Semilinear parabolic problems

Master's thesis

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Introduction

Parabolic problems are one of the fields of mathematics which undergoes a detailed investigation, due to the many problems which rely on this theory. Parabolic equations describe various time dependent models of many physical, chemical and biological phenomena. Such problems are reaction-diffusion and convection-diffusion systems, air pollution and meteorological models, flame propagation, superconductivity or Navier-Stokes and nonlinear heat equation, for more examples see [5]. Since these kind of systems can be quite large, numerical (and parallel) methods are very important in this context.

The parabolic semilinear problems can be treated as abstract ordinary differential equations, hence semigroup theory is used. For related monographs see [3] and [8, 13].

During the solution of time dependent problems it is essential to efficiently handle the elliptic problems arising from the time discretization. Elliptic problems are often treated, just as in this thesis, by preconditioning techniques, see e.g. [6].

The thesis has three major parts: at first we make some introduction to theory of semigroups of operators, then applications to abstract Cauchy problems and partial differential equations are detailed, and finally we discuss numerical methods and investigations.

In Chapter 1 we discuss some nice, but sometimes nontrivial, examples for semilinear parabolic problems, such as Navier-Stokes equation, nonlinear Schrödinger equation, nonlinear heat equation and wave equation, finally we formulate the common abstract form of these problems.

In Chapter 2 first we detail the basics of semigroup theory, we also discuss operators generating nice strongly continuous semigroups. After that we make some ODE motivation of semigroups corresponding to linear Cauchy problems, i.e. nice exponential formulas and the variation of constants formula also remain true in the abstract context and help us to gain mild solutions. We describe existence and uniqueness theorems.

The main part of the thesis, besides the last chapter, is Chapter 3. This part also starts with proper motivations, then we gradually switch to the abstract case. We describe some nice theorems with Lipschitzian conditions (see [13]). Based on these theorems and semigroup theory we state and prove existence and uniqueness theorems for a quite large class of semilinear parabolic partial differential equations.

In Chapter 4 we turn to numerical aspects. First we introduce the time discretization we used – the method of lines or Rothe's method [11] – and the auxiliary elliptic problems arise from it in each time step. We made some comments on the numerical solution of such problems, for this –based on the investigations of the author in [12]– we used quasi-Newton method with stepwise variable preconditioning with a piecewise constant preconditioner [9]. The elliptic problems were numerically solved using a finite element discretization. The chapter ends with numerical experiments and comparison of first and second order time discretizations. These investigations were done using our own Matlab codes.

Acknowledgement

I would like to dedicate this thesis to my late Father.

I would like to express my gratitude to my supervisor János Karátson for his enormous support, patience and encouragement and for those great discussions on functional analysis.

I am also very thankful to my family and friends for their endless understanding and favour.

Chapter 1

Preface

1.1 Semilinear problems

Semilinear parabolic problems are a special kind of nonlinear equations. They arise in various physical and chemical problems, as well as their abstract form in applied mathematics. One of the most typical examples are reaction-diffusion equations, some nontrivial examples are the nonlinear heat equation, time dependent Schrödinger equation, Navier-Stokes equation. They can also be treated by these techniques.

1.1.1 Our problems

Let $\Omega \subset \mathbf{R}^N$ denote a bounded set. We use the notations $Q_T := [0, T) \times \Omega$, where $0 < T < +\infty$. Let $\Gamma_T := [0, T) \times \partial\Omega$, i.e. the lateral of Q_T, and finally we also use $\Omega_s := \{s\} \times \Omega$. We use these notations throughout this thesis.

Above all, we focus on the following semilinear problems. First a semilinear equation is studied:

$$\begin{cases} \partial_t u - \operatorname{div}\Big(A(x)\nabla u\Big) + q(t,x,u) = g(t,x) & ((t,x) \in Q_T) \\ u(t,x) = 0 & ((t,x) \in \Gamma_T) \\ u(t,x) = \gamma(x) & ((t,x) \in \Omega_0), \end{cases} \tag{1.1}$$

and we also consider the system:

$$\begin{cases} \partial_t u_j - \operatorname{div}\Big(A_j(x)\nabla u_j\Big) + q_j(t,x,u_1,\ldots u_M) = g_j(t,x) & ((t,x) \in Q_T) \\ u_j(t,x) = 0 & ((t,x) \in \Gamma_T) \\ u_j(t,x) = \gamma_j(x) & ((t,x) \in \Omega_0), \end{cases} \tag{1.2}$$

for $j = 1, 2, \ldots, M$. Clearly these equations are linear in their principle part. The system is only coupled in the zero order terms, by the usually nonlinear functions q_j. Sometimes q or q_j depends on ∇u or ∇u_j, respectively.

1.2 Some examples

We detail here a few more examples which fits into this theory, however, the semilinearity and/or the parabolicity of these equations is sometimes nontrivial.

There are many examples, we detail some of them below. A lot of nice examples can be found in [5].

1.2.1 Chemical reactions

An immediate example for semilinear systems are the reaction-diffusion equations describing chemical reactions.

Suppose we have N chemical species $(M_1, \ldots, M_N)$ involved in R reactions $(\sum_{i=1}^N \nu_{ij} M_i = 1$ $(j = 1, \ldots, R))$. If c_i is the concentration of the i^{th} compound and T is the temperature, then

$$\begin{aligned} \varepsilon_p\ \partial_t c_i &= \operatorname{div}\big(d_i \nabla c_i\big) + \sum_{j=1}^{R} \nu_{ij} f_j(c_1, \ldots, c_N, T) \qquad (i = 1, \ldots, N) \\ \varrho c_p\ \partial_t T &= \operatorname{div}\big(k \nabla T\big) - \sum_{i=1}^{N} \sum_{j=1}^{R} \nu_{ij} h_i f_j(c_1, \ldots, c_N, T), \end{aligned} \tag{1.3}$$

where $f_j = f_j(c_1, \ldots, c_N, T)$ is the rate of the j^{th} reaction and h_i is the partial molar enthalpy of the species i (assumed constant).

See more details in [8, Section 2.4.].

1.2.2 Nonlinear Schrödinger equation

Consider the initial value problem for the semilinear problem:

$$\begin{cases} \partial_t \Psi - i\Delta\Psi + ik|\Psi|^2\Psi = 0 & \text{in } (0, +\infty) \times \mathbf{R}^2 \\ \Psi(0, x) = \Psi_0(x) & \text{in } \mathbf{R}^2, \end{cases}$$

where $\Psi : \mathbf{R}^2 \to \mathbf{C}$ and k is a real constant. The function $\Psi = \psi_1 + i\psi_2$ is complex valued, hence the system form of the equation is

$$\begin{cases} \partial_t \psi_1 + \Delta\psi_2 - k(\psi_1^2 + \psi_2^2)\psi_2 = 0 \\ \partial_t \psi_2 - \Delta\psi_1 + k(\psi_1^2 + \psi_2^2)\psi_1 = 0 \end{cases} \tag{1.4}$$

which is a semilinear parabolic problem.

The interested reader can find a more detailed description in [13, Section 8.1.].

