Extending partial homomorphisms of relational structures

Master's thesis

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Contents

1 Introduction

Let X be a finite graph and $f : A \to B$ be an isomorphism between induced subgraphs A and B of X. Then there exists a greater finite graph Z such that X is an induced subgraph of Z, and f has an extension f^*, which is an automorphism of Z. This theorem has been proved by Truss, for more general results, see Hrushovski [5] and Herwig [3].

The main results of this work are the following: we generalize this theorem as writing partial homomorphism h instead of partial isomorphism f, we get an extension h^* which is an endomorphism of Z (Theorem 3.5). Furthermore, using Hrushovski's theorem, we show another proof of Truss's theorem (4.2), which says that the space $Aut(R)$ has a dense conjugacy class, where R is the Rado graph (also known as countable random graph). The definitions and the topology can be found in the 4th chapter. Similarly, we prove it for $End(R)$, too (Theorem 4.4). Then we show that different versions of density is equivalent in this space (Theorem 4.5). Finally, we deduce a weaker version of Hrushovski's theorem from the existence of a dense conjugacy class (Theorem 4.7).

Introduction

In this first chapter we will attempt to summarize the basic definitions and theorems that are crucial for understanding theorems we prove later in this work. We will also set the notation that will be used.

The second chapter is structured around a pair of notions: the homogeneity of a structure, and its age. We will specify two properties, namely hereditary property and joint embedding property, that characterize whether a class of structures is the age of a structure. Then with the help of a third property - the amalgamation property - we will succeed in finding a unique homogeneous structure (called Fraïssé limit) belonging to a class of structures. (The ideas treated here will recur later in the fourth chapter.)

Next, applying this theory we will construct the universal homogeneous structure of the class of finite graphs, namely the Rado graph. This graph will be characterized by the separation property. Finally, we will point out the connection between the Rado graph and random graphs. In addition to this, we will prove an interesting theorem about limit probability of sentences in a graph, as well.

This chapter has a survey character and it is based on Hodges's book ([4]), Cameron's lecture notes ([2]) and Sági's textbook ([7]). In particular, all the results in this chapter are well known.

The third chapter is based on the 1992 findings of E. Hrushovski ([5]). His theorem says that every finite graph can be embedded to a greater finite graph such that every partial isomorphism of the original graph extends to an automorphism of the greater one. Here we will present his original proof. This proof has two major steps. First the original graph will be embedded to an intermediate graph such that the edges/no-edges departs from the domain of some partial isomorphism will be preserved. As second step the greater graph will be constructed with a masterly algebraic construction. We will be generalize this theorem for partial homomorphism and endomorphism. To do this we must modify the conditions, and find another way as the first step.

In the fourth chapter, we will evoke the notion of generic automorphism (which has a dense conjugacy class in the automorphism group). Of course, because of denseness we will need to introduce a topology. Then we give another proof of Truss's theorem: we create a generic automorphism of the Rado graph – using the theorem of Hrushovski. We will also prove the similar theorem for generic endomorphism. Then we will prove the equivalency of some of properties of its conjugacy class. Finally, we will deduce a weaker version of the theorem of Hrushovski from the existence of generic automorphism.

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Notation and preliminaries

We summarize the notation of this paper, and the basic definitions and theorems we will use.

In this paper ω denotes the set of natural numbers. The set of integer and rational numbers will be denoted by $\mathbb{Z}$ and $\mathbb{Q}$, respectively, and $[a, b]$ will denote the set $\{a, a+1, \ldots, b\}$ for $a \leq b$ natural numbers. We will use the standard notation of operation of sets as $\cup, \cap, \subseteq$, etc. We will denote the set of function from A to B with ${}^{A}B$. As usual $\mathcal{P}(X)$ denotes the power set of a set X. The sign of the cardinality of a set X is $|X|$. Finally, $\aleph_0$ is the cardinality of ω.

The functions are considered as sets (of pairs). So we will use set operation and relation on its. E.g. $f \subseteq g$ means g is an extension of f. The domain and range of a function f is denoted by $Dom(f)$ and $Ran(f)$, respectively. The image of a set X by a function f will be denoted by $f[X]$.

In this paper we deal only with relational structures. So a signature $\mathscr{S}$ contains (at most countable, but almost always finite number of) relational symbols, and an arity function ar on the symbols: $\mathscr{S} = (Rel, ar)$. If we say $\mathscr{S}$-structure A, we think a set with some relation $r^A \subseteq {}^{ar(r)}A$ for some $r \in Rel$. We will not differentiate in notation between a structure and its underlying set, it will be clear which to think about. For the relation we will use infix notation xRy or $(x, y) \in R$.

We also use the standard phrases and notation of mathematical logic, like A is a model of a set of sentences T ($A \models T$).

In most cases the relational structure will be a graph: this means a structure with an irreflexive and symmetric "adjacency" relation E or $E(X)$ (if the graph is X). Graph theory has an own tradition of notation, and partially this will be appeared here. So, the elements of a graph structure will be called vertices (sometimes nodes), the elements of the relation set will be called edges. The vertices adjacent to a fixed vertex a will be called neighbours, and will be denoted by $X(a)$ for a graph X.

We will say that A is a substructure of B (denoted by $A \leq B$), if A and B structures with the same signature, and A (as a set) is a subset of B, and all the relations constraining to A will be unchanged. Let us note that the case of graphs this means that A is an induced subgraph of B.

Definition. *If A and B are $\mathscr{S}$-structures for a fixed $\mathscr{S}$, then we call a one-to-one function $i : A \to B$ relational isomorphism or isomorphism iff for every $r \in Rel$ we have $(x_1, \ldots, x_{ar(r)}) \in r^A$ if and only if $(i(x_1), \ldots, i(x_{ar(r)})) \in r^B$.*

If there is an isomorphism between A and B, then we denote it with $A \cong B$. Similarly,

Definition. *If A and B are $\mathscr{S}$-structures for a fixed $\mathscr{S}$, then we call a function $h : A \to B$ relational homorphism or homomorphism iff for every $r \in Rel$ we have $(x_1, \ldots, x_{ar(r)}) \in r^A$ if and only if $(h(x_1), \ldots, h(x_{ar(r)})) \in r^B$.*

Note that according to our definition, homomorphism for graphs preserves not only the edges but also the not edges!

An isomorphism between the same structure will be called automorphism, and a homomorphism between the same structure will be called endomorphism. If for an isomorphism (homomorphism) $m : A \to B$ the set $A \neq Dom(f)$, then it is called a partial isomorphism (respectively, partial homomorphism). The group of automorphisms of a structure X will be denoted by $Aut(X)$, the semigroup of endomorphisms of a structure X will be denoted by $End(X)$.

Recall that the kernel of a function $A \to B$ is an equivalence relation on A, in which two elements are equivalent iff they have the same homomorphic image. A congruence is the kernel of a homomorphism. The quotient structure by an equivalence R will be denoted by X/R. We will use the 'join' function of two equivalence relation, it will be denoted by $\vee$. This notions are precisely the same as introduced in [1].

