbbR^2|a > 0, b < 0\}$. Then D is a convex cone, and for $\mathbf{x},\mathbf{y} \in S$ it is $\mathbf{x} \prec \mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x} \in D$. So if C_i is a $\prec$ chain $L(C_i)$ can be defined with the help of D. After that it can be assumed by *Lemma 6* that $L(C_1),...,L(C_k)$ are disjoint. Let

$$\partial S = \{(a,b) \in S\big|a = 1 \vee b = 1 \vee a = n \vee b = n\}.$$

The bisector of D is $(1,-1)$, so by the definition of L the broken line $L(C_i)$ intersects ∂S at two points, one of them is on the left or upper sides of S, let this be $\mathbf{x}_i = (a_i,b_i)$ (then $a_i = 1$ or $b_i = n$), the other is on the top or right side, let that be $\mathbf{y}_i = (c_i,d_i)$ (then $c_i = n$ or $d_i = 1$). Then $|C_i| \leq \min\{c_i - a_i + 1, b_i - d_i + 1\} \leq \min\{n + 1 - a_i, d_i\}$, because the first coordinates of the elements of C_i forms a strictly increasing series of integers, and the second coordinates form a strictly monotone decreasing series of integers. So

$$\left|\bigcup_{i=1}^{k} C_i\right| \leq \sum_{i=1}^{k} |C_i| \leq \sum_{i=1}^{k} \min\{n+1-a_i, d_i\}.$$

But because of $L(C_1),...,L(C_n)$ are pairwise disjoint, the points $\mathbf{x}_1,...,\mathbf{x}_n$ are pairwise different, so

$$\sum_{i=1}^{k} \min\{n+1-a_i, d_i\}$$

takes its maximum, if we choose the most points possible from the top right corner of ∂S. More preciously, it takes its maximum if

$$\{\mathbf{x}_1,...,\mathbf{x}_n\} =$$

$$= \left\{(1,n),(1,n-1),...,\left(1,n-\left\lfloor\frac{k-1}{2}\right\rfloor\right),(2,n),(3,n),...,\left(\left\lceil\frac{k-1}{2}\right\rceil,n\right)\right\}$$

so

$$\sum_{i=1}^{k} \min\{n+1-a_i, d_i\} \leq$$

$$\leq n+(n-1)+...+\left(n-\left\lfloor\frac{k-1}{2}\right\rfloor\right)+(n-1)+(n-2)+...+\left(n-\left\lceil\frac{k-1}{2}\right\rceil\right).$$

It is easy to check, that the right side is equal to the formula given in the Lemma.■

Second proof of lemma For $i = 1, ..., n$ let

$$A_i = \{(n+1-i, j) \big| j = 1, ..., i\} \cup \{(n+1-j, i) \big| j = 1, ..., i\}.$$

Then $|A_i| = 2i - 1$ and the disjoint union of $A_1, ..., A_n$ is S. Furthermore A_i is a $\prec$ anti chain, which means, that for $l = 1, ..., k$ the intersection of C_l and A_i contains maximum one point. That means, that

$$\left|A_i \cap \bigcup_{l=1}^{k} C_l\right| \leq \min\{|A_i|, k\} = min\{2i-1, k\}.$$

But then

$$\left|\bigcup_{l=1}^{k} C_l\right| = \left|\bigcup_{i=1}^{n} A_i \cap \bigcup_{l=1}^{k} C_l\right| \leq$$

$$\leq \sum_{i=1}^{n} \left|A_i \cap \bigcup_{l=1}^{k} C_l\right| \leq \sum_{i=1}^{n} \min\{2i-1, k\}.$$

If k is odd, then

$$\sum_{i=1}^{n} \min\{2i-1, k\} = 1 + 3 + ... + k - 2 + k + k\left(n - \frac{k+1}{2}\right) =$$

$$= \left(\frac{k+1}{2}\right)^2 + k\left(n - \frac{k+1}{2}\right) = kn - \frac{k^2-1}{4},$$

and if k is even, then

$$\sum_{i=1}^{n} \min\{2i-1, k\} = 1+3+...+k-1+k\left(n-\frac{k}{2}\right) = \left(\frac{k}{2}\right)^2 + k\left(n-\frac{k}{2}\right) = kn - \frac{k^2}{4}.$$