1.2.3 Wave equation

In numerical methods an often used trick is that we reformulate our equation to a system and we solve it instead of the original equation. Here we use this to rewrite the wave equation into a semilinear problem.

Consider the heat equation of the form

$$\begin{cases} \partial_t^2 u = \Delta u & (\text{in } Q_T) \\ u(0, x) = g_1(x), \quad \partial_t u(0, x) = g_2(x) & (x \in \Omega) \\ u(t, x) = 0 & ((t, x) \in \Gamma_T). \end{cases}$$

This is equivalent to

$$\begin{cases} \partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & (\text{in } Q_T) \\ \begin{pmatrix} u_1(0, x) \\ u_2(0, x) \end{pmatrix} = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} & (x \in \Omega) \\ u_1(t, x) = 0 & ((t, x) \in \Gamma_T) \end{cases} \tag{1.5}$$

To apply semigroup theory we have to verify that the operator $\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ generates a strongly continuous semigroup over some properly chosen Banach space. (The right choice is $H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ which is a Hilbert space.)

A detailed discussion (e.g. the generation) is in [13, Section 7.4.]

1.2.4 Nonlinear heat equation

At first glance nonlinear heat equation are not semilinear, but a simple trick transforms it to a semilinear system.

The heat conduction is described by the well known, thoroughly investigated equation:

$$\varrho c_p \partial_t T = \operatorname{div}\big(K \nabla T\big) + \varrho q.$$

Here T denotes temperature, ϱ is density, c_p is specific heat, and q is the rate of production of heat per unit mass. The temperature dependence of q is highly based on the type of heat source, e.g. if it is a radioactive decay then it will be independent of T, if it is a chemical reaction then it strongly depends on T (by the Arrhenius factor).

A convection term can be added to the equation if the medium moves:

$$\varrho c_p \Big(\partial_t T + \langle v, \nabla T\rangle\Big) = \operatorname{div}\big(K \nabla T\big) + \varrho q.$$

Usually K depends on the concentration of the fluid, hence leading us to quasilinear equations rather then semilinear.

By an easy transformation the problem can be turned into a semilinear system of equations. Suppose that $K := K(T)$ is a smooth and positive scalar function and we change the time variable t to a new one s, by calculating the divergence we have

$$\varrho c_p \partial_t T = K(T)\Delta T + K'(T)|\nabla T|^2 + \varrho q,$$

dividing both sides by $K(T) > 0$ and introducing the new time variable $s = \varphi^{-1}(t)$, and

$$\partial_s T(\varphi(s), x) = \partial_0 T(\varphi(s), x)\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = \partial_t T(t, x)\frac{\mathrm{d}t}{\mathrm{d}s},$$

yields the condition $\frac{\mathrm{d}t}{\mathrm{d}s} = K(T)^{-1}$ and we have the following semilinear system of coupled ODE and PDE:

$$\begin{cases} \varrho c_p \partial_s T = \Delta T + K'(T)K(T)^{-1}|\nabla T|^2 + \varrho q K(T)^{-1} \\ \frac{\mathrm{d}t}{\mathrm{d}s} = K(T)^{-1}. \end{cases} \tag{1.6}$$

A short description can also be found in [8, Section 2.1.].

1.2.5 Navier-Stokes equation

A nontrivial example for semilinear system is the Navier-Stokes equation. Again a "little" trick will help us in the reformulation.

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a smooth boundary and consider this form of the Navier-Stokes equation

$$\begin{cases} \partial_t u - \frac{1}{Re}\Delta u = -(u \cdot \nabla)u - \nabla p + f, & (\text{in } Q_{+\infty}) \\ \operatorname{div}\big(u(t,x)\big) = 0 & ((t,x) \in \Omega_t), \\ u(t,x) = 0 & ((t,x) \in \Gamma_{+\infty}), \\ u(t,x) = u_0(x) & ((t,x) \in \Omega_0), \end{cases}$$

where Re is the Reynolds number.

At first glance, the theory of semilinear problems is not applicable since the time derivative of the pressure p does not appear and one of the equations is also free from time derivative.

Therefore we will choose the space X so that $\mathrm{div} u = 0$ is satisfied automatically, and then we drop out the pressure term.

If $u : \Omega \to \mathbf{R}^3$ is continuously differentiable, $\mathrm{div} u = 0$, and the normal component $\partial_\nu u$ vanishes on $\partial\Omega$, then for any scalar $\phi \in C^1(\Omega)$,

$$\int_\Omega u \cdot \nabla\phi = 0.$$

Conversely a smooth vector field which is orthogonal to all gradients must satisfy $\mathrm{div} u = 0$ in Ω and $\partial_\nu u = 0$ on $\partial\Omega$. Based on this motivation we define the two following sets:

$$H_g = \Big\{ \nabla\phi \,|\, \phi \in C^1(\Omega) \Big\}$$
$$H_d = \Big\{ u \in C^1(\Omega, \mathbf{R}^3) \,|\, \mathrm{div}(u) = 0 \text{ in } \Omega,\ \partial_\nu u = 0 \text{ on } \partial\Omega \Big\}.$$

The $\|.\|_{L^2(\Omega,\mathbf{R}^3)}$ closure of the above sets are also denoted by the same.

Using Green's formula, it is clear that for arbitrary $v \in H_g$ and $u \in H_d$

$$\langle v, u \rangle_{L^2(\Omega,\mathbf{R}^3)} = \int_\Omega vu = \int_\Omega \nabla\phi u = -\int_\Omega \phi\, \mathrm{div} u + \int_{\partial\Omega} \phi \partial_\nu u \mathrm{d}\sigma = 0,$$

hence the two sets are orthogonal, i.e.

$$H_g \perp H_d \qquad \text{and} \qquad L^2(\Omega, \mathbf{R}^3) = H_g \oplus H_d \quad (\text{see [7]}).$$

Now every $u : \Omega \to \mathbf{R}^3$ has the form $u = v + \nabla\phi$ with $v \in H_d$ and $\nabla\phi \in H_g$. Let P denote the orthogonal projection of $L^2(\Omega, \mathbf{R}^3)$ onto H_d, and project the Navier-Stokes equation onto H_d, we have

$$\partial_t v + \frac{1}{Re} Av = N(v) + f_P(t), \tag{1.7}$$

where $A = -P\Delta$ with zero boundary conditions, $N(v) = -P(v \cdot \nabla)v$ and $f_P = Pf(t, .)$.

This form of the equation is clearly semilinear. This ideas were first discussed by Fujita and Kato [7]. A brief discussion can be found in [8, Section 2.7.]

1.2.6 Abstract form

All of these problems can be formulated as an abstract semilinear problem:

$$\begin{cases} \dot{u}(t) + Au(t) = F\big(t, u(t)\big), \\ u(0) = u_0 \in X, \end{cases} \tag{1.8}$$

where X is a given Banach or Hilbert space and the boundary and other conditions are incorporated either in the operator $\big(A, D(A)\big)$, or in the space X.

This work will look for the solution and numerical approximation of such problems.

