DIPLOMAMUNKA

MULTISYMMETRIC POLYNOMIALS IN DIMENSION THREE

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Abstract

The symmetric group S_3 acts on the three-dimensional complex vector space V by permutation of the coordinates. $\mathbb{C}[V^m]^{S_3}$ is the algebra of polynomial invariants corresponding to the diagonal action of S_3 on the direct sum of m copies of V. $\mathbb{C}[V^m]^{S_3}$ is the algebra of multisymmetric polynomials, which is not a polynomial ring if m is greater than one. A presentation of this algebra by generators and relations is summarized in the first chapter. The second chapter provides some sufficient conditions on a set that imply that the set is a system of secondary generators of the algebra, or that it minimally generates the ideal of relations among chosen generators. A minimal generating system of the ideal of relations is determined in cases $m \leq 4$. In the last chapter it is proved that the polarizations of one relation of degree 5 and two relations of degree 6 generate the ideal of relations among chosen generators of $\mathbb{C}[V^m]^{S_3}$ for any $m \geq 4$.

Contents

List of Tables

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Chapter 1

Introduction

1.1 The algebra of multisymmetric polynomials

This section is cited from [1]. There a more general case is discussed, some alterations have been made accordingly.

Let n and m be natural numbers, and $\mathbb{C}$ the field of complex numbers. Consider the following action of S_n on $V = \mathbb{C}^n$: for arbitrary elements $g \in S_n$ and $v = (x_1, \dots, x_n) \in V$, $gv := (x_{\pi(1)}, \dots, x_{\pi(n)}) \in V$. As an abstract group, S_n is isomorphic to the subgroup G of $\mathrm{GL}(n, \mathbb{C})$ consisting of permutation matrices (monomial matrices with all nonzero entries equal to 1). The action of S_n on V is isomorphic to the natural action of G on V. According to the fundamental theorem of symmetric polynomials, the algebra $\mathbb{C}[V]^G$ is generated by algebraically independent elements.

Now consider the diagonal action of G on $V^m = V \oplus \cdots \oplus V$, the direct sum of m copies of V. This is isomorphic to the natural action of the diagonal subgroup $\bar{G} = \{(g, \dots, g) \mid g \in G)\} \leq G \times \cdots \times G \leq \mathrm{GL}(nm, \mathbb{C})$ on V^m. If $m \geq 2$ G does not contain pseudo-reflections. Hence, according to the Sheppard-Todd-Chevalley Theorem, the algebra $\mathbb{C}[V^m]^G$ of multisymmetric functions is not a polynomial ring. A presentation of the algebra of multisymmetric polynomials by generators and relations is discussed in [1]. Our goal is to give a (minimal) generating system of the ideal of relations among the chosen generators for arbitrary m in the case $n = 3$.

Let $x(i)_j$ denote the function on V^m mapping $v \in V^m$, $v = (v(1), \dots, v(m))$ to the jth coordinate of $v(i)$. The coordinate ring $\mathbb{C}[V^m]$ is a polynomial ring generated by the mn variables $x(i)_j$, $i = 1, \dots, m$, $j = 1, \dots n$. Consider the following multigrading on $\mathbb{C}[V^m]$: a polynomial $f \in \mathbb{C}[V^m]$ has multidegree $\alpha = (\alpha_1, \dots, \alpha_m)$ if it has a total degree α_i in the variables $x(i)_1, \dots, x(i)_n$. The polynomial f is multihomogeneous of multidegree α if all monomials in f have the multidegree α, write $\mathbb{C}[V^m]^G_\alpha$ for the set of these elements in

$\mathbb{C}[V^m]^G$. Clearly the action of G preserves this multigrading, hence $f \in \mathbb{C}[V^m]$ is G-invariant if and only if all multihomogeneous components in f are G-invariant.

The following notation will be used in all computations. Consider the symbols $x(1), \dots, x(m)$ as commuting variables and let $\mathcal{M}$ denote the set of nonempty monomials in the variables $x(i)$. For $w = x(1)^{\alpha_1} \cdots x(m)^{\alpha_m} \in \mathcal{M}$ $((\alpha_1, \dots \alpha_m) \neq (0, \dots, 0))$, let $w_{\langle j \rangle} = x(1)_j^{\alpha_1} \cdots x(m)_j^{\alpha_m}$, and define

$$[w] = \sum_{j=1}^{n} w_{\langle j \rangle}.$$

The following proposition is the special case $q = 1$ of Proposition 2.1 in [1].

Proposition 1.1.1 *The products* $[w_1] \cdots [w_r]$ *for* $r \leq n$, $w_i \in \mathcal{M}$ *constitute a* $\mathbb{C}$*-vector space basis of* $\mathbb{C}[V^m]^G$.

Proof An arbitrary monomial $u \in \mathbb{C}[V^m]$ in the variables $x(i)_j$ can be written as $u = u^1_{\langle 1 \rangle} \cdots u^n_{\langle n \rangle}$ with a unique n-tuple $(u^1, \dots, u^n)$ of monomials in $\mathcal{M} \cup \{1\}$. The action of $G \cong S_n$ permutes these monomials, thus the S_n-orbit sums of such monomials form a basis in $\mathbb{C}[V^m]^G_\alpha$. For a multiset $\{w_1, \dots, w_r\}$ with $r \leq n$, $w_i \in \mathcal{M}$, denote by $O_{\{w_1, \dots, w_r\}}$ the S_n-orbit sum of the monomial $w_{1\langle 1 \rangle} \cdots w_{r\langle r \rangle}$. Call r the height of this monomial multisymmetric function. An element $\pi \in S_n$ acts on the monomial in the following way: $\pi(w_{1\langle 1 \rangle} \cdots w_{r\langle r \rangle}) = w_{1\langle \pi^{-1}(1) \rangle} \cdots w_{r\langle \pi^{-1}(r) \rangle}$. Assume that the multiset $\{w_1, \dots, w_r\}$ contains d distinct elements with multiplicities $r_1, \dots, r_d$ (so $r_1 + \cdots + r_d = r$), then

$$O_{\{w_1, \dots, w_r\}} = \frac{1}{r_1! \cdots r_d!} \cdot \sum_{\pi \in S_n} w_{1\langle \pi(1) \rangle} \cdots w_{r\langle \pi(r) \rangle}$$

Set $T_{\{w_1, \dots, w_r\}} = [w_1] \cdots [w_r]$.

Since the $O_{\{w_1, \dots, w_r\}}$ with $\mathrm{multideg}(w_1 \cdots w_r) = \alpha$ form a $\mathbb{C}$-basis in $\mathbb{C}[V^m]^G_\alpha$, to prove the proposition, it is sufficient to show by induction on r that $O_{\{w_1, \dots, w_r\}}$ can be expressed as a linear combination of such T-s.

According to the definition

$$T_{\{w_1, \dots, w_r\}} = [w_1] \cdots [w_r] = \prod_{i=1}^{r} \left(\sum_{j=1}^{n} w_{i\langle j \rangle} \right);$$

this can be extended as a linear combination of monomial multisymmetric functions. When extending $T_{\{w_1, \dots, w_r\}}$, the coefficient of $O_{\{w_1, \dots, w_r\}}$ is $r_1! \cdots r_d! \neq 0$, and all other monomial multisymmetric functions have height strictly less than r. This completes the proof. □

Associate with $w \in \mathcal{M}$ a variable $t(w)$, and consider the polynomial ring

$$\mathcal{F} = \mathbb{C}[t(w) \mid w \in \mathcal{M}]$$

in infinitely many variables. Denote by

$$\varphi : \mathcal{F} \to \mathbb{C}[V^m]^G$$

the $\mathbb{C}$-algebra homomorphism induced by the map $t(w) \mapsto [w]$. This is a surjection by 1.1.1. There is a uniform set of elements in its kernel. For a multiset $\{w_1, \dots, w_{n+1}\}$ of $n+1$ monomials from $\mathcal{M}$, an element can be associated as follows. Write $\mathcal{P}_{n+1}$ for the set of partitions $\lambda = \lambda_1 \cup \dots \cup \lambda_h$ of the sets $\{1, \dots, n+1\}$ into the disjoint union of non-empty subsets λ_i, and denote $h(\lambda) = h$ the number of parts of the partition λ. Set

$$\Psi(w_1, \dots, w_{n+1}) = \sum_{\lambda \in \mathcal{P}_{n+1}} \prod_{i=1}^{h(\lambda)} (-1)(|\lambda_i| - 1)! \cdot t\left(\prod_{s \in \lambda_i} w_s\right).$$

Remark In the special case when $n = 3$, this element has the following form:

$$\Psi(w_1, w_2, w_3, w_4) = -6t(w_1w_2w_3w_4) +$$

$$+2\big(t(w_1w_2w_3)t(w_4) + t(w_1w_2w_4)t(w_3) + t(w_1w_3w_4)t(w_2) + t(w_2w_3w_4)t(w_1)\big) +$$

$$+\big(t(w_1w_2)t(w_3w_4) + t(w_1w_3)t(w_2w_4) + t(w_1w_4)t(w_2w_3)\big) +$$

$$-\big(t(w_1w_2)t(w_3)t(w_4) + t(w_1w_3)t(w_2)t(w_4) + t(w_1w_4)t(w_2)t(w_3) +$$

$$+t(w_2w_3)t(w_1)t(w_4) + t(w_2w_4)t(w_1)t(w_3) + t(w_3w_4)t(w_1)t(w_2)\big) +$$

$$+t(w_1)t(w_2)t(w_3)t(w_4).$$

Proposition 1.1.2 *The kernel of the homomorphism φ contains the element $\Psi(w_1, \dots, w_{n+1})$ for arbitrary $w_1, \dots, w_{n+1} \in \mathcal{M}$.*

Proof In the case $n = 3$.

Write $\pi_k = v_1^k + v_2^k + v_3^k + v_4^k$ for the kth power sum function, and σ_k for the kth elementary symmetric function of 4 variables. According to the Newton-Girard formula:

$$0 = (-1)^k \cdot (k) \cdot \sigma_k + \sum_{i=1}^{k} (-1)^{k-i} \pi_i \cdot \sigma_{k-i}.$$

In the case $n = 3$, these formulas are the following:

$$0 = \pi_4 - \sigma_1\pi_3 + \sigma_2\pi_2 - \sigma_3\pi_1 + 4\sigma_4 \tag{1.1}$$

$$0=\pi_3-\sigma_1\pi_2+\sigma_2\pi_1-3\sigma_3 \tag{1.2}$$

$$0=\pi_2-\sigma_1\pi_1+2\sigma_2 \tag{1.3}$$

$$0=\pi_1-\sigma_1 \tag{1.4}$$

Using (1.1)-(1.4) σ_4 can be expressed in terms of π_1, π_2, π_3 and π_4 as follows:

$$(1.4)\Rightarrow\sigma_1=\pi_1 \tag{1.5}$$

$$(1.5),(1.3)\Rightarrow\sigma_2=\frac{1}{2}(\pi_1^2-\pi_2) \tag{1.6}$$

$$(1.5),(1.6),(1.2)\Rightarrow 0=\pi_3-\pi_1\pi_2+\frac{\pi_1}{2}(\pi_1^2-\pi_2)-3\sigma_3\Rightarrow$$

$$\Rightarrow\sigma_3=\frac{1}{3}\left(\frac{1}{2}\pi_1^3-\frac{3}{2}\pi_1\pi_2+\pi_3\right) \tag{1.7}$$

$$(1.5),(1.6),(1.7),(1.1)\Rightarrow$$

$$\Rightarrow 0=\pi_4-\pi_1\pi_3+\frac{\pi_2}{2}(\pi_1^2-\pi_2)-\frac{1}{3}\left(\frac{1}{2}\pi_1^3-\frac{3}{2}\pi_1\pi_2+\pi_3\right)\pi_1+4\sigma_4\Rightarrow$$

$$\Rightarrow\sigma_4=-\frac{1}{4}\pi_4+\frac{1}{3}\pi_1\pi_3+\frac{1}{8}\pi_2^2-\frac{1}{4}\pi_1^2\pi_2+\frac{1}{24}\pi_1^4 \tag{1.8}$$

Consider the equation (1.8) as an identity of the coordinates of the vector $v=(v_1,v_2,v_3,v_4)$. Write $x^{(i)}$ for the vector $(x_1^{(i)},x_2^{(i)},x_3^{(i)},x_4^{(i)})$ $(1\leq i\leq 4)$ Set $v=x^{(1)}+x^{(2)}+x^{(3)}+x^{(4)}$. If f is a polynomial of the coordinates of $x^{(1)},x^{(2)},x^{(3)},x^{(4)}$, write $f\mid_{(1,1,1,1)}$ for the multilinear part of f, that is, the sum of terms of the form $x_{j_1}^{(1)}x_{j_2}^{(2)}x_{j_3}^{(3)}x_{j_4}^{(4)}$. The identity (1.8) holds between to polynomials of the coordinates of $x^{(1)},x^{(2)},x^{(3)},x^{(4)}$, thus the the multilinear parts of the two sides are equal. This gives an identity in the variables $x_j^{(i)}$. Now for arbitrary $w_1,w_2,w_3,w_4\in\mathcal{M}$, set $x_j^{(i)}=(w_i)_{\langle j\rangle}$ for $j=1,2,3$ and $x_4^{(i)}=0$ $(1\leq i\leq 4)$. Using the fact that for $a,b\in\mathcal{M}$, $(ab)_{\langle j\rangle}=a_{\langle j\rangle}\cdot b_{\langle j\rangle}$, multilinear parts can be written as follows.

$$(\sigma_4)\mid_{(1,1,1,1)}=0 \tag{1.9}$$

$$(\pi_4)\mid_{(1,1,1,1)}=24\cdot[w_1w_2w_3w_4] \tag{1.10}$$

$$\begin{aligned}(\pi_1\pi_3)_{(1,1,1,1)}= \quad & 6([w_1][w_2w_3w_4]+[w_2][w_1w_3w_4]+\\ & +[w_3][w_1w_2w_4]+[w_4][w_1w_2w_3])\end{aligned} \tag{1.11}$$

$$(\pi_2^2)_{(1,1,1,1)}=8([w_1w_2][w_3w_4]+[w_1w_3][w_2w_4]+[w_1w_4][w_2w_3]) \tag{1.12}$$

$$\begin{aligned}(\pi_1^2\pi_2)_{(1,1,1,1)}= \quad & 4([w_1][w_2][w_3w_4]+[w_1][w_3][w_2w_4]+\\ & +[w_1][w_4][w_2w_3]+[w_2][w_3][w_1w_4]+\\ & +[w_2][w_4][w_1w_3]+[w_3][w_4][w_1w_2])\end{aligned} \tag{1.13}$$

$$(\pi_1^4)_{(1,1,1,1)} = 24[w_1][w_2][w_3][w_4] \tag{1.14}$$

Now from (1.8), (1.9), (1.10), (1.11), (1.12), (1.13) and (1.14) it follows that

$$\begin{aligned} 0 = \quad & -6[w_1w_2w_3w_4]+ \\ & +2([w_1][w_2w_3w_4] + [w_2][w_1w_3w_4] + [w_3][w_1w_2w_4] + [w_4][w_1w_2w_3])+ \\ & +([w_1w_2][w_3w_4] + [w_1w_3][w_2w_4] + [w_1w_4][w_2w_3])- \\ & -([w_1][w_2][w_3w_4] + [w_1][w_3][w_2w_4] + [w_1][w_4][w_2w_3]+ \\ & +[w_2][w_3][w_1w_4] + [w_2][w_4][w_1w_3] + [w_3][w_4][w_1w_2])+ \\ & +[w_1][w_2][w_3][w_4] \end{aligned}$$

This proves the claim. Call $\varphi(\Psi(w_1, \ldots, w_{n+1})) = 0$ the *fundamental identity*. □

Theorem 1.1.3

(i) The kernel of the K-algebra homomorphism φ is the ideal generated by the $\Psi(w_1, \ldots, w_{n+1})$, where $w_1, \ldots, w_{n+1} \in \mathcal{M}$.

(ii) The algebra $\mathbb{C}[V^m]^G$ is minimally generated by the $[w]$, where $w \in \mathcal{M}$ with $\deg(w) \leq n$.

Proof

(i) The coefficient in $\Psi(w_1, \ldots, w_{n+1})$ of the term $t(w_1) \cdots t(w_n)$ is $(-1)^{n+1}$, and all other terms are products of at most n variables $t(u)$. So the relation $\varphi(\Psi(w_1, \ldots, w_{n+1})) = 0$ can be used to rewrite $[w_1] \cdots [w_{n+1}]$ as a linear combination of products of at most n variables of the form $[u]$ (where $u \in \mathcal{M}$). So these relations are sufficient to rewrite an arbitrary product of the generators $[w]$ in terms of the basis given by 1.1.1. This implies the statement about the kernel of φ.

(ii) If $w \in \mathcal{M}$ and $\deg(w) > n$, then w can be factored as $w = w_1 \cdots w_{n+1}$ where $w_i \in \mathcal{M}$. The term $t(w)$ appears in $\Psi(w_1, \ldots, w_{n+1}$ with coefficient $-(n!)$, therefore the relation $\varphi(\Psi(w_1, \ldots, w_{n+1}) = 0$ shows that $[w]$ can be expressed as a polynomial of strictly smaller degree. It follows that $K[V^m]^G$ is generated by the $[w]$, where $w \in \mathcal{M}$ with $\deg(w) \leq n$. This is a minimal generating system, because if $\deg(w) \leq n$ for some $w \in \mathcal{M}$, then $[w]$ can not be expressed by invariants of lower degree, since according to (i), there is no relation among the generators whose degree is smaller than $n + 1$. □

For a natural number d, consider the finitely generated subalgebra of $\mathcal{F}$ given by

$$\mathcal{F}(d) = \mathbb{C}[t(w) \mid w \in \mathcal{M},\ \deg(w) \leq d].$$

According to Theorem 3.2 in [1], the $\mathbb{C}$-algebra homomorphism $\mathcal{F}(n(n+1)-2n+2) \to \mathbb{C}[V^m]^G$ induced by $t(w) \mapsto [w]$ (that is, $\varphi\mid_{\mathcal{F}(n(n+1)-2n+2)}$) is a surjection and its kernel is generated by the elements $\Psi(w_1,\dots,w_{n+1})$, where $w_1,\dots,w_{n+1} \in \mathcal{M}$ and the degree of the product $w_1\cdots w_{n+1}$ is not greater than $n(n+1)-2n+2$.

Corollary 1.1.4 *From the theorem cited above and 1.1.3 it follows that in the case $n=3$, the algebra $\mathbb{C}[V^m]^G$ is minimally generated by the $[w]$, where $w \in \mathcal{M}$ with $\deg(w) \leq 3$; the ideal of relations among these generators of $\mathbb{C}[V^m]^G$ is generated by relations of degree at most 8.*

1.2 Cohen-Macaulay property

1.2.1 General facts about Cohen-Macaulay rings and Hironaka decomposition

A detailed discussion of the concepts introduced in this section can be found in section 2.3 of [6].

Let R be a finitely generated commutative graded $\mathbb{C}$-algebra,

$$R = \bigoplus_{\alpha\in\mathbb{N}} R_\alpha, \qquad R_0 = \mathbb{C}$$

and let k be the maximal number of algebraically independent elements in R.

Definition The system $\{z_1,\dots,z_k\}$ is called a homogeneous system of parameters if $z_1,\dots,z_k$ are homogeneous and R is a finitely generated module over the subalgebra $\mathbb{C}[z_1,\dots,z_k]$.

Remark (i) According to Noether Normalisation lemma, such a system does exist.
(ii) The definition implies that $z_1,\dots,z_k$ are algebraically independent.

Definition Let R be a ring like mentioned above, and $\{z_1,\dots,z_k\}$ a homogeneous system of parameters. The ring R is Cohen-Macaulay if it is a (finitely generated) free module over $\mathbb{C}[z_1,\dots,z_k]$, that is, there exist $y_1,\dots y_t$ homogeneous elements in R such that

$$R = \bigoplus_{i=1}^{t} \mathbb{C}[z_1,\dots,z_k]\cdot y_i.$$

(This is called the Hironaka decomposition of R.)

Remark (i) A ring R is Cohen-Macaulay if and only if it is a free $\mathbb{C}[z'_1,\dots,z'_k]$-module for any $\{z'_1,\dots,z'_k\}$ homogeneous system of parameters.
(ii) Moreover, in this case if $(z_1,\dots,z_k)$ denotes the ideal in R generated by the elements

$\{z_1,\dots,z_k\}$, the system $\{y_1,\dots y_t\}$ is a homogeneous free $\mathbb{C}[z_1,\dots,z_k]$-module generating system of R if and only if $\{y_1+(z_1,\dots,z_k),\dots y_t+(z_1,\dots,z_k)\}$ is a $\mathbb{C}$-basis in the factor space $R/(z_1,\dots,z_k)$.

Theorem 1.2.1 *If H is a finite group, $\mathbb{C}[W]^H$ is the ring of invariants corresponding to a representation of H on a $\mathbb{C}$-vector space W, then $\mathbb{C}[W]^H$ is Cohen-Macaulay.*

1.2.2 The case of multisymmetric polynomials

We continue to use the notation introduced in 1.1. In addition (using notation of Section 6 in [1]), write

$$P=\{[x(i)],[x(i)^2],\dots,\ [x(i)^n]\mid i=1,\dots,m\}.$$

Denote by $\langle P\rangle$ the ideal of $\mathbb{C}[V^m]^G$ generated by P and by $\mathbb{C}[P]$ the polynomial algebra (subalgebra of $\mathbb{C}[V^m]^G$) generated by P ($1\in\mathbb{C}[P]$).

The following lemma is the citation of 6.1 in [1]. (Specialised to the case $K=\mathbb{C}$.)

Lemma 1.2.2 *Let w be a monomial having degree $\geq n$ in one of the variables $x(1),\dots,x(n)$, and having total degree $\geq n+1$. Then $[w]$ belongs to $\langle P\rangle$.*

Proof Assume for example that w has degree $\geq n$ in $x(1)$. Then w can be written as a product of $n+1$ factors as $x(1)x(1)\cdots x(1)u$ for some nonempty monomial u. The relation $\varphi(\Psi_{n+1}(x(1),x(1),\dots,x(1),u))=0$ verifies the statement, since each non-trivial partition of the multiset $\{x(1),x(1),\dots,x(1),u\}$ contains a part consisting solely of $x(1)$-s. □

Proposition 1.2.3 *The set P is a homogeneous system of parameters in the Cohen-Macaulay algebra $\mathbb{C}[V^m]^G$ ([1]).*

Proof From Theorem 1.2.1 it immediately follows, that $\mathbb{C}[V^m]^G$ is Cohen-Macaulay. We need to prove that $\mathbb{C}[V^m]^G$ is a finitely generated $\mathbb{C}[P]$-module. According to 1.1.1, the products $[w_1]\cdots[w_r]$ for $r\leq n$, $w_i\in\mathcal{M}$ constitute a $\mathbb{C}$-vector space basis of $\mathbb{C}[V^m]^G$. By 1.2.2, if the total degree of w is at least $m(n-1)+1$, then $[w]=p\cdot u$, where $p\in\langle P\rangle$ and $u\in\mathbb{C}[V^m]^G$, and the total degree of u is smaller than that of w. From this it follows that the ring $\mathbb{C}[V^m]^G$ is generated as a $K[P]$-module by the products $[w_1]\cdots[w_r]$ for $r\leq n$, $w_i\in\mathcal{M}$, $\deg(w_i)\leq m(n-1)$. □

Remark According to 1.2.1, a system of secondary generators is the same as a $\mathbb{C}$-vector space basis in $\mathbb{C}[V^m]^G$ modulo $\langle P\rangle$.

1.3 Hilbert series

1.3.1 Hilbert-series of S_3-modules in general

Let $\mathbf{t} = (t_1, \ldots, t_m)$ be a set of formal variables, and for $\alpha = (\alpha_1, \ldots, \alpha_m)$ write $\mathbf{t}^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$. For an R multigraded $\mathbb{C}$-algebra,

$$R = \bigoplus_{\alpha \in \mathbb{N}^m} R_\alpha, \qquad R_0 = \mathbb{C},$$

R_α is the multihomogeneous component of R with multidegree α.