Let us remind some of basic theorems of logic. These theorems can be found practically all the textbooks of logic.

Theorem 1.1 (Compactness). *A (possibly infinite) set of first order sentences has a model if and only if every finite subset of it has a model.*

Theorem 1.2 (Löwenheim–Skolem). *Let T be a theory. If T has an infinite model, then it has some infinite model K such that $|K| = \kappa$ where $\kappa \geq |\mathscr{S}| \cdot \aleph_0$ is an arbitrary cardinality.*

2 Fraïssé-limit and random graphs

If there are finite relational structures, e.g. linear orderings, then an interesting question arises: are there a (possibly countable) structure, in which these finite structures can be embedded? We can think of the linear ordering of the natural numbers. A more general question is, whether there are structures, in which every partial embedding can be extended? We can think of $(\mathbb{Q}, <)$ as a linear ordering. We will prove that these examples are right.

This chapter relies on W. Hodges's book ([4]), P. J. Cameron's notes ([2]) and Gábor Sági's textbook ([7]).

Age and Fraïssé-limit

In this section let $\mathscr{S}$ be a fixed signature. First, here is a definition about the class of "embeddable" structures.

Definition. *A class $\mathcal{K}$ of $\mathscr{S}$-structures is said to be the age of a countable structure M iff it contains all the structures (up to isomorphism) which are isomorphic with some finite substructure of M. Denote this class by $Age(M)$.*

We can also say that a class $\mathcal{K}$ of $\mathscr{S}$-structures is the age of M iff for every $K \in \mathcal{K}$ K is finite and can be embedded to M as substructure, and it contains all the finite substructures of M.

The question is, what condition are necessary and sufficient to establish the existence and uniqueness of such a model for a class $\mathcal{K}$ of finite structures? Now let us introduce three conditions we will need.

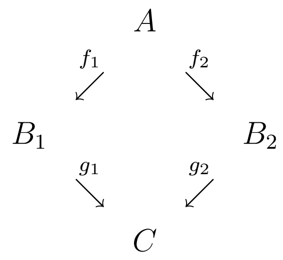
Hereditary Property (HP) If $A \in \mathcal{K}$ is a structure and $B \leq A$ is a finite structure, then $B \in \mathcal{K}$, too.

Joint Embedding Property (JEP) If $A, B \in \mathcal{K}$ are structures, then there exists a structure C in $\mathcal{K}$ in which A and B are embeddable.

$$\begin{array}{ccc} A & & B \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

Amalgamation property (AP) If $A, B_1, B_2 \in \mathcal{K}$ are $\mathscr{S}$-structures and $f_1 : A \to B_1$ and $f_2 : A \to B_2$ are embeddings, then there are $C \in \mathcal{K}$ and $g_1 : B_1 \to C$,

$g_2 : B_2 \to C$ embeddings such that $g_1 \circ f_1 = g_2 \circ f_2$, i.e. the diagram



commutes.

If $\mathcal{K}$ is the age of M, then (HP) obviously holds for $\mathcal{K}$. It is clear that (JEP) is also true because if we embed A and B to M then the induced substructure of the images satisfies (JEP). The third property will be proved later.

Fraïssé has proven that two of these conditions are also sufficient for the existence of such an M.

Proposition 2.1. *If $\mathcal{K}$ is a class of finite $\mathscr{S}$-structures which has (HP) and (JEP), then there exists a countable structure M such that $\mathcal{K} = Age(M)$.*

Proof There are at most countably many elements in $\mathcal{K}$, because the structures are finite and there is only one from every isomorphism class. So there is an enumeration $(A_i)_{i<\omega}$ of elements of $\mathcal{K}$.

Then we define another list of structures. Let $B_0 \overset{def}{=} A_0$. Because (JEP) there is a joint embedding of B_{i-1} and A_i for every $i < \omega$. Let B_i be the image of this embedding. Then we can define

$$M = B_\omega = \bigcup_{i<\omega} B_i.$$

It can be seen that every $A_i \in \mathcal{K}$ can be embedded to M, and M is countable as it is the countable union of finite sets. •

This theorem is not enough for us: we need the third condition to prove the uniqeness of M. But what does uniqueness mean in this situation? The answer is homogeneity.

Definition. *An $\mathscr{S}$-structure M is homogeneous iff every partial isomorphism f between its finite substructures can be extended to an automorphism of M.*

Let us note that W. Hodges called this concept ultrahomogeneous. Next, here is a more concrete definition:

Definition. *An $\mathscr{S}$-structure is weakly homogeneous iff whenever $A, B \in Age(M)$ $\quad A \subseteq B$ and $|A| + 1 = |B|$, then every embedding $g : A \to M$ can be extended to an embedding $g^* : B \to M$.*

Lemma 2.2. *A countable structure M is homogeneous if and only if it is weakly homogeneous.*

Proof Necessity comes from homogeneity: if $h : B \to M$ is an embedding then gh^{-1} is a partial isomorphism of M (between $g[A]$ and $h[A]$). By homogeneity there is an automorphism i which extends gh^{-1}. So it is clear that ih is an extension of g and is an embedding of B.

$$\begin{array}{ccc} A & \subseteq & B \\ \downarrow_g & & \downarrow_h \\ M & \overset{i}{\leftarrow} & M \end{array}$$

As for sufficiency, we apply recursion: let $f = f_0$ be the partial isomorphism which should be extended to an automorphism. Let $m_1, m_2, \ldots$ be an enumeration of those elements of M which do not belong to $Dom(f)$. Having defined f_{i-1}, we can set f_i using weak homogeneity:

$$Dom(f_{i-1}) \cup \{m_i\} \quad \text{as } B, \qquad Dom(f_{i-1}) \quad \text{as } A, \qquad \text{and} \qquad f_{i-1} \text{ as } g.$$

Thus we get a partial isomorphism on $Dom(f_{i-1}) \cup \{m_i\}$. Let this partial isomorphism be f_i. This extends f_{i-1}. Finally, we can define the desired automorphism as $\bigcup_{i<\omega} f_i$. It is an extension of f (we defined it this way), so M is homogeneous. •

This proof is very important because its idea plays an essential role in the proofs of the fourth section.

Now, we are ready to prove Fraïssé's theorem.

Theorem 2.3 (Fraïssé)**.** *1. The class $\mathcal{K}$ of $\mathscr{S}$-structures is the age of some countable homogeneous structure M if and only if $\mathcal{K}$ has (HP), (JEP) and (AP).*

2. If M and N are countable homogeneous structures with the same age ($Age(M) = Age(N)$) then M and N are isomorphic.

Proof

1. For the "only if" part, we only need to prove that $\mathcal{K}$ has (AP) follows from homogeneity of M (as we promise above).

We may assume that A, B_1, B_2 are subsets of M such that $A \subseteq B_1$. Then the partial isomorphism f_2 of M (between A and $f_2[A]$) can be extended to an automorphism i by homogeneity.

Then let $C \overset{def}{=} i[B_1] \cup B_2$. Choosing $g_1 = i|_{B_1}$ and $g_2 = \text{id}$ completes the proof.