Chapter 2

Semigroups and corresponding abstract linear problems

In this chapter we briefly summarize some basic properties and results for semigroups based on [3, 4, 13]. We also introduce the notations we will use throughout the thesis. For different approaches the reader is referred to Engel and Nagel [3, 4] for basic semigroup theory, to Pazy [13] for applications to PDEs and some basics and to Henry [8] for basic and geometric theory.

In our context the semigroup theory is used to guarantee existence and uniqueness of solutions for our problems found in Section 1.2 and hence (1.8) above.

Let X be a vector space, usually some Banach or a Hilbert space, and a family of linear mappings $\big(T(t)\big)_{t\geq 0}$ satisfying

$$\begin{cases} T(t+s) = T(t)T(s) \quad \text{for all}\, t, s \geq 0, \\ T(0) = I, \end{cases} \tag{2.1}$$

then $\big(T(t)\big)_{t\geq 0}$ is called a one-parameter semigroup.

An easy example of semigroups is the mapping $f : \mathbf{R} \to \mathbf{R}$, $f(t) := e^{at}$ for some fixed $a \in \mathbf{R}$. Clearly this family is a semigroup.

The early problem for real functions, was investigated by Cauchy in 1982, he derived that the only continuous function satisfying (2.1) is e^{at}. See [2, p. 100-102.].

Definition 2.1 *(i) If the family satisfies* (2.1) *and the orbit maps*

$$t \to T(t)x$$

are continuous from $\mathbf{R}_+$ into X for all $x \in X$, then $\big(T(t)\big)_{t\geq 0}$ is called a strongly continuous semigroup or C_0 semigroup.

(ii) The semigroup is said to be uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

Example 2.1 The above two definitions do not coincide.

For the semigroup $T(t)$ generated by the multiplication operator M_q defined by some continuous function q with compact support, $T(t)$ is uniformly continuous if and only if q is bounded. If q is unbounded, but $\sup_{s\in\Omega} \operatorname{Re}\, q(s) \leq +\infty$ then $T(t)$ is just strongly continuous. For more see [3, Section 1.4.a.].

We define the infinitesimal generator or simply generator A of a semigroup, by

Definition 2.2 *Let X be a Banach space. The linear operator $A : X \supset\!\!\to X$ is the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and it is defined by*

$$Ax := \lim_{h\to 0} h^{-1}\big(T(h)x - x\big), \qquad D(A) = \big\{x \in X \,\text{where the limit exists}\big\}. \tag{2.2}$$

The generator should be denoted by the pair $(A, D(A))$, but for convenience we drop the domain and implicitly assume it with the above definition.

The following lemma is very important in our discussion.

Lemma 2.1 *Every strongly continuous semigroup of bounded operators $\big(T(t)\big)_{t\geq 0}$ has a generator A, which satisfies the following properties.*

(i) $A : D(A) \subseteq X \to X$ is a linear operator;

(ii) If $x \in D(A)$, then $T(t)x \in D(A)$ and

$$\dot{T}(t)x = T(t)Ax = AT(t)x \qquad (\forall t \geq 0).$$

Proof: Assertion (i) is trivial.

(ii) For $x \in D(A)$

$$\lim_{h\to 0+} h^{-1}\big(T(t+h)x - T(t)x\big) = \lim_{h\to 0+} T(t)h^{-1}\big(T(h)x - x\big) = T(t)Ax.$$

For $-t \leq h < 0$, we have

$$h^{-1}\big(T(t+h)x - T(t)x\big) - T(t)Ax = T(t+h)\Big(h^{-1}(x - T(-h)x) - Ax\Big) + T(t+h)Ax - T(t)Ax.$$

Since $\|T(t+h)\|$ remains bounded, this yields that the first term converges to zero, and by strong continuity the second term also converges to zero. This also means that the mapping $t \mapsto T(t)x \in X$ is differentiable on $\mathbf{R}^+$. The limit

$$\lim_{h\to 0+} h^{-1}\big(T(h)T(t)x - T(t)x\big)$$

also exists, hence $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax = \dot{T}(t)x$. ∎

Remark 2.1 If $T(t)$ is uniformly continuous in the above lemma, then A is not just linear but also bounded. Moreover, $T(t) = e^{tA} := \sum_{n=0}^{+\infty} \frac{(tA)^n}{n!}$ and there exists a constant $\omega > 0$ such that $\|T(t)\| \leq e^{\omega t}$.

Here we state some useful lemmas on semigroups, but we do not include proofs, for more information the reader is referred to [3, 4, 13].

Lemma 2.2 *Let $T(t)$ be a strongly continuous semigroup. Then there exists constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}.$$

Lemma 2.3 (Properties of generators) *If A is the infinitesimal generator of a strongly continuous semigroup, then the domain $D(A)$ is dense in X and A is a closed linear operator.*

Lemma 2.4 (Connection of A and $T(t)$) *Let $T(t)$ be a strongly continuous semigroup and let A be its generator. Then*

(i) For $x \in X$

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)x\mathrm{d}s = T(t)x.$$

(ii) For $x \in X$, $\int_0^t T(s)x\mathrm{d}s \in D(A)$ and

$$A\bigg(\int_0^t T(s)x\mathrm{d}s\bigg) = \int_0^t T(s)Ax\mathrm{d}s = T(t)x - x.$$

(iii) For $x \in D(A)$

$$T(t)x - T(s)x = \int_s^t T(\xi)Ax\mathrm{d}\xi = \int_s^t AT(\xi)x\mathrm{d}\xi.$$

The straightforward way to calculate A is by (2.2). An important and different way to obtain A, to be precise the resolvent of A, is as

Lemma 2.5 *The equation*

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t}T(t)\mathrm{d}t \tag{2.3}$$

holds, for $x \in X$, $Re\lambda > \omega$, if $\|T(t)\| \le Me^{\omega t}$. For more information see [13, Remark 1.5.4].

Since we are interested in applications to partial differential equations, it is more interesting to obtain $T(t)$ from its infinitesimal generator A. The motivation of this can be seen in the first part of Section 2.2, i.e. for $x \in D(A)$, the semigroup $T(t)u_0$ generated by A is clearly the solution of the problem:

$$\dot{u} = Au, \qquad u(0) = u_0.$$

2.1 Semigroups generated by differential operators

One of our main goals is the application of semigroup theory to various partial differential equations, hence we need that some well known differential operator generates a semigroup. There are different approaches to prove that a certain operator generates a semigroup (e.g. sectoriality, Hille-Yoshida and Lumer-Philips theorems, etc.), but we can ensure this result with very simple assumptions, which are specialized for differential operators. This section is devoted to these results.

Lemma 2.6 ([10]) *Let $L : D(L) \subset H \to H$ be a densely defined strictly positive operator on a Hilbert space H, with $\mathrm{ran}(L) = H$, and let the inverse of the operator be compact. Let λ_n denote the eigenvalues of L corresponding to the eigenvectors e_n such that (e_n) is a complete orthonormal system. Then $-L$ generates a semigroup $\big(T(t)\big)_{t\ge 0}$ in $B(H)$:*

$$T(t) = e^{-Lt} \qquad (t \ge 0),$$

where, for

$$x = \sum_{n=1}^{+\infty} c_n e_n \in H, \qquad e^{-Lt}x := \sum_{n=1}^{+\infty} e^{-\lambda_n t}c_n e_n.$$

PROOF: We will show that the semigroup property is satisfied, i.e. we prove that e^{-Lt} satisfies (2.1) and the semigroup is strongly continuous (Definition 2.1 (i) is satisfied).