Definition The multigraded Hilbert-series of R in the variables $\mathbf{t}$ is the formal multivariate power-series

$$H(R; \mathbf{t}) = \sum_{\alpha \in \mathbb{N}^m} \dim(R_\alpha) \cdot \mathbf{t}^\alpha.$$

The action of $G \cong S_3$ on a $\mathbb{C}$-vector space W induces an action of G on $\mathbb{C}[W]$. This latter action preserves the natural multigrading of $\mathbb{C}[W]$. A subspace of given multigrade is invariant under the action and finite dimensional. The action on R_α is isomorphic to the direct sum of irreducible G-modules. Hence the action on $\mathbb{C}[W]$ is isomorphic to the direct sum of irreducible G-modules. Thus as an S_3-module, $\mathbb{C}[W]$ has three isotypical components corresponding to the three irreducible representations of S_3. According to Schur's lemma, $\mathbb{C}[W]$ is the direct sum of the three isotypical components. Write χ_0, χ_1 and χ_2 for the three irreducible characters of S_3 (χ_0 the trivial, χ_1 the alternating and χ_2 the third one). Write $\mathbb{C}[W]_{\chi_i}$ ($i = 0, 1, 2$) for the corresponding isotypical components of $\mathbb{C}[W]$ ($\mathbb{C}[W]^{S_3} = \mathbb{C}[W]_{\chi_0}$). Hence

$$\mathbb{C}[W] = \mathbb{C}[W]_{\chi_0} \oplus \mathbb{C}[W]_{\chi_1} \oplus \mathbb{C}W]_{\chi_2}.$$

Now if $W = W^{(1)} \oplus W^{(2)}$, then

$$\mathbb{C}[W] = \mathbb{C}[W^{(1)}] \otimes \mathbb{C}[W^{(2)}]$$

$$\begin{aligned} \mathbb{C}[W] &= \left(\mathbb{C}[W^{(1)}]_{\chi_0} \oplus \mathbb{C}[W^{(1)}]_{\chi_1} \oplus \mathbb{C}[W^{(1)}]_{\chi_2}\right) \otimes \\ &\otimes \left(\mathbb{C}[W^{(2)}]_{\chi_0} \oplus \mathbb{C}[W^{(2)}]_{\chi_1} \oplus \mathbb{C}[W^{(2)}]_{\chi_2}\right) \end{aligned}$$

$$\mathbb{C}[W] = \bigoplus_{0 \le i,j \le 2} \left(\mathbb{C}[W^{(1)}]_{\chi_i} \otimes \mathbb{C}[W^{(2)}]_{\chi_j}\right). \tag{1.15}$$

The S_3-module structure of $\mathbb{C}[W^{(1)}]_{\chi_i} \otimes \mathbb{C}[W^{(2)}]_{\chi_j}$ can be calculated by calculating the corresponding character. The S_3-module $\mathbb{C}[W^{(1)}]_{\chi_i}$ is the sum of irreducible S_3-representations of character χ_i, and $\mathbb{C}[W^{(2)}]_{\chi_j}$ is the sum of irreducible S_3-representations

$\cdot$	χ_0	χ_1	χ_2
χ_0	χ_0	χ_1	χ_2
χ_1	χ_1	χ_0	χ_2
χ_2	χ_2	χ_2	$\chi_0+\chi_1+\chi_2$

Table 1.1: Multiplication table of irreducible characters of S_3

of character χ_j. From this it follows, that $\mathbb{C}[W^{(1)}]_{\chi_i} \otimes \mathbb{C}[W^{(2)}]_{\chi_j}$ is the direct sum of S_3-representations of character $\chi_i \cdot \chi_j$. Table 1.1 shows the multiplication table of S_3-characters.

Let $H(U_{\chi_i}, t)$ denote the Hilbert-series of the χ_i-isotypical component of an S_3-module U. From the above considerations it follows that the Hilbert-series of different isotypical components of $\mathbb{C}[W]$ can be expressed in terms of the series of components of $\mathbb{C}[W^{(1)}]$ and $\mathbb{C}[W^{(2)}]$ as follows:

$$\begin{aligned} H(\mathbb{C}[W]_{\chi_0}, t_1, t_2) &= H(\mathbb{C}[W^{(1)}]_{\chi_0}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_0}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_1}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_1}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_2}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_2}, t_2); \end{aligned} \tag{1.16}$$

$$\begin{aligned} H(\mathbb{C}[W], t_1, t_2)_{\chi_1} &= H(\mathbb{C}[W^{(1)}]_{\chi_0}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_1}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_1}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_0}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_2}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_2}, t_2); \end{aligned} \tag{1.17}$$

$$\begin{aligned} H(\mathbb{C}[W]_{\chi_2}, t_1, t_2) &= H(\mathbb{C}[W^{(1)}]_{\chi_0}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_2}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_2}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_0}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_1}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_2}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_2}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_1}, t_2)+ \\ &+ H(\mathbb{C}[W^{(1)}]_{\chi_2}, t_1) \cdot H(\mathbb{C}[W^{(2)}]_{\chi_2}, t_2); \end{aligned} \tag{1.18}$$

1.3.2 Hilbert-series of the ring of invariants

Later in the course of calculations we will need the Hilbert-series $H(\mathbb{C}[V^m]^{S_3}, \mathbf{t})$. The equations (1.16), (1.17) and (1.18) show that it can be determined by induction on m. To follow the multigrading defined in 1.1, assign the formal variable t_i to the i-th component V.

In the case $m = 1$ the Hilbert-series are:

$$H(\mathbb{C}[V]_{\chi_0}, t_1) = \frac{1}{(1-t_1)(1-t_1^2)(1-t_1^3)} \tag{1.19}$$

$$H(\mathbb{C}[V]_{\chi_1}, t_1) = \frac{t_1^3}{(1-t_1)(1-t_1^2)(1-t_1^3)} \tag{1.20}$$

$$H(\mathbb{C}[V]_{\chi_2}, t_1) = \frac{t_1 + t_1^2}{(1-t_1)(1-t_1^2)(1-t_1^3)} \tag{1.21}$$

From (1.16), (1.17), (1.18) and (1.19), (1.20), (1.21) it follows that the Hilbert-series $H(\mathbb{C}[V^m], \mathbf{t})_{\chi_i}$ is a rational expression with denominator

$$q(m, \mathbf{t}) := \prod_{i=1}^{m} (1-t_i)(1-t_i^2)(1-t_i^3).$$

Write $E_m(\mathbf{t})$, $A_m(\mathbf{t})$, $B_m(\mathbf{t})$ for the enumerator (a polynomial), that is,

$$E_m(\mathbf{t}) := H(\mathbb{C}[V^m]_{\chi_0}, \mathbf{t}) \cdot q(m, \mathbf{t});$$

$$A_m(\mathbf{t}) := H(\mathbb{C}[V^m]_{\chi_1}, \mathbf{t}) \cdot q(m, \mathbf{t});$$

$$B_m(\mathbf{t}) := H(\mathbb{C}[V^m]_{\chi_2}, \mathbf{t}) \cdot q(m, \mathbf{t}).$$

From (1.19), (1.20) and (1.21), $E_1(t_1) = 1$, $A_1(t_1) = t_1^3$ and $B_1(t_1) = t_1 + t_1^2$.

Write $\mathbf{x}$ for $x_1, \ldots, x_k$, $\mathbf{y}$ for $y_1, \ldots, y_l$, $k + l = m$. From (1.16), (1.17), (1.18) it follows that the polynomials E_m, A_m and B_m satisfy the following recursion:

$$E_m(\mathbf{x}, \mathbf{y}) = E_k(\mathbf{x}) \cdot E_l(\mathbf{y}) + A_k(\mathbf{x}) \cdot A_l(\mathbf{y}) + B_k(\mathbf{x}) \cdot B_l(\mathbf{y}) \tag{1.22}$$

$$A_m(\mathbf{x}, \mathbf{y}) = E_k(\mathbf{x}) \cdot A_l(\mathbf{y}) + A_k(\mathbf{x}) \cdot E_l(\mathbf{y}) + B_k(\mathbf{x}) \cdot B_l(\mathbf{y}) \tag{1.23}$$

$$\begin{aligned} B_m(\mathbf{x}, \mathbf{y}) &= E_k(\mathbf{x}) \cdot B_l(\mathbf{y}) + B_k(\mathbf{x}) \cdot E_l(\mathbf{y}) + \\ &+ A_k(\mathbf{x}) \cdot B_l(\mathbf{y}) + B_k(\mathbf{x}) \cdot A_l(\mathbf{y}) + \\ &+ B_k(\mathbf{x}) \cdot B_l(\mathbf{y}) \end{aligned} \tag{1.24}$$

From this recursion and E_1, A_1 and B_1,

$$H(\mathbb{C}[V^m]^{S_3}, \mathbf{t}) = \frac{E_m(\mathbf{t})}{q(m, \mathbf{t})} \tag{1.25}$$

can easily be determined.

1.3.3 The Hilbert-series of the ideal of relations

As seen in 1.1.3 and 1.1.4, the $\mathbb{C}$-algebra homomorphism

$$\varphi : \mathcal{F}(3) \to \mathbb{C}[V^m]^G$$

induced by $t(w) \mapsto [w]$ is a surjection, and its kernel is generated by the elements $\Psi(w_1, w_2, w_3, w_4)$, where $w_1, w_2, w_3, w_4 \in \mathcal{M}$ and $\deg(w_1 w_2 w_3 w_4) \leq 8$. The kernel is an ideal $\ker(\varphi) = I_m$ in $\mathcal{F}(3)$.

Let $\mathrm{multideg}(t_w) := \mathrm{multideg}(w)$, this induces a multigrading on $\mathcal{F}(3)$ preserved by φ. Since $\mathcal{F}(3)$ is a polynomial ring generated by the elements t_w for $w \in \mathcal{M}$, $\deg(w) \leq 3$, the Hilbert-series of $\mathcal{F}(3)$ with respect to this multigrading is

$$H(\mathcal{F}(3), \mathbf{t}) = \frac{1}{q(m, \mathbf{t})} \cdot \prod_{\substack{i,j=1 \\ i<j}}^{m} \frac{1}{(1 - t_i t_j)(1 - t_i^2 t_j)(1 - t_i t_j^2)} \cdot \prod_{\substack{i,j,k=1 \\ i<j<k}}^{m} \frac{1}{(1 - t_i t_j t_k)}$$

By writing

$$h_2(t_i, t_j) = (1 - t_i t_j)(1 - t_i^2 t_j)(1 - t_i t_j^2); \qquad h_3(t_i, t_j, t_k) = (1 - t_i t_j t_k) \tag{1.26}$$

this can be written as

$$H(\mathcal{F}(3), \mathbf{t}) = \frac{1}{q(m, \mathbf{t})} \cdot \prod_{\substack{i,j=1 \\ i<j}}^{m} \frac{1}{h_2(t_i, t_j)} \cdot \prod_{\substack{i,j,k=1 \\ i<j<k}}^{m} \frac{1}{h_3(t_i, t_j, t_k)} \tag{1.27}$$

The Hilbert-series of the kernel $\ker(\varphi) = I_m$ is

$$H(I_m, \mathbf{t}) = H(\mathcal{F}(3), \mathbf{t}) - H(\mathbb{C}[V^m]^{S_3}, \mathbf{t}),$$

and according to (1.25) and (1.27) is equal to the following:

$$H(I_m, \mathbf{t}) = \left(1 - E_m(\mathbf{t}) \cdot \prod_{\substack{i,j=1 \\ i<j}}^{m} h_2(t_i, t_j) \cdot \prod_{\substack{i,j,k=1 \\ i<j<k}}^{m} h_3(t_i, t_j, t_k) \right) \cdot H(\mathcal{F}(3), \mathbf{t}). \tag{1.28}$$

Chapter 2

Secondary generators and relations

In the next chapter we perform some calculations in the cases $n = 3$, $m = 2, 3, 4$ respectively to give a minimal generating system of I_m, and in the cases $m = 2$ and $m = 3$ to determine a system of secondary generators. In this chapter results of the previous parts are applied to prepare these calculations.

Write

$$Q = \{[w] \mid w \in \mathcal{M},\ \deg(w) \leq 3\} \tag{2.1}$$

for a minimal generating system of the algebra $\mathbb{C}[V^m]^G$ (1.1.4), and also (using the notation introduced in 1.2.1 and 1.3.3) write

$$P = \{[x(i)], [x(i)^2], [x(i)^3] \mid i = 1, \ldots, m\} \tag{2.2}$$

for a system of homogeneous generators (1.2.3), and I_m for the ideal of relations among the elements of Q. (That is, the kernel of the $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{F}(3) \to \mathbb{C}[V^m]^G$).

According to 1.2, $\mathbb{C}[V^m]^G$ is a finitely generated free $\mathbb{C}[P]$-module. A free $\mathbb{C}[P]$-module generating system of $\mathbb{C}[V^m]^G$ is called a system of secondary generators. By the last remark in 1.2.1, a system of secondary generators is the same as a $\mathbb{C}$-vector space basis in $\mathbb{C}[V^m]^G$ modulo $\langle P \rangle$.

2.1 Secondary generators

Consider a finite set S. Our goal is to set some conditions on S such that these conditions together imply that S is a system of secondary generators. The conditions should guarantee that the module $\mathbb{C}[P] \cdot S$ is the same as $\mathbb{C}[V^m]^G$, and that the elements of S generate a free $\mathbb{C}[P]$-module.

Proposition 2.1.1 *Let S be a subset of $\mathbb{C}[V^m]^G$ such that the elements*

$$\{s + \langle P \rangle \mid s \in S\}$$

generate $\mathbb{C}[V^m]^G/\langle P \rangle$ as a $\mathbb{C}$-vector space. Then $\mathbb{C}[P] \cdot S = \mathbb{C}[V^m]^G$.

Proof In this case there exists a subset $S' \subset S$ such that the set

$$\{s' + \langle P \rangle \mid s' \in S'\}$$

is a $\mathbb{C}$-vector space basis of $\mathbb{C}[V^m]^G/\langle P \rangle$. According to the last remark in 1.2.2, this implies that S' is a system of secondary generators. Hence

$$\mathbb{C}[V^m]^G = \mathbb{C}[P] \cdot S' \subseteq \mathbb{C}[P] \cdot S \subseteq \mathbb{C}[V^m]^G \Rightarrow$$

$$\Rightarrow \mathbb{C}[P] \cdot S = \mathbb{C}[V^m]^G.$$

This shows the claim. □

Proposition 2.1.2 *Assume that the elements of S are monomials of the elements of $Q \setminus P$, $\{1\} \cup (Q \setminus P) \subseteq S$, and that for any $r \in S$ and any $q \in Q \setminus P$ the congruence*

$$r \cdot q \equiv \sum_{s \in S} c_s \cdot s \mod \langle P \rangle \tag{2.3}$$

holds for some $c_s \in \mathbb{C}$. From this it follows that $\mathbb{C}[P] \cdot S = \mathbb{C}[V^m]^G$.

Proof Consider any two elements $s_1, s_2 \in S$. If $s_2 = 1$, clearly $s_1 s_2 = s_1 \in S$. If $s_2 \in S \setminus \{1\}$, then s_2 is a nonempty monomial $s_2 = q_1 \cdots q_k$ of the elements $q_i \in Q \setminus P$ $(1 \leq i \leq k)$. From (2.3) it follows that if $k = 1$ then $s_1 s_2$ can be rewritten as a linear combination of the elements of S modulo $\langle P \rangle$. By induction on k this is true for any k. That is, for arbitrary elements $s_1, s_2 \in S$,

$$s_1 \cdot s_2 \equiv \sum_{s \in S} c_s \cdot s \mod \langle P \rangle.$$

Thus any monomial of the elements of S can be rewritten as a linear combination of the elements of S modulo $\langle P \rangle$. (This easily follows from the above statement by induction on the length of the monomial.)

On the other hand, as Q generates $\mathbb{C}[V^m]^G$ as a $\mathbb{C}$-algebra, the elements

$$\{q + \langle P \rangle \mid q \in (Q \setminus P) \cup \{1\}\}$$

generate the $\mathbb{C}$-algebra $\mathbb{C}[V^m]^G/\langle P\rangle$. The images of the monomials of $Q \setminus P$ in $\mathbb{C}[V^m]^G/\langle P\rangle$ form a vector-space generating system. Since $(Q \setminus P) \subseteq S$, any monomial of the elements of $(Q \setminus P)$ is a linear combination of the elements of S modulo $\langle P\rangle$. Therefore the elements

$$\{s + \langle P\rangle \mid s \in S\}$$

generate $\mathbb{C}[V^m]^G/\langle P\rangle$ as a $\mathbb{C}$-vector space. Now 2.1.1 clearly shows the claim. $\square$

Note that all sets mentioned (e.g. P, Q) bear the natural multigrading described in 1.1. By 1.2.3 and the first remark in 1.2, the elements of P are algebraically independent. From this it follows that (using the notations introduced in 1.3)

$$H(\mathbb{C}[P], \mathbf{t}) = \prod_{i=1}^{m}(1-t_i)(1-t_i^2)(1-t_i^3) = \frac{1}{q(m, \mathbf{t})}. \tag{2.4}$$

For a multigraded set $S \subset \mathbb{C}[V^m]^G$ the set $\mathrm{Span}\,(S)$ is a multigraded $\mathbb{C}$-vector-space. Clearly the $\mathbb{C}[P]$-module $\mathbb{C}[P] \cdot S$ is a free module generated by the elements of S if and only if

$$H(\mathbb{C}[P] \cdot S, \mathbf{t}) = H(\mathbb{C}[P], \mathbf{t}) \cdot H(\mathrm{Span}\,(S), \mathbf{t}). \tag{2.5}$$

Proposition 2.1.3 *If for a finite set S the conditions*

$$\mathbb{C}[P] \cdot S = \mathbb{C}[V^m]^G \tag{2.6}$$

and

$$H(\mathrm{Span}\,(S), \mathbf{t}) = E_m(\mathbf{t}) \tag{2.7}$$

hold, then S is a system of secondary generators.

Proof According to (2.6), S generates $\mathbb{C}[V^m]^G$ as a $\mathbb{C}[P]$-module. By (2.4), (2.7), (1.25) and (2.6) it follows that

$$H(\mathbb{C}[P], \mathbf{t}) \cdot H(\mathrm{Span}\,(S), \mathbf{t}) = \frac{E_m(\mathbf{t})}{q(m, \mathbf{t})} = H(\mathbb{C}[V^m]^G, \mathbf{t}) = H(\mathbb{C}[P] \cdot S, \mathbf{t}).$$

As seen above from this it follows that S generates $\mathbb{C}[V^m]^G$ as a free $\mathbb{C}[P]$-module, that is, S is a system of secondary generators. $\square$

Now the conclusion of these considerations is the following theorem:

Theorem 2.1.4 *Let the (finite) set $S \subset \mathbb{C}[V^m]^G$ satisfy the following conditions:*

1. $\{1\} \cup (Q \setminus P) \subseteq S$;

2. *all elements of S are monomials of the elements of $Q \setminus P$;*

3. *for any $r \in S$ and any $q \in Q \setminus P$ the congruence*

$$r \cdot q \equiv \sum_{s \in S} c_s \cdot s \mod \langle P \rangle \tag{2.8}$$

holds for some $c_s \in \mathbb{C}$;

4. $H(\mathrm{Span}\,(S), \mathbf{t}) = E_m(\mathbf{t})$.

Then S is a secondary generating system of $\mathbb{C}[V^m]^G$ (with the primary generating system P).

Proof By 2.1.2 the first three conditions imply that $\mathbb{C}[P] \cdot S = \mathbb{C}[V^m]^G$. From this fact and the fourth condition 2.1.3 shows the claim. □

Lemma 2.1.5 *There exists a set S such that S satisfies the conditions of Theorem 2.1.4.*

Proof According to 1.1.3, the elements defined in 1.1.2 generate the ideal of relations among the elements of Q. All such relations have degree at least 4, hence the elements

$$\{v_q = q + \langle P \rangle \mid q \in \{1\} \cup (Q \setminus P)\}$$

in the $\mathbb{C}$-vector space $\mathbb{C}[V^m]^G / \langle P \rangle$ are linearly independent. As Q generates $\mathbb{C}[V^m]^G$ as an algebra, there exists a set S for which $\{1\} \cup (Q \setminus P) \subset S$, all elements of S are monomials of $Q \setminus P$ and the elements

$$\{s + \langle P \rangle \mid s \in S\}$$

constitute a $\mathbb{C}$-vector space basis of $\mathbb{C}[V^m]^G / \langle P \rangle$. By the last remark in 1.2, S is a secondary generating system, and conditions 1 and 2 of 2.1.4 hold for S. Hence $\mathbb{C}[V^m]^G = \mathbb{C}[P] \cdot S$ is a free $\mathbb{C}[P]$-module generated by S. From this it follows that condition 3 of 2.1.4 holds. Also,

$$H(\mathbb{C}[V^m]^G; \mathbf{t}) = H(\mathbb{C}[P]; \mathbf{t}) \cdot H(\mathrm{Span}\,(S); \mathbf{t}).$$

By (2.4) $H(\mathbb{C}[P]; \mathbf{t}) = q(m, \mathbf{t})^{-1}$, thus by (1.25)

$$H(\mathrm{Span}\,(S); \mathbf{t}) = \frac{E_m(\mathbf{t})}{q(m, \mathbf{t})} \cdot q(m, \mathbf{t}) = E_m(\mathbf{t}).$$

Thus condition 4 of 2.1.4 holds. This proves the claim. □

Remark An appropriate set S exists even if condition 1. is strengthened. Let S' be a set of monomials of $Q \setminus P$. The proof shows that if the elements

$$\{v_q = q + \langle P \rangle \mid q \in \{1\} \cup (Q \setminus P) \cup S'\}$$

are linearly independent in $\mathbb{C}[V^m]^G / \langle P \rangle$, then a secondary generating system S can be found such that the conditions of 2.1.4 holds for S and $S' \subseteq S$ is true.

2.2 The ideal of relations

Our goal is to give a set of relations that generate I_m (as an ideal of $\mathcal{F}(3)$), and prove that I_m cannot be generated as an ideal by a smaller set. An element of I_m is by definition an element of $\mathcal{F}(3)$, that is, a polynomial r in the variables $\{t(w) \mid w \in M,\ \deg(w) \le 3\}$ such that $r \in \ker(\varphi)$. Similarly to the case of a secondary generating system, our goal is to find conditions on a set $R \subset I_m$ that imply that R minimally generates the set of relations.

2.2.1 Conditions that imply that $\langle R \rangle = I_m$.

Write $P_0 := \{t(w) | [w] \in P\}$ and $Q_0 := \{t(w) | [w] \in Q\}$, hence $\varphi(P_0) = P$, $\varphi(Q_0) = Q$ and Q_0 is exactly the set of generators of $\mathcal{F}(3)$, that is, $\mathcal{F}(3) = \mathbb{C}[Q_0]$. Write $\langle P_0 \rangle$ for the ideal generated by P_0. Let S be a system of secondary generators in $\mathbb{C}[V^m]^G$ such that the conditions of 2.1.4 hold for S. Write $S_0 := \{t(w_1) \cdots t(w_k) | [w_1] \cdots [w_k] \in S\}$, hence $\varphi(S_0) = S$. From this it follows that the elements of S_0 are monomials of the elements of Q_0 and $1 \cup (Q_0 \setminus P_0) \subseteq S_0$.

Proposition 2.2.1 *The homomorphism φ is injective on the $\mathbb{C}[P_0]$-submodule $\mathbb{C}[P_0] \cdot S_0$, that is,*

$$\varphi \mid_{\mathbb{C}[P_0] \cdot S_0} : \mathbb{C}[P_0] \cdot S_0 \to \mathbb{C}[V^m]^G$$

is a module-isomorphism.