The idea we use to prove the "if" part is very similar to the one in the proof of Proposition 2.1.

There are at most countably many elements in $\mathcal{K}$, because the structures are finite and there is only one from every isomorphism class. So we can form an enumeration $(A_i, B_i)_{i<\omega}$ of pairs of structures from $\mathcal{K}$ (where $A_i \subseteq B_i$ and $|A_i| + 1 = |B_i|$) such that if $A_j \cong B_i$ then $i < j$. (E.g. if one orders structures by its cardinality, it will be appropriate.)

Next we define another list of structures using recursion. Let $M_0 \overset{def}{=} A_0$. Suppose that there are an embedding $A_i \to M_j$ for some $i < j$. Since there is an embedding $A_i \to B_i$, by (AP) there exist a structure in which B_i and M_j can be embedded. Let us call this structure M_{j+1}. (We may consider $M_j \leq M_{j+1}$.) So B_i can be embedded to M_{j+1}.

Finally, we define

$$M = M_\omega = \bigcup_{i<\omega} M_i.$$

It can be seen that every $A_i \in \mathcal{K}$ can be embedded to M. It is countable because it is the countable union of finite sets.

It can be seen from this construction, that M is weakly homogeneous therefore homogeneous.

2. We would like to define a chain of partial isomorphisms between N and M, so we do this by recursion.

   Let f_0 be the empty function, $n_0, n_1, \ldots$ be an enumeration of the elements of N and $m_0, m_1, \ldots$ be an enumeration of the elements of M.

   Suppose that the partial isomorphism f_i between N and M has been already defined. Then f_i^{-1} is also a partial isomorphism. Let $A_i = Ran(f_i)$ $(\subseteq M)$ and $B_i = Ran(f_i) \cup \{m_i\}$. By weak homogeneity, we can extend f_i^{-1} to m_i. Denote this extension by g_i.

   Similarly, suppose that the partial isomorphism g_i has been already defined. Then g_i^{-1} is also a partial isomorphism. Let $A'_i = Ran(g_i)$ $(\subseteq N)$ and $B'_i = Ran(f_i) \cup \{n_i\}$. By weak homogeneity, we can extend g_i^{-1} to n_i. Denote this extension by f_{i+1}.

As the last step, define f as the union of f_i-s for all $i < \omega$. It is surjective, because the domain of f^{-1} contains all the elements of M. And f is also injective, because $Dom(f)$ contains all the elements of N. Then f is an isomorphism between N and M, hence uniqeness is proved.

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Let us call the unique homogeneous structure corresponding to a class $\mathcal{K}$ of structures the Fraïssé-limit of $\mathcal{K}$.

Random graphs

In this section we will present a graph which is a model of an interesting theory. Then we will show, that this graph is the Fraïssé limit of the class of finite graphs. Finally, we will show that this graph is isomorphic to the random graph on ω (with probability 1).

In this section $\mathscr{S}$ denotes the signature of the graphs: there is only one relation symbol E in it (excluding the equality symbol =). So all the structures in this section are considered to be graphs.

Let T_R be the following theory:

$$T_R \quad \stackrel{def}{=} \quad \{\forall v\ (v,v) \notin E\} \cup \{\forall u \forall v((u,v) \in E \rightarrow (v,u) \in E)\} \cup \{\varphi_{n,m} : n,m \in \omega\}.$$

The first sentence expresses irreflexivity, the second expresses symmetry. For n, m natural numbers abbreviate $\varphi_{n,m}$ the following: for every $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_m$ there is a z different from x-s and y-s such that $x_i E z$ and $\neg y_j E z$ (where $i \leq n,\ j \leq m$). So the third set of sentences above gives us a graph property called

Separation property For every $n, m \in \omega$ if N, M are disjoint finite graphs such that $|N| = n$ and $|M| = m$ then there is a vertex z which is adjacent to all the vertices of N and none of the vertices of M.

At this point, it is not obvious, whether T_R is consistent. The next construction gives a model wittily.

Proposition 2.4. *T_R is consistent.*

Proof We give an explicit model of T_R.

Let R be a graph with underlying set ω. If $i = j$ then let $(i, j) \notin E(R)$. So assume $i < j$ Then let $(i, j) \in E(R)$ iff the ith digit of j is 1 (in base 2).

This graph is a model of T_R, because it is irreflexive, symmetric and has the separation property: for every $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_m$ there is a $z \in \omega$ such that the correspondent digits are 1 for every x_i and 0 for every y_i. •

So T_R has (at least one) model, but are there any more? We can observe that the separation property implies homogeneity, hence a model of T_R is the Fraïssé limit of the class of finite graphs!

Proposition 2.5. *If a graph M has the separation property, then it is a homogeneous.*

Proof It is clear that if a graph has the separation property, then it contains all the finite graphs. •

Corollary 2.6. *T_R is an $\aleph_0$-categorical theory.*

So it is a unique graph on ω. Call this graph the Rado graph, and denote it by R.

Proposition 2.7. *T_R has only infinite models.*

Proof Suppose there is a finite graph G satisfying T_R. It is obvious that in this graph $\varphi_{|G|,0}$ is false. •

Recall the following theorem:

Theorem 2.8 (Łoś–Vaught test)**.** *If theory T has only infinite models and T is κ-categorical for some $\kappa \geq \aleph_0$, then T is complete.*

Proof Suppose T is not complete. Then there exists a(n independent) sentence ψ and A, B models of T such that $A \models T \cup \{\psi\}$ and $B \models T \cup \{\neg\psi\}$. Because of Löwenheim–Skolem theorems, there are also models $A' \models T \cup \{\psi\}$ and $B' \models T \cup \{\neg\psi\}$ such that $|A'| = |B'| = \kappa$. By the κ-categoricity A' and B' are isomorphic, but it is a contradiction. •

Corollary 2.9. *T_R is complete.*

We already know that T_R has only infinite models. But what about the finite slices of T_R? That is why we define the graph construction random graph. Let R_k be a graph: let the set of vertices be the numbers $0, \dots, k-1$. Then decide for each (i, j) to be an edge or not, with coin flipping (probability $\frac{1}{2}$) where $i < j < k$.

Proposition 2.10. *If $n, m \in \omega$, then $\lim_{k\to\infty} Pr(R_k \not\models \varphi_{n,m}) = 0$.*

Proof Let $E_{N,M,z}$ be the event for which the sentence 'z is adjacent to all the elements of N and to none the elements of M' is *not* true. So

$$Pr(E_{N,M,z}) = 1 - \frac{1}{2^{|N|+|M|}}.$$

Then denote by $E_{N,M}$ the event that 'there is no z vertex such that z is adjacent'. For different z-s the events $E_{N,M,z}$ are independent, thus

$$Pr(E_{N,M}) = (1 - \frac{1}{2^{n+m}})^{k-n-m}.$$

Finally, denote the event '$R_k \not\models \varphi_{n,m}$' by E. Because $E = \bigcup\limits_{N,M} E_{N,M}$,

$$0 \leq Pr(E) \leq k^{n+m}(1 - \frac{1}{2^{n+m}})^{k-n-m},$$

estimated the choice of M and N with k^{n+m}. The limit of this expression is 0. •

Proposition 2.11. *If $T \subseteq T_R$ is finite, then*

$$\lim_{k\to\infty} Pr(R_k \models T) = 1$$

in a random graph R_k on k vertices.