It follows form the definition that (2.1) is clearly true. To prove that the orbit maps are continuous: let $x \in H$ be arbitrary and $t \geq 0$, then $\lim_{h\to 0} \|(T(t+h) - T(t))x\| = 0$, by looking at the definition we have:

$$\|(T(t+h) - T(t))x\|^2 = \sum_{n=1}^{\infty} (e^{-(t+h)\lambda_n} - e^{-t\lambda_n})^2 |c_n|^2.$$

Since for $r, s \geq 0$ the estimates $|e^{-r} - e^{-s}| \leq |r - s|$ and $|e^{-r} - e^{-s}| \leq 1$ are true, we have $|e^{-(t+h)\lambda_n} - e^{-t\lambda_n}| \leq \min\{\lambda_n |h|,\, 1\}$, this yields

$$\|(T(t+h) - T(t))x\|^2 \leq \sum_{n=1}^{\infty} \min\{\lambda_n^2 h^2,\, 1\} |c_n|^2.$$

Let $h \in \mathbf{R}$ be arbitrary, if $\lambda_n \leq 1/\sqrt{|h|}$, then $\lambda_n^2 h^2 \leq |h|$, therefore $\min\{\lambda_n^2 h^2,\, 1\} \leq |h|$, else $\lambda_n > 1/\sqrt{|h|}$, then $\min\{\lambda_n^2 h^2,\, 1\} \leq 1$. From these inequalities

$$\|(T(t+h) - T(t))x\|^2 \leq \quad |h| \sum_{\lambda_n \leq \frac{1}{\sqrt{|h|}}} |c_n|^2 \; + \sum_{\lambda_n > \frac{1}{\sqrt{|h|}}} |c_n|^2.$$

Now by letting $h \to 0$, then the first term converges to 0, since it can be estimated by $|h|\|x\|^2$, the second term also converges to 0, since we drop the first segments of the convergent series of $\|x\|^2$.

We only have to prove that the generator of $\big(T(t)\big)_{t\geq 0}$ is $-L$, i.e. (2.2) is satisfied. For arbitrary but fixed $u_0 = \sum\limits_{n=1}^{\infty} c_n e_n \in D(L)$,

$$\Big\|\frac{T(h) - I}{h} u_0 + Lu_0\Big\|^2 = \Big\|\sum_{n=1}^{\infty} \Big(\frac{e^{-h\lambda_n} - 1}{h} + \lambda_n\Big) c_n e_n\Big\|^2 = \sum_{n=1}^{\infty} \Big(\frac{e^{-h\lambda_n} - 1}{h\lambda_n} + 1\Big)^2 \lambda_n^2 |c_n|^2.$$

Let $f(r) := \frac{e^{-r}-1}{r} + 1$ $(r > 0)$. It is clear that the estimates $1 - r \leq e^{-r} \leq 1 - r + \frac{r^2}{2}$ hold, hence $0 \leq f(r) \leq \frac{r}{2}$ $(\forall r > 0)$, and $\lim_0 f = 0$ and $\lim_\infty f = 1$, yields the boundedness of f. Let M denote $\sup f$, then $|f(r)| \leq \min\{\frac{r}{2},\, M\}$. We can estimate the above equation as follows

$$\Big\|\frac{T(h) - I}{h} u_0 + Lu_0\Big\|^2 \leq \sum_{n=1}^{\infty} \min\Big\{\frac{1}{4} h^2 \lambda_n^2,\; M^2\Big\}\; \lambda_n^2 |c_n|^2.$$

Again we follow the same chain of ideas as before. For $h > 0$, if $\lambda_n \leq 1/\sqrt{h}$ then $\lambda_n^2 h^2 \leq h$, and the above minimum is smaller than $h/4$, else if $\lambda_n > 1/\sqrt{h}$, then the above minimum is smaller than M^2, hence

$$\Big\|\frac{T(h) - I}{h} u_0 + Lu_0\Big\|^2 \leq \quad \frac{h}{4} \sum_{\lambda_n \leq \frac{1}{\sqrt{h}}} \lambda_n^2 |c_n|^2 \; + \; M^2 \sum_{\lambda_n > \frac{1}{\sqrt{h}}} \lambda_n^2 |c_n|^2,$$

again by letting $h \to 0$, then the first term converges to 0, since it can be estimated by $\frac{h}{4}\|Lu_0\|^2$, the second term also converges to 0, since we drop the first segments of the convergent series of $M^2\|Lu_0\|^2$. ■

It is well known that the operator $-\Delta$ on $H^2(\Omega) \cap H_0^1(\Omega)$ satisfies the following conditions over the Hilbert space $H := L^2(\Omega)$:

(1) It is densely defined in H and strictly positive.

(2) If Ω is C^2 diffeomorphic to a convex set, then from the solvability of Poisson's equation, $\operatorname{ran}(L) = H$. (We note here that weaker assumptions are also enough for this.)

(3) The inverse of the Laplacian is compact.

Corollary 2.1 *(i) By the above properties and using Lemma 2.6 it is clear that $-L = \Delta$ generates a strongly continuous semigroup over $H = L^2(\Omega)$.*

(ii) Again from the above properties of the Laplacian it is also follows that the operator with $D(A) := \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^k$,

$$\begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix}$$

also generates a strongly continuous semigroup.

2.2 Linear Cauchy problems

In this section we give a brief summary on linear abstract initial value problems. This section also serve as a motivation as well as basics of application of semigroup theory on Cauchy problems and PDEs.

The semilinear case will be thoroughly discussed in the next chapter.

2.2.1 Initial value problems

Consider the initial value problem in one dimension

$$\dot{x}(t) + ax(t) = 0 \quad \text{with} \quad x(0) = x_0 \in \mathbf{R},$$

for some fixed $a \in \mathbf{R}$.

We know that for arbitrary $x_0 \in \mathbf{R}$ the problem has a unique solution defined by

$$x(t) = x_0 e^{-at}.$$

We can define a semigroup $\left(T(t)\right)_{t\geq 0}$ on $\mathbf{R}$, in the sense of the previous section, which is defined by the mapping

$$T(t)x_0 := x_0 e^{-at}.$$

Let X be a given Banach space and $-A$ be the generator of a strongly continuous semigroup $\left(T(t)\right)_{t\geq 0}$. By Lemma 2.1 we can easily solve the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) + Au(t) = 0, \\ u(0) = u_0 \in X, \end{cases}$$

namely the solution, analogously to the ODE case, is $T(t)u_0$ for every $u_0 \in D(A)$. By the usual techniques it is easy to show that this solution is also unique.

2.2.2 Inhomogeneous problems

If we consider the non-homogeneous problem, we can also express the solution using the *variation of constants* formula. Again we use the scalar equations as a motivation.