Proof By the definition of P and S, $\mathbb{C}[V^m]^G = \mathbb{C}[P] \cdot S$ is a finitely generated $\mathbb{C}[P]$-module. Clearly $\mathbb{C}[P_0] \cong \mathbb{C}[P]$ (both are polynomial rings over $\mathbb{C}$ generated by the same number of variables), $|S_0| = |S|$ and $\mathbb{C}[P_0] \cdot S_0$ is a free $\mathbb{C}[P_0]$-module generated by S_0. This proves the claim. □

The third condition in 2.1.4 ensures that all elements of $\mathbb{C}[V^m]^G$ can be rewritten to a normal form according to the Hironaka-decomposition $\mathbb{C}[V^m]^G = \mathbb{C}[P] \cdot S$. The following proposition shows that if a set R of relations can be used to reduce any element of $\mathbb{C}[V^m]^G$ to its normal form, then R generates the ideal of relations.

Proposition 2.2.2 *Let $R = \{r_1, \ldots, r_k\} \subset I_m$ be a set of relations and assume that for any polynomial $f \in \mathbb{C}[Q_0]$ there exist polynomials $g_s \in \mathbb{C}[P_0]$ ($s \in S_0$) and $h_i \in \mathbb{C}[Q_0]$ ($1 \leq i \leq k$) such that*

$$f - \sum_{s \in S_0} g_s \cdot s = \sum_{i=1}^{k} h_i \cdot r_i.$$

Then R generates the ideal I_m.

Proof The ideal of $\mathbb{C}[Q_0]$ generated by $R \subset I_m$ is

$$\langle R \rangle = \mathbb{C}[Q_0] \cdot r_1 + \cdots + \mathbb{C}[Q_0] \cdot r_k \subseteq I_m.$$

According to 2.2.1, $I_m \cap \mathbb{C}[P_0] \cdot S_0 = \{0\}$. From this it follows that since $\langle R \rangle \subset I_m$ (and hence the right side of the equation above is in I_m), the choice of polynomials g_s is unique. To prove $\langle R \rangle = I_m$, it suffices to show that $\mathbb{C}[Q_0]/\langle R \rangle \cong \mathbb{C}[V^m]^G$. The map

$$f \mapsto \sum_{s \in S_0} g_s \cdot s$$

shows that $\mathbb{C}[Q_0]/\langle R \rangle \cong \mathbb{C}[P_0] \cdot S_0$, and as seen above, $\mathbb{C}[P_0] \cdot S_0 \cong \mathbb{C}[P] \cdot S = \mathbb{C}[V^m]^G$. This shows the claim. □

The condition given in 2.2.2 is formulated more transparently in the proposition below.

Proposition 2.2.3 *Let $R = \{r_1, \ldots, r_k\} \subset I_m$ be a set of relations, and assume that for any $q \in Q_0 \setminus P_0$ and $s_0 \in S_0$ there exist polynomials $g_s \in \mathbb{C}[P_0]$ ($s \in S_0$) and $h_i \in \mathbb{C}[Q_0]$ ($1 \leq i \leq k$) such that*

$$q \cdot s_0 - \sum_{s \in S_0} g_s \cdot s = \sum_{i=1}^{k} h_i \cdot r_i.$$

Then R generates the ideal I_m.

Proof The condition is equivalent to the following: for any $q \in Q_0 \setminus P_0$ and $s_0 \in S_0$ there exists a polynomial $m_{q \cdot s_0} \in \mathbb{C}[P_0] \cdot S_0$ such that $q \cdot s_0 - m_{q \cdot s_0} \in \langle R \rangle$. Consider the following set:

$$H := \{f \in \mathbb{C}[Q_0] \mid \exists\, m_f \in \mathbb{C}[P_0] \cdot S_0 : \; f - m_f \in \langle R \rangle\}.$$

According to 2.2.2 it suffices to show that $H = \mathbb{C}[Q_0]$. This follows if H contains any monomial of the variables $Q_0 \setminus P_0$ and is a $\mathbb{C}[P_0]$-submodule of $\mathbb{C}[Q_0]$. It trivially is a $\mathbb{C}[P_0]$-submodule, since

$$\begin{aligned} f \in H,\; g \in \mathbb{C}[P_0] \Rightarrow \exists\, m_f \in \mathbb{C}[P_0] \cdot S_0 : \; f - m_f = r_f \in \langle R \rangle \Rightarrow \\ \Rightarrow g \cdot f - g \cdot m_f \in \langle R \rangle,\; g \cdot m_f \in \mathbb{C}[P_0] \cdot S_0 \Rightarrow gf \in H \end{aligned} \tag{2.9}$$

$$\begin{gathered} f_1, f_2 \in H \Rightarrow \exists\, m_{f_1}, m_{f_2} \in \mathbb{C}[P_0] \cdot S_0 : \\ f_1 - m_{f_1} \in \langle R \rangle,\ f_2 - m_{f_2} \in \langle R \rangle \Rightarrow \\ \Rightarrow (f_1 + f_2) - (m_{f_1} + m_{f_2}) \in \langle R \rangle,\ (m_{f_1} + m_{f_2}) \in \mathbb{C}[P_0] \cdot S_0 \Rightarrow \\ \Rightarrow f_1 + f_2 \in H. \end{gathered} \tag{2.10}$$

Let $q_1 \dots q_l$ be a monomial, $q_1, \dots, q_l \in Q_0 \setminus P_0$. Since $\{1\} \cup (Q_0 \setminus P_0) \subseteq S_0$, this monomial is in H according to the condition if $l = 1$ or $l = 2$. By induction, we show that the monomial is in H for any l. Assume that the monomials of length $l - 1$ are in H, that is, there exists $m \in \mathbb{C}[P_0] \cdot S_0$ such that $q_1 \dots q_{l-1} - m = r \in \langle R \rangle$. Let

$$m = \sum_{s \in S_0} g_s \cdot s$$

where $g_s \in \mathbb{C}[P_0]$ $(s \in S_0)$. Now from the condition it follows that

$$m \cdot q_l = \sum_{s \in S_0} g_s \cdot (q_l \cdot s) = \sum_{s \in S_0} g_s \cdot (m_{q_l s} + r_{q_l s}),$$

where $m_{q_l s} \in \mathbb{C}[P_0] \cdot S_0$ and $r_{q_l s} \in \langle R \rangle$. From this it follows that

$$q_1 \dots q_{l-1} q_l = (m + r) q_l = \left(\sum_{s \in S_0} g_s \cdot m_{q_l s} \right) + \left(r q_l + \sum_{s \in S_0} g_s q_l cdot r_{q_l s} \right),$$

$$\left(\sum_{s \in S_0} g_s \cdot m_{q_l s} \right) \in \mathbb{C}[P_0] \cdot S_0, \qquad \left(r q_l + \sum_{s \in S_0} g_s q_l \cdot r_{q_l s} \right) \in \langle R \rangle,$$

thus $q_1 \dots q_l \in H$. From this it follows that monomials of $Q_0 \setminus P_0$ of arbitrary length are in H. As seen above, from this it follows that $H = \mathbb{C}[Q_0]$, thus by 2.2.2, R generates I_m. □

2.2.2 Conditions to prove the minimality of R.

As seen in 1.3.3, the Hilbert-series of the ideal I_m is the following (with the same notation):

$$H(I_m, \mathbf{t}) = \left(1 - E_m(\mathbf{t}) \cdot \prod_{\substack{i,j=1 \\ i<j}}^{m} h_2(t_i, t_j) \cdot \prod_{\substack{i,j,k=1 \\ i<j<k}}^{m} h_3(t_i, t_j, t_k) \right) \cdot H(\mathcal{F}(3), \mathbf{t}).$$

Write

$$J_m(\mathbf{t}) := 1 - E_m(\mathbf{t}) \cdot \prod_{\substack{i,j=1 \\ i<j}}^{m} h_2(t_i, t_j) \cdot \prod_{\substack{i,j,k=1 \\ i<j<k}}^{m} h_3(t_i, t_j, t_k), \tag{2.11}$$

hence $J_m(\mathbf{t})$ is a polynomial and

$$H(I_m, \mathbf{t}) = J_m(\mathbf{t}) \cdot H(\mathcal{F}(3), \mathbf{t}). \tag{2.12}$$

(Note that (2.12) gives $H(I_m, \mathbf{t})$ as a rational expression, where $J_m(\mathbf{t})$ is the numerator.)

Remark As seen earlier, the ring $\mathcal{F}(3) = \mathbb{C}[Q_0]$ bears a natural multigrading. The element $\Psi(w_1, \ldots, w_{n+1})$ defined in 1.1 is multihomogeneous. Moreover, the definition of the multigrading on $\mathcal{F}(3)$ ensures that φ is a multihomogeneous $\mathbb{C}$-algebra homomorphism. From this it follows that $\ker(\varphi) = I_m$ is multihomogeneous, that is, a polynomial is in I_m if and only if all multihomogeneous parts of it are in I_m.

Therefore in the following if a set R of relations is mentioned, we assume that every element of R is multihomogeneous. A grading corresponds to the multigrading: an element of multidegree $\alpha = (\alpha_1, \ldots, \alpha_m)$ has (total) degree $\alpha_1 + \ldots + \alpha_m$.

Proposition 2.2.4 *Let $R \subset I_m$ be a finite set. Write $|R_l|$ for the number of elements in R with degree l and j_l for the coefficient of t^l in $J_m(t, \ldots, t)$ (j_0 is the constant term). Assume that R generates I_m as an ideal, $1 \leq d$ and $0 \leq j_l$ for $0 \leq l \leq d$, and $j_l = |r_l|$ for $0 \leq l \leq d-1$. Then $j_d \leq |R_d|$.*

Proof In this proof change the notation defined in 1.3 as follows: let $t_1 = \ldots = t_m = t$, thus $\mathbf{t} = (t, \ldots, t)$.

The ideal generated by $R = \{r_1, \ldots r_k\}$ is $\langle R \rangle = \mathbb{C}[Q_0] \cdot r_1 + \ldots + \mathbb{C}[Q_0] \cdot r_k$. From $\langle R \rangle = I_m$ it follows that

$$\sum_{i=1}^{k} H(\mathbb{C}[Q_0], \mathbf{t}) \cdot t^{\deg(r_i)} \geq H(I_m, \mathbf{t}),$$

that is, the coefficient of t^l on the left side is at least the coefficient on the right side for every $0 \leq l$. (Coefficients on the left side are greater or equal as elements of R are not necessarily algebraically independent.) Hence, by (2.12)

$$\left(\sum_{l=1}^{\infty} |R_l| \cdot t^l \right) \cdot H(\mathbb{C}[Q_0], \mathbf{t}) \geq J_m(\mathbf{t}) \cdot H(\mathbb{C}[Q_0], \mathbf{t}). \tag{2.13}$$

Write c_l for the coefficient of t^l in $H(\mathbb{C}[Q_0], \mathbf{t})$ (c_0 for the constant term), since $H(\mathbb{C}[Q_0], \mathbf{t})$ is a polynomial ring, c_l is a nonnegative integer. Now for the coefficient of t^d in (2.13) we get

$$\sum_{l=1}^{d} |R_i| \cdot c_{d-l} \geq \sum_{l=1}^{l} j_l \cdot c_{d-l}.$$

Now as $j_l = |r_l|$ for $0 \leq l \leq d-1$, this can be rewritten as:

$$\left(\sum_{l=1}^{d-1} |R_i| \cdot c_{d-l}\right) + |R_d| \cdot c_0 \geq \left(\sum_{l=1}^{d-1} j_l \cdot c_{d-l}\right) + j_d \cdot c_0 \Rightarrow$$

$$\Rightarrow |R_d| \cdot c_0 \geq j_d \cdot c_0.$$

As $c_0 = 1$, the proposition is true. □

2.2.3 Simplifying relations

Proposition 2.2.3 and 2.13 can be used to prove that a set $R \subseteq \mathcal{F}(3)$ generates I_m as an ideal, and that R is minimal (that is, by omitting an element the generated ideal becomes strictly smaller). The result 1.1.2 can be used to produce elements of I_m. However, these elements are usually polynomials with a huge number of terms. To simplify notation, instead of presenting relations as elements of $\mathcal{F}(3) = \mathbb{C}[Q_0]$, relations are written as congruences modulo $\langle P_0 \rangle$.

Let f and g be polynomials of the elements of $Q_0 \setminus P_0$, $f - g = r$. In the ring $\mathbb{C}[V^m]^G$, $\varphi(f) \equiv \varphi(g) \mod \langle P \rangle$ if the element $\varphi(r)$ for $r = f - g$ is equal to a polynomial of the generators which contains an element of P in every monomial. That is, $\varphi(f) \equiv \varphi(g) \mod \langle P \rangle$ if and only if

$$\varphi(f - g) = \varphi(r) = \sum_{p \in P} h_p \cdot p$$

holds for some $h_p \in \mathbb{C}[V^m]^G$. According to 2.2.1 there exists a unique $\overline{h_p} \in \mathbb{C}[P_0] \cdot S_0$ for which $\varphi(\overline{h_p}) = h_p$ (for every $p \in P$). Write p_0 for the element of P_0 mapped to an element $p \in P$ by φ, and set

$$r^* := r - \left(\sum_{p_0 \in P_0} \overline{h_p} \cdot p_0\right).$$

Then $\varphi(r^*) = 0$, that is, $r^* \in I_m$.

This leads to the following terminology: call the element r of $\mathbb{C}[Q_0 \setminus P_0]$ a relation mod $\langle P_0 \rangle$ if there exist $\overline{h_p} \in \mathbb{C}[P_0] \cdot S_0$ polynomials ($p \in P_0$) such that for the element r^* defined above $r^* \in I_m$. The above consideration shows that in this case the $\overline{h_p}$ polynomials are uniquely determined by r (because S is a system of secondary generators).

Let $R = \{r_1, \ldots r_k\} \subset \mathbb{C}[Q_0 \setminus P_0]$ be a set of relations modulo $\langle P_0 \rangle$. Write r_i^* for the element corresponding to r_i as described above, that is,

$$r_i^* := r_i - \left(\sum_{p_0 \in P_0} \overline{h_p^{(i)}} \cdot p_0\right) \in I_m,$$

where $\overline{h_p^{(i)}} \in \mathbb{C}[P_0] \cdot S_0$. Let $R^* = \{r_1^*, \dots r_k^*\}$. Then $R^* \subset I_m$.

Proposition 2.2.5 *Let $R = \{r_1, \dots r_k\} \subset \mathbb{C}[Q_0 \setminus P_0]$ be a set of relations $\operatorname{mod} \langle P_0 \rangle$ as described above. Assume that for any $s_0 \in S_0$ and any $q_0 \in Q_0 \setminus P_0$ there exist some $c_s \in \mathbb{C}$ ($s \in S_0$) such that*

$$s_0 \cdot q_0 - \left(\sum_{s \in S_0} c_s \cdot s \right) = \sum_{i=1}^{k} f_i \cdot r_i$$

holds for some $f_i \in \mathbb{C}[P_0] \cdot S_0$ ($1 \leq i \leq k$). Then the set R^* generates the ideal I_m of relations.

Proof According to 2.2.3 it suffices to show that for arbitrary $q_0 \in Q_0 \setminus P_0$ and $s_0 \in S_0$ there exists an $m_{q_0 s_0} \in \mathbb{C}[P_0] \cdot S_0$ such that $q_0 s_0 - m_{q_0 s_0} \in \langle R^* \rangle$. Induction on $\deg(q_0 s_0)$ shows that this follows from the condition on R.

If $\deg(q_0 s_0) = 0$, that is, $q_0 = s_0 = 1$, the claim is trivial, $1 \in \mathbb{C}[P_0] \cdot S_0$, and clearly $1 - 1 \in \langle R^* \rangle$. Now let $q_0 \in Q_0 \setminus P_0$ and $s_0 \in S_0$ be arbitrary elements such that $\deg(q_0 s_0) > 0$ and assume that for such products of smaller degree the claim is true.

According to the condition there exist some $c_s \in \mathbb{C}$ ($s \in S_0$) and $f_i \in \mathbb{C}[P_0] \cdot S_0$ ($1 \leq i \leq k$) such that

$$s_0 \cdot q_0 - \left(\sum_{s \in S_0} c_s \cdot s \right) = \sum_{i=1}^{k} f_i \cdot r_i. \tag{2.14}$$

Replacing r_i by

$$r_i^* + \left(\sum_{p_0 \in P_0} \overline{h_p^{(i)}} \cdot p_0 \right),$$

2.14 can be rewritten as

$$s_0 \cdot q_0 - \left(\sum_{s \in S_0} c_s \cdot s \right) = \sum_{i=1}^{k} f_i \cdot r_i^* + \sum_{i=1}^{k} f_i \cdot \left(\sum_{p_0 \in P_0} \overline{h_p^{(i)}} \cdot p_0 \right). \tag{2.15}$$

To show the claim about $s_0 \cdot q_0$, it suffices to show that the right side is equal to a sum of two terms: one in $\langle R^* \rangle$ and the other in $\mathbb{C}[P_0] \cdot S_0$. The polynomials $f_i, \overline{h_p^{(i)}}$ are all elements of $\mathbb{C}[P_0] \cdot S_0$. Thus there exist $a_s^{(i)}, b_s^{(i,p)} \in \mathbb{C}[P_0]$, for every $0 \leq i \leq k$, $s \in S_0$ such that

$$f_i = \sum_{s \in S_0} a_s^{(i)} \cdot s, \qquad \overline{h_p^{(i)}} = \sum_{s \in S_0} b_s^{(i,p)} \cdot s.$$

Rewriting the last term in (2.15) accordingly gives

$$\sum_{i=1}^{k} \left(\sum_{s_1 \in S_0} a_{s_1}^{(i)} \cdot s_1 \right) \cdot \left(\sum_{p_0 \in P_0} \left(\sum_{s_2 \in S_0} b_{s_2}^{(i,p)} \cdot s_2 \right) \cdot p_0 \right) =$$

$$= \sum_{s_1,s_2\in S_0} \left(\sum_{i=1}^{k} \left(a_{s_1}^{(i)} \cdot \sum_{p_0\in P_0} p_0 \cdot b_{s_2}^{(i,p)} \right) \right) \cdot s_1 s_2$$

Our goal is to prove that this is a sum of a term in $\langle R^* \rangle$ and one in $\mathbb{C}[P_0] \cdot S_0$. The assumption is that any product from $(Q_0 \setminus P_0) \cdot S_0$ of degree strictly smaller than $\deg(q_0 s_0)$ is equal to some element of $\mathbb{C}[P_0] \cdot S_0$ modulo $\langle R^* \rangle$. Now since $s_1 \in S$ is a monomial of elements of $Q_0 \setminus P_0$, if $\deg(s_1 s_2) < \deg(q_0 s_0)$ for every s_1, s_2 for which the coefficient is nonzero, the assumption shows that

$$s_1 s_2 \equiv \sum_{s\in S} d_{s_1,s_2}^{s} \cdot s \mod \langle R^* \rangle$$

for some $d_{s_1,s_2}^{s} \in \mathbb{C}[P_0]$. Consider the coefficient of $s_1 s_2$ in (2.15). The product $s_1 s_2$ might be equal to $s_3 s_4$ for some other elements of S_0, but the coefficient of a product of two elements of S is always the sum of expressions of the following form:

$$\sum_{i=1}^{k} \left(a_{s_1}^{(i)} \cdot \sum_{p_0\in P_0} p_0 \cdot b_{s_2}^{(i,p)} \right).$$

These expressions have constant term 0 because of the factor $p_0 \in P$ in every term. Hence the coefficient of $s_1 s_2$ on the right side of (2.15) has constant term 0. Clearly the two sides have the same degree, hence if the coefficient is nonzero, then $\deg(s_1 s_2) < \deg(q_0 s_0)$. According to the argument above (and the assumption of the induction) from this it follows that there exist polynomials $d_{s_1,s_2}^{s} \in \mathbb{C}[P_0]$ such that

$$s_1 s_2 \equiv \sum_{s\in S} d_{s_1,s_2}^{s} \cdot s \mod \langle R^* \rangle.$$

Rewriting (2.15) accordingly gives

$$s_0 \cdot q_0 - \left(\sum_{\overline{s}\in S_0} c_s \cdot \overline{s} \right) =$$

$$= \sum_{i=1}^{k} \overline{f_i} \cdot r_i^* + \sum_{s_1,s_2\in S_0} \left(\sum_{i=1}^{k} \left(a_{s_1}^{(i)} \cdot \sum_{p_0\in P_0} p_0 \cdot b_{s_2}^{(i,p)} \right) \right) \cdot s_1 s_2 \equiv$$

$$\equiv \sum_{s_1,s_2\in S_0} \left(\sum_{i=1}^{k} \left(a_{s_1}^{(i)} \cdot \sum_{p_0\in P_0} p_0 \cdot b_{s_2}^{(i,p)} \right) \right) \cdot \left(\sum_{s\in S} d_{s_1,s_2}^{s} \cdot s \right) \equiv$$

$$\equiv \sum_{s\in S} \left(\sum_{s_1,s_2\in S_0} d_{s_1,s_2}^{s} \cdot \left(\sum_{i=1}^{k} \left(a_{s_1}^{(i)} \cdot \sum_{p_0\in P_0} p_0 \cdot b_{s_2}^{(i,p)} \right) \right) \right) \cdot s \mod \langle R^* \rangle.$$

Now since

$$\sum_{s_1,s_2\in S_0} d^s_{s_1,s_2}\cdot\left(\sum_{i=1}^{k}\left(a^{(i)}_{s_1}\cdot\sum_{p_0\in P_0} p_0\cdot b^{(i,p)}_{s_2}\right)\right)\in\mathbb{C}[P_0],$$

the claim is true for $q_0 \cdot s_0$. Therefore the induction proves the proposition. □

In the following we summarize the conditions which imply that a set R^* is a minimal generating system of the ideal I_m.

Theorem 2.2.6 *Let $R = \{r_1, \dots r_k\} \subset \mathbb{C}[Q_0 \setminus P_0]$ be a set of relations* $\bmod \langle P_0\rangle$, *$R^* \subset I_m$ the corresponding set as described above. Write $|R_l|$ for the number of elements in R with degree l and j_l for the coefficient of t^l in $J_m(t, \dots, t)$ (j_0 is the constant term), and d for the maximum degree of elements of R. Assume that for every $s_0 \in S$ and any $q \in Q_0 \setminus P_0$ there exist some $c_s \in \mathbb{C}$ ($s \in S_0$) such that*

$$s_0\cdot q-\left(\sum_{s\in S_0} c_s\cdot s\right)=\sum_{i=1}^{k} f_i\cdot r_i$$

holds for some $f_i \in \mathbb{C}[P_0]\cdot S_0$ ($1 \le i \le k$). Assume moreover that $0 \le j_l = |r_l|$ for $0 \le l \le d$. Then R^* minimally generates the ideal of relations.

Proof According to 2.2.5 the conditions imply that R^* generates I_m. Clearly $\deg(r_i) = \deg(r_i^*)$ for $1 \le i \le k$. For any $1 \le i \le k$, $R^* \setminus \{r_i^*\}$ has less than j_d elements in degree $d = \deg(r_i)$, thus by 2.2.4 and the conditions it follows that $R^* \setminus \{r_i^*\}$ does not generate I_m. Thus R^* is minimal. □

2.3 Finding S and R at the same time

The results of the previous chapters can be summarized as follows. Let S_0 be a set of monomials of $Q_0 \setminus P_0$, such that $\{1\} \cup (Q_0 \setminus P_0) \subseteq S_0$; $S := \varphi(S)$ (assume φ is injective on S_0). Let R be a set of (homogeneous) relations modulo $\langle P_0\rangle$. Write $|R_l|$ for the number of elements in R with degree l and j_l for the coefficient of t^l in $J_m(t, \dots, t)$ (j_0 is the constant term), d for the maximum degree of elements of R and R^* for the corresponding subset of I_m.