Proof

$$Pr(R_k \not\models T) \leq Pr(R_k \not\models \phi_1) + \cdots + Pr(R_k \not\models \phi_l),$$

where $\phi_1, \dots, \phi_l$ are the elements of T. The limit of right hand side is 0. •

This proposition (and completeness of T_R) has an interesting corollary:

Corollary 2.12 (0-1 law)**.** *If ϕ is an arbitratry formula then $\lim_{k\to\infty} Pr(R_k \models \phi)$ is 1 or 0.*

Proof $T_R \models \phi$ or $T_R \models \neg\phi$, because of completeness of T_R. From the compactness theorem we get a finite theory T from which either ϕ or $\neg\phi$ follows. Applying Proposition 2.11 to T, we get a probability value 1 or 0. •

Finally, we prove that a random graph R_ω of the set ω is isomorphic with the Rado graph (with probability 1).

Proposition 2.13. *With probability 1, a random graph is isomorphic to R.*

Proof Let R_i be the subgraph of R_ω induced by the set $[0, i-1]$. From Proposition 2.10 we get that for any fixed n, m: $\lim_{k\to\infty} Pr(R_k \not\models \varphi_{n,m}) = 0$. Since the union of ω many measure-0-set is a measure-0-set, $Pr(R_\omega \not\models T_R) = 0$. •

3 Hrushovski-type theorems

A theorem of Hrushovski

For completeness, we start this chapter by including Hrushovski's original proof (appeared in [5]).

Definition. *We say that $\mathcal{F} \subseteq \mathcal{P}(X)$ is a statistically independent family of sets on X, if for every $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathcal{F}$*

$$|A_1 \cap \cdots \cap A_n \setminus B_1 \setminus \cdots \setminus B_m| = \frac{|X|}{2^{n+m}}.$$

Proposition 3.1. *Every finite graph X can be embedded to a finite graph Y such that the family of sets $\{Y(a) : a \in X\}$ forms a statistically independent family of sets on Y.*

Proof We may assume that $a \mapsto X(a)$ is one-to-one on X, else we can easily embed X to a finite graph, where $a \mapsto X(a)$ is one-to-one. Let Y be a graph on the set $\mathcal{P}(X)$ as nodes, and define $(y_1, y_2) \in E(Y) \iff y_1 = X(a_2)$ for some $a_2 \in y_2$ or $y_2 = Y(a_1)$ for some $a_1 \in y_1$. Note that for any $a \in X$ and $Y' \in \mathcal{P}(X)$ we have $(X(a), Y') \in E(Y)$ iff $a \in Y$. Particularly, if $a_1, a_2 \in X$, then

$$(X(a_1), X(a, 2)) \in E(Y) \Leftrightarrow a_1 \in X(a_2) \Leftrightarrow a_2 \in X(a_1) \Leftrightarrow (a_1, a_2) \in E(X).$$

So the map $a \mapsto a^* = X(a)$ is an embedding to Y. We have to show that the family of sets below is statistically independent:

$$\{Y(a) : a \in X\} = \{Y(a^*) : a \in X\} = \{Y(X(a))\} : a \in X\}.$$

We have that if $Y' \in \mathcal{P}(X)$ and $a \in X$ then Y' is a neighbour of $X(a)$ iff $a \in Y'$. So the intersection of sets $Y(a^*) = \{X' : a \in X'\}$ and $Y(b^*) = \{X' : b \in X'\}$ halves both of them, so the family $\{Y(a) : a \in X\}$ is statistically independent. •

Theorem 3.2 (Hrushovski)**.** *Let X be a finite graph. Then there exists a finite graph Z containing X as an induced subgraph ($X \leq Z$), such that every partial isomorphism $f : A \to B$, (where $A, B \leq X$) extends to an automorphism $f^* : Z \to Z$.*

Let U be the set of partial isomorphisms on X, and let Y be as in the previous proposition. For the proof we will need the following lemma:

Lemma 3.3. *If $f \in U$ then there exists an $f^* \in Sym(Y)$ such that $f^*|_{Dom(f)} = f$ and $f^*[Y(a)] = Y(f^*(a))$ for all $a \in Dom(f)$. If $Dom(f) = Ran(f)$ and $f^2 = \mathrm{id}$ then we can choose an f^* such that $f^{*2} = \mathrm{id}$.*

Proof of the Lemma. Let $D = Dom(f)$ and $R = Ran(f)$. If $\nu : D \to \{0, 1\}$ is a function then write

$$D_\nu = \{y \in Y : (d, y) \in E(Y) \Leftrightarrow \nu(d) = 1 \quad (d \in D)\} \qquad \text{and}$$

$$R_\nu = \{y \in Y : (r, y) \in E(Y) \Leftrightarrow \nu(r) = 1 \quad (r \in R)\}.$$

Since $\{Y(a) : a \in X\}$ is statistically independent (and by Proposition 3.1) $|D_\nu| = |R_\nu| = |Y| \cdot 2^{-|D|}$. Then $|D \cap D_\nu| = |R \cap R_\nu|$, because f is an isomorphism between D and R, so $|D_\nu \setminus D| = |R_\nu \setminus R|$.
Let f^* be a permutation on Y extending f and mapping $D_\nu \backslash D$ onto $R_\nu \backslash R$ arbitrarily. (Except if $D = R$ and $f^2 = \mathrm{id}$ then we must choose f^* such that $f^{*2} = \mathrm{id}$.) This permutation respects the adjecency relation on D, so it is sufficient for the lemma.
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Proof of Hrushovski's theorem. Let U be as above, $U^* = \{f^* : f \in U\}$ and $G = \langle U^* \rangle$ be the group generated by U^*.

Notation: if there exist $x_0, x_1, x_2 \in Y$ with $x_0 \in Dom(f_1)$, $x_1 \in Ran(f_1) \cap Dom(f_2)$ and $x_2 \in Ran(f_2)$, and in addition $f_1 x_0 = x_1$ and $f_2 x_1 = x_2$, then we write $f_2 f_1 x_0 = x_2$.

Define a relation $\approx$ on $G \times X$ as follows:

$(g, x) \approx (g', x')$ iff there are $h_1, \ldots, h_n \in U$ with

(i) $h_n \ldots h_1 x = x'$;

(ii) $g = g' h_n^* \ldots h_1^*$ (in G).

The relation $\approx$ is an equivalence relation on $G \times Y$: identity funtion witnesses reflexivity. For symmetry we can take the partial isomorphisms $h_1^{-1}, \ldots, h_n^{-1}$. Finally, if the equivalences of $(g, x) \approx (g', x')$ and $(g', x') \approx (g'', x'')$ are demonstrated by $h_1, \ldots, h_n \in U$ and $h_1', \ldots, h_n' \in U$, respectively, then transitivity follows by choosing $h_1, \ldots, h_n, h_1', \ldots, h_n'$.