Consider the problem

$$\begin{cases} \dot{x}(t) + ax(t) = f(t), \\ x(0) = x_0 \in \mathbf{R}, \end{cases} \tag{2.4}$$

for some integrable function f. The function

$$x(t) = x_0 e^{-at} + \int_0^t e^{-a(t-s)} f(s)\mathrm{d}s = T(t)x_0 + \int_0^t T(t-s)f(s)\mathrm{d}s \tag{2.5}$$

is the solution of (2.4).

The derivative of x can be calculated as

$$\dot{x}(t) = -ax_0e^{-at} - a\int_0^t e^{-a(t-s)}f(s)\mathrm{d}s + f(t) = -aT(t)x_0 - a\int_0^t T(t-s)f(s)\mathrm{d}s + f(t),$$

hence x is clearly a solution.

Now let us consider the inhomogeneous abstract problem:

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), \\ u(0) = u_0 \in X, \end{cases} \tag{2.6}$$

where $f : \mathbf{R}^+ \to X$ is a given integrable function, i.e. $\int_0^t \|f(s)\|\mathrm{d}s < +\infty$.

Definition 2.3 *The function $u : [0, T) \to X$ is called a (classical) solution of (2.6) on $[0, T)$ if $u \in C^1$ and $u(t) \in D(A)$ for all $0 < t < T$ and the problem is satisfied on $[0, T)$.*

Let $T(t)$ be the semigroup generated by $-A$, and let u be a solution of the above problem. We define the function $g : \mathbf{R}^+ \to X$ as $g(s) := T(t-s)u(s)$.

$$\begin{aligned} \dot{g}(s) &= AT(t-s)u(s) + T(t-s)\dot{u}(s) = \\ &= AT(t-s)u(s) - T(t-s)Au(s) + T(t-s)f(s) = T(t-s)f(s). \end{aligned} \tag{2.7}$$

Integrating both sides on $(0, t)$ yields

$$u(t) - T(t)u(0) = \int_0^t \dot{g}(s)\mathrm{d}s = \int_0^t T(t-s)f(s)\mathrm{d}s,$$

and $u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\mathrm{d}s$ which is exactly the same formula as (2.5), i.e. the *variation of constants* formula.

If $f \in L^1([0, T); X)$ is a given function and we define $g(s) := T(t-s)u(s)$ just as we done before and hence $\dot{g}(s) = T(t-s)f(s)$ which is integrable since f is.

Corollary 2.2 Let $f \in L^1([0, T); X)$ and $u_0 \in X$. If (2.6) has a solution, then it is given by the formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\mathrm{d}s. \tag{2.8}$$

This implies that the solution u is unique also.

If (2.8) yields a continuous function but it is not differentiable, then a solution, is defined as follows.

Definition 2.4 *Let $-A$ be the generator of a strongly continuous semigroup $T(t)$ and let $u_0 \in X$ and $f \in L^1([0,T);X)$. The function $u \in C([0,T),X)$ is called a mild solution if (2.8) is satisfied on $[0,T]$.*

Theorem 2.1 ([13]) *Let $-A$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$, let $f \in L^1([0,T);X)$ be continuous on $(0,T]$ and let*

$$v(t) = \int_0^t T(t-s)f(s)\mathrm{d}s \qquad (0 \le t \le T).$$

Then the initial value problem (2.6) has a classical solution u on $[0,T)$ for arbitrary $u_0 \in D(A)$ if either of the following condition is satisfied.

(i) The function v is continuously differentiable on $(0,T)$.

(ii) For all $t \in (0,T)$ $\;v(t) \in D(A)$ holds and $Av(t)$ is continuous on $(0,T)$.

And conversely if there exists a solution u on $[0,T)$ of problem (2.6) for some $u_0 \in D(A)$, then v satisfies both (i) and (ii).

From this theorem the corollary below clearly follows.

Corollary 2.3 *Let $-A$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$, and f is continuously differentiable on $[0,T]$ then the Cauchy problem (2.6) has a classical solution on $[0,T)$ for all $x \in D(A)$.*

Chapter 3

Parabolic semilinear equations

In this chapter we will study the semilinear Cauchy problem:

$$\begin{cases} \dot{u}(t) + Au(t) = f(t, u(t)) \qquad (t > 0), \\ u(0) = u_0 \in X, \end{cases} \tag{3.1}$$

where $-A$ is the infinitesimal generator of $\big(T(t)\big)_{t\geq 0}$, a strongly continuous semigroup over a given Banach space X. In our context $f : [0,T] \times X \to X$ is a continuous function in t and Lipschitz continuous in the second variable. If F is said to be integrable it is meant as $\int_0^t \|f(s,.)\|\mathrm{d}s < +\infty$. This kind of problems arises in many applications, see Section 1.2. The abstract discussion of this kind of problems can be found in (and this section is also based on) [13].

We can analogously define the (classical) solution of (3.1) as Definition 2.3.

A similar argument as found in the previous section shows that a classical solution u still satisfies the integral equation:

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))\mathrm{d}s.$$

If the problem has a solution then it satisfies the integral equation above. Hence, it is natural that we define the mild solution as:

Definition 3.1 *A continuous solution u of the above integral equation*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))\mathrm{d}s,$$

will be called the mild solution of (3.1).

3.1 Solution of abstract problems

The following theorem is the classical result which assures the existence and uniqueness of mild solutions for Lipschitz continuous f.

Theorem 3.1 ([13]) *Let $f : [0,T] \times X \to X$ be a continuous function in t and uniformly Lipschitz continuous (with constant L) on X. If $-A$ is the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$, then for every $u_0 \in X$ the initial value problem* (3.1) *has a unique mild solution $u \in C\big([0,T]; X\big)$. Moreover the mapping $u_0 \mapsto u$ is Lipschitz continuous from X to $C\big([0,T]; X\big)$.*

Proof: For a given $u_0 \in X$ we define the following function:

$$F : C\big([0,T];X\big) \to C\big([0,T];X\big)$$

by

$$(Fu)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))\mathrm{d}s \qquad \text{on } [0,T]. \tag{3.2}$$

Denoting by $\|u\|_\infty$ the norm of u as an element of $C\big([0,T];X\big)$. From the definition of F it clearly follows that

$$\|(Fu)(t) - (Fv)(t)\| \leq MLt\|u-v\|_\infty, \tag{3.3}$$

where M is a bound of $\|T(t)\|$ on $[0,T]$.

Using (3.2) and (3.3), by induction over n it easily follows that

$$\|(F^n u)(t) - (F^n v)(t)\| \leq ML\int_0^t \frac{(MLs)^{n-1}}{(n-1)!}\|u-v\|_\infty \mathrm{d}s \leq \frac{(MLt)^n}{n!}\|u-v\|_\infty,$$

this yields

$$\|F^n u - F^n v\| \leq \frac{(MLT)^n}{n!}\|u-v\|_\infty.$$

If n is large enough $(MLT)^n/n! < 1$, hence by a known extension of the contraction principle F has a unique fixed point $u \in C\big([0,T];X\big)$. This fixed point is the solution of the integral equation and hence the mild solution of the problem.