Theorem 2.3.1 *Assume that for any $s_0 \in S_0$ and any $q \in Q_0 \setminus P_0$*

$$s_0\cdot q-\left(\sum_{s\in S_0} c_s\cdot s\right)=\sum_{r\in R} f_r\cdot r \tag{2.16}$$

holds for some $c_s \in \mathbb{C}$ and some $f_r \in \mathbb{C}[P_0] \cdot S_0$. Assume moreover that $H(\mathrm{Span}\,(S), \mathbf{t}) = E_m(\mathbf{t})$ and that $0 \leq j_l = |r_l|$ for $0 \leq l \leq d$. Then S is a secondary generating system of $\mathbb{C}[V^m]^G$ (with the primary generating system P) and R^* minimally generates the ideal of relations.

Proof For any $s_1 \in S$ and $q_1 \in Q \setminus P$ choose c_s and f_r such that (2.16) holds for the elements $s_0 \in S_0$ and $q \in Q_0$ for which $s_1 = \varphi(s_0)$ and $q_1 = \varphi(q)$. Since every $r \in R$ is a relation modulo $\langle P_0 \rangle$, $\varphi(r) \in \langle P \rangle$ holds. Thus the conditions imply that condition 3 of 2.1.4 holds on S. Thus S is a secondary generating system of $\mathbb{C}[V^m]^G$. Then the theorem follows by 2.2.6. □

In the next chapter, calculations are performed in $\mathbb{C}[V^m]^G$ by writing $\varphi(r)$ (a relation modulo $\langle P \rangle$) instead of a relation r modulo $\langle P_0 \rangle$. Monomials of the elements of $Q \setminus P$ can be rewritten as φ-images of monomials of $Q_0 \setminus P_0$. Therefore given a set of relation mod $\langle P \rangle$, it is clear which set R of $\mathrm{mod}\ \langle P_0 \rangle$ relations it corresponds to. Sometimes both sets are denoted by R.

To produce relations $r \bmod \langle P \rangle$, the result 1.1.2 is used. As seen in 1.1.2, the image $\varphi(\Psi_4(w_1, w_2, w_3, w_4))$ where $w_1, w_2, w_3, w_4 \in \mathcal{M}$ gives an identity in $(\mathbb{C}[V^m]^{S_3}$. According to 1.1.3 and 1.1.4, the ideal I_m of the algebra $\mathcal{F}(3)$ is generated by the elements $\Psi_4(w_1, w_2, w_3, w_4)$ where $w_1, w_2, w_3, w_4 \in \mathcal{M}$ and $\deg(w_1 w_2 w_3 w_4) \leq 8$. Thus considering identities of this form gives a sufficient supply of relations.

The identity of 1.1.2 takes the following form (writing x, y, z, w instead of w_1, w_2, w_3, w_4)

$$\begin{aligned} 6[xyzw] \;&=\; 2\big([xyz][w] + [xyw][z] + [xzw][y] + [yzw][x]\big)+ \\ &+\; \big([xy][zw] + [xz][yw] + [xw][yz] + [x][y][z][w]\big)+ \\ &-\; \big([xy][z][w] + [xz][y][w] + [xw][y][z]+ \\ &\quad +[zw][x][y] + [yw][x][z] + [yz][x][w]\big) \end{aligned} \tag{2.17}$$

Chapter 3

Calculations

In this chapter we perform some calculations in the cases $n = 3$, $m = 2, 3, 4$ respectively. A minimal generating system of I_m is given in each case. In the cases $m = 2$ and $m = 3$ a system of secondary generators is determined.

The notation and terminology used is the same as in 2. Our purpose is to determine an appropriate set S (in cases $m = 2$ and $m = 3$) and R (in all three cases), and prove (by 2.3.1) that S is a system of secondary generators, and R is a set of relations modulo $\langle P_0 \rangle$ for which R^* is a minimal generating system of I_m. Therefore the sets S and R are built such that they satisfy the conditions of 2.1.4 and 2.2.6 respectively. According to 2.1.5 such an S exists, this is applied in the case $m = 4$, where not all elements of S are determined. As in 2.2.3, the elements of R are relations modulo $\langle P_0 \rangle$, but to simplify notation, they are written as relations modulo $\langle P \rangle$. Congruences throughout the calculations are congruences modulo $\langle P \rangle$.

3.1 The case $m = 2$

This case is discussed in detail in the subsection 6.2 of [1]. By (1.22), the Hilbert-series can be easily computed:

$$H(\mathbb{C}[V^m]^G, x, y) = \frac{1 \cdot 1 + (x + x^2)(y + y^2) + x^3 \cdot y^3}{(1-x)(1-x^2)(1-x^3)(1-y)(1-y^2)(1-y^3)} \Rightarrow$$

$$\Rightarrow H(\mathbb{C}[V^m]^G,\ x, y) = \frac{1 + xy + xy^2 + x^2y + x^2y^2 + x^3y^3}{(1-x)(1-x^2)(1-x^3)(1-y)(1-y^2)(1-y^3)} \tag{3.1}$$

By the third condition in 2.1.4 from this it follows that elements of S should have multidegrees $(0, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 3)$ (each with multiplicity 1). Some secondary generators are going to be chosen for each multidegree using identities of the form (2.17).

According to 1.2.2, if w is a monomial having degree ≥ 3 in one of the two variables, and having total degree ≥ 4, $[w] \equiv 0$. To make calculations more transparent, x and y are written instead of the variables $x(1)$ and $x(2)$.

Multidegrees $(0,0)$, $(1,1)$, $(1,2)$, $(2,1)$, **and** $(2,2)$**.** For each of these multidegrees there exists only one polynomial of elements of Q that is not in $\langle P\rangle$: 1, $[xy]$, $[xy^2]$, $[x^2y]$ and $[xy]^2$ respectively. Let $\{1, [xy], [xy^2], [x^2y][xy]^2\} \subset S$.

Multidegree $(3,3)$**.** There are two choices for a secondary generator of this degree: $[xy]^3$ or $[x^2y][xy^2]$. From the identity $\varphi(\Psi_4(w_1, w_2, w_3, w_4)) = 0$ it follows that

$$\begin{aligned} 6[x^3y^3] &= 6[xy \cdot xy \cdot x \cdot y] \equiv \\ &\equiv 4 \cdot [x^2y^2][xy] + (2[x^2y][xy^2] + [x^2y^2][xy]) - [xy]^3 = \\ &= 5 \cdot [x^2y^2][xy] + 2[x^2y][xy^2] - [xy]^3 \end{aligned} \tag{3.2}$$

$$1.2.2 \Rightarrow 6[x^3y^3] \equiv 0 \tag{3.3}$$

$$6[x^2y^2] = 6[x \cdot x \cdot y \cdot y] \equiv 2[xy]^2 \Rightarrow [x^2y^2] \equiv \frac{1}{3}[xy]^2 \tag{3.4}$$

$$(3.2),\ (3.4), (3.3) \Rightarrow 6[x^3y^3] \equiv \frac{2}{3}[xy]^3 + 2[x^2y][xy^2] \equiv 0 \tag{3.5}$$

$$(3.5) \Rightarrow [xy]^3 \equiv -3[x^2y][xy^2] \tag{3.6}$$

We assert that the set $S = \{1, [xy], [xy^2], [x^2y][xy]^2, [x^2y][xy^2]\}$ (notation taken from [1]) is a secondary generating system, that is, the algebra $\mathbb{C}[V^m]^G$ is generated as a (free) $\mathbb{C}[P]$-module by S. It is enough to prove that S satisfies the conditions of 2.1.4. Conditions 1., 2. and 4. are trivially satisfied. Hence, it suffices to show that 3. holds, which follows if the product of any two elements of S can be rewritten as a linear combination of elements of S modulo $\langle P\rangle$. To show such congruences, some relations are needed. Consider some consequences of $\varphi(\Psi_4(w_1, w_2, w_3, w_4)) = 0$.

$$6[x^2y^3] \equiv 0 \tag{3.7}$$

$$6[x^2y^3] = 6[xy \cdot x \cdot y \cdot y] \equiv 2[xy][xy^2] + 2[xy^2][xy] = 4[xy][xy^2] \tag{3.8}$$

$$(3.7),\ (3.8) \Rightarrow [xy][xy^2] \equiv 0 \tag{3.9}$$

Clearly from (3.9) and the logical symmetry it follows that $[xy][x^2y] \equiv 0$. An other consequence is the following:

$$6[x^2y^4] = 6[xy \cdot xy \cdot y \cdot y] \equiv 4[xy][xy^3] + 2[xy^2]^2 \tag{3.10}$$

$$1.2.2 \Rightarrow [xy^3] \equiv 0 \tag{3.11}$$

$$(3.10),\ (3.11) \Rightarrow [xy^2]^2 \equiv 0 \tag{3.12}$$

Again by symmetry it follows that $[x^2y]^2 \equiv 0$. So far the following relations have been found (by (3.6), (3.9), (3.12) and symmetry):

$$[xy][xy^2]\ (r_{2,3}),\ [xy][x^2y]\ (r_{3,2}),$$
$$[xy]^3 + 3[x^2y][xy^2]\ (r_{3,3}),$$
$$[xy^2]^2\ (r_{2,4}),\ [x^2y]^2\ (r_{4,2})$$

The following table of multiplication shows how to rewrite any product of two elements in S as a linear combination of elements of S modulo $\langle P\rangle$. (The row and column corresponding to 1 is trivial, hence omitted. The table is symmetric with respect to the main diagonal. Write $s_{1,1} := [xy]$, $s_{2,1} := [x^2y]$, $s_{1,2} := [xy^2]$, $s_{2,2} := [xy]^2$ and $s_{3,3} := [x^2y][xy^2]$ for the elements of S).

$\cdot$	$s_{1,1}$	$s_{2,1}$	$s_{1,2}$	$s_{2,2}$	$s_{3,3}$
$s_{1,1}$	$s_{2,2}$	$0\ (r_{3,2})$	$0\ (r_{2,3})$	$-3s_{3,3}\ (r_{3,3})$	$0\ (r_{3,2})$
$s_{2,1}$	0	$0\ (r_{4,2})$	$s_{3,3}$	$0\ (r_{3,2})$	$0\ (r_{4,2})$
$s_{1,2}$	0	$s_{3,3}$	$0\ (r_{2,4})$	$0\ (r_{2,3})$	$0\ (r_{2,4})$
$s_{2,2}$	$-3s_{3,3}$	0	0	$0\ (r_{3,3} \wedge r_{3,2})$	$0\ (r_{3,2})$
$s_{3,3}$	0	0	0	0	$0\ (r_{4,2})$

Table 3.1: Multiplication table of $S \bmod \langle P\rangle$ in case $m = 2$

Hence in the case $m = 2$ S is a system of secondary generators. Consider the set $R = \{r_{2,3}, r_{3,2}, r_{2,4}, r_{4,2}, r_{3,3}\}$. To prove that the set of relations R^*corresponding to R minimally generates I_m, it is enough to prove that R satisfies the conditions of 2.2.6. As seen above, products of $(Q \setminus P) \cdot S$ can be formally rewritten as elements of $\mathbb{C}[P] \cdot S$ using elements of R. On the other hand, R has 2 elements of degree 5 and 3 elements of degree 6, hence $|R_0| = |R_1| = |R_2| = |R_3| = |R_4| = 0$, $|R_5| = 2$ and $|R_6| = 3$, the maximal degree of the elements is $d = 6$. By (2.11), the numerator $J_2(\mathbf{t})$ of the Hilbert-series of the ideal I_2 is

$$J_2(t,t) = 1 - E_2(t,t) \cdot \prod_{\substack{i,j=1 \\ i<j}}^{2} h_2(t,t) \cdot \prod_{\substack{i,j,k=1 \\ i<j<k}}^{2} h_3(t,t,t) = 1 - E_2(t,t) \cdot h_2(t,t)$$

By (3.1) $E_2(t,t) = 1 + t^2 + 2t^3 + t^4 + t^6$, and by (1.26) $h_2(t,t) = (1-t^2)(1-t^3)^2$, hence

$$J_2(t,t) = 1 - (1 + t^2 + 2t^3 + t^4 + t^6) \cdot (1-t^2)(1-t^3)^2 = 2t^5 + 3t^6 - 2t^9 - 3t^8 + t^{14}.$$

Thus $j_0 = j_1 = j_2 = j_3 = j_4 = 0$, $j_5 = 2$ and $j_6 = 3$. As $j_l = |R_l|$ for $0 \leq l \leq d = 6$, the conditions of 2.2.6 hold for R. This proves that the set R^* derived from R as discussed in 2.2.3 is a minimal generating system of I_2.

Relations to eliminate the superfluous generators. During the calculations above, some useful $\bmod \langle P \rangle$ relations have been found that are not elements of R :

$$\text{(3.4)} \Rightarrow [x^2y^2] \equiv \frac{1}{3}[xy]^2 \tag{3.13}$$

$$1.2.2 \Rightarrow [xy^3] \equiv 0 \tag{3.14}$$

These are referred to in the cases $m > 2$.

3.2 The case $m = 3$

Similarly to the previous case, the set S is determined by choosing secondary generators by multidegrees such that $H(\text{Span}\,(S); t_1, t_2, t_3) = E_3(t_1, t_2, t_3)$ holds.

According to 1.3.2, the numerator of the Hilbert-series of $\mathbb{C}[V^3]^G$ is $E_3(t_1, t_2, t_3)$, which can be computed using the recursion (1.22), (1.23) and (1.24). The polynomial $E_3(t_1, t_2, t_3)$, is obviously symmetric in the variables t_1, t_2 and t_3. Thus it is uniquely determined by giving one coefficient for all descending multidegrees, that is, multidegrees $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ for which $\alpha_1 \geq \alpha_2 \geq \alpha_3$. As all nonzero coefficients happen to be 1, $E_3(t_1, t_2, t_3)$ is determined by the list of descending multidegrees with nonzero coefficients. This list is the following: $(3, 2, 2)$; $(3, 3, 0)$; $(3, 2, 1)$; $(2, 2, 2)$; $(3, 1, 1)$; $(2, 2, 1)$; $(2, 2, 0)$; $(2, 1, 1)$; $(1, 1, 1)$; $(2, 1, 0)$; $(1, 1, 0)$; $(0, 0, 0)$. To satisfy the fourth condition in 2.1.4, S should have one element for each multidegree appearing with positive coefficient in $E_3(t_1, t_2, t_3)$. The logical symmetry in the variables is used to reduce the number of cases checked: if $\alpha := (\alpha_1, \alpha_2, \alpha 3)$ is a descending multidegree, the expressions of descending multidegree α are all expressions of multidegree $(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)})$, where $\sigma \in S_3$. Also, x, y and z are written instead of the variables $x(1)$, $x(2)$ and $x(3)$.

Multidegrees with a 0 coordinate. Since $\mathbb{C}[V^2]^G$ is a subalgebra of $\mathbb{C}[V^3]^G$, the secondary generators and relations found in the case $m = 2$ hold in the present case. The table below shows the secondary generators chosen accordingly, listed by the descending multidegrees with $\alpha_3 = 0$.

The relations found in the case $m = 2$ are obviously valid in the case $m = 3$ too. In notation a relation r is used to denote not only r as it was defined, but also all relations that

Total degree	Multidegree	Secondary generators
6	$(3,3,0)$	$[x^2y][xy^2]$, $[y^2z][yz^2]$, $[z^2x][zx^2]$
4	$(2,2,0)$	$[xy]^2$, $[yz]^2$, $[zx]^2$
3	$(2,1,0)$	$[x^2y]$, $[xy^2]$, $[y^2z]$, $[yz^2]$, $[z^2x]$, $[zx^2]$
2	$(1,1,0)$	$[xy]$, $[yz]$, $[zx]$
0	$(0,0,0)$	1

Table 3.2: Secondary generators for $m = 3$ from the case $m = 2$

can be derived from r by permuting the variables x, y, and z. For example the relation $r_{4,2}$ may refer either to $[x^2y]^2 \equiv 0 \bmod \langle P \rangle$ or to $[yz^2]^2 \equiv 0 \bmod \langle P \rangle$.

Multidegrees $(1,1,1)$ **and** $(2,1,1)$**.** The only monomials in $\mathbb{C}[Q \setminus P]$ of multidegree $(1,1,1)$ and $(2,1,1)$ are $[xyz]$ and $[xy][xz]$. Hence the secondary generators corresponding to these descending multidegrees should be $[xyz]$, $[xy][xz]$, $[xy][yz]$ and $[xz][yz]$.

Descending multidegree $(3,1,1)$**.** The possible choices for a secondary generator of multidegree $(3,1,1)$ are $[x^2y][xz]$ and $[x^2z][xy]$. According to 1.2.2, $[x^3yz] \equiv 0$. From the identity $\varphi(\Psi_4(x^2,x,y,z)) = 0$ it follows that

$$0 \equiv 6[x^3yz] = 6[x^2 \cdot x \cdot y \cdot z] \equiv [x^2y][xz] + [x^2z][xy] \Rightarrow$$

$$\Rightarrow [x^2y][xz] + [x^2z][xy] \equiv 0 \ (r_{3,1,1}). \tag{3.15}$$

Thus $[x^2z][xy] \equiv -[x^2y][xz]$. Choose the following secondary generators for this descending multidegree: $[x^2y][xz]$, $[y^2z][xy]$ and $[z^2x][yz]$.

Descending multidegree $(2,2,1)$**.** Monomials of elements of $Q \setminus P$ of multidegree $(2,2,1)$ are $[x^2y][yz]$, $[xy^2][xz]$ and $[xy][xyz]$. From different substitutions into the fundamental identity it follows that

$$6[x^2y^2z] = 6[x \cdot y \cdot z \cdot xy] \equiv 3[xy][xyz] + [xz][xy^2] + [x^2y][yz] \tag{3.16}$$

$$6[x^2y^2z] = 6[x^2 \cdot y \cdot y \cdot z] \equiv 2[x^2y][yz] \tag{3.17}$$

$$6[x^2y^2z] = 6[x \cdot x \cdot y^2 \cdot z] \equiv 2[xy^2][xz] \tag{3.18}$$

$$\text{(3.17), (3.18)} \Rightarrow 0 \equiv [x^2y][yz] - [xy^2][xz] \ (r^{(1)}_{2,2,1}) \tag{3.19}$$

$$\text{(3.16), (3.17), (3.18)} \Rightarrow 6[x^2y^2z] \equiv 3[xy][xyz] + 6[x^2y^2z] \Rightarrow$$

$$\Rightarrow 0 \equiv [xy][xyz] \ (r^{(2)}_{2,2,1}) \tag{3.20}$$

Thus $[xy][xyz] \equiv 0$ and $[x^2y][yz] \equiv [xy^2][xz]$. Choose the following secondary generators for $(2,2,1)$ descending multidegree: $[xy^2][xz]$, $[yz^2][zy]$ and $[zx^2][yz]$.

Descending multidegree $(3,2,1)$**.** Monomials of elements of $Q \setminus P$ of multidegree $(3,2,1)$ are $[x^2y][xyz]$, $[x^2z][xy^2]$ and $[xy]^2[xz]$. By 1.2.2 $[x^3y^2z] \equiv 0$. Consider the following consequences of $\varphi(\Psi_4(w_1,w_2,w_3,w_4)) = 0$:

$$0 \equiv 6[x^3y^2z] = 6[x^2 \cdot x \cdot y^2 \cdot z] \equiv [x^2y^2][xz] + [x^2z][xy^2] \tag{3.21}$$

$$\text{(3.21), (3.13)} \Rightarrow 0 \equiv 3[x^2z][xy^2] + [xy]^2[xz] \ (r^{(1)}_{3,2,1}) \tag{3.22}$$

$$0 \equiv 6[x^3y^2z] = 6[xyz \cdot x \cdot x \cdot y] \equiv 2[x^2y][xyz] + 2[x^2yz][xy] \tag{3.23}$$

$$[x^2yz] = [x \cdot x \cdot y \cdot z] \equiv \frac{1}{3}[xy][yz] \tag{3.24}$$

$$\text{(3.23), (3.24)} \Rightarrow 0 \equiv 3[x^2y][xyz] + [xy]^2[xz] \Rightarrow$$

$$0 \equiv [x^2y][xyz] - [x^2z][xy^2] \ (r^{(2)}_{3,2,1}) \tag{3.25}$$

Thus $[xy]^2[xz] \equiv -3[x^2z][xy^2]$ and $[x^2y][xyz] \equiv [x^2z][xy^2]$. Let the secondary generators of $(3,2,1)$ descending multidegree be $[x^2z][xy^2]$, $[x^2y][xz^2]$, $[y^2z][x^2y]$, $[y^2x][yz^2]$, $[xy^2][yz^2]$, $[xz^2][y^2z]$.

Multidegree $(2,2,2)$**.** Monomials in $\mathbb{C}[Q \setminus P]$ of this multidegree are $[xy][yz][zx]$, $[xyz]^2$, $[xy^2][xz^2]$, $[x^2y][yz^2]$ and $[x^2z][y^2z]$. Some consequences of $\varphi(\Psi_4(w_1,w_2,w_3,w_4)) = 0$ are:

$$6[x^2y^2z^2] = 6[x \cdot x \cdot y^2 \cdot z^2] \equiv 2[xy^2][xz^2] \tag{3.26}$$

$$6[x^2y^2z^2] = 6[x^2 \cdot y \cdot y \cdot z^2] \equiv 2[x^2y][yz^2] \tag{3.27}$$

$$6[x^2y^2z^2] = 6[x^2 \cdot y \cdot y \cdot z^2] \equiv 2[x^2z][y^2z] \tag{3.28}$$

$$6[x^2y^2z^2] = 6[x \cdot y \cdot z \cdot xyz] \equiv 2[xyz]^2 + [x^2yz][yz] + [xy^2z][xz] + [xyz^2][xy] \tag{3.29}$$

$$\text{(3.24),(3.29)} \Rightarrow 6[x^2y^2z^2] \equiv 2[xyz]^2 + [xy][yz][zx] \tag{3.30}$$

$$6[x^2y^2z^2] = 6[x \cdot y \cdot xy \cdot z^2] \equiv 3[xy][xyz^2] + [x^2y][yz^2] + [xy^2][xz^2] \tag{3.31}$$

$$2 \cdot \text{(3.31)} - \text{(3.26)} - \text{(3.27), (3.24)} \Rightarrow 0 \equiv [xy][yz][zx] \ (r^{(1)}_{2,2,2}) \tag{3.32}$$

$$\text{(3.30), (3.32)} \Rightarrow 3[x^2y^2z^2] \equiv [xyz]^2 \tag{3.33}$$

$$\text{(3.26), (3.33)} \Rightarrow 0 \equiv [xyz]^2 - [xy^2][xz^2] \ (r^{(2)}_{2,2,2}) \tag{3.34}$$

Thus $[xy][yz][zx] \equiv 0$, $[xy^2][xz^2] \equiv [xyz]^2$ and clearly $[x^2y][yz^2] \equiv [x^2z][y^2z] \equiv [xyz]^2$. Let $[xyz]^2$ be the secondary generator of $(2,2,2)$ multidegree.

Multidegree $(3,2,2)$**.** Monomials in $\mathbb{C}[Q \setminus P]$ of this multidegree are the following: $[x^2y][xz][yz]$, $[x^2z][xy][yz]$, $[xyz][xy][xz]$, $[xy^2][xz]^2$, $[xz^2][xy]^2$. Now from the relations found above it follows that

$$r^{(1)}_{2,2,1} \Rightarrow [xy^2][xz]^2 \equiv [x^2y][yz][xz]$$

$$r^{(1)}_{2,2,1},\ r_{3,1,1} \Rightarrow [xz^2][xy]^2 \equiv [x^2z][zy][xy] \equiv -[x^2y][xz][yz]$$

$$r^{(2)}_{2,2,1} \Rightarrow [xyz][xy][xz] \equiv 0.$$

Let the secondary generators of $(3,2,2)$ descending multidegree be $[x^2y][xz][yz]$, $[y^2z][xy][xz]$ and $[z^2x][yz][xy]$.