Next we can define a group action by G on Y as follows: $h(g, x) = (hg, x)$. This action preserves $\approx$ because of the following. If $(g, x) \approx (g', x')$, then there exists

$h_1, \dots, h_n \in U$ with properties (i) and (ii). Since $hg = hg'h_n^* \dots h_1^*$, the second condition is true for (hg, x) and (hg', x'). The first condition is obviously true.

Let $Z = G \times Y / \approx$, and turn Z into a graph: put $\big((g,a)/\approx, (g,b)/\approx\big) \in E(Z)$ if $g \in G$, $(a,b) \in E(Y)$. This way Z can be considered to be a graph, on which G acts by automorphisms, and every $f \in U$ extends to an $f^* \in G$. The only thing we need to prove for the theorem is that Y can be embedded to Z as an induced subgraph.

If $(\mathrm{id}, x) = (\mathrm{id}, y)$ then there exists $h_1, \dots h_n$ with $h_n \dots h_1 x = y$ and $h_n^* \dots h_1^* = \mathrm{id}$, so $x = y$. Hence Y is embeddable to Z naturally (as a set).

It is easy to see that Y is a subgraph of Z. But is it an induced subgraph? Suppose that $\big((\mathrm{id},x)/\approx, (\mathrm{id},y)/\approx\big)$ is an edge of Z for any $x, y \in Y$. Then there exists a $g \in G$ and $(x', y') \in E(Y)$ such that

$$(\mathrm{id}, x) \approx (g, x') \qquad \text{and} \qquad (\mathrm{id}, y) \approx (g, y').$$

Then there exists $f_1, \dots f_m, \quad h_1, \dots h_n \in U$ where $g = f_m^* \dots f_1^* = h_n^* \dots h_1^*$,

$$h_n \dots h_1 x' = x \qquad \text{and} \qquad f_m \dots f_1 y' = y.$$

Let $x_0, \dots, x_n \in Y$ be such that $x_0 = x'$ and $x_n = x$,

$$x_i \in Dom(h_{i+1}), \qquad h_{i+1} x_i = x_{i+1}.$$

Then $h_{i+1}^*[Y(x_i)] = Y(h_{i+1}^* x_i) = Y(x_{i+1})$ so $g[Y(x')] = Y(x)$. Moreover $y' \in Y(x')$ (because $x'y'$ is an edge), hence $gy' = f_m^* \dots f_1^* y' = y \in Y(x)$ so $(x,y) \in E(Y)$. •

Note, that this theorem has been generalized for relational structures:

Theorem 3.4 (B. Herwig, [3]). *Let X be a finite relational structure. Then there exists a finite relational structure Z containing X as an substructure ($X \leq$), such that every partial isomorphism $f : A \to B$, (where $A, B \leq X$) extends to an automorphism $f^* : Z \to Z$.*

Homomorphism instead of isomorphism

We would like to generalize Hrushovski's theorem for partial homomorphism instead of isomorphism. But there is a problem with this plan. If there are nodes b, c in the domain of a partial homomorphism and a node a outside of the domain such that (a, b) is an edge and (a, c) is not an edge, and if the partial homomorphism maps b

and c to the same node, than we cannot find an appropriate endomorphism which extends our original partial homomorphism. Therefore we avoid this situation.

Definition. *We call b and c (both nodes in $Dom(h)$) incompatible with respect to a partial homomorphism h of a finite graph X, if there is a node $a \in X$ such that $(a, b) \in E(X)$ and $(a, c) \notin E(X)$, and $h(b) = h(c)$.*

Definition. *We say that a finite graph X is allowed for a partial homomorphism h, if for h there are no incompatible nodes with respect to h.*

Theorem 3.5. *Let X be a finite graph. Then there exists a finite graph Z containing X as an induced subgraph ($X \leq Z$), such that every partial homomorphism f, for which X is allowed, extends to an endomorphism $f^* : Z \to Z$.*

An essential idea of the proof of Hrushovski's theorem is the embedding the graph X to Y such that the edges are preserved not only in the domain, but also between a pair of nodes, where at least one of them is in the domain. We would like to carry out a similar argument. But we should choose another way, because we cannot complement a homomorphism to a permutation.

Let U be the set of partial homomorphisms on X, for which X is allowed.

Lemma 3.6. *There exists a finite graph $Y \geq X$ such that every $f \in U$ extends to an $f^* \in Sym(Y)$ such that $f^*|_{Dom(f)} = f$ and $f^*[Y(a)] = Y(f^*(a))$ for all $a \in Dom(f)$. Even Y is allowed.*

Proof of the Lemma.

It is clear that every $h \in U$ is a congruency (on its domain), because X is allowed. So we can consider the kernel of each partial homomorphism.

So define an equivalence relation:

$$H \overset{def}{=} \bigvee_{h \in U} Ker(h),$$

namely H is the supremum of the kernels of partial homomorphisms.

Note that H is a congruence for all f^*.

We will define Y by adding vertices and edges to X such that X will remains an induced subgraph. Let

$$V(Y) \overset{def}{=} V(X) \cup \bigcup_{\substack{h \in U \\ a \in V(X) \setminus Dom(h)}} \{u_{a,h}\}.$$

The set of edges of Y will be the set of edges of X together with some new ones that we add according to the rule below:

$$(u_{a,h}, b) \in E(Y) \quad \text{iff} \quad b \in f[X(a)]/H,$$

where $a, b \in V(X)$, $h \in U$.

Then we can extend an $f \in U$ such that for every $a \in V(X) \smallsetminus Dom(f)$,

$$f^*(a) \stackrel{def}{=} u_{a,f},$$

and $f^*(u) \stackrel{def}{=} u$ for each new vertex u. So f^* extend to the whole $V(Y)$.

Let us make an observation. Let u be a new node and $x, y \in V(X)$. If xHy, then u is connected to x and also to y, or neither to x, nor to y. This is true, because edges between old and new nodes are defined up to H.

Finally, Y is allowed because of the following. Suppose that there are nodes $b, c \in V(X)$ and $u_{a,i} \in V(Y) \smallsetminus V(X)$ such that $(u_{a,i}, b) \in E(Y)$ and $(u_{a,i}, c) \notin E(Y)$. If b' and c' are the nodes, for which $i^*(b') = b$ and $i^*(c') = c$, then $(a, b') \in E(X)$ and $(a, c') \notin E(X)$ by the observation above. But then X is not allowed, which is a contradiction, so Y is allowed. •

Proof of the Theorem.

Let U be as above, $U^* = \{f^* : f \in U\}$ and $S = \langle U^* \rangle$ be the semigroup generated by U^*. This semigroup has an identity element: $\mathrm{id} = \mathrm{id}^* \in U^*$.

Notation: if there exist $x_0, x_1, x_2 \in Y$ with $x_0 \in Dom(f_1)$, $\quad x_1 \in Ran(f_1) \cap Dom(f_2)$ and $x_2 \in Ran(f_2)$, in addition $f_1 x_0 = x_1$ and $f_2 x_1 = x_2$, then we write $f_2 f_1 x_0 = x_2$.