This proves the lemma. ■

Let $C_1, ..., C_{2s-1}$ be $<_3$ chains, whose union is H. Define the relation $\prec$ on $\{z_{j,l}\}$ as follows: $z_{a,b} \prec z_{c,d}$ if $a < b$ and $d < c$. Then $<_3 \subset \prec$ (which means $x <_3 y \Rightarrow x \prec y$), so $C_1, ..., C_{2s-1}$ are $\prec$ chains too. Let $S = \{(a,b) \big| a, b = 1, ..., 2s-1\}$ and $\phi : H \to S$ be the function, that $\phi(z_{j,l}) = (j, l)$ if $z_{j,l} \in H$. Then

$$\phi : (H, \prec) \to (\phi(H), \prec)$$

is an isomorphism, where $(S, \prec)$ is defined as in the lemma above. Applying the lemma for $n = 2s-1$,$k = 2s-1$ and $\phi(C_1), ..., \phi(C_{2s-1})$ it is

$$\left|\bigcup_{i=1}^{2s-1} \phi(C_i)\right| \leq (2s-1)(2s-1) - \frac{(2s-1)^2-1}{4} = 3s^2 - 3s + 1 < |H|,$$

so $C_1, ..., C_{2s-1}$ cannot cover H, which is a contradiction. So the theorem is proven.■

Now I will show a construction, which proves, that the previous theorem is strict.

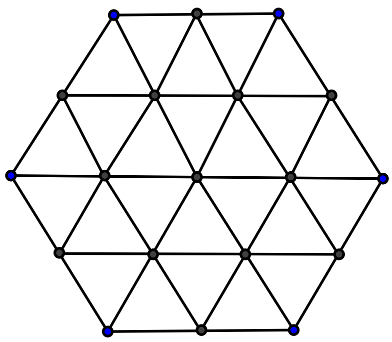
Theorem 16 *Let s be a positive integer and for $i = 1, 2, 3$ let*

$$D_i = \left\{(r\cos\alpha, r\sin\alpha)\middle| r \geq 0; \frac{(i-1)\pi}{3} \leq \alpha < \frac{i\pi}{3}\right\}.$$

Then there exists an $H \subset \mathbb{R}^2$, that $|H| = 3s^2 - 3s + 1$ and in $\mathfrak{H}(H, D_1, D_2, D_3) = (H, <_1, <_2, <_3)$ the biggest anti chain due to any of the relations has at most $2s-1$ elements.

Proof Let $\mathbf{v} = \left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) \in D_1$ and $\mathbf{w} = \left(\cos\frac{5\pi}{6}, \sin\frac{5\pi}{6}\right) \in D_3$, then $\mathbf{v} + \mathbf{w} \in D_2$. Let

$$H = \{a\mathbf{v} + b\mathbf{w} \middle| a, b \in \{0, ..., 2s-2\} \vee |a-b| \leq s-1\}.$$



It will be proved, that H satisfies the condition. Firstly, the size of H will be determined. $\mathbf{v}$ and $\mathbf{w}$ are independent, so if $a\mathbf{v} + b\mathbf{w} = c\mathbf{v} + d\mathbf{w}$, then $a = c, b = d$. If a is given, then there are $\min\{a, 2s-2-a\}$ pieces of (a, b) pairs, such that $b \in \{0, ..., 2s-2\}$ and $|a-b| \leq s-1$, so $|H| = (s-1) + s + ... + (2s-2) + (2s-3) + ... + (s-1) = 3s^2 - 3s + 1$.
Secondly, it will be proved, that the biggest anti chain due to $<_1, <_2, <_3$ has at most $2s-1$ elements. For that, it is enough to prove, that H is the union of $2s-1$ pieces of $<_i$ chains $(i = 1, 2, 3)$. If $i = 1$, then for $j = 1, ..., 2s-1$ let

$$A_j = \{a\mathbf{v} + (j-1)\mathbf{w} \middle| a \in \{0, ..., 2s-2\} \vee |a-j+1| \leq s-1\}.$$