To prove that the monomials of $\mathbb{C}[Q \setminus P]$ chosen above form a secondary generating system, one more relation is needed. Consider the following consequences of $\varphi(\Psi_4(w_1,w_2,w_3,w_4)) = 0:$

$$1.2.2 \Rightarrow [x^4yz] \equiv 0$$

$$0 \equiv [x^4yz] = [x^2 \cdot x^2 \cdot y \cdot z] \equiv \frac{1}{3}[x^2y][x^2z] \Rightarrow 0 \equiv [x^2y][x^2z]\ (r_{4,1,1}).$$

3.2.1 The secondary generator system.

Table 3.3 shows the elements chosen as secondary generators in the calculations above. The set S of these generators satisfies the conditions 1, 2 and 4 of 2.1.4. According to the argument below proving that the set of relations defined generates I_3, S satisfies condition 3 too and is hence a secondary generating system.

3.2.2 Minimal generating system of the ideal I_3.

Table 3.4 shows the types of relations defined above. The relations are listed by descending multidegree, and the third column shows the number of relations of a certain type. Let R be the set of relations of the types listed above (every element of R can be derived from a relation above by permuting the variables). To prove that R^* is a minimal generating system of the ideal I_3, it is sufficient to prove that the conditions of 2.2.6 hold for R.

According to table 3.4, R contains elements of total degree 5 and 6 only, $|R_5| = 15$ and $|R_6| = 28$. The numerator of the Hilbert-series of the ideal I_3 is (by (2.11)):

$$J_3(t,t,t) = 1 - E_3(t,t,t) \cdot \prod_{\substack{i,j=1 \\ i<j}}^{3} h_2(t,t) \cdot \prod_{\substack{i,j,k=1 \\ i<j<k}}^{3} h_3(t,t,t) \Rightarrow$$

$$\Rightarrow 1 - E_3(t,t,t) \cdot (h_2(t,t))^3 \cdot h_3(t,t,t).$$

Degree	Descending multidegree	Secondary generators
0	$(0,0,0)$	1
2	$(1,1,0)$	$[xy]$, $[yz]$, $[zx]$
3	$(2,1,0)$	$[x^2y]$, $[xy^2]$, $[y^2z]$, $[yz^2]$, $[z^2x]$, $[zx^2]$
3	$(1,1,1)$	$[xyz]$
4	$(2,2,0)$	$[xy]^2$, $[yz]^2$, $[zx]^2$
4	$(2,1,1)$	$[xy][xz]$, $[xy][yz]$, $[xz][yz]$
5	$(3,1,1)$	$[x^2y][xz]$, $[y^2z][xy]$, $[z^2x][yz]$
5	$(2,2,1)$	$[xy^2][xz]$, $[yz^2][zy]$, $[zx^2][yz]$
6	$(3,3,0)$	$[x^2y][xy^2]$, $[y^2z][yz^2]$, $[z^2x][zx^2]$
6	$(3,2,1)$	$[x^2z][xy^2]$, $[x^2y][xz^2]$, $[y^2z][x^2y]$, $[xy^2][yz^2]$, $[yz^2][zx^2]$, $[xz^2][y^2z]$
6	$(2,2,2)$	$[xyz]^2$
7	$(3,2,2)$	$[x^2y][xz][yz]$, $[y^2z][xy][xz]$ $[z^2x][yz][xy]$

Table 3.3: Secondary generators in the case $m = 3$

Degree	Descending multidegree	Type of relation	Name	$\sharp$
5	$(3,2,0)$	$[xy][x^2y] \equiv 0$	$r_{3,2}$	6
5	$(3,1,1)$	$[x^2y][xz] + [x^2z][xy] \equiv 0$	$r_{3,1,1}$	3
5	$(2,2,1)$	$[x^2y][yz] - [xy^2][xz] \equiv 0$	$r^{(1)}_{2,2,1}$	3
5	$(2,2,1)$	$[xy][xyz] \equiv 0$	$r^{(2)}_{2,2,1}$	3
6	$(4,2,0)$	$[x^2y]^2 \equiv 0$	$r_{4,2}$	6
6	$(4,1,1)$	$[x^2y][x^2z] \equiv 0$	$r_{4,1,1}$	3
6	$(3,3,0)$	$[xy]^3 + 3[x^2y][xy^2] \equiv 0$	$r_{3,3}$	3
6	$(3,2,1)$	$3[x^2z][xy^2] + [xy]^2[xz] \equiv 0$	$r^{(1)}_{3,2,1}$	6
6	$(3,2,1)$	$[x^2y][xyz] - [x^2z][xy^2] \equiv 0$	$r^{(2)}_{3,2,1}$	6
6	$(2,2,2)$	$[xy][yz][zx] \equiv 0$	$r^{(1)}_{2,2,2}$	1
6	$(2,2,2)$	$[xyz]^2 - [xy^2][xz^2] \equiv 0$	$r^{(2)}_{2,2,2}$	3

Table 3.4: Relations in the case $m = 3$

Calculation (using (1.22) and (1.26)) shows that the first coefficients in $J_3(t,t,t)$ are $j_0 = j_1 = j_2 = j_3 = j_4 = 0$, $j_5 = 15$ and $j_6 = 28$. From this it follows that $0 \leq j_l \leq |R_l|$ for $0 \leq l \leq 6$. The maximum degree of elements of R is $d = 6$. From this it follows that if R^* generates I_3, then it is minimal.

Lemma 3.2.1 *If for any $q_1, q_2, q_3 \in Q_0 \setminus P_0$ the product $q_1q_2q_3$ can be reduced to a linear combination of S_0 modulo $\langle P_0 \rangle$, then S_0 is a secondary generating system, and R^* generates the ideal of relations.*

Proof Both parts of the claim are true if any product $q \cdot s$ ($q \in Q_0 \setminus P_0$, $s \in S_0$) can be reduced to a linear combination of S_0 using the relations in R. As elements of S are monomials of elements of $Q_0 \setminus P_0$, it suffices to show that any monomial in $\mathbb{C}[Q_0 \setminus P_0]$ can be reduced to $\mathrm{Span}\,(S_0)$ modulo $\langle P_0 \rangle$. If this is true for a monomial of length three, it follows for a monomial of arbitrary length by induction. □

During calculations, we continue to work in $\mathbb{C}[V^3]^G$.

Products of length less than three. If one or two q_i-s are equal to 1 in the product $q_1q_2q_3$, the monomial is a product of one or two elements of $Q \setminus P$. Since $Q \setminus P \subset S$, the reduction is trivial if the product has length 1. If the product has length two, it is either an element of S as listed in 3.3, or equal (up to permutations of x, y and z) to one of the following: $[xy][x^2y]$, $[xy][x^2z]$, $[xy][xz^2]$, $[xy][xyz]$, $[x^2y]^2$, $[x^2y][yz^2]$, $[x^2y][x^2z]$, $[x^2y][xyz]$. Modulo $\langle P \rangle$ these are equal to 0 or $c \cdot s$ for some $c \in \mathbb{C}$ and $s \in S$ by relations of the type $r_{3,2}$, $r_{3,1,1}$, $r^{(1)}_{2,2,1}$, $r^{(2)}_{2,2,1}$, $r_{4,2}$, $r^{(2)}_{2,2,2}$, $r_{4,1,1}$, $r^{(2)}_{3,2,1}$ respectively.

Products of length three. In the following we systematically check that the products $q_1q_2q_3$ ($1 \notin \{q_1, q_2, q_3\}$) can be rewritten in a similar way by using elements of R. As before, not all products are checked separately, only the ones that are different up to the permutation of variables. The types of relations used are listed after every chain of steps.

If $[\mathbf{xyz}] \in \{\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}\}$. The product has one, two or three $[xyz]$ factors. If $q_1 = q_2 = q_3 = [xyz]$, then

$$q_1q_2q_3 = [xyz]^3 \equiv [xyz] \cdot [xy^2][xz^2] \equiv [y^2x][yz^2] \cdot [xz^2] \equiv 0$$

holds by $r^{(1)}_{2,2,1} \cdot [xyz]$, $r^{(2)}_{3,2,1} \cdot [xz^2]$ and $r_{4,1,1} \cdot [y^2x]$. If $q_1 = q_2 = [xyz] \neq q_3$, , then q_3 is $[x^2y]$ or $[xy]$ up to symmetry. If $q_3 = [x^2y]$, then

$$q_1q_2q_3 = [xyz]^2[x^2y] \equiv [xyz] \cdot \left(-\frac{1}{3}\right) [xy]^2 \cdot [xz] \equiv 0$$

holds by $(r^{(2)}_{3,2,1} + \frac{1}{3} r^{(1)}_{3,2,1}) \cdot [xyz]$ and $r^{(2)}_{2,2,1} \cdot [xy][xz]$. If $q_3 = [xy]$, then $q_1 q_2 q_3 = [xyz]^2[xy] \equiv 0$ by $r^{(2)}_{2,2,1} \cdot [xyz]$. If only one factor is equal to $[xyz]$, then by relations $[xyz][xy] \equiv 0$ $(r^{(2)}_{2,2,1})$ or $[xyz][x^2y] \equiv [x^2z][xy^2]$ $(r^{(2)}_{3,2,1})$ it can be rewritten into a product without $[xyz]$. These are discussed in the following cases.

If $\deg(\mathbf{q_1}) = \deg(\mathbf{q_2}) = \deg(\mathbf{q_3}) = \mathbf{3}$. In the following we may assume that the product has no factor $[xyz]$. If $\deg(q_1) = \deg(q_2) = \deg(q_3) = 3$, we may assume that $q_1 = [x^2y]$ and q_1, q_2 and q_3 are all different, since $[x^2y]^2 \equiv 0$ by $r_{4,2}$. As $[x^2y][x^2z] \equiv 0$ by $r_{4,1,1}$, we may assume that q_2 and q_3 are not equal to $[x^2z]$. This leaves us 6 possible $q_1q_2q_3$ products to examine. The following lines show how to rewrite these using R.

$$[x^2y][xy^2][y^2z] \equiv 0 \ (r_{4,1,1} \cdot [x^2y])$$

$$[x^2y][yz^2][xz^2] \equiv 0 \ (r_{4,1,1} \cdot [x^2y])$$

$$[x^2y][xy^2][yz^2] \equiv -[x^2y] \cdot \frac{1}{3}[yz]^2[xy] \equiv 0 \ \left(\frac{[x^2y]}{3} r^{(1)}_{3,2,1},\ r_{3,2} \cdot \frac{[yz]^2}{3} \right)$$

$$[x^2y][xy^2][xz^2] \equiv -[xy^2] \cdot \frac{1}{3}[xz]^2[xy] \equiv 0 \ \left(\frac{[xy^2]}{3} r^{(1)}_{3,2,1},\ r_{3,2} \cdot \frac{[xz]^2}{3} \right)$$

$$[x^2y][y^2z][yz^2] \equiv -\frac{1}{3}[xy]^2[yz][yz^2] \equiv 0 \ \left(\frac{[yz^2]}{3} r^{(1)}_{3,2,1},\ r_{3,2} \cdot \frac{[xy]^2}{3} \right)$$

$$[x^2y][y^2z][xz^2] \equiv -\frac{1}{3}[xy]^2[yz][xz^2] = -\frac{[xy]^2}{3}[yz][xz^2] \equiv$$

$$\equiv \frac{[xy]^2}{3}[yz^2][xz] = \frac{1}{3}[yz^2][xy][xy][xz] \equiv \frac{1}{3}[y^2z][xz][xy][xz] \equiv$$

$$\equiv -[y^2z][x^2y][xz^2] \Rightarrow [x^2y][y^2z][xz^2] \equiv 0$$

$$\left(\frac{[xz^2]}{3} \cdot r^{(1)}_{3,2,1},\ \frac{[xy]^2}{3} \cdot r_{3,1,1},\ \frac{[xy][xz]}{3} \cdot r^{(1)}_{2,2,1},\ \frac{[y^2z]}{3} \cdot r^{(1)}_{3,2,1} \right)$$

If $\deg(\mathbf{q_1}) = \deg(\mathbf{q_2}) = \mathbf{3}$, $\deg(\mathbf{q_3}) = \mathbf{2}$. Similarly to the previous case, we may assume that $q_1 = [x^2y]$, and by $r_{4,2}$, $r_{4,1,1}$ and $r_{3,2}$ we may also assume that $q_2 \notin \{[x^2y], [x^2z]\}$ and $q_3 \neq [xy]$. All cases left are checked below.

$$[x^2y][xy^2][yz] \equiv -[x^2y][xy][y^2z] \equiv 0 \ (r_{3,1,1} \cdot [x^2y],\ r_{3,2} \cdot [y^2z])$$

$$[x^2y][xy^2][xz] \equiv [x^2y][x^2y][yz] \equiv 0 \ (r^{(1)}_{2,2,1} \cdot [x^2y],\ r_{4,2} \cdot [yz])$$

$$[x^2y][y^2z][yz] \equiv 0 \ (r_{3,2} \cdot [x^2y])$$

$$[x^2y][y^2z][xz] \equiv [x^2y][yz^2][xy] \equiv 0 \ (r^{(1)}_{2,2,1} \cdot [x^2y],\ r_{3,2} \cdot [yz^2])$$

$$[x^2y][yz^2][yz] \equiv 0 \; (r_{3,2} \cdot [x^2y])$$

$$[x^2y][yz^2][xz] \equiv [xyz]^2[xz] \equiv 0 \; (r^{(2)}_{2,2,2} \cdot [xz], \; r^{(2)}_{2,2,1} \cdot [xyz])$$

$$[x^2y][xz^2][yz] \equiv [x^2z][xyz][yz] \equiv 0 \; (r^{(2)}_{3,2,1} \cdot [yz], \; r^{(2)}_{2,2,1} \cdot [x^2z])$$

$$[x^2y][xz^2][xz] \equiv 0 \; (r_{3,2} \cdot [x^2y])$$

If $\deg(\mathbf{q_1}) = \mathbf{3}$, $\deg(\mathbf{q_2}) = \deg(\mathbf{q_3}) = \mathbf{2}$. As before, it may be assumed that $q_1 = [x^2y]$, and by $r_{3,2}$ that $[xy] \notin \{q_2, q_3\}$. The possible products left are $[x^2y][yz]^2$, $[x^2y][xz]^2$ and $[x^2y][yz][zx]$.

$$[x^2y][yz]^2 \equiv [y^2x][xz][yz] \equiv -[y^2z][xy][xz] \; (r^{(1)}_{2,2,1} \cdot [yz], \; r_{3,1,1} \cdot [xz]),$$

and $[y^2z][xy][xz]$ and $[x^2y][yz][zx]$ are in S. As for the third product:

$$[x^2y][xz]^2 \equiv -[x^2z][xy][xz] \equiv 0, \; (r_{3,1,1} \cdot [xz], \; r_{3,2} \cdot [xy]).$$

If $\deg(\mathbf{q_1}) = \deg(\mathbf{q_2}) = \deg(\mathbf{q_3}) = \mathbf{2}$. All products different up to symmetry are the following: $[xy][yz][zx] \equiv 0$ $(r^{(1)}_{2,2,2})$, $[xy]^2[xz] \equiv -3[x^2z][xy^2]$ $(r^{(1)}_{3,2,1})$ and $[xy]^3 \equiv -3[x^2y][xy^2]$ $(r_{3,3})$. As $[x^2z][xy^2]$ and $[x^2y][xy^2]$ are elements of S, these products can be reduced by R too.

From 3.2.1and the cases discussed above, it follows that for any $q \in Q$ and $s \in S$, the product qs can be reduced to a linear combination of elements of S by using relations from R, and thus S is a secondary generating system of $\mathbb{C}[V^m]^G$ with primary generators P, and the set R^* corresponding to R minimally generates the ideal of relations.

Apart from the relations in R, the following relation is useful during the calculations in the case $m = 4$:

$$\text{(3.24)} \Rightarrow [x^2yz] \equiv \frac{1}{3}[xy][yz] \tag{3.35}$$

3.3 The case $m = 4$

In this case our goal is to find a set R such that the corresponding R^* minimally generates the ideal of relations. Let S be a system of secondary generators satisfying the conditions of 2.1.4. By 2.1.5 such S does exist. Unlike for cases $m = 2$ and $m = 3$, in this case not all elements of S are determined, although determining some elements of S is necessary to find R. To reduce the amount of calculation needed to prove that for the set R defined below R^* is a minimal generating system of I_4, consider the following lemma.

Lemma 3.3.1 *Let R be a set of relations modulo $\langle P_0\rangle$. Assume that for any $q \in Q_0 \setminus P_0$ and $s \in S_0$, $\deg(qs) \leq 8$ the product qs can be reduced to a linear combination of elements of S_0 by relations in R, that is, there exist $c_{s'} \in \mathbb{C}$, $f_i \in \mathbb{C}[Q_0 \setminus P_0]$ and $r_i \in R$ ($s' \in S_0$, $1 \leq i \leq k$) such that*

$$qs_0 - \sum_{s' \in S_0} c_s \cdot s' = \sum_{i=1}^{k} f_i \cdot r_i.$$

Then the R^* corresponding to R as described in 2.2.3 generates the ideal of relations I_m.

Proof According to 1.1.4, the ideal of relations is generated by the relations of degree ≤ 8. Hence if the ideal generated by R^* contains all elements of I_m up to degree 8, then it contains a set of generators of I_m, hence $\langle R^*\rangle = I_m$. The ideal $\langle R^*\rangle$ is the same as I_m up to degree 8 if any two polynomials $g_1, g_2 \in \mathcal{F}(3)$, $\deg(g_1), \deg(g_2) \leq 8$ are the same modulo $\langle R^*\rangle$ if and only if they are the same modulo I_m. This holds for every polynomial if it holds for monomials. R is a set of relations ($R^* \subset I_m$), thus it suffices to prove that monomials of degree at most 8 different modulo $\langle R^*\rangle$ are different modulo $\langle P_0\rangle$. The elements of $\mathrm{Span}\,(S_0)$ are all different modulo $\langle P_0\rangle$. Thus this follows if every monomial $g \in \mathbb{C}[Q_0 \setminus P_0]$, $\deg(g) \leq 8$ can be reduced to a linear combination of elements of S_0 modulo $\langle P_0\rangle$ using only the relations from R. If for any $q \in Q_0 \setminus P_0$ and $s \in S_0$, $\deg(qs) \leq 8$ the product qs can be reduced to such normal form, than clearly every monomial of elements of $Q_0 \setminus P_0$ can. □

Theorem 2.2.4 proves the minimality of an R for which $\langle R^*\rangle = I_m$ if the number of elements in R of certain degree is the same as certain coefficients in the numerator $J_4(t,t,t,t)$ of the Hilbert-series of I_4. By (2.11) and (1.26)

$$J_4(t,t,t,t) = 1 - E_4(t,t,t,t) \cdot (1-t^2)^6(1-t^3)^{6+4}.$$

Calculation (and the recursion in (1.22)) shows that the first few coefficients are $j_0 = j_1 = j_2 = j_3 = j_4 = 0$, $j_5 = 60$ and $j_6 = 136$. Thus if a set R has 60 elements of degree 5 and 136 elements of degree 6, and R^* generates the ideal of relations, then it generates it minimally.

The types of relations found in the cases $m = 2$ and $m = 3$ are valid if $m = 4$. Since the variables (denoted by x, y, z, w instead of $x(1)$, $x(2)$, $x(3)$, $x(4)$) have more permutations than in those cases, more relations belong to a certain type. The table 3.5 shows the number of relations in each type.

According to the table, the case $m = 3$ gives 48 relations of degree 5 and 94 of degree 6. As seen above, at least 12 more relations are needed of degree 5 and 42 of degree 6. As the relations of the type above were generated the ideal of relations in the case $m = 3$, the new relations should have a multidegree $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that $\alpha_1\alpha_2\alpha_3\alpha_4 \neq 0$. Thus we look for new relations in descending multidegree $(2,1,1,1)$, $(3,1,1,1)$ and $(2,2,1,1)$.

Degree	Descending multidegree	Type of relation	Name	$\sharp$
5	$(3,2,0)$	$[xy][x^2y] \equiv 0$	$r_{3,2}$	12
5	$(3,1,1)$	$[x^2y][xz]+[x^2z][xy] \equiv 0$	$r_{3,1,1}$	12
5	$(2,2,1)$	$[x^2y][yz]-[xy^2][xz] \equiv 0$	$r^{(1)}_{2,2,1}$	12
5	$(2,2,1)$	$[xy][xyz] \equiv 0$	$r^{(2)}_{2,2,1}$	12
6	$(4,2,0)$	$[x^2y]^2 \equiv 0$	$r_{4,2}$	12
6	$(4,1,1)$	$[x^2y][x^2z] \equiv 0$	$r_{4,1,1}$	12
6	$(3,3,0)$	$[xy]^3+3[x^2y][xy^2] \equiv 0$	$r_{3,3}$	6
6	$(3,2,1)$	$3[x^2z][xy^2]+[xy]^2[xz] \equiv 0$	$r^{(1)}_{3,2,1}$	24
6	$(3,2,1)$	$[x^2y][xyz]-[x^2z][xy^2] \equiv 0$	$r^{(2)}_{3,2,1}$	24
6	$(2,2,2)$	$[xy][yz][zx] \equiv 0$	$r^{(1)}_{2,2,2}$	4
6	$(2,2,2)$	$[xyz]^2-[xy^2][xz^2] \equiv 0$	$r^{(2)}_{2,2,2}$	12

Table 3.5: Number of relations in types from the case $m=3$ when $m=4$

3.3.1 Relations of degree 5.

To obtain multihomogeneous relations of degree 5, consider the following consequences of $\varphi(\Psi_4(w_1,w_2,w_3,w_4))=0$ with $\mathrm{multideg}(w_1w_2w_3w_4)=(2,1,1,1)$:

$$6[x^2yzw] \equiv [x^2y][zw]+[x^2z][yw]+[x^2w][yz] \tag{3.36}$$

$$6[x^2yzw] \equiv 6[x\cdot xy\cdot z\cdot w] \equiv [x^2y][zw]+2[xy][xzw]+[xz][xyw]+[xw][xyz] \tag{3.37}$$

$$6[x^2yzw] \equiv 6[x\cdot y\cdot xz\cdot w] \equiv [x^2z][yw]+[xy][xzw]+2[xz][xyw]+[xw][xyz] \tag{3.38}$$

$$6[x^2yzw] \equiv 6[x\cdot y\cdot z\cdot xw] \equiv [x^2w][yz]+[xy][xzw]+[xz][xyw]+2[xw][xyz] \tag{3.39}$$

$$\frac{1}{2}((3.36)+(3.37)-(3.38)-(3.39)) \Rightarrow 0 \equiv [x^2y][zw]-([xz][xyw]+[xw][xyz])\ (r_{2,1,1,1}) \tag{3.40}$$

$$\frac{1}{2}((3.36)+(3.38)-(3.37)-(3.39)) \Rightarrow 0 \equiv [x^2z][yw]-([xy][xzw]+[xw][xyz]) \tag{3.41}$$

$$\frac{1}{2}((3.36)+(3.39)-(3.37)-(3.38)) \Rightarrow 0 \equiv [x^2w][yz]-([xz][xyw]+[xy][xzw]) \tag{3.42}$$

Thus $0 \equiv [x^2y][zw]-([xz][xyw]+[xw][xyz])$ and there are $3\cdot 4=12$ similar relations of descending multidegree $(2,1,1,1)$. These relations are of type $r_{2,1,1,1}$.

3.3.2 Relations of degree 6.

Our goal is to find 42 multihomogeneous relations modulo $\langle P\rangle$ of degree 6. We look for relations of descending multidegrees $(2,2,1,1)$ and $(3,1,1,1)$.