Define a relation $\sim$ on $S \times X$ by $(\alpha, a) \sim (\alpha', a')$ iff there are $\pi_1, \ldots, \pi_n, \varrho_1, \ldots, \varrho_m \in U$ with

(i) $\pi_n \ldots \pi_1 a = a', \quad \varrho_m \ldots \varrho_1 a' = a,$

(ii) $\alpha = \alpha' \pi_n^* \ldots \pi_1^*, \alpha' = \alpha \varrho_m^* \ldots \varrho_1^*$ (in S).

The relation $\sim$ is an equivalence relation on $S \times Y$: identity function shows reflexivity. For symmetry we can take the partial isomorphisms $\pi_1^{-1}, \ldots, \pi_n^{-1}$ for π_i's and $\varrho_1^{-1}, \ldots, \varrho_m^{-1}$ for the ϱ_i's. Then the corresponding U-elements for transitivity will be $\pi_1, \ldots, \pi_n, \pi_1', \ldots, \pi_n'$, and $\varrho_1, \ldots, \varrho_m, \varrho_1', \ldots, \varrho_m'$.

We can define a semigroup action by S on Y as follows: $\beta(\alpha, x) = (\beta\alpha, x)$. This action preserves $\sim$. If $(\alpha, x) \sim (\alpha', x')$, then there exists $\pi_1, \ldots, \pi_n, \varrho_1, \ldots, \varrho_m \in U$

with properties (i) and (ii). Then we can get the second condition from the original (ii) by multiplying by β. The first condition is the same as the original.

Let $Z = S \times X/\sim$, and let Z be the graph where we let $\big((\alpha, a)/\sim, (\beta, b)/\sim\big) \in E(Z)$ if $\alpha = \beta \in S$, $(a, b) \in E(X)$. So Z is a graph, on which S acts by endomorphisms, and every $f \in U$ extends to an $f^* \in S$. The only thing we need to prove is that X can be embedded to Z as an induced subgraph.

If $(\mathrm{id}, x) \sim (\mathrm{id}, y)$ then there exist $\pi_1, \dots, \pi_n \in U$ such that $\mathrm{id} = \pi_n^* \dots \pi_1^*$ and $\pi_n \dots \pi_1 x = y$ hence $x = y$, so $x \mapsto (\mathrm{id}, x)$ is a natural embedding from X to Z (as a set). Then X is a subgraph of Z, because this embedding respects the edges.

Suppose that $\big((\mathrm{id}, x)/\sim, (\mathrm{id}, y)/\sim\big)$ is a Z-edge for any $x, y \in X$. We must show that $(x, y) \in E(X)$. To do so, observe that there exist $\alpha \in S, \quad x', y' \in X$ such that

$$(\mathrm{id}, x) \sim (\alpha, x') \quad \text{and} \quad (\mathrm{id}, y) \sim (\alpha, y')$$

where $(x', y') \in E(X)$. By the definition of $\sim$ there are $\pi_1, \dots, \pi_n, \varrho_1, \dots, \varrho_m \in U$ such that

$$\pi_n \dots \pi_1 x' = x, \quad \varrho_m \dots \varrho_1 y' = y,$$

$$\pi_n^* \dots \pi_1^* = \alpha = \varrho_m^* \dots \varrho_1^*.$$

Let $x_0, \dots, x_n \in Y$ be such that $x_0 = x'$ and $x_n = x$,

$$x_i \in Dom(\pi_{i+1}), \qquad \pi_{i+1} x_i = x_{i+1}.$$

Then $\pi_{i+1}^*[Y(x_i)] = Y(\pi_{i+1}^* x_i) = Y(x_{i+1})$ so $\alpha[Y(x')] = Y(x)$. Moreover $y' \in Y(x')$ (because $x'y'$ is an edge), hence $\alpha y' = \varrho_m^* \dots \varrho_1^* y' = y \in Y(x)$ so $(x, y) \in E(Y)$. •

4 Orbits

The generic automorphism

In this section we would like to find a generic automorphism of the Rado graph. To recall the definition of generic automorphism, we need to introduce a topology on the automorphism group. If the conjugacy class of an automorphism is big enough in this topology (namely it is dense), then this automorphism is used to call generic. To prove the existence of such automorphism we need the following lemma.

Lemma 4.1. *If f is a partial isomorphism of the Rado graph R and g is a partial isomorphism of a finite graph $G \leq R$, then we can create a partial isomorphism f^* of R such that for a suitable automorphism $h \in Aut(R)$*

$$f^* \supseteq f \cup h^{-1}gh.$$

If $Dom(f) = Ran(f)$, then the domain and range of f^* are also the same.

Proof If G is a finite graph and g is a partial isomorphism of it, then we can apply Hrushovski's theorem (3.2), so there is a finite graph $G^* \geq G$ and an automorphism $g^* \in Aut(G^*)$ such that g^* is an extension of g. We may assume that $G^* \leq R$ (when it is not true then we can find an embedding $\nu : G^* \to R$, and instead of G^* we can write $\nu^{-1}G^*\nu$).

First we define a partial isomorphism $h' : G^* \to R$ using recursion, then complete it to an automorphism h.

By the separation property of the Rado graph, we can find an image vertex in each step of the recursion, as follows. Let the elements of G^* be $(x_i)_{i\leq|G^*|}$. Assume that $h'(x_i)$ has been defined for all $i < j$ for some $j \leq |G^*|$ such that there are no edges between any $h'(x_i)$ and nodes belonging to $Dom(f) \cup Ran(f)$. Let X_j be the subset of G^* which contains the neighbors of x_j from $x_1, \ldots, x_{j-1}$ and let Y_j be the subset of G^* which contains the vertices not connected to x_j from the elements $x_1, \ldots, x_{j-1}$. Using the separation property (see page 12.) we can find a vertex $y_j \in R$ such that y_j is adjacent to the elements of $h'[X_j]$ and not adjacent to the elements of $h'[Y_j] \cup Dom(f) \cup Ran(f)$. Define $h'(x_j) \overset{def}{=} y_j$.
So we defined h' for all the vertices of G^*.

Since R is homogeneous, the partial isomorphism h' extends to an automorphism h of R. This function respects the graph structure of G^* hence the graph structure of G too. Therefore,

$$f^* \overset{def}{=} f \cup h^{-1}g^*h$$

is suitable for the lemma. Because g^* is an automorphism on G^*, the domain and range of the obtained partial isomorphism are the same. •

Let R be the Rado graph on the set ω of vertices. Taking the discrete topology on ω, the space ${}^{\omega}\omega$ is endowed with the product topology. Because $Aut(R) \subseteq {}^{\omega}\omega$, it also has an inherited topology, so we can talk about the densitity of a subset of the space $Aut(R)$.