The uniqueness of u and the Lipschitz continuity of the map $u_0 \mapsto u$ follows from the argument below.

Let v be a mild solution corresponding to the initial value v_0. Then

$$\begin{aligned}\|u(t)-v(t)\| &\leq \|T(t)u_0 - T(t)v_0\| + \int_0^t \|T(t-s)\Big(f(s,u(s)) - f(s,v(s))\Big)\|\mathrm{d}s \\ &\leq M\|u_0 - v_0\| + ML\int_0^t \|u(s)-v(s)\|\mathrm{d}s,\end{aligned}$$

implying, by Gronwall's inequality,

$$\|u(t)-v(t)\| \leq Me^{MLT}\|u_0 - v_0\|$$

and finally

$$\|u-v\|_\infty \leq Me^{MLT}\|u_0-v_0\|.$$

This means that the solution u is unique and the mapping is Lipschitz continuous. ■

This theorem can be generalized: if $g \in C\big([0,T];X\big)$ and the definition of F is modified to

$$(Fu)(t) = g(t) + \int_0^t T(t-s)f(s,u(s))\mathrm{d}s.$$

The generalization is as follows

Corollary 3.1 If A and f satisfy the conditions of Theorem 3.1 then for every $g \in C\big([0,T];X\big)$ the integral equation

$$w(t) = g(t) + \int_0^t T(t-s)f(s,w(s))\mathrm{d}s$$

has a unique solution $w \in C\big([0,T];X\big)$.

Proof: The simple modification of the proof of Theorem 3.1 shows the desired result. ■

In the usual ordinary differential equation context the existence and uniqueness is derived from the local Lipschitz continuity in the second variable. Here it is also true in some sense if f is locally Lipschitz continuous in u and uniformly in t for bounded intervals. To be more precise, if for every $t' > 0$ and constant $c \geq 0$ there exists a constant $L(c, t')$ such that

$$\|f(t, u) - f(t, v)\| \leq L(c, t')\|u - v\|$$

holds for all $u, v \in X$ with $\|u\|, \|v\| < c$ and $t \in [0, t']$, than we have the following version of Theorem 3.1.

Theorem 3.2 ([13]) *Let $f : [0, +\infty) \times X \to X$ be a continuous in t for $t \geq 0$ and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If $-A$ is the generator of a strongly continuous semigroup $T(t)$, then for every $u_0 \in X$ there is a $t_{\max} \leq +\infty$ such that the initial value problem* (3.1) *has a unique mild solution on* $[0, t_{\max})$. *If $t_{\max} < +\infty$ then*

$$\lim_{t \to t_{\max}} \|u(t)\| = +\infty.$$

The proof of this theorem goes similarly as the one before, but it is more technical.

Theorem 3.3 ([13]) *Let $-A$ be the generator of a strongly continuous semigroup $T(t)$ on X, and let $f : [0, T] \times X \to X$ be a continuously differentiable function then the mild solution of* (3.1) *for $u_0 \in D(A)$ is always a classical solution.*

Theorem 3.4 ([13]) *Let $f : [0, T] \times Y \to Y$ be uniformly Lipschitz in $Y := \big(D(A), \|.\|_A\big)$, and for each $y \in Y$ let $f(t, y)$ be a continuous function from $[0, T]$ to Y. If $u_0 \in D(A)$ the problem* (3.1) *has a unique classical solution on* $[0, T]$.

The proofs of the last three theorems are highly based on Theorem 3.1 or 3.2.

We note here that there are some theorems with slightly weaker assumptions for reflexive Banach spaces, see [13, Section 6.1].

3.2 Solutions of semilinear PDE problems

This section contains some of our main results. Using semigroup theory we conclude that a quite large class of semilinear parabolic PDE has a unique solution in some sense.

Theorem 3.5 *Let q'_ξ be bounded and q be continuous in all other variables and let Ω be C^2 diffeomorphic to a convex set, just as before Corollary 2.1. Then the problem*

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + q(t,x,u(t,x)) = g(t,x) & ((t,x) \in Q_T) \\ u(t,x) = 0 & ((t,x) \in \Gamma_T) \\ u(t,x) = u_0(x) & ((t,x) \in \Omega_0), \end{cases} \tag{3.4}$$

has a unique mild solution over some interval $[0, T)$.

Proof: First we formulate our problem in the usual abstract form.

The corresponding abstract space X is the Hilbert space $L^2(\Omega)$, the domain of the operator $A = -\Delta$ is $H^2(\Omega) \cap H_0^1(\Omega)$, and by Lemma 2.6 and Corollary 2.1 (i) the operator $-A$ generates a strongly continuous semigroup.

The abstract problem is

$$\begin{cases} \dot{u}(t) + Au(t) = F\big(t, u(t)\big) \\ u(0) = u_0 \in L^2(\Omega), \end{cases} \tag{3.5}$$

where $F\big(t, u(t)\big) = -q\big(t, ., u(t, .)\big) + g(t, .)$ and q'_ξ is bounded, i.e. for some constant K $|q'_\xi\big(t, x, \xi\big)| \leq K$.

To use Theorem 3.2 we have to prove that the mapping F is locally Lipschitz in u, uniformly in t on bounded intervals:

$$\begin{aligned} \|F(t, u) - F(t, v)\|^2_{L^2} &= \int_\Omega | - q\big(t, x, u(t, x)\big) + g(t, x) - (-q\big(t, x, v(t, x)\big) + g(t, x))|^2 = \\ &= \int_\Omega |q'_\xi\big(t, x, \eta(t, x)\big)|^2 |u(t, x) - v(t, x)|^2 \leq K^2 \|u - v\|^2_{L^2}. \end{aligned}$$

Therefore, by Theorem 3.2, equation (3.4) has a unique mild solution. ■

We now turn to the investigation of systems. The above theorem can easily be generalized for system of equations coupled in the nonlinearity.

Consider the problem

$$\begin{cases} \partial_t u_j(t, x) - \Delta u_j(t, x) + q_j(t, x, u_1(t, x), \ldots, u_M(t, x)) = g_j(t, x) & ((t, x) \in Q_T, \quad j = 1, 2, \ldots, M) \\ u_j(t, x) = 0 & ((t, x) \in \Gamma_T) \\ u_j(0, x) = \gamma_j(x) & ((t, x) \in \Omega_0), \end{cases} \tag{3.6}$$

and its vector form is

$$\partial_t \mathbf{u} + A\mathbf{u} + \mathbf{q}(t, x, \mathbf{u}) = \mathbf{g}.$$

Theorem 3.6 *If $\|q'_\xi\|$ is bounded in some matrix norm and q_j is continuous in all other variables for all $j = 1, 2, \ldots, M$. Then problem (3.6) has a unique mild solution.*

Proof: The proof of this theorem goes similarly to the one before. Again we only have to prove the local Lipschitz continuity of the mapping F, which follows using the Lagrange inequality:

$$\begin{aligned} \|F(t, \mathbf{u}) - F(t, \mathbf{v})\|^2_{L^2} &= \sum_{j=1}^{k} \int_\Omega | - q_j\big(t, x, \mathbf{u}(t, x)\big) + g_j(t, x) - (-q_j\big(t, x, \mathbf{v}(t, x)\big) + g_j(t, x))|^2 = \\ &= \int_\Omega |\mathbf{q}(t, x, \mathbf{u}) - \mathbf{q}(t, x, \mathbf{v})|^2 \leq \sup \|q'_\xi\|^2 \int_\Omega |\mathbf{u} - \mathbf{v}| \leq K^2 \|\mathbf{u} - \mathbf{v}\|^2_{L^2}. \end{aligned}$$

■

Chapter 4

Numerical methods and experiments

4.1 Discretization, the method of lines

There are various ways to solve these problems numerically. Here the numerical solution is obtained by a time discretization, i.e. in the space variables the equation remains continuous and we approximate the solution on each time step, hence the elliptic theory of solutions (see e.g. [6]) is applicable. We discuss here the method of lines (it is also called Rothe's method). The time discretization yields usually nonlinear elliptic problems which we solve with some finite element based iterative method.