3.3.2.1 Multidegree $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$.

To obtain relations of multidegree $(2,2,1,1)$, consider the consequences of the fundamental identity with substitutions w_1, w_2, w_3, w_4 such that $\mathrm{multideg}(w_1w_2w_3w_4) = (2,2,1,1)$. Calculations below are simplified using symmetry of the variables, and relations found in earlier cases are used to expand every term to monomials of elements of $Q \setminus P$.

$$6[x^2y^2zw] \equiv [x^2y^2][zw] + [x^2z][y^2w] + [x^2w][y^2z] \equiv \frac{1}{3}[xy]^2[zw] + [x^2z][y^2w] + [x^2w][y^2z]$$

$$6[x^2y^2zw] = 6[xy \cdot xy \cdot z \cdot w] = 4[xyzw][xy] + [x^2y^2][zw] + 2[xyz][xyw] - [xy]^2[zw] \equiv$$

$$\equiv 4[xy] \cdot \frac{1}{6}\left([xy][zw] + [xz][yw] + [xw][yz]\right) + \frac{1}{3}[xy]^2[zw] + 2[xyz][xyw] - [xy]^2[zw] \equiv$$

$$\equiv \frac{2}{3}[xy][xz][yw] + \frac{2}{3}[xy][xw][yz] + 2[xyz][xyw]$$

$$6[x^2y^2zw] = 6[x \cdot y \cdot xz \cdot yw] \equiv 2\left([x^2yz][yw] + [xy^2w][xz]\right) +$$

$$+[xy][xyzw] + [x^2z][y^2w] + [xyw][xyz] - [xy][xz][yw] \equiv$$

$$\equiv \frac{1}{2}[xy][xz][yw] + \frac{1}{6}[xy]^2[zw] + \frac{1}{6}[xy][xw][yz] + [x^2z][y^2w] + [xyw][xyz]$$

$$6[x^2y^2zw] = 6[x \cdot y \cdot xw \cdot yz] \equiv$$

$$\equiv \frac{1}{2}[xy][xw][yz] + \frac{1}{6}[xy]^2[zw] + \frac{1}{6}[xy][xz][yw] + [x^2w][y^2z] + [xyw][xyz]$$

$$6[x^2y^2zw] = 6[x^2 \cdot y \cdot yz \cdot w] \equiv 2[x^2yw][yz] + [x^2y][yzw] + [x^2yz][yw] + [x^2w][y^2z] \equiv$$

$$\equiv \frac{2}{3}[xy][xw][yz] + [x^2y][yzw] + \frac{1}{3}[xy][xz][yw] + [x^2w][y^2z]$$

$$6[x^2y^2zw] = 6[x^2 \cdot y \cdot z \cdot yw] \equiv \frac{2}{3}[xy][xz][yw] + [x^2y][yzw] + \frac{1}{3}[xy][xw][yz] + [x^2z][y^2w]$$

$$6[x^2y^2zw] = 6[x \cdot y^2 \cdot xz \cdot w] \equiv \frac{2}{3}[xy][yw][xz] + [xy^2][xzw] + \frac{1}{3}[xy][yz][xw] + [y^2w][x^2z]$$

$$6[x^2y^2zw] = 6[x \cdot y^2 \cdot z \cdot xw] \equiv \frac{2}{3}[xy][yz][xw] + [xy^2][xzw] + \frac{1}{3}[xy][yw][xz] + [y^2z][x^2w]$$

$$6[x^2y^2zw] = 6[xy \cdot y \cdot xz \cdot w] \equiv$$

$$\equiv 2[xy][xyzw] + 2[xz][xy^2w] + [xy^2][xzw] + [x^2yz][yw] + [xyz][xyw] - [xy][xz][yw] \equiv$$

$$\equiv \frac{1}{3}([xy]^2[zw] + [xy][xz][yw] + [xy][xw][yz]) + \frac{2}{3}[xz][xy][yw] + [xy^2][xzw] +$$

$$+\frac{1}{3}[xy][xz][yw] + [xyz][xyw] - [xy][xz][yw] \equiv$$

$$\equiv \frac{1}{3}[xy]^2[zw] + \frac{1}{3}[xy][xw][yz] + \frac{1}{3}[xy][xz][yw] + [xy^2][xzw] + [xyz][xyw]$$

$$6[x^2y^2zw] = 6[xy \cdot x \cdot yz \cdot w] \equiv$$
$$\equiv \frac{1}{3}[xy]^2[zw] + \frac{1}{3}[xy][yw][xz] + \frac{1}{3}[xy][yz][xw] + [x^2y][yzw] + [xyz][xyw]$$

To find appropriate relations to choose as elements of R, these relations need to be simplified. To make calculations more transparent, consider the following notation of all monomials of $Q \setminus P$ of multidegree $(2,2,1,1)$: $a_1 = [xy]^2[zw]$, $a_2 = [xy][xz][yw]$, $a_3 = [xy][yz][xw]$, $b_1 = [x^2y][yzw]$, $b_2 = [xy^2][xzw]$, $c_1 = [x^2z][y^2w]$, $c_2 = [x^2w][y^2z]$, $d = [xyz][xyw]$. Using this notation, the above congruences can be rewritten as:

$$\begin{aligned}
6[x^2y^2zw] &\equiv \frac{1}{3}a_1 + c_1 + c_2 \\
6[x^2y^2zw] &\equiv \frac{2}{3}a_2 + \frac{2}{3}a_3 + 2d \\
6[x^2y^2zw] &\equiv \frac{1}{6}a_1 + \frac{1}{2}a_2 + \frac{1}{6}a_3 + c_1 + d \\
6[x^2y^2zw] &\equiv \frac{1}{6}a_1 + \frac{1}{6}a_2 + \frac{1}{2}a_3 + c_2 + d \\
6[x^2y^2zw] &\equiv \frac{1}{3}a_2 + \frac{2}{3}a_3 + b_1 + c_2 \\
6[x^2y^2zw] &\equiv \frac{2}{3}a_2 + \frac{1}{3}a_3 + b_1 + c_1 \\
6[x^2y^2zw] &\equiv \frac{2}{3}a_2 + \frac{1}{3}a_3 + b_2 + c_1 \\
6[x^2y^2zw] &\equiv \frac{1}{3}a_2 + \frac{2}{3}a_3 + b_2 + c_2 \\
6[x^2y^2zw] &\equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + b_2 + d \\
6[x^2y^2zw] &\equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + b_1 + d
\end{aligned}$$

By subtracting every relation from the previous one, relations of the form $r \equiv 0 \bmod \langle P \rangle$ can be obtained. (This gives 9 relations, but some of them appear multiple times. A relation is omitted if it is the constant multiple of one of the others.)

$$0 \equiv \frac{1}{3}a_1 - \frac{2}{3}a_2 - \frac{2}{3}a_3 + c_1 + c_2 - 2d \tag{3.43}$$
$$0 \equiv -\frac{1}{6}a_1 + \frac{1}{6}a_2 + \frac{1}{2}a_3 - c_1 + d \tag{3.44}$$
$$0 \equiv \frac{1}{3}a_2 - \frac{1}{3}a_3 + c_1 - c_2 \tag{3.45}$$
$$0 \equiv \frac{1}{6}a_1 - \frac{1}{6}a_2 - \frac{1}{6}a_3 - b_1 + d \tag{3.46}$$
$$0 \equiv b_1 - b_2 \tag{3.47}$$
$$0 \equiv -\frac{1}{3}a_1 + \frac{1}{3}a_3 + c_2 - d \tag{3.48}$$

Thus $[x^2y][yzw] \equiv [xy^2][xzw]$ ($b_1 \equiv b_2$). As (3.45) = $-2 \cdot$ (3.44) − (3.43), (3.45) is superfluous.

$$(3.43) + 4 \cdot (3.44) - 4 \cdot (3.48) \Rightarrow 0 \equiv a_1 - 3c_1 - 3c_2 + 6d \Rightarrow a_1 \equiv 3c_1 + 3c_2 - 6d \tag{3.49}$$

$$2 \cdot (3.43) + 2 \cdot (3.44) + (3.48) \Rightarrow 0 \equiv -a_2 + 3c_2 - 3d \Rightarrow a_2 \equiv 3c_2 - 3d \tag{3.50}$$

$$(3.43) + 4 \cdot (3.44) - (3.48) \Rightarrow 0 \equiv a_3 - 3c_1 + 3d \Rightarrow a_3 \equiv 3c_1 - 3d \tag{3.51}$$

Now from (3.46), (3.49), (3.50), and (3.51) it follows that

$$b_1 \equiv \frac{1}{6}(3c_1 + 3c_2 - 6d) - \frac{1}{6}(3c_2 - 3d) - \frac{1}{6}(3c_1 - 3d) + d \Rightarrow b_1 \equiv d \tag{3.52}$$

From (3.47), (3.49), (3.50), (3.51) and (3.52) it follows that all monomials of $Q \setminus P$ of multidegree $(2,2,1,1)$ can be reduced to a linear combination of the expressions $c_1 = [x^2z][y^2w]$, $c_2 = [x^2w][y^2z]$, $d = [xyz][xyw]$. The relations used are similar to one of the following:

$$(3.49) \Rightarrow 0 \equiv [xy]^2[zw] - 3[x^2z][y^2w] - 3[x^2w][y^2z] + 6[xyz][xyw] \ (r^{(1)}_{2,2,1,1}) \tag{3.53}$$

$$(3.50) \Rightarrow 0 \equiv -[xy][xz][yw] + 3[x^2w][y^2z] - 3[xyz][xyw] \ (r^{(2)}_{2,2,1,1}) \tag{3.54}$$

$$(3.52) \Rightarrow 0 \equiv [x^2y][yzw] - [xyz][xyw] \ (r^{(3)}_{2,2,1,1}) \tag{3.55}$$

There are 6 relations of type $r^{(1)}_{2,2,1,1}$ and 12 relations of type $r^{(2)}_{2,2,1,1}$ and$r^{(3)}_{2,2,1,1}$. This gives 30 relations.

3.3.2.2 Multidegree $(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})$.

Consider the consequences of $\varphi(\Psi_4(w_1, w_2, w_3, w_4)) = 0$ of multidegree $(3,1,1,1)$. According to 1.2.2, $6[x^3yzw] \equiv 0$ modulo $\langle P \rangle$.

$$0 \equiv 6[x^3yzw] = 6[x \cdot xy \cdot xz \cdot w] \equiv$$

$$\equiv 2[x^2yw][xz] + 2[x^2zw][xy] + [x^2y][xzw] + [x^2z][xyw] + [xw][x^2yz] - [xy][xz][xw] \equiv$$

$$\equiv \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2z][xyw] \Rightarrow$$

$$\Rightarrow 0 \equiv \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2z][xyw] \tag{3.56}$$

and similarly (by permutations of y, z and w):

$$0 \equiv \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2w][xyz] \tag{3.57}$$

$$0 \equiv \frac{2}{3}[xy][xz][xw] + [x^2z][xyw] + [x^2w][xyz] \tag{3.58}$$

$$(3.56) + (3.57) - (3.58) \Rightarrow 0 \equiv [xy][xz][xw] + 3[x^2y][xzw] \ (r_{3,1,1,1}) \tag{3.59}$$

There are 3 relations of type $r_{3,1,1,1}$ of multidegree $(3,1,1,1)$ and $3 \cdot 4 = 12$ corresponding to this descending multidegree.

Let R be the set of $\mod \langle P \rangle$ relations of the types found above. That is, R contains every relation of the types listed in Table 3.5, and also the relations of type $r_{2,1,1,1}$, $r^{(1)}_{2,2,1,1}$, $r^{(2)}_{2,2,1,1}$, $r^{(3)}_{2,2,1,1}$ and $r_{3,1,1,1}$. From this it follows that $|R_5| = 48 + 12 = 60$ and $|R_6| = 94 + 6 + 12 + 12 + 12 = 136$. As seen above, from this it follows that if R^* generates the ideal of relations, than it generates it minimally.

3.3.3 The set given generates the ideal of relations.

Our goal is to prove that $\langle R^* \rangle = I_4$. The proof involves some calculation, and the argument below. The method is the following. A set S' is built such that $\{1\} \cup (Q \setminus P) \subset S'$ and the elements

$$\{s + \langle P \rangle \mid s \in S'\}$$

are linearly independent in $\mathbb{C}[V^4]^G / \langle P \rangle$. At the same time, every product qs for $\deg(qs) \leq 8$, $q \in Q \setminus P$ and $s \in S'$ is reduced ($\mod \langle P \rangle$) to a linear combination of elements of S', using relations from R. This is done by induction on the degree. One step of the induction determines elements of S' and reduces the products qs in one descending multidegree.

Using secondary generators from the case $\mathbf{m = 3}$. A secondary generating system has been chosen in the case $m = 3$. The algebra $\mathbb{C}[V^3]^G$ can be embedded into $\mathbb{C}[V^4]^G$ as a subalgebra, by considering only the polynomials that do not contain the variable w. Similarly, it can be embedded by considering only the polynomials that do not contain the variable x, or the ones without y, or the ones without z. Each embedding gives a set of secondary generators. These sets intersect, but the union is a set of linearly independent variables modulo $\langle P \rangle$. Let S' contain the union of these sets, that is, the secondary generators from the case $m = 3$. (From this it follows that $\{1\} \cup (Q \setminus P) \subset S'$). R contains a generating system of the ideal of relations for $m = 3$, and is closed under the permutation of variables. From this it follows that any product qs as described above can be reduced by R to a linear combination of S' if the multidegree of qs has a zero coordinate.

One step of the induction. During the calculations below, descending multidegrees are examined one by one. A multidegree $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is examined when all multidegrees of smaller total degree have already been examined, and the elements of S' corresponding to those multidegrees have been chosen, such that S' contains a maximal set of $\mod \langle P \rangle$

linearly independent elements in those multidegrees. For α, consider every product qs for which $q \in Q \setminus (P \cup \{1\})$, s an element of S' already chosen, and $\text{multideg}(qs) = \alpha$. Let c be the coefficient of $E_4(\mathbf{t})$ corresponding to the multidegree α. Our goal is to choose a set $(S')_\alpha$ of such qs products, such that $|(S')_\alpha| = c$, and any of the products not in $(S')_\alpha$ can be reduced to a linear combination of elements of $(S')_\alpha$ modulo $\langle P \rangle$. Since any secondary generating system does have c elements of multidegree α, there exist $c \bmod \langle P \rangle$ linearly independent elements in multidegree α. From this it follows that the elements of $(S')_\alpha$ are linearly independent $\bmod \langle P \rangle$. Let $(S')_\alpha \subseteq S'$. Similarly, let S' contain the similar set corresponding to the multidegrees $(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}, \alpha_{\sigma(4)})$ where $\sigma \in S_4$. This ends the step that belongs to α.

End of the induction. The induction ends when all multidegrees with total degree ≤ 8 are examined. According to the remark following 2.1.5, there exists a secondary generating system S such that S satisfies the conditions of 2.1.4 and $S' \subset S$. From the way S' is built it follows that all elements of S of degree at most 8 are in S'.

According to 3.3.1, to prove that $\langle R^* \rangle = I_4$ it is enough to prove that every product qs for $\deg(qs) \leq 8$, $q \in Q \setminus P$ and $s \in S$ can be reduced to a linear combination of elements of S using relations from R. This is clear from the way S' was built.

As seen above, certain coefficients of the polynomial $E_4(\mathbf{t})$ are needed. These can be calculated using 1.22. As $E_4(\mathbf{t})$ is a symmetric polynomial, terms with the same descending multidegree have the same coefficient. Nonzero coefficients are 1 or 3. The terms of which the descending multidegree is

$$(3,3,3,3);\ (3,3,2,2);\ (3,3,2,1);\ (3,2,2,2);\ (3,3,1,1);$$

$$(3,2,2,1);\ (3,2,2,0);\ (3,2,1,1);\ (3,3,0,0);\ (3,2,1,0);$$

$$(3,1,1,1);\ (2,2,2,0);\ (3,1,1,0);\ (2,2,1,0);\ (2,2,0,0);$$

$$(2,1,1,0);\ (2,1,0,0);\ (1,1,1,0);\ (1,1,0,0);\ (0,0,0,0)$$

have coefficient 1, the terms with descending multidegree

$$(2,2,2,2);\ (2,2,2,1);\ (2,2,1,1);\ (2,1,1,1);\ (1,1,1,1)$$

have coefficient 3.

The calculation that completes the proof is contained in A of the appendix. Thus for the set R given, R^* minimally generates the ideal of relations.

Chapter 4

$\mathrm{GL_m}(\mathbb{C})$-structure

In this chapter our goal is to prove that in the case $n = 3$, the ideal of relations for arbitrary m is generated by the polarizations of the relations found in the previous chapter. From this it follows that the ideal of relations is generated with relations of degree at most six. To obtain such a result, we introduce a graded $\mathrm{GL}_m(\mathbb{C})$-module structure on $\mathbb{C}[V^m]^G$, $\mathcal{F}(3)$ and I_m. With this structure, the degree d homogeneous component of each is a degree d polynomial representation of $\mathrm{GL}_m(\mathbb{C})$. Consider the following result.

Proposition 4.0.2 *There is a one-to-one correspondence between the irreducible polynomial representations of* $\mathrm{GL}_m(\mathbb{C})$ *of degree d and the Young-diagrams of type* λ, *where* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$, $\lambda_1 + \lambda_2 + \cdots + \lambda_m = d$. *The dimension of the irreducible representation corresponding to the Young-diagram of type* λ *is*

$$d_\lambda = \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Proof Details can be found in [3] □

Throughout the chapter, irreducible submodules of polynomial $\mathrm{GL}_m(\mathbb{C})$-modules are examined. We shall use the convention that for a partition λ we write also λ for the corresponding irreducible polynomial $GL_m(C)$-module. In the arguments we use the fact that polynomial representations of $\mathrm{GL}_m(\mathbb{C})$ are completely reducible.

4.1 The $\mathrm{GL}_{\mathbf{m}}(\mathbb{C})$-module structure of $\mathbb{C}[\mathbf{V}^{\mathbf{m}}]^{\mathbf{G}}$

The vector space V^m is isomorphic to the vector space of $n \times m$ matrices. Using notation from 1.1, the homomorphism ϱ that maps a vector $v \in V^m$ to the matrix

$$\varrho(v) = (x(i)_j(v))_{\substack{1\le i\le m\\ 1\le j\le n}} \in \mathbb{C}^{n\times m}$$

is an isomorphism.

Consider the following action ($\theta : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}(V^m)$) of the group $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ on V^m : for any $(g, h) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ (that is, $g \in \mathrm{GL}_n(\mathbb{C})$ and $h \in \mathrm{GL}_m(\mathbb{C})$) and any $v \in V^m$ the image of v by (g, h) is

$$\theta(g, h)v = \varrho^{-1}(g \cdot \varrho(v) \cdot h^{-1}).$$

From this it follows that $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ acts on the algebra $\mathbb{C}[V^m]$ too. Let θ_n : $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V^m)$ be the diagonal action of $\mathrm{GL}_n(\mathbb{C})$ on $V^m \cong (\mathbb{C}^n)^m$ and $\theta_m :\ \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}(V^m)$ the diagonal action of $\mathrm{GL}_m(\mathbb{C})$ on $V^m \cong \mathbb{C}^{n\times m} \cong (\mathbb{C}^m)^n$. With these notations

$$\big(\theta(g,h)\big)(v) = \big(\theta_m(h)\big)\big((\theta_n(g))\,(v)\big) = \big(\theta_n(g)\big)\big((\theta_m(h))\,(v)\big)$$

holds for any $g \in \mathrm{GL}_n(\mathbb{C})$, $h \in \mathrm{GL}_m(\mathbb{C})$ and $v \in V^m$. That is, the actions θ_m and θ_n are commutable. As seen in 1.1, $S_n \cong G \le \mathrm{GL}_n(\mathbb{C})$, thus $G \times \mathrm{GL}_m(\mathbb{C})$ acts on $\mathbb{C}[V^m]$ too. From the fact that $\theta = \theta_m \circ \theta_n = \theta_n \circ \theta_m$, it follows that the invariant ring $\mathbb{C}[V^m]^G$ is a $\mathrm{GL}_m(\mathbb{C})$-submodule of the polynomial algebra $\mathbb{C}[V^m]$. The action of $\mathrm{GL}_m(\mathbb{C})$ on $\mathbb{C}[V^m]^G$ clearly preserves the degree of a polynomial. Thus $\mathbb{C}[V^m]^G$ is a graded $\mathrm{GL}_m(\mathbb{C})$-module. For simplicity denote by θ_m the action of $\mathrm{GL}_m(\mathbb{C})$ on $\mathbb{C}[V^m]^G$.

4.2 The $\mathrm{GL}_{\mathbf{m}}(\mathbb{C})$-module structure of $\mathcal{F}(3)$

Recall the notation introduced in 1.1. The group $\mathrm{GL}_m(\mathbb{C})$ acts on the polynomial ring $\mathbb{C}[x(1), \dots, x(m)]$ by linear substitution of the variables. Namely, $g = (g_{ij})_{m\times m} \in \mathrm{GL}_m(\mathbb{C})$ sends the variable $x(i)$ to

$$\sum_{j=1}^{m} g_{ji}x(j) =: x(i)^g,$$

i.e. the ith entry of the row vector $[x(1), \dots, x(m)] \cdot g$ (matrix multiplication). This gives a representation τ of $\mathrm{GL}_m(\mathbb{C})$ on $\mathbb{C}[x(1), \dots, x(m)]$, where

$$\tau(g)(f) = f(x(1)^g, \dots, x(m)^g).$$

The degree d homogeneous component of the polynomial algebra is a τ-invariant subspace. In fact the representation of $\mathrm{GL}_m(\mathbb{C})$ on the degree d homogeneous component is the irreducible polynomial representation associated to the partition (d) having a single non-zero part. This representation is called the *dth symmetric tensor power of the defining representation*.

As seen in 1.1, $\mathcal{F}(3)$ is the polynomial $\mathbb{C}$-algebra generated by the formal variables $t(w)$, where w is a non-empty monomial of the variables $x(1),\dots,x(m)$ and $\deg(w) \le 3$. The desired representation $\overline{\theta}$ of the group $\mathrm{GL}_m(\mathbb{C})$ on $F(3)$ is again an action on polynomials by linear substitution of the variables $t(w)$. Namely, set

$$g \cdot t(w) := t(\tau(g)w),$$

where for a linear combination $\sum_i a_i w_i$ of monomials of degree at most three in the variables $x(1),\dots,x(m)$, we define $t(\sum_i a_i w_i) := \sum_i a_i t(w_i)$. Now

$$\overline{\theta}(g)f = f(g \cdot t(w))$$

for any $g \in \mathrm{GL}_m(\mathbb{C})$ and polynomial f in the variables $t(w)$. In other words, denoting by Q_l the subspace of $\mathcal{F}(3)$ spanned by the variables $t(w)$ where $\deg(w) = l$ ($1 \le l \le 3$), the subspace Q_l is $\overline{\theta}$-invariant, and the representation of $\mathrm{GL}_m(\mathbb{C})$ on this subspace is isomorphic to (l). The generators of $\mathcal{F}(3)$ span the subspace $Q_1 + Q_2 + Q_3$, isomorphic as a $\mathrm{GL}_m(\mathbb{C})$-representation to $(1) + (2) + (3)$, and $\mathcal{F}(3)$ is the symmetric tensor algebra

$$\bigoplus_{k=0}^{\infty} \mathrm{Sym}^k((1) + (2) + (3))$$

(where $\mathrm{Sym}^k(U)$ stands for the kth symmetric tensor power of the $\mathrm{GL}_m(\mathbb{C})$-module U). Consider the grading of $\mathcal{F}(3)$ defined earlier, and write $(\mathcal{F}(3))_d$ for the degree d homogeneous component. The $\mathrm{GL}_m(\mathbb{C})$-module structure of $(\mathcal{F}(3))_d$ is

$$(\mathcal{F}(3))_d = \bigoplus (\mathrm{Sym}^{d_1}((1)) \otimes \mathrm{Sym}^{d_2}((2)) \otimes \mathrm{Sym}^{d_3}((3))) \tag{4.1}$$

where the summation is over the triples (d_1, d_2, d_3) of non-negative integers with $d_1 + 2d_2 + 3d_3 = d$.

Remark The module-structure and the algebra-structure of $\mathcal{F}(3)^{(m_2)}$ are compatible. That is, for any $h \in \mathrm{GL}_m(\mathbb{C})$ and $l_1, l_2 \in \mathcal{F}(3)$

$$\big(\overline{\theta}(h)\big)(l_1 l_2) = \big(\overline{\theta}(h)\big)(l_1) \cdot \big(\overline{\theta}(h)\big)(l_2)$$

holds.