Definition. *An automorphism f of R called generic, if it has a dense conjugacy class (in $Aut(R)$).*

Theorem 4.2. *There is a generic automorphism $f \in Aut(R)$. For every $x \in V(R)$ the orbit $\{f^{(n)}(x) : n \in \mathbb{Z}\}$ is finite.*

Proof We would like to define the generic automorphism as adding all the partial isomorphisms of finite graphs (by the previous lemma).

Let $(G_i, g_i)_{i\in\omega}$ be an enumeration of all the pairs of finite graphs G_i and partial isomorphisms g_i on G_i. We may assume that $G_i \leq R$ for all $i < \omega$. Define a sequence $(f_n)_{n\in\omega}$ of partial isomorphisms with the following recursion such that all the stipulations below are satisfied.

(i) f_n is an isomorphism between its domain and range for every $n \in \omega$, so that $Dom(f_n) = Ran(f_n)$,

(ii) $n \in Dom(f_n) \cap Ran(f_n)$

(iii) if $n < \omega$ then there exists $D \subseteq Dom(f_n)$, such that $(D, f_n|_D) \cong (G_n, g_n)$.

Let f_0 be the empty function. It is trivial that f_0 satisfies (i)–(iii).

Then assume that f_j has been defined for all $j < i$ for some $i \in \omega$.

As first step, we would like i to be in the domain. So if $i \in Dom(f_{i-1})$, then define $f'_{i-1} \stackrel{def}{=} f_{i-1}$. Else using Lemma 4.1 to the one-node-graph i with the identical isomorphism, we get a partial isomorphism, whose domain contains i. Let this partial isomorphism be f'_{i-1}. It is clear that $Dom(f'_{i-1}) = Ran(f'_{i-1})$.

As second step, applying Lemma 4.1 to f'_{i-1} and (G_i, g_i), we get that there exist a partial isomorphism f'^*_{i-1} of R such that

$$f'^*_{i-1} \supseteq f'_{i-1} \cup h^{-1} g_i h.$$

Define f_i as f'^*_{i-1}.

Since (i) is true for f_{i-1}, Lemma 4.1 implies that $Dom(f_i) = Ran(f_i)$, so (i) remains true. The first step guarantees that (ii) remains true. The way as f_i has been constructed ensures the state of (iii).

Finally we can define

$$f = \bigcup_{i \in \omega} f_i.$$

Then f is a partial isomorphism, because (i) is true for every $i \in \omega$. From (ii) we get that $Dom(f) = \omega$, so f is an automorphism.

Next we show that f is generic: we must to prove that this automorphism has a dense conjugacy class. Consider an arbitrary nonempty open set $A \subseteq Aut(R)$. We may assume that it is a basic open set: there is a partial isomorphism $d : \omega \to \omega$ where $Dom(d)$ is finite and $A = \{f : d \subseteq f\}$. Since d is a partial isomorphism of R, there exists n such that $(Dom(d) \cup Ran(d), d) \cong (G_n, g_n)$, specially $Dom(d) \cong G_n \subseteq R$. So there is an automorphism $h : \omega \to \omega$ which maps $Dom(d)$ to G_n. Let $g = h^{-1} \cdot f \cdot h$. Because of $g \in Aut(R)$, h shows that g and f are conjugate, $d \subseteq g$ (because $g_n \subseteq f$), so $g \in A$.

Finally, if $x \in R$, then choosing a number $n < \omega$ such that $x \in Dom(f_n)$ we can get a finite orbit of f_n in x, because $Dom(f_n) = Ran(f_n)$ is finite. So $f (\supseteq f_n)$ has also a finite orbit on x. •

The generic endomorphism

We would like to find also the generic endomorphism of R. As we will see, the thread of proof is the same as in the previous section.

To prove the existence of such endomorphism we need the following lemma.

Lemma 4.3. *If f is a partial homomorphism of the Rado graph R and g is a partial homomorphism of a finite graph $G \leq R$, for which G is allowed, then we can create a partial homomorphism f^* of R such that for a suitable automorphism $h \in Aut(R)$*

$$f^* \supseteq f \cup h^{-1} g h.$$

Proof If G is a finite graph and g is a partial homomorphism of it, then we can apply Theorem (3.5), so there is a finite graph $G^* \geq G$ and an endomorphism $g^* \in End(G^*)$ such that g^* is an extension of g. We may assume that $G^* \leq R$

(when it is not true then we can find an embedding $\nu : G^* \to R$, and instead of G^* we can write $\nu^{-1}G^*\nu$).

First we define a partial isomorphism $h' : G^* \to R$ using recursion, then complete it to an automorphism h.

By the separation property of the Rado graph, we can find an image vertex in each step of the recursion, as follows. Let the elements of G^* be $(x_i)_{i \le |G^*|}$. Assume that $h'(x_i)$ has been defined for all $i < j$ for some $j \le |G^*|$ such that there are no edges between any $h'(x_i)$ and nodes belonging to $Dom(f) \cup Ran(f)$. Let X_j be the subset of G^* which contains the neighbors of x_j from $x_1, \ldots, x_{j-1}$ and let Y_j be the subset of G^* which contains the vertices not connected to x_j from the elements $x_1, \ldots, x_{j-1}$. Using the separation property (see page 12.) we can find a vertex $y_j \in R$ such that y_j is adjacent to the elements of $h'[X_j]$ and not adjacent to the elements of $h'[Y_j] \cup Dom(f) \cup Ran(f)$. Define $h'(x_j) \stackrel{def}{=} y_j$.
So we defined h' for all the vertices of G^*.

Since R is homogeneous, the partial isomorphism h' extends to an automorphism h of R. This function respects the graph structure of G^* hence the graph structure of G too. Therefore,

$$f^* \stackrel{def}{=} f \cup h^{-1}g^*h$$

is suitable for the lemma. •

Let R be the Rado graph on the set ω of vertices. Taking the discrete topology on ω, the space ${}^{\omega}\omega$ is endowed with the product topology. Because $End(R) \subseteq {}^{\omega}\omega$, it also has an inherited topology, so we can talk about the densitity of a subset of the space $End(R)$.

Definition. *An endomorphism f of R called generic, if it has a dense conjugacy class (in $End(R)$).*

Theorem 4.4. *There is a generic endomorphism $f \in End(R)$. For every $x \in V(R)$ the orbit $\{f^{(n)}(x) : n \in \omega\}$ is finite.*

Proof The way, as we created the generic endomorphism, is the same as the previous section: we add all the partial homomorphisms by recursion

Let $(G_i, g_i)_{i \in \omega}$ be an enumeration of all the pairs of finite subgraphs G_i of R and partial homomorphisms g_i on G_i such that there are no incompatible nodes in G_i with respect to g_i. Define a sequence $(f_n)_{n \in \omega}$ of partial homomorphisms with the following recursion such that all the stipulations below are satisfied.

(i) $n \in Dom(f_n)$

(ii) if $n < \omega$ then there exists $D \subseteq Dom(f_n)$, such that $(D, f_n|_D) \cong (G_n, g_n)$.

Let f_0 be the empty function. It is trivial that f_0 satisfies (i)–(iii).

Then assume that f_j has been defined for all $j < i$ for some $i \in \omega$.