Some convergence results of the method of lines can be found in e.g. [11].

We introduce the following notations. For an arbitrary function $\varphi : [0,T] \times \Omega \to X$

$$\varphi^k(x) := \varphi(k\tau, x) \tag{4.1}$$

for $k = 0, 1, \dots, n$, where $n\tau = T$ and the problem is usually considered on $[0,T] \times \Omega$.

Let us consider the problem

$$\begin{cases} \dot{u}(t) + Au(t) = F(t, u(t)) \\ u(0) = \gamma \in X, \end{cases}$$

where X is some Banach space.

4.1.1 The exponential formula and differential approximation

We have seen that a C_0 semigroup, with generator A, is in close relation with e^{tA} (for bounded linear A they are equal). If A is unbounded then the interpretation of the sense in which they are "equal" is little more complicated, see [3, 3.4.10. Yoshida Approx.] or [13, Theorem 1.5.5].

Here we would like to detail an important result following from this relation.

Lemma 4.1 (The exponential formula, [13]) *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup on X, with generator A. Then*

$$T(t)x = \lim_{n\to+\infty} \Big(I - \frac{t}{n}A\Big)^{-n} x = \lim_{n\to+\infty} \Big[\frac{n}{t} R\Big(\frac{n}{t}, A\Big)\Big]^n x \qquad (x \in X) \tag{4.2}$$

and the limit is uniform in t on any bounded interval.

Proof: Assume that $\|T(t)\| \leq Me^{\omega t}$. By Lemma 2.5 we have

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda s} T(s)x \, \mathrm{d}s \qquad (x \in X).$$

After differentiating it n times with respect to λ and by substituting $s = vt$ and taking $\lambda = nt^{-1}$ we have:

$$R\Big(\frac{n}{t}, A\Big)^{(n)} x = (-1)^n t^{n+1} \int_0^{+\infty} \big(ve^{-v}\big)^n T(tv)x \, \mathrm{d}v,$$

and

$$R(\lambda, A)^{(n)} = (-1)^n n! R(\lambda, A)^{n+1}$$

and therefore

$$\Big[\frac{n}{t} R\Big(\frac{n}{t}, A\Big)\Big]^{n+1} x = \frac{n^{n+1}}{n!} \int_0^{+\infty} \big(ve^{-v}\big)^n T(tv)x \, \mathrm{d}v.$$

Noting that

$$\frac{n^{n+1}}{n!} \int_0^{+\infty} \big(ve^{-v}\big)^n \mathrm{d}v = 1$$

we obtain

$$\Big[\frac{n}{t} R\Big(\frac{n}{t}, A\Big)\Big]^{n+1} x - T(t)x = \frac{n^{n+1}}{n!} \int_0^{+\infty} \big(ve^{-v}\big)^n \Big(T(tv)x - T(t)x\Big) \mathrm{d}v.$$

For a given $\varepsilon > 0$, we choose $0 < a < 1 < b < +\infty$ such that $t \in [0, t_0]$ implies

$$\|T(tv)x - T(t)x\| < \varepsilon \qquad \text{for} \quad a \leq v \leq b.$$

Now we split the right hand side of the above equation into three parts I_1, I_2, I_3, on the intervals $[0, a]$, $[a, b]$, $[b, +\infty]$, respectively. We have

$$\begin{aligned}
\|I_1\| &\leq \frac{n^{n+1}}{n!} \big(ae^{-a}\big)^n \int_0^a \|T(tv)x - T(t)x\| \mathrm{d}v \\
\|I_2\| &\leq \varepsilon \frac{n^{n+1}}{n!} \int_a^b \big(ve^{-v}\big)^n \, \mathrm{d}v < \varepsilon \\
\|I_3\| &= \frac{n^{n+1}}{n!} \Big\| \int_b^{+\infty} \big(ve^{-v}\big)^n \Big(T(tv)x - T(t)x\Big) \mathrm{d}v \Big\|.
\end{aligned}$$

Here we used the facts: $ve^{-v} \geq 0$ is monotonically nondecreasing on $[0, 1]$, and nonincreasing for $v \geq 1$. Since furthermore $ve^{-v} < e^{-1}$ for $v \neq 1$, $\|I_1\| \to 0$ uniformly in $t \in [0, t_0]$ as $n \to +\infty$. Choosing $n > \omega t$ in I_3, we see that the integral in the estimate of I_3 converges and that $\|I_3\| \to 0$ uniformly in $t \in [0, t_0]$ as $n \to +\infty$. Consequently,

$$\limsup_{n \to +\infty} \Big\| \Big[\frac{n}{t} R\Big(\frac{n}{t}, A\Big)\Big]^{n+1} x - T(t)x \Big\| \leq \varepsilon$$

and since $\varepsilon > 0$ was arbitrary we have

$$\lim_{n \to +\infty} \Big[\frac{n}{t} R\Big(\frac{n}{t}, A\Big)\Big]^{n+1} x = T(t)x$$

and we know that

$$\lim_{n \to +\infty} \frac{n}{t} R\Big(\frac{n}{t}, A\Big) y = y$$

for all y,in particular $y := \left[\frac{n}{t}R\left(\frac{n}{t}, A\right)\right]^n$. It is straightforward from the definition of resolvents that the second equality of (4.2) is true, this completes the proof. ■

In our context this formula has a very interesting and important interpretation. Let A be the generator of a strongly continuous semigroup $T(t)$. We would like to solve the problem:

$$\dot{u} = Au, \qquad u(0) = u_0. \tag{4.3}$$

A standard way of solve this equation is applying implicit Euler method, i.e.

$$\frac{u^n\left(\dfrac{jt}{n}\right) - u^n\left(\dfrac{(j-1)t}{n}\right)}{\dfrac{t}{n}} = Au^n\left(\frac{jt}{n}\right), \qquad u^n(0) = u_0.$$

These equations can be solved explicitly and their solution $u^n(t)$ is given by

$$u^n(t) = \left(I - \frac{t}{n}A\right)^{-n} u_0,$$

where $u^n(t)$ is the approximation of the solution of the problem at time t. Lemma 4.1 guarantees that as $n \to +\infty$, $u^n(t) \to T(t)u_0$. We now that for $u_0 \in D(A)$ the solution $T(t)u_0$ is unique, thus the solutions of the difference equations converge to the solution of the differential equation.