To examine the irreducible summands of this representation, consider the following lemma:

Lemma 4.2.1 *(Pieri's rule) Consider two Young-diagrams, λ and μ, where μ consists of only one line. Let m be at least the number of lines in λ. Write θ_λ and θ_μ for the corresponding $\mathrm{GL}_m(\mathbb{C})$-representations. Then the irreducible components of the representation $\theta_\lambda \otimes \theta_\mu$ correspond to the diagrams that can be made from λ by adding the number of boxes in μ, in a way that no two of the boxes are added to the same column and the number of rows is still at most m. Each of these representations has multiplicity one.*

Proof Pieri's rule is a special case of the Littlewood-Richardson rule. See [3] for a proof. □

According to 1.1.4, in the case $n = 3$, the ideal of relations is generated by the relations of degree at most eight. Therefore in the following we examine the irreducible direct summands of $(\mathcal{F}(3))_d$ where $d \leq 8$. Denote by $h(\lambda)$ the number of non-zero parts of the partition λ (height of λ). Recall that $\mathrm{Sym}^{d_1}((1)) \cong (d_1)$ is irreducible.

Lemma 4.2.2 *The irreducible summands λ occurring in $\mathrm{Sym}^k((2))$ or $Sym^k((3))$ all satisfy $h(\lambda) \leq k$.*

Proof Recall that $\mathrm{Sym}^k(U)$ is a direct summand in the k tensor power $U \otimes \cdots \otimes U$. Since both $h((2)) = h((3)) = 1$, Pieri's rule proves the lemma. □

Using again Pieri's rule one easily checks the following:

Proposition 4.2.3 *Up to degree 8, all irreducible summands λ occurring in $\mathcal{F}(3)$ satisfy $h(\lambda) \leq 4$.*

Proof By (4.1), the irreducible summands in $(\mathcal{F}(3))_d$ are summands of $\mathrm{Sym}^{d_1}((1)) \otimes \mathrm{Sym}^{d_2}((2)) \otimes \mathrm{Sym}^{d_3}((3))$, where $d_1 + 2d_2 + 3d_3 = d$. By 4.2.2, irreducible summands of $\mathrm{Sym}^{d_2}((2))$ and $\mathrm{Sym}^{d_3}((3))$ have height at most d_2 and d_3 respectively. If $d = d_1 + 2d_2 + 3d_3 \leq 8$, then $d_2 \leq 1$ or $d_3 \leq 1$, and $h(d_1) = 1$, thus Pieri's rule can be used to compute irreducible summands of $(\mathcal{F}(3))_d$. If $d_3 > 1$, then $d_3 = 2$, and summands are of height at most $2+1+1 = 4$. If $d_2 > 1$, then if $d_2 = 4$, then $d_1 = d_3 = 0$, and thus heights are at most $d_2 = 4$. If $d_2 = 3$, then $d_3 = 0$, thus heights are at most $d_2 + 1 = 4$. If $d_2 = 2$, then heights are at most $d_2 + 1 + 1 = 4$. If both d_2 and d_3 is 1, then summands are of height at most $1 + 1 + 1 = 3$. This completes the proof. □

4.3 The $\mathrm{GL}_\mathbf{m}(\mathbb{C})$-module structure of the ideal of relations

The way $\mathrm{GL}_m(\mathbb{C})$ acts on the formal variables $x(i)$ as defined in 4.2 is isomorphic to its action on the polynomials $[x(i)] \in \mathbb{C}[V^m]^G$. From this it follows that the $\mathbb{C}$-algebra homomorphism

$$\varphi : \mathcal{F}(3) \to \mathbb{C}[V^m]^G$$

is a $\mathrm{GL}_m(\mathbb{C})$-module homomorphism. Therefore the kernel of this homomorphism, I_m is a submodule of $\mathcal{F}(3)$. The irreducible summands of I_m are among the irreducible summands of $\mathcal{F}(3)$.

Our goal is to prove that for any m the elements of I_m are consequences of relations that are polarizations of the relations found earlier. By 1.1.4, every element of I_m is a consequence of relations of degree at most eight. That is, if all relations in $(I_m)_k$ are consequences of elements of $R \subset I_m$ for every $k \leq 8$, then R generates I_m as an ideal of $\mathcal{F}(3)$.

Write $\mathcal{F}(3)^{(m)}$ for the polynomial ring $\mathcal{F}(3)$ corresponding to $\mathbb{C}[V^m]^G$ as defined in 1.1. Then $\mathcal{F}(3)^{(m_1)}$ is a subalgebra of $\mathcal{F}(3)^{(m_2)}$ if $m_1 \leq m_2$. Since I_{m_1} is an ideal in $\mathcal{F}(3)^{(m_1)}$, it is also a subset of $\mathcal{F}(3)^{(m_2)}$.

The following proposition shows that if $(I_m)_k$ is the $\mathrm{GL}_m(\mathbb{C})$-submodule generated by $(I_4)_k$, then the polarizations of relations found in the case $m = 4$ generate $(I_m)_k$.

Lemma 4.3.1 *Let $m_1 \leq m_2$, $R^{(m_1)} \subseteq I_{m_1}$ be a set of relations such that every element of $(I_{m_1})_k$ is a consequence of $R^{(m_1)}$. Denote by $R^{(m_2)} \subseteq I_{m_2}$ the set of polarizations of $R^{(m_1)}$, that is, the $\mathrm{GL}_{m_2}(\mathbb{C})$-submodule generated by $R^{(m_1)}$. Assume that $(I_{m_1})_k$ generates $(I_{m_2})_k$ as a $\mathrm{GL}_{m_2}(\mathbb{C})$-submodule. Then every element of $(I_{m_2})_k$ is a consequence of $R^{(m_2)}$.*

Proof The lemma follows from the fact that the module-structure and the algebra-structure of $\mathcal{F}(3)^{(m_2)}$ are compatible. □

Therefore it is sufficient to prove that for any m and any $k \leq 8$, $(I_m)_k$ is generated by $(I_4)_k$.

Lemma 4.3.2 *If W is an irreducible summand of $\mathcal{F}(3)$ isomorphic to λ with $h(\lambda) \leq 4$, then W contains a non-zero element in $\mathcal{F}(3)^{(4)}$ (necessarily generating W as a $\mathrm{GL}_m(\mathbb{C})$-submodule).*

Proof This follows from the theory of *highest weights* (see for example [2]). Denote by U_m the subgroup of $\mathrm{GL}_m(\mathbb{C})$ consisting of unipotent upper triangular matrices. Then it is well known that any irreducible polynomial $\mathrm{GL}_m(\mathbb{C})$-module contains a unique (up to non-zero scalar multiple) element u fixed by U_m (a so-called highest weight vector), on which the diagonal subgroup of $\mathrm{GL}_m(\mathbb{C})$ acts by the character labeled by λ. The assumption $h(\lambda) \leq 4$ forces that u belongs to $\mathcal{F}(3)^{(4)}$. □

Corollary 4.3.3 *For $k \leq 8$ and $m \geq 4$, $(I_m)_k$ is generated by $(I_4)_k$ as a $\mathrm{GL}_m(\mathbb{C})$-submodule of $\mathcal{F}(3)^{(m)}$.*

Proof Decompose $(I_m)_k$ as a direct sum of irreducible $\mathrm{GL}_m(\mathbb{C})$-modules (where $k \leq 8$). Take an arbitrary irreducible summand W. Then by 4.2.3 $W \cong \lambda$ with $h(\lambda) \leq 4$, hence W has a nontrivial intersection with $\mathcal{F}(3)^{(4)}$ by 4.3.2 above. Clearly this intersection is contained in $(I_4)_k$, so W contains a non-zero element u in I_4. Since W is irreducible, u generates W as a $\mathrm{GL}_m(\mathbb{C})$-module, so W is contained in the $\mathrm{GL}_m(\mathbb{C})$-submodule of $\mathcal{F}(3)$ generated by $(I_4)_k$. Since this holds for all irreducible summands of $(I_m)_k$, we conclude that $(I_m)_k$ is the $\mathrm{GL}_m(\mathbb{C})$-submodule of $\mathcal{F}(3)$ generated by $(I_4)_k$. □

4.4 Conclusion

Write $R^{(4)}$ for the set of relations found in the case $m = 4$. These generate I_4 as an ideal. If $4 < m_2$, $\mathcal{F}(3)^{(4)} \subset \mathcal{F}(3)^{(m_2)}$, thus $R^{(4)} \subset \mathcal{F}(3)^{(m_2)}$. Denote by $R^{(m_2)}$ the set of polarizations of relations in $R^{(4)}$, that is, the $\mathrm{GL}_{m_2}(\mathbb{C})$-submodule generated by $R^{(4)}$.

Proposition 4.4.1 *$R^{(m_2)}$ generates I_{m_2} as an ideal in $\mathcal{F}(3)^{(m_2)}$.*

Proof By 1.1.4, it is sufficient to prove that every element of $(I_{m_2})_k$ is a consequence of $R^{(m_2)}$ if $k \leq 8$. By 4.3.1, this follows if $(I_4)_k$ generates $(I_{m_2})_k$ as a $\mathrm{GL}_{m_2}(\mathbb{C})$-module. This is true according to 4.3.3. Thus the proof is complete. □

Some relations can be chosen that generate $R^{(4)}$ as a $\mathrm{GL}_4(\mathbb{C})$-submodule. Note that if polynomial f generates a $\mathrm{GL}_{m_1}(\mathbb{C})$-submodule of type λ, and $m_1 \leq m_2$, then the $\mathrm{GL}_{m_2}(\mathbb{C})$-submodule generated by f is of type λ as well.

In the case $m = 2$, the relations $r_{3,2}$ and $r_{2,3}$ span $(I_2)_5$, and are clearly in the same $\mathrm{GL}_2(\mathbb{C})$-submodule. Since the $\mathrm{GL}_2(\mathbb{C})$-representations of type (5), $(4,1)$ and $(3,2)$ have dimension 6, 4 and 2 respectively, $(I_2)_5 = (3,2)$. From this it follows that the polarizations of $r_{3,2}$ generate $(R^{(4)})_5$. The generators of degree 6 are $r_{4,2}$, $r_{2,4}$ and $r_{3,3}$. These span a

$\mathrm{GL}_2(\mathbb{C})$-submodule. Clearly $r_{4,2}$ and $r_{2,4}$ are in the same $\mathrm{GL}_2(\mathbb{C})$-orbit. Since the $\mathrm{GL}_2(\mathbb{C})$-representations of type (6), $(5,1)$, $(4,2)$ and $(3,3)$ have dimension 7, 5, 3 and 1 respectively, $r_{4,2}$, $r_{2,4}$ and $r_{3,3}$ span an irreducible component of type $(4,2)$.

In the case $m = 3$ the degree 5 generators span a component of type $(3,2)$. The degree 6 generators span a submodule of dimension 28. It has a component of type $(4,2)$, this has dimension 27. Thus it has a trivial irreducible component too. This has type $(2,2,2)$. Since there is only one relation that is unique in its type (3.2), the component of type $(2,2,2)$ is generated by the unique relation of type $r^{(1)}_{2,2,2}$.

Consider the relations $r_{3,2}$ (choose one of the type), $r_{4,2}$ and $r^{(1)}_{2,2,2}$. In the case $m = 4$, these generate $\mathrm{GL}_4(\mathbb{C})$ submodules of type $(3,2)$, $(4,2)$ and $(2,2,2)$, these span subspaces of dimension 60 in degree 5 and $126 + 10 = 136$ in degree 6. From this it follows that R^4 is contained by the submodule generated by these relations.

Theorem 4.4.2 *The ideal of relations among the generators Q of $\mathbb{C}[V^m]^{S_3}$ is generated by the polarizations of the relations $r_{3,2}$, $r_{4,2}$ and $r^{(1)}_{2,2,2}$ for any $m \geq 3$, and by the polarizations of $r_{3,2}$ and $r_{4,2}$ if $m = 2$.*

Proof The theorem follows trivially from 4.4.1 and the above considerations. □

Corollary 4.4.3 *If $n = 3$, then the ideal of relations among the elements of a minimal generating system Q of $\mathbb{C}[V^m]^{S_3}$ is generated by elements of degree at most six.*

Proof Since $r_{3,2}$, $r_{4,2}$ and $r^{(1)}_{2,2,2}$ have degree 5 and 6, and polarization preserves total degree, the corollary is true. □

4.5 Further remarks about the ideal of relations

In 4.3.3 it was not necessary to determine the exact irreducible components of $(I_m)_k$ to prove it is the polarization of $(I_4)_k$. However, this is not impossible, and gives another way to prove that $(I_m)_k$ and $(I_4)_k$ have the same structure (without using the lemma 4.3.2). Clearly if $k \leq m_1$, then the $\mathrm{GL}_{m_2}(\mathbb{C})$-submodule generated by $(I_{m_1})_k$ is $(I_{m_2})_k$. Therefore (by 1.1.4) it suffices to prove that $(I_m)_k$ has the same components as $(I_4)_k$ for $5 \leq m \leq k \leq 8$.

Since I_4 is the kernel of the $\mathrm{GL}_4(\mathbb{C})$-module homomorphism φ, the irreducible summands in $(I_4)_k$ are exactly the summands of $(\mathcal{F}(3))_k$ that do not appear in $(\mathbb{C}[V^4]^G)_k$. The irreducible components of $\mathcal{F}(3)$ can be computed by (4.1) and Pieri's rule. The components of $(\mathbb{C}[V^4]^G)_k$ can be computed using Cauchy and Jacobi-Trudi formulae ([2]). The difference of the two structures shows that if $m = 4$ the ideal of relations has the following irreducible $\mathrm{GL}_4(\mathbb{C})$-module components:

$$
\begin{aligned}
(I_4)_5 &= (3,2) \\
(I_4)_6 &= 2(4,2) + (3,3) + (3,2,1) + (2,2,2) \\
(I_4)_7 &= 3(5,2) + 3(4,3) + 3(4,2,1) + (3,3,1) + 3(3,2,2) + (2,2,2,1) \\
(I_4)_8 &= 5(6,2) + 5(5,3) + 5(5,2,1) + 3(4,4) + 5(4,3,1) + 7(4,2,2) + \\
&\quad 2(3,3,2) + (4,2,1,1) + 2(3,2,2,1) + 2(2,2,2,2).
\end{aligned}
$$

Now to prove that $(I_m)_k$ has the same structure when $k \le 8$, it suffices to show that the dimension of the subspace spanned by these types of $\mathrm{GL}_m(\mathbb{C})$-modules is equal to the dimension of $(I_m)_k$ (if $5 \le m \le k \le 8$). The tables in B show the dimensions of the irreducible submodules. The coefficients of the Hilbert-series of I_m corresponding to these (m,k) pairs is contained in C. Some calculation completes the argument.

Appendix A

Calculations for the case $m = 4$.

The calculation below, together with the argument in 3.3.3, proves that for the set of relations found in the case $m = 4$ $\langle R^* \rangle = I_4$. As seen above, the induction step for multidegrees with a zero coordinate is done by the case $m = 3$. (End the elements of S' in these multidegrees are chosen accordingly.) Hence only those descending multidegrees are left, which do not have a zero coordinate, and have total degree at most eight. Let c_α denote the coefficient of $E_4(\mathbf{t})$ corresponding to the elements of descending multidegree α

Multidegree $\alpha = (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \{[xy][zw],\ [xz][yw],\ [xw][yz]\}$ ($c_\alpha = 3$). There are no other products of multidegree α.

Multidegree $\alpha = (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \{[xy][xzw],\ [xz][xyw],\ [xw][xyz]\}$ ($c_\alpha = 3$). The other products of multidegree α are eliminated using $r_{2,1,1,1} \in R$ the following way:

$$[x^2y][zw] \equiv [xz][xyw] + [xw][xyz]$$

$$[x^2z][yw] \equiv [xy][xzw] + [xw][xyz]$$

$$[x^2w][yz] \equiv [xz][xyw] + [xy][xzw]$$

Multidegree $\alpha = (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \{[xy][xz][xw]\}$ ($c_\alpha = 1$). The other products of multidegree α are eliminated using $r_{3,1,1,1} \in R$ the following way:

$$[x^2y][xzw] \equiv -\frac{1}{3}[xy][xz][xw]$$

$$[x^2z][xyw] \equiv -\frac{1}{3}[xy][xz][xw]$$

$$[x^2w][xyz] \equiv -\frac{1}{3}[xy][xz][xw]$$

Multidegree $\alpha = (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \{[x^2z][y^2w],\ [x^2w][y^2z],\ [xyz][xyw]\}$ ($c_\alpha = 3$). The other products of multidegree α are eliminated using R the following way:

$$[xy]^2[zw] \equiv 3[x^2z][y^2w] + 3[x^2w][y^2z] - 6[xyz][xyw]\ (r^{(1)}_{2,2,1,1})$$

$$[xy][xz][yw] \equiv 3[x^2w][y^2z] - 3[xyz][xyw]\ (r^{(2)}_{2,2,1,1})$$

$$[xy][xw][yz] \equiv 3[x^2z][y^2w] - 3[xyz][xyw]\ (r^{(2)}_{2,2,1,1})$$

$$[x^2y][yzw] \equiv [xyz][xyw]\ (r^{(3)}_{2,2,1,1})$$

$$[xy^2][xzw] \equiv [xyz][xyw]\ (r^{(3)}_{2,2,1,1})$$

Multidegree $\alpha = (\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \emptyset$ ($c_\alpha = 0$). The only product of multidegree α is eliminated using $r_{3,1,1} \in R$ the following way:

$$[x^2y][xz][xw] \equiv -[x^2z][xy][xw] \equiv \pm[x^2w][xy][xz] \equiv 0$$

Multidegree $\alpha = (\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \{[xy^2][xz][xw]\}$ ($c_\alpha = 1$). The other products of multidegree α are eliminated using R the following way:

$$[x^2y][xz][yw] \equiv -[x^2z][xy][yw] \equiv -[xy]^2[xzw] - [xy][xw][xyz] \equiv$$

$$\equiv -[xy]^2[xzw] \equiv -[xy]^2[xzw] - [xy][xz][xyw] \equiv -[x^2w][yz][xy] \equiv$$

$$\equiv [x^2y][xw][yz] \equiv [xy^2][xz][xw]$$

$$(r^{(1)}_{2,1,1} \cdot [yw] \wedge r_{2,1,1,1} \cdot [xy] \wedge r^{(2)}_{2,1,1} \wedge r_{2,1,1,1} \cdot [xy])$$

$$[x^2y][xy][zw] \equiv 0\ (r_{3,2} \cdot [zw])$$

$$[xyz][xy][xw] \equiv 0\ (r^{(2)}_{2,1,1} \cdot [xw])$$

$$[xyw][xy][xz] \equiv 0\ (r^{(2)}_{2,1,1} \cdot [xz])$$

Multidegree $\alpha = (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1})$. Let $(S')_\alpha := \{[xy][yz][xzw],\ [xz][yz][xyw],\ [xz][xy][yzw]\}$ ($c_\alpha = 3$). The other products of multidegree α are eliminated using R the following way:

$$[yz][xw][xyz] \equiv [xz][yw][xyz] \equiv [xy][zw][xyz] \equiv 0\ (r^{(2)}_{2,1,1})$$

$$[xy^2][xz][zw] \equiv [x^2y][yz][zw] \equiv [xz][yz][xyw] + [xw][yz][xyz] \equiv [xz][yz][xyw]$$

$$(r^{(1)}_{2,1,1} \cdot [zw] \wedge r_{2,1,1,1} \cdot [yz] \wedge r^{(2)}_{2,1,1} \cdot [xw])$$

$$[x^2z][yw][yz] \equiv [xy][yz][xzw] + [xw][yz][xyz] \equiv [xy][yz][xzw]$$

$$(r_{2,1,1,1} \cdot [yz] \wedge r^{(2)}_{2,1,1} \cdot [xw])$$

$$[yz^2][xy][xw] \equiv [y^2z][xz][xw] \equiv [xz][xy][yzw] + [xz][yw][xyz] \equiv [xz][xy][yzw]$$

$$(r^{(1)}_{2,1,1} \cdot [xw] \wedge r_{2,1,1,1} \cdot [xz] \wedge r^{(2)}_{2,1,1})$$

$$[x^2w][yz][yz] \equiv [xz][yz][xyw] + [xy][yz][xzw] \ (r_{2,1,1,1} \cdot [yz])$$

$$[xy]^2[z^2w] \equiv [xy][zx][yzw] + [xy][zy][xzw] \ (r_{2,1,1,1} \cdot [xy])$$

$$[xz]^2[y^2w] \equiv [xz][xy][yzw] + [xz][yz][xyw] \ (r_{2,1,1,1} \cdot [xz])$$

$$[yz]^2[x^2w] \equiv [yz][xz][xyw] + [yz][xy][xzw] \ (r_{2,1,1,1} \cdot [yz])$$

Multidegree $\alpha = (\mathbf{5}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \emptyset$ ($c_\alpha = 0$). The only product of multidegree α is $[x^2y][x^2z][xw] \equiv 0$ by $r_{4,1,1} \in R$.

Multidegree $\alpha = (\mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \emptyset$ ($c_\alpha = 0$). The products of multidegree α are eliminated using R the following way:

$$[xy]^2[xz][xw] \equiv -3[xy][x^2y][xzw] \equiv 0 \ (r_{3,1,1,1} \cdot [xy] \wedge r_{3,2} \cdot [xzw])$$

$$[x^2z][xy^2][xw] \equiv -[x^2w][xy^2][xz] \equiv -[x^2w][x^2y][yz] \equiv 0$$

$$(r_{3,1,1} \cdot [xy^2] \wedge r^{(1)}_{2,1,1} \cdot [x^2w] \wedge r_{4,1,1})$$

$$[x^2y][xy][xzw] \equiv 0 \ (r_{3,2} \cdot [xzw])$$

$$[x^2y][xz][xyw] \equiv -[xy][x^2z][xyw] \equiv 0 \ (r_{3,1,1} \cdot [xyw] \wedge r^{(2)}_{2,1,1} \cdot [x^2z])$$

$$[x^2y][xw][xyz] \equiv -[x^2w][xy][xyz] \equiv 0 \ (r_{3,1,1} \cdot [xyz] \wedge r^{(2)}_{2,1,1} \cdot [x^2w])$$

Multidegree $\alpha = (\mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1})$. Let $(S')_\alpha := \{[xw][xyz][xy^2]\}$ ($c_\alpha = 1$). The other products of multidegree α are eliminated using R the following way:

$$[xy^2][xy][xzw] \equiv [xy][x^2y][yzw] \equiv 0 \ (r_{3,2})$$

$$[x^2y][yw][xyz] \equiv [x^2y][xy][yzw] + [x^2y][yw][xyz] \equiv [x^2y][xw][y^2z] \equiv$$

$$\equiv -[xy][x^2w][y^2z] \equiv [x^2w][xy^2][yz] \equiv [x^2y][yz][xyw] \equiv [xy^2][xyw][xz] \equiv$$

$$\equiv [y^2w][x^2y][xz] \equiv -[x^2z][xy][y^2w] \equiv [x^2z][xy^2][yw] \equiv [x^2y][yw][xyz] \equiv$$

$$\equiv [xw][xyz][xy^2]$$

$$(r_{3,2} \wedge r_{2,1,1,1} \cdot [x^2y] \wedge r_{3,1,1} \cdot [y^2z] \wedge r_{3,1,1} \cdot [x^2w] \wedge r^{(2)}_{3,2,1} \cdot [yz] \wedge r^{(1)}_{2,1,1} \cdot [xyw] \wedge$$

$$\wedge r^{(2)}_{3,2,1} \cdot [xz] \wedge r_{3,1,1} \cdot [y^2w] \wedge r_{3,1,1} \cdot [x^2z] \wedge r^{(2)}_{3,2,1} \cdot [zw] \wedge r^{(1)}_{2,1,1} \cdot [xyz])$$

$$[x^2y][xy^2][zw] \equiv [x^2y][yz][xyw] + [x^2y][yw][xyz] \equiv$$

$$\equiv 2 \cdot [x^2y][yw][xyz] \equiv 2[xw][xyz][xy^2]$$

(This follows from $r_{2,1,1,1} \cdot [x^2y]$ and the relation above.)