Then the difference of any two elements of A_j is a multiple of $\mathbf{v}$, so A_j is a $<_1$ chain and clearly H is the union of $A_1, ..., A_{2s-1}$. If $i = 3$, then for similar reasons the $<_3$ chains

$$C_j = \{(j-1)\mathbf{v} + b\mathbf{w} \big| b \in \{0, ..., 2s-2\} \vee |b - j + 1| \leq s - 1\}$$

prove, that the biggest $<_3$ chain has maximum $2s - 1$ elements. If $i = 2$ then let

$$B_j = \{(a + j - s)\mathbf{v} + a\mathbf{w} \big| a \in \{0, ..., 2s-2\} \vee a + j - s \in \{0, ..., 2s-2\}\}.$$

Then the difference of any two element in B_j is the multiple of $\mathbf{v} + \mathbf{w}$, so B_j is a $<_2$ chain. Plus if $\mathbf{x} = a\mathbf{v} + b\mathbf{w} \in H$, then for $j = a - b + s$ it is $1 \leq j \leq 2s - 1$, because of $|a - b| \leq s - 1$ and so $\mathbf{x} \in B_j$, which means, that $\bigcup_{j=1}^{2s-1} B_j = H$. So the construction of H satisfies the condition.■

6 Applications

There is not known polynomial algorithm for finding the biggest clique or the biggest empty set in an ordinary graph yet. But there is polynomial algorithm for finding the biggest chain and the biggest anti chain in a partially ordered set. If we look at the graph of a partially ordered set (the graph, whose vertices are the points of the POSET and there is an edge between two vertices if and only if the two points can be compared), cliques are equivalent to chains and empty sets are equivalent to anti chains. So if we could order a graph's edges, that we get a POSET, it will be easy to find the biggest cliques and anti chains. Unluckily, the next theorem will show, that it is very unlikely, and there are graphs, which are not the union of "few" POSETs.

Definition 8 *Let's call a simple graph G POSET graph, if there exists a partially ordering of the vertices of G, $(V(G), <)$, such that if $x, y \in V(G)$, then*

$$(x < y) \vee (y < x) \Leftrightarrow \{x, y\} \in E(G).$$

Let's call such an ordering of $V(G)$ good.

Theorem 17 *Let k be a given positive integer. Then there exists a simple graph G, such that G is not the union of k POSET graphs, so there are no k POSET graphs $P_1, ..., P_k$, that $V(P_i) = V(G)$ $(i = 1, ..., k)$ and $\bigcup_{i=1}^{k} E(P_i) = E(G)$.*

Proof Due to Erdős and Szekeres[5] there exists a G_n graph with at least $(1 + o(1))\frac{n}{e\sqrt{2}}2^{\frac{n}{2}}$ vertices, such that nor the graph, and nor its complement contain a clique with n vertices. We will use, that $(1 + o(1))\frac{n}{e\sqrt{2}}2^{\frac{n}{2}} > 2^{\frac{n}{2}}$ if

$n > N$ for some N. It will be shown, that if n is big, then G_n or $\bar{G}_n$ is not the union of k POSET graphs.
Let assume indirectly, that both of them is the union of k POSET graphs. Let be $<_1, ..., <_k$ good orderings for the k POSET graphs, whose union is G_n, and let be $<_{k+1}, ..., <_{2k}$ good orderings for the POSET graphs covering $\overline{G}_n$. Then $(V(G_n), <_1, ..., <_{2k})$ is a Hyper POSET, because if $x, y \in V(G_n)$ and $\{x, y\} \in E(G_n)$, then there exists $1 \leq i \leq k$, that $x <_i y$ or $y <_i x$ and if $\{x, y\} \in E(\overline{G}_n)$, then there exists $k+1 \leq j \leq 2k$, that $x <_j y$ or $y <_j x$. Applying *Theorem 1* on $(V(G_n), <_1, ..., <_{2k})$, there is a chain with at least