As first step, we would like i to be in the domain. So if $i \in Dom(f_{i-1})$, then define $f'_{i-1} \stackrel{def}{=} f_{i-1}$. Else using Lemma 4.1 to the one-node-graph i with the identical isomorphism, we get a partial isomorphism, whose domain contains i. Let this partial isomorphism be f'_{i-1}. It is clear that $Dom(f'_{i-1}) = Ran(f'_{i-1})$.

As second step, applying Lemma 4.3 to f'_{i-1} and (G_i, g_i), we get that there exist a partial homomorphism f'^{*}_{i-1} of R such that

$$f'^{*}_{i-1} \supseteq f'_{i-1} \cup h^{-1} g_i h.$$

Define f_i as f'^{*}_{i-1}.

The first step guarantees that (i) remains true. The way as f_i has been constructed ensures the state of (ii).

Finally we can define

$$f = \bigcup_{i \in \omega} f_i.$$

Then f is a partial homomorphism. From (i) we get that $Dom(f) = \omega$, so f is an endomorphism.

Next we show that f is generic: we must to prove that this endomorphism has a dense conjugacy class. Consider an arbitrary nonempty open set $A \subseteq End(R)$. We may assume that it is a basic open set: there is a partial homomorphism $d : \omega \longrightarrow \omega$ where $Dom(d)$ is finite and $A = \{f : d \subseteq f\}$. Since d is a partial homomorphism of R, there exists n such that $(Dom(d) \cup Ran(d), d) \cong (G_n, g_n)$, specially $Dom(d) \cong G_n \subseteq R$. So there is an automorphism $h : \omega \longrightarrow \omega$ which maps $Dom(d)$ to G_n. Let $g = h^{-1} \cdot f \cdot h$. Because of $g \in Aut(R)$, h shows that g and f are conjugate, $d \subseteq g$ (because $g_n \subseteq f$), so $g \in A$.

Finally, if $x \in R$, then choosing a number $n < \omega$ such that $x \in Dom(f_n)$ we can get a finite orbit of f_n in x, because $Dom(f_n)$ is finite. So $f (\supseteq f_n)$ has also a finite orbit on x. •

Equivalent definitions of dense conjugacy classes

Theorem 4.5. *Let $f \in Aut(R)$ and let K be its conjugacy class. Then the following are equivalent:*

(1) K is dense somewhere
(there is a nonempty open set $G \subseteq Aut(R)$ such that for every (nonempty, open) $H \subseteq G$ we have $H \cap K \neq \emptyset$),

(2) K is dense
(for every open set H we have $H \cap K \neq \emptyset$),

(3) K is co-meager
(there is a family of sets $\{L_i : i \in \omega\}$ where L_i is nowhere dense for every $i \in \omega$, L_i-s are disjoint, and $Aut(R) \setminus K \subseteq \bigcup_{i\in\omega} L_i$).

It is known that (3) $\Rightarrow$ (2). It is obvious that (2) $\Rightarrow$ (1).

Proof of direction (1) $\Rightarrow$ (2).

Suppose that k is a partial isomorphism such that K is dense in $\{j \in Aut(R) : k \subseteq j\}$. Let l be another partial isomorphism. By Lemma 4.1 there is an embedding $h : Dom(l) \cup Ran(l) \to R$ such that $k \cup h^{-1}lh$ is also a partial isomorphism. Because (1) we can take an automorphism $f' \in K$, which is an extension of $k \cup h^{-1}lh$. So $h \cdot f' \cdot h^{-1} \supseteq l$ is an appropriate automorphism for proving implication (1) $\Rightarrow$ (2). •

We insert here a lemma we will need.

Lemma 4.6. *If K and K' are dense conjugacy classes then $K = K'$.*

Proof Let $f \in K$ and $g \in K'$ be automorphisms. Then we have to show that they are conjugate: we should find an automorphism h such that $f = h^{-1}gh$.

Because of Theorem 4.2, the basic open set $\{f' \in Aut(R) : f'(0) = f(0)\}$ contains an automorphism f_0 such that $f_0 = h_0^{-1}gh_0$ for some automorphism h_0.

Suppose that f_{i-1} and h_{i-1} has been defined for any i.

First, if i is an odd number. Because of Theorem 4.2, the basic open set $\{f' \in Aut(R) : f'|_{[0,i]} = f|_{[0,i]}\}$ contains an automorphism f_i such that $f_i = h_i'^{-1}gh_i'$ for some automorphism h_i'. Then there is an isomorphism k' between $h_{i-1}[0, i-1]$ and $h_i'[0, i]$. Then k' has an extension k, which is an automorphism of R, because R is homogeneous. Define $h_i \stackrel{def}{=} k^{-1}h'_i$. Then it is an extension of h_{i-1}, and it witnesses that f_i is conjugate to g.

Next, changing the role of f and g, we can define h_i for even numbers such that it is an extension of the previous h_i-s, and the range of h_i contains $[0, i] \subseteq Dom(g)$.

So $h \stackrel{def}{=} \bigcup_{i<\omega} h_i|_{[0,i]}$ is an automorphism, we get $f = h^{-1}gh$, f and g are conjugate, so $K = K'$. •

Proof of direction (2) $\Rightarrow$ (3).

Let $g \notin K$ be an automorphism, and let K' be its conjugacy class. Then K' is nowhere dense because of the following. Supposing K' is dense somewhere then it is dense (because (1) $\Rightarrow$ (2) has already been proved). By the Lemma 4.6 $K' = K$, so it would be a contradiction.

So, for such a g there is partial isomorphism b_g, for which

$$K \cap \{j \in Aut(R) : b_g \subseteq j\} = \emptyset.$$

(Call this basic open set L_g). Then the family of sets $\{L_g : g \notin K\}$ covers $Aut(R) \setminus K$, it is disjoint from K, and has at most countable many members (because $|^{<\omega}\omega| \leq \aleph_0$). •

Back to the extending property

Finally, we show that the existence of the generic automorphism implies a weaker version of Hrushovksi's theorem.

Theorem 4.7. *Suppose $f \in Aut(R)$ is the generic automorphism of the Rado graph, namely*

(i) the conjugacy class of f is dense, and

(ii) for every $x \in V(R)$ $\{f^{(n)}(x) : n \in \mathbb{Z}\}$ is a finite set.

Then the weak extension theorem follows: for every pair of finite graph X and partial isomorphism h on it, there exists a finite graph Z and an automorphism $h^* \in Aut(Z)$ such that $X \leq Z$ and h^* extends h.

Proof Let X be a finite graph and h a partial isomorphism between induced subgraphs of X. We may assume that $X \leq R$.
We know that there is an automorphism g which is conjugate to f and $h \subseteq g$, because the conjugacy class of f is dense.
Then define the underlying set of the graph Z as follows:

$$V(Z) \quad = \quad \bigcup \quad \{g^{(n)}(b) : b \in V(X) \text{ and } n \in \mathbb{Z}\}.$$

This is a finite set (see (ii)). The graph structure comes from R. So choosing g^* to be $g|_{V(Z)}$ the proof is completed. •

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