Multidegree $\alpha = (\mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1})$. Let $(S')_\alpha := \{[xyz]^2[xw]\}$ ($c_\alpha = 1$). The other products of multidegree α are eliminated using R the following way:

$$[x^2w][yz^2][xy] \equiv [x^2w][y^2z][xz] \equiv [xw][y^2z][x^2z] \equiv [xyz]^2[xw]$$

$$(r^{(1)}_{2,1,1} \cdot [x^2w] \wedge r^{(1)}_{2,1,1} \cdot [y^2z] \wedge r^{(2)}_{2,2,2} \cdot [xw])$$

$$[yw][xyz][x^2z] \equiv [xy][xzw][xyz] + [xw][xyz]^2 \equiv [xyz]^2[xw]$$

$$(r_{2,1,1,1} \cdot [xyz] \wedge r^{(2)}_{2,1,1} \cdot [xzw])$$

$$[zw][xyz][x^2y] \equiv [xz][xyw][xyz] + [xw][xyz]^2 \equiv [xyz]^2[xw]$$

$$(r_{2,1,1,1} \cdot [xyz] \wedge r^{(2)}_{2,1,1} \cdot [xyw])$$

$$[xy^2][xw][xz^2] \equiv [x^2y][xw][yz^2] \equiv [x^2z][xw][y^2z] \equiv [xyz]^2[xw]$$

$$(r^{(2)}_{2,2,2} \cdot [xw])$$

$$[x^2y][yzw][xz] \equiv -[x^2z][yzw][xy] \equiv -[xz^2][xyw][xy] \equiv 0$$

$$(r_{3,1,1} \cdot [yzw] \wedge (r^{(3)}_{2,2,1,1} - r^{(3)'}_{2,2,1,1})' \cdot [xy] \wedge r^{(2)}_{2,1,1} \cdot [xz^2])$$

$$[x^2z][y^2w][xz] \equiv 0 \ (r_{3,2} \cdot [y^2w])$$

$$[x^2y][z^2w][xy] \equiv 0 \ (r_{3,2} \cdot [z^2w])$$

$$[xy][xz][xw][yz] \equiv 0 \ (r^{(1)}_{2,2,2} \cdot [xw])$$

$$[xy][xzw][xyz] \equiv 0 \ (r^{(2)}_{2,1,1} \cdot [xzw])$$

$$[xz][xyw][xyz] \equiv 0 \ (r^{(2)}_{2,1,1} \cdot [xyw])$$

$$[yz][xyw][x^2z] \equiv [xz^2][xy][xyw] \equiv 0 \ (r^{(1)}_{2,1,1} \cdot [xyw] \wedge r^{(2)}_{2,1,1} \cdot [xz^2])$$

$$[yz][xzw][x^2y] \equiv [xy^2][xz][xzw] \equiv 0 \ (r^{(1)}_{2,1,1} \cdot [xzw] \wedge r^{(2)}_{2,1,1} \cdot [xy^2])$$

$$[xy^2][xz][xzw] \equiv 0 \ (r^{(2)}_{2,1,1} \cdot [xy^2])$$

$$[x^2z][zw][xy^2] \equiv -\frac{1}{3}[zw][xy]^2[xz] \equiv$$

$$\equiv -[xz][x^2z][y^2w] - [xz][x^2w][y^2z] + 2[xz][xyz][xyw] \equiv$$

$$\equiv -[xz][x^2w][y^2z] \equiv -[xyz]^2[xw]$$

(This follows from $r^{(1)}_{3,2,1} \cdot [zw] \wedge r^{(1)}_{2,2,1,1} \cdot [xz] \wedge r_{3,2} \cdot [y^2w] \wedge r^{(2)}_{2,1,1} \cdot [xyw]$ and the relation above.)

$$[x^2y][xz^2][yw] \equiv -\frac{1}{3}[xy][xz]^2[yw] \equiv$$

$$\equiv -[xy][x^2y][z^2w] - [xy][x^2w][z^2y] + 2[xy][xyz][xzw] \equiv$$

$$\equiv -[xy][x^2w][z^2y] \equiv [xyz]^2[xw]$$

(This follows from $r^{(1)}_{3,2,1} \cdot [yw] \wedge r^{(1)}_{2,2,1,1} \cdot [xy] \wedge r_{3,2} \cdot [z^2w] \wedge r^{(2)}_{2,1,1} \cdot [xzw]$ and the relation above.)

Multidegree $\alpha = (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$. Let $(S')_\alpha := \{[xz^2][yw^2][xy],\ [y^2z][xw^2][xz],\ [x^2y][zw^2][yz]\}$ ($c_\alpha = 3$). The other products of multidegree α are eliminated using R the following way:

$$[y^2w][xz^2][xw] \equiv [x^2z][y^2w][zw] \equiv [x^2z][yz][yw^2] \equiv [xz^2][yw^2][xy]$$

$$(r^{(1)}_{2,1,1} \cdot [y^2w] \wedge r^{(1)}_{2,1,1} \cdot [x^2z] \wedge r^{(1)}_{2,1,1} \cdot [yw^2])$$

$$[x^2w][y^2z][zw] \equiv [x^2w][yz^2][yw] \equiv [yz^2][xw^2][xy] \equiv [y^2z][xw^2][xz]$$

$$(r^{(1)}_{2,1,1} \cdot [x^2w] \wedge r^{(1)}_{2,1,1} \cdot [yz^2] \wedge r^{(1)}_{2,1,1} \cdot [xw^2])$$

$$[x^2y][z^2w][yw] \equiv [xy^2][z^2w][xw] \equiv [xy^2][zw^2][xz] \equiv [x^2y][zw^2][yz]$$

$$(r^{(1)}_{2,1,1} \cdot [z^2w] \wedge r^{(1)}_{2,1,1} \cdot [xy^2] \wedge r^{(1)}_{2,1,1} \cdot [zw^2])$$

$$[xy][xzw][yzw] \equiv [xy][z^2w][xyw] \equiv 0 \ (r^{(3)}_{2,2,1,1} \cdot [xy] \wedge r^{(2)}_{2,1,1} \cdot [z^2w])$$

$$[xz][xyw][yzw] \equiv [xz][y^2w][xzw] \equiv 0 \ r^{(3)}_{2,2,1,1} \cdot [xz] \wedge r^{(2)}_{2,1,1} \cdot [y^2w])$$

$$[xw][xyz][yzw] \equiv [xw][y^2z][xzw] \equiv 0 \ (r^{(3)}_{2,2,1,1} \cdot [xw] \wedge r^{(2)}_{2,1,1} \cdot [y^2z])$$

$$[yz][xyw][xzw] \equiv [yz][x^2w][wyz] \equiv 0 \ (r^{(3)}_{2,2,1,1} \cdot [yz] \wedge r^{(2)}_{2,1,1} \cdot [x^2w])$$

$$[zw][xyz][xyw] \equiv [zw][x^2y]yzw[] \equiv 0 \ (r^{(3)}_{2,2,1,1} \cdot [zw] \wedge r^{(2)}_{2,1,1} \cdot [x^2y])$$

$$[yw][xzw][xyz] \equiv [yw][x^2z][zyw] \equiv 0 \ (r^{(3)}_{2,2,1,1} \cdot [yw] \wedge r^{(2)}_{2,1,1} \cdot [x^2z])$$

Since there are no other descending multidegrees with all coordinates nonzero and total degree ≤ 8. This completes the proof.

Appendix B

Young-diagrams and dimensions

This section contains tables with Young-diagrams of k boxes and m lines, and the dimensions of corresponding irreducible $\mathrm{GL}_m(\mathbb{C})$-representations.

m	Dimension
2	$\frac{3-2+1}{1} = 2$
3	$2 \cdot \frac{3+2}{2} \cdot \frac{2+1}{1} = 15$
4	$15 \cdot \frac{3+3}{3} \cdot \frac{2+2}{2} = 60$
5	$60 \cdot \frac{3+4}{4} \cdot \frac{2+3}{3} = 175$

Table B.1: Dimension of the $\mathrm{GL}_m(\mathbb{C})$-representation of type $(3, 2)$ by m

Type	$m=2$	$m=3$	$m=4$
$(4,2)$	$\frac{4-2+1}{1}=3$	$3\cdot\frac{4+2}{2}\cdot\frac{2+1}{1}=27$	$27\cdot\frac{4+3}{3}\cdot\frac{2+2}{2}=126$
$(3,3)$	$\frac{3-3+1}{1}=1$	$1\cdot\frac{3+2}{2}\cdot\frac{3+1}{1}=10$	$10\cdot\frac{3+3}{3}\cdot\frac{3+2}{2}=50$
$(3,2,1)$	$-$	$\frac{1+1}{1}\cdot\frac{2+2}{2}\cdot\frac{1+1}{1}=8$	$8\cdot\frac{3+3}{3}\cdot\frac{2+2}{2}\cdot\frac{1+1}{1}=64$
$(2,2,2)$	$-$	1	$1\cdot\frac{2+3}{3}\cdot\frac{2+2}{2}\cdot\frac{2+1}{1}=10$

Type	$m=5$	$m=6$
$(4,2)$	$126\cdot\frac{4+4}{4}\cdot\frac{2+3}{3}=420$	$420\cdot\frac{4+5}{5}\cdot\frac{2+4}{4}=1134$
$(3,3)$	$50\cdot\frac{3+4}{4}\cdot\frac{3+3}{3}=175$	$175\cdot\frac{3+5}{5}\cdot\frac{3+4}{4}=490$
$(3,2,1)$	$64\cdot\frac{3+4}{4}\cdot\frac{2+3}{3}\cdot\frac{1+2}{2}=280$	$280\cdot\frac{3+5}{5}\cdot\frac{2+4}{4}\cdot\frac{1+3}{3}=896$
$(2,2,2)$	$10\cdot\frac{2+4}{4}\cdot\frac{2+3}{3}\cdot\frac{2+2}{2}=50$	$50\cdot\frac{2+5}{5}\cdot\frac{2+4}{4}\cdot\frac{2+3}{3}=175$

Table B.2: Dimension of the $\mathrm{GL}_m(\mathbb{C})$-representations of degree 6 by m

Type	$m = 2$	$m = 3$
$(5, 2)$	$\frac{3+1}{1} = 4$	$4 \cdot \frac{5+2}{2} \cdot \frac{2+1}{1} = 42$
$(4, 3)$	$\frac{1+1}{1} = 2$	$2 \cdot \frac{4+2}{2} \cdot \frac{3+1}{1} = 24$
$(4, 2, 1)$	–	$\frac{2+1}{1} \cdot \frac{3+2}{2} \cdot \frac{1+1}{1} = 15$
$(3, 3, 1)$	–	$\frac{2+2}{2} \cdot \frac{2+1}{1} = 6$
$(3, 2, 2)$	–	$\frac{1+1}{1} \cdot \frac{1+2}{2} = 3$
$(2, 2, 2, 1)$	–	–

Type	$m = 4$	$m = 5$
$(5, 2)$	$42 \cdot \frac{5+3}{3} \cdot \frac{2+2}{2} = 224$	$224 \cdot \frac{5+4}{4} \cdot \frac{2+3}{3} = 840$
$(4, 3)$	$24 \cdot \frac{4+3}{3} \cdot \frac{3+2}{2} = 140$	$140 \cdot \frac{4+4}{4} \cdot \frac{3+3}{3} = 560$
$(4, 2, 1)$	$15 \cdot \frac{4+3}{3} \cdot \frac{2+2}{2} \cdot \frac{1+1}{1} = 140$	$140 \cdot \frac{4+4}{4} \cdot \frac{2+3}{3} \cdot \frac{1+2}{2} = 700$
$(3, 3, 1)$	$6 \cdot \frac{3+3}{3} \cdot \frac{3+2}{2} \cdot \frac{1+1}{1} = 60$	$60 \cdot \frac{3+4}{4} \cdot \frac{3+3}{3} \cdot \frac{1+2}{2} = 315$
$(3, 2, 2)$	$3 \cdot \frac{3+3}{3} \cdot \frac{2+2}{2} \cdot \frac{2+1}{1} = 36$	$36 \cdot \frac{3+4}{4} \cdot \frac{2+3}{3} \cdot \frac{2+2}{2} = 210$
$(2, 2, 2, 1)$	$\frac{1+3}{3} \cdot \frac{1+2}{2} \cdot \frac{1+1}{1} = 4$	$4 \cdot \frac{2+4}{4} \cdot \frac{2+3}{3} \cdot \frac{2+2}{2} \cdot \frac{1+1}{1} = 40$

Type	$m = 6$	$m = 7$
$(5, 2)$	$840 \cdot \frac{5+5}{5} \cdot \frac{2+4}{4} = 2520$	$2520 \cdot \frac{5+6}{6} \cdot \frac{2+5}{5} = 6468$
$(4, 3)$	$560 \cdot \frac{4+5}{5} \cdot \frac{3+4}{4} = 1764$	$1764 \cdot \frac{4+6}{6} \cdot \frac{3+5}{5} = 4704$
$(4, 2, 1)$	$700 \cdot \frac{4+5}{5} \cdot \frac{2+4}{4} \cdot \frac{1+3}{3} = 2520$	$2520 \cdot \frac{4+6}{6} \cdot \frac{2+5}{5} \cdot \frac{1+4}{4} = 7350$
$(3, 3, 1)$	$315 \cdot \frac{3+5}{5} \cdot \frac{3+4}{4} \cdot \frac{1+3}{3} = 1176$	$1176 \cdot \frac{3+6}{6} \cdot \frac{3+5}{5} \cdot \frac{1+4}{4} = 3528$
$(3, 2, 2)$	$210 \cdot \frac{3+5}{5} \cdot \frac{2+4}{4} \cdot \frac{2+3}{3} = 840$	$840 \cdot \frac{3+6}{6} \cdot \frac{2+5}{5} \cdot \frac{2+4}{4} = 2646$
$(2, 2, 2, 1)$	$40 \cdot \frac{2+5}{5} \cdot \frac{2+4}{4} \cdot \frac{2+3}{3} \cdot \frac{1+2}{2} = 210$	$210 \cdot \frac{2+6}{6} \cdot \frac{2+5}{5} \cdot \frac{2+4}{4} \cdot \frac{1+3}{3} = 784$

Table B.3: Dimension of the $\mathrm{GL}_m(\mathbb{C})$-representations of degree 7 by m

Type	$m=2$	$m=3$	$m=4$
$(6,2)$	$\frac{6-2+1}{1}=5$	$5\cdot\frac{6+2}{2}\cdot\frac{2+1}{1}=60$	$60\cdot\frac{6+3}{3}\cdot\frac{2+2}{2}=360$
$(5,3)$	$\frac{2+1}{1}=3$	$3\cdot\cdot\frac{5+2}{2}\frac{3+1}{1}=42$	$42\cdot\frac{5+3}{3}\cdot\frac{3+2}{2}=280$
$(4,4)$	1	$1\cdot\frac{4+2}{2}\cdot\frac{4+1}{1}=15$	$15\cdot\frac{4+3}{3}\cdot\frac{4+2}{2}=105$
$(5,2,1)$	-	$\frac{4+2}{2}\cdot\frac{3+1}{1}\cdot\frac{1+1}{1}=24$	$24\cdot\frac{5+3}{3}\cdot\frac{2+2}{2}\cdot\frac{1+1}{1}=256$
$(4,3,1)$	-	$\frac{3+2}{2}\cdot\frac{1+1}{1}\cdot\frac{2+1}{1}=15$	$15\cdot\frac{4+3}{3}\cdot\frac{3+2}{2}\cdot\frac{1+1}{1}=175$
$(4,2,2)$	-	$\frac{2+2}{2}\cdot\frac{2+1}{1}=6$	$6\cdot\frac{4+3}{3}\cdot\frac{2+2}{2}\cdot\frac{2+1}{1}=84$
$(3,3,2)$	-	$\frac{1+2}{2}\cdot\frac{1+1}{1}=3$	$3\cdot\frac{3+3}{3}\cdot\frac{3+2}{2}\cdot\frac{2+1}{1}=45$
$(4,2,1,1)$	-	-	$\frac{3+3}{3}\cdot\frac{3+2}{2}\cdot\frac{2+1}{1}\cdot\frac{1+2}{2}\cdot\frac{1+1}{1}=45$
$(3,2,2,1)$	-	-	$\frac{2+3}{3}\cdot\frac{1+2}{2}\cdot\frac{1+1}{1}\cdot\frac{1+2}{2}\cdot\frac{1+1}{1}=15$
$(2,2,2,2)$	-	-	1

Type	$m=5$	$m=6$
$(6,2)$	$360\cdot\frac{6+4}{4}\cdot\frac{2+3}{3}=1500$	$1500\cdot\frac{6+5}{5}\cdot\frac{2+4}{4}=4950$
$(5,3)$	$280\cdot\frac{5+4}{4}\cdot\frac{3+3}{3}=1260$	$1260\cdot\frac{5+5}{5}\cdot\frac{3+4}{4}=4410$
$(4,4)$	$105\cdot\frac{4+4}{4}\cdot\frac{4+3}{3}=490$	$490\cdot\frac{4+5}{5}\cdot\frac{4+4}{4}=1764$
$(5,2,1)$	$256\cdot\frac{5+4}{4}\cdot\frac{2+3}{3}\cdot\frac{1+2}{2}=1440$	$1440\cdot\frac{5+5}{5}\cdot\frac{2+4}{4}\cdot\frac{1+3}{3}=5760$
$(4,3,1)$	$175\cdot\frac{4+4}{4}\cdot\frac{3+3}{3}\cdot\frac{1+2}{2}=1050$	$1050\cdot\frac{4+5}{5}\cdot\frac{3+4}{4}\cdot\frac{1+3}{3}=4410$
$(4,2,2)$	$84\cdot\frac{4+4}{4}\cdot\frac{2+3}{3}\cdot\frac{2+2}{2}=560$	$560\cdot\frac{4+5}{5}\cdot\frac{2+4}{4}\cdot\frac{2+3}{3}=2520$
$(3,3,2)$	$45\cdot\frac{3+4}{4}\cdot\frac{3+3}{3}\cdot\frac{2+2}{2}=315$	$315\cdot\frac{3+5}{5}\cdot\frac{3+4}{4}\cdot\frac{2+3}{3}=1470$
$(4,2,1,1)$	$45\cdot\frac{4+4}{4}\cdot\frac{2+3}{3}\cdot\frac{1+2}{2}\cdot\frac{1+1}{1}=450$	$450\cdot\frac{4+5}{5}\cdot\frac{2+4}{4}\cdot\frac{1+3}{3}\cdot\frac{1+2}{2}=2430$
$(3,2,2,1)$	$15\cdot\frac{3+4}{4}\cdot\frac{2+3}{3}\cdot\frac{2+2}{2}\cdot\frac{1+1}{1}=175$	$175\cdot\frac{3+5}{5}\cdot\frac{2+4}{4}\cdot\frac{2+3}{3}\cdot\frac{1+2}{2}=1050$
$(2,2,2,2)$	$1\cdot\frac{2+4}{4}\cdot\frac{2+3}{3}\cdot\frac{2+2}{2}\cdot\frac{2+1}{1}=15$	$15\cdot\frac{2+5}{5}\cdot\frac{2+4}{4}\cdot\frac{2+3}{3}\cdot\frac{2+2}{2}=105$

Type	$m=7$	$m=8$
$(6,2)$	$4950\cdot\frac{6+6}{6}\cdot\frac{2+5}{5}=13860$	$13860\cdot\frac{6+7}{7}\cdot\frac{2+6}{6}=34320$
$(5,3)$	$4410\cdot\frac{5+6}{6}\cdot\frac{3+5}{5}=12936$	$12936\cdot\frac{5+7}{7}\cdot\frac{3+6}{6}=33264$
$(4,4)$	$1764\cdot\frac{4+6}{6}\cdot\frac{4+5}{5}=5292$	$5292\cdot\frac{4+7}{7}\cdot\frac{4+6}{6}=13860$
$(5,2,1)$	$5760\cdot\frac{5+6}{6}\cdot\frac{2+5}{5}\cdot\frac{1+4}{4}=18480$	$18480\cdot\frac{5+7}{7}\cdot\frac{2+6}{6}\cdot\frac{1+5}{5}=50688$
$(4,3,1)$	$4410\cdot\frac{4+6}{6}\cdot\frac{3+5}{5}\cdot\frac{1+4}{4}=14700$	$14700\cdot\frac{4+7}{7}\cdot\frac{3+6}{6}\cdot\frac{1+5}{5}=41580$
$(4,2,2)$	$2520\cdot\frac{4+6}{6}\cdot\frac{2+5}{5}\cdot\frac{2+4}{4}=8820$	$8820\cdot\frac{4+7}{7}\cdot\frac{2+6}{6}\cdot\frac{2+5}{5}=25872$
$(3,3,2)$	$1470\cdot\frac{3+6}{6}\cdot\frac{3+5}{5}\cdot\frac{2+4}{4}=5292$	$5292\cdot\frac{3+7}{7}\cdot\frac{3+6}{6}\cdot\frac{2+5}{5}=15876$
$(4,2,1,1)$	$2430\cdot\frac{4+6}{6}\cdot\frac{2+5}{5}\cdot\frac{1+4}{4}\cdot\frac{1+3}{3}=9450$	$9450\cdot\frac{4+7}{7}\cdot\frac{2+6}{6}\cdot\frac{1+5}{5}\cdot\frac{1+4}{4}=29700$
$(3,2,2,1)$	$1050\cdot\frac{3+6}{6}\cdot\frac{2+5}{5}\cdot\frac{2+4}{4}\cdot\frac{1+3}{3}=4410$	$4410\cdot\frac{3+7}{7}\cdot\frac{2+6}{6}\cdot\frac{2+5}{5}\cdot\frac{1+4}{4}=14700$
$(2,2,2,2)$	$105\cdot\frac{2+6}{6}\cdot\frac{2+5}{5}\cdot\frac{2+4}{4}\cdot\frac{2+3}{3}=490$	$490\cdot\frac{2+7}{7}\cdot\frac{2+6}{6}\cdot\frac{2+5}{5}\cdot\frac{2+4}{4}=1764$

Table B.4: Dimension of the $\mathrm{GL}_m(\mathbb{C})$-representations of degree 8 by m

Appendix C

Hilbert-series of $\mathbf{I_m}$

The Hilbert-series of the ideal of relations can be computed using the results in 1.3.3. Substitute every variable $t_1, \ldots, t_m$ by the same variable t and write $\mathbf{t}$ for the vector $(t, \ldots, t)$. The coefficient of t^k in $H(I_m, \mathbf{t})$ shows the dimension of $(I_m)_k$. In 4.5 these coefficients are needed only for $5 \leq m, k \leq 8$. (Every coefficient is 0 if $k \leq 4$.) The table below shows these coefficients by m and k

	$k = 5$	$k = 6$	$k = 7$	$k = 8$
$m = 4$	60	376	1684	6425
$m = 5$	175	1345	7285	33100
$m = 6$	420	3829	24318	128262
$m = 7$	882	9310	67816	407330
$m = 8$	1680	20160	165648	1116324

Table C.1: The dimension of $(I_m)_k$ when $2 \leq m \leq 8$ and $5 \leq k \leq 8$.

Selected Bibliography

[1] M. Domokos. Vector invariants of a class of pseudo-reflection groups and multisymmetric syzygies. *Arxiv preprint arXiv:0706.2154*, 2007.

[2] W. Fulton and J. Harris. *Representation theory: a first course*. Springer, 1991.

[3] I.G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, USA, 1995.

[4] L. Solomon. Partition identities and invariants of finite groups. *J. Combin. Theory Ser. A*, 23:148–175, 1977.

[5] R.P. Stanley. Invariants of finite groups and their applications to combinatorics. *American Mathematical Society*, 1(3), 1979.

[6] B. Sturmfels. *Algorithms in invariant theory*. Springer, 2008.