$$\sqrt[2k]{|V(G_n)|} > \sqrt[2k]{2^{\frac{n}{2}}} = 2^{\frac{n}{4k}}$$

elements due to one of the relations $<_l$ $(1 \leq l \leq 2k)$. Let's choose n, such that $\frac{n}{4\log_2 n} > k$ (such an n always exists), then

$$2^{\frac{n}{4k}} > n$$

so there is a $<_l$ chain with at least n elements. But if $l \leq k$, then a $<_l$ chain is a clique in G_n, and if $l > k$, then it is a clique in $\overline{G}_n$, but due to the definition of G_n every clique in G_n and $\overline{G}_n$ has a size less then n, which is a contradiction. So if $\frac{n}{4\log_2 n} > k$, then G_n or its complement is not the union of k POSET graphs.■

It is a well known problem, that for any d there exists an N_d, such that in d dimensional space if a set has N_d points, then it contains an obtuse angle. This problem has been already solved and the fact, that the smallest such N_d is 2^d was proved by Ludwig Danzer and Branko Grünbaum[4].
Without giving the smallest possible limit, I will prove the following generalization of this problem:

Theorem 18 *If $d, n \in \mathbb{N}$, $d \geq 2$,$n \geq 3$ and $\alpha \in \mathbb{R}^+$ is given, then there exists an $N_{d,n,\alpha}$, such that if $H \subset \mathbb{R}^d$ and $|H| > N_{d,n,\alpha}$ then there exists a subset G of H, that $|G| = n$ and every triangle whose vertices are from G has an angle greater then $\pi - \alpha$.*

Remark If $n = 3$ and $\alpha = \frac{\pi}{2}$, then it is the same problem as above. If $\alpha = \frac{\pi}{2}$ then it states, that every enough big set in $\mathbb{R}$ contains an n element subset, that every three points in that determine an obtuse angle.

Proof Let B be the unit sphere with center 0 in $\mathbb{R}$. Let $s = 2sin\frac{\alpha}{4}$. Because B is compact, there is a finite s-net in B, let the points of it be $v_1, ..., v_k \in B$. So it means, that $B \subset \bigcup_{i=1}^{k} B_s(v_i)$, where $B_s(v_i)$ is the open sphere with center v_i and radius s. Let $D_i = \{r \in \mathbb{R}^d \big| \|v_i - \frac{r}{\|r\|}\| < s\}$. Then D_i is the set of vectors, whose preserved angle with v_i is smaller than $\frac{\alpha}{2}$, so D_i is a convex cone. Plus

$\bigcup_{i=1}^{k} D_i = \mathbb{R}^d \setminus 0$, because $\frac{r}{\|r\|} \in B$ and $v_1, ..., v_k$ is an s-net of B, so there is an i for every r, such that $\|v_i - \frac{r}{\|r\|}\| < s$.

If H is a subset of $\mathbb{R}^d$, then define the Geometric Hyper POSET $\mathfrak{H}(H, D_1, ..., D_n) = (H, <_1, ..., <_k)$. Let $N_{d,n,\alpha} = (n-1)^k$. If $|H| > N_{d,n,\alpha} = (n-1)^k$, then by *Theorem 1* there exist an $1 \leq m \leq k$ and $A \subset H$ with n elements, that A is a $<_m$ chain.

It will be shown, that $G = A$ satisfies the conditions. Let x, y, z be different elements of A, then it can be assumed without the loss of generality, that $x <_m y <_m z$. Then $y - x \in C_m$ and $z - y \in C_m$, which means that the angle of v_i and $y - x$ is smaller than $\frac{\alpha}{2}$, and the angle of v_i and $z - y$ is also smaller than $\frac{\alpha}{2}$. But then the angle of $y - x$ and $z - y$ is smaller than α, so $xyz\angle > \pi - \alpha$. So $G = A$ really satisfies the conditions. That means, that the constant $N_{d,n,\alpha} = (n-1)^k$ is a good choice.■

Remark With the idea of this solution, one may count, that $N_{d,3,\frac{\pi}{2}} \leq 2^{2^{d-1}}$, which means, that there is an obtuse angle in every set with more than $2^{2^{d-1}}$ elements, which is unluckily far from the strict.

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