If $u_0 \notin D(A)$ then equation (4.3) does not have a solution. The solutions of the difference equations do converge to $T(t)u_0$, which should be considered a generalized solution of the differential equation.

4.2 Semilinear parabolic problems

By a simple first order approximation of the time derivative, the difference equation corresponding to the problems is:

$$\begin{cases} \frac{u^{k+1}-u^k}{\tau} + Au^{k+1} = F^{k+1}(u^{k+1}) \\ u^0 = u(0) = \gamma, \end{cases}$$

where F^{k+1} is defined analogously as (4.1). The implicit time integration is used for stability reasons.

The arising elliptic problem is

$$Au^{k+1} - F^{k+1}(u^{k+1}) + \frac{u^{k+1}}{\tau} = \frac{u^k}{\tau},$$

this problem can be formulated in a more compact form: $F_0(u) = b$.

For time integration, somehow we have to solve these elliptic problems. To solve them we could apply efficient iterative techniques. Newton-like methods are very appropriate to do so. By the investigations found in [12] we use the quasi-Newton method with stepwise variable preconditioning with a piecewise constant preconditioner.

4.2.1 The quasi-Newton method

Although the reader can find a detailed description of stepwise variable preconditioning in [9], we briefly formulate the algorithm and convergence results here.

In order to apply the theorem below, we rewrite our elliptic problem using the nonlinear operator $F(u) := F_0(u) - b$. Now we need the solution of

$$F(u) = 0,$$

which is numerically obtained by

Theorem 4.1 ([9]) *Let $\big(H, \langle .,. \rangle\big)$ be a real Hilbert space with induced norm $\|.\|$. Let the operator $F : H \to H$ have a Gâteaux derivative satisfying the following conditions:*

(i) $F'(u)$ is self-adjoint for all $u \in H$.

(ii) There exists constants $\lambda_2 \geq \lambda_1 > 0$ satisfying

$$\lambda_1 \|h\|^2 \leq \langle F'(u)h, h \rangle \leq \lambda_2 \|h\|^2 \qquad (u, h \in H).$$

(iii) There exists $L > 0$ such that

$$\|F'(u) - F'(v)\| \leq L\|u - v\| \qquad (u, v \in H).$$

We introduce the following notations:

$$\|h\|_n^2 := \langle F'(u_n)^{-1} h, h \rangle \quad (n \in \mathbf{N}), \quad \|h\|_*^2 := \langle F'(u^*)^{-1} h, h \rangle,$$

$$\text{and} \quad \mu(u_n) := L\lambda^{-2} \|F(u_n)\|.$$

Let u^* be the unique solution of the equation $F(u) = 0$. For arbitrary u_0 we define the sequence (u_n) as follows:

$$u_{n+1} = u_n - \frac{2\tau_n}{M_n + m_n} {A_n}^{-1} F(u_n) \qquad (n \in \mathbf{N}),$$

where the following conditions hold:

(iv) Let A_n be self-adjoint linear operators and $M_n \geq m_n > 0$ be such that

$$m_n \langle A_n h, h \rangle \leq \langle F'(u_n)h, h \rangle \leq M_n \langle A_n h, h \rangle \qquad (h \in H^1(\Omega),\ n \in \mathbf{N}), \tag{4.4}$$

further, there exists constants $K > 1$ and $\varepsilon > 0$ such that $M_n/m_n \leq 1 + 2/(\varepsilon + K\mu(u_n))$.

(v) We define

$$\tau_n = \min \left\{ 1, \frac{1 - Q_n}{2\rho_n} \right\},$$

where $\mu(u_n)$ and $\|.\|_n$ are defined above, $Q_n = \frac{M_n - m_n}{M_n + m_n}(1 + \mu(u_n))$, and $\rho_n = 2LM_n^2 \lambda^{-3/2}(M_n + m_n)^{-2} \|F(u_n)\|(1 + \mu(u_n))^{1/2}$.

Then there holds

$$\|u_n - u^*\| \leq \frac{1}{\lambda}\|F(u_n)\| \to 0,$$

namely

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq \limsup \frac{M_n - m_n}{M_n + m_n} < 1. \tag{4.5}$$

Moreover, if in addition we assume $\frac{M_n}{m_m} \leq 1 + c_1\|F(u_n)\|^\gamma \quad (n \in \mathbf{N})$ *with some constants* $c_1 > 0$ *and* $0 < \gamma \leq 1$*, then*

$$\|F(u_{n+1})\|_* \leq d_1\|F(u_n)\|_*^{1+\gamma} \qquad (n \in \mathbf{N}),$$

with positive constant d_1*.*

Since the norms $\|.\|$ *and* $\|.\|_*$ *are equivalent, then the last estimate is also true with the original norm on* H*.*

The proof of this theorem and more details can be found in [9].

4.3 Applications to partial differential equations

In this section we apply the theory of semigroups and the above discretization techniques for PDEs. We consider the equations (1.1), (1.2) and the examples found in Section 1.2 in the case when the elliptic operator is the Laplacian, and we formulate them here:

$$\begin{cases} \partial_t u - \Delta u + q(t,x,u) = g(t,x) & ((t,x) \in Q_T) \\ u(t,x) = 0 & ((t,x) \in \Gamma_T) \\ u(t,x) = \gamma(x) & ((t,x) \in \Omega_0), \end{cases} \tag{4.6}$$

and

$$\begin{cases} \partial_t u_j - \Delta u_j + q_j(t,x,u_1,\ldots,u_M) = g_j(t,x) & ((t,x) \in Q_T, \qquad j = 1,2,\ldots,M) \\ u_j(t,x) = 0 & ((t,x) \in \Gamma_T) \\ u_j(0,x) = \gamma_j(x) & ((t,x) \in \Omega_0), \end{cases} \tag{4.7}$$

respectively.

The corresponding Banach space is $X := L^2(\Omega)$ and the domain of the operator $A = -\Delta$ is $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and $X := L^2(\Omega) \times \ldots \times L^2(\Omega)$ and the operator is

$$\begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix},$$

with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times \ldots \times H^2(\Omega) \cap H_0^1(\Omega)$, respectively.

In Theorem 3.5 we have seen that the above problem has a unique weak solution.

4.3.1 Semidiscretization and solvability of the elliptic problems

After the semidiscretization the given elliptic problem is

$$-\Delta u^{k+1} + q^{k+1}(x, u^{k+1}) + \frac{u^{k+1}}{\tau} = \frac{u^k}{\tau} + g^{k+1},$$

or in weak form, it is

$$\int_\Omega \Big(\nabla u^{k+1} \nabla v + q^{k+1}(x, u^{k+1})v + \frac{u^{k+1}}{\tau} v\Big) = \int_\Omega \Big(\frac{u^k}{\tau} v + gv\Big) \qquad (\forall v \in V).$$

For solving this equation we need a simple modification of the well known assumptions see e.g. [6, Theorem 6.5 or Assumptions 7.3].

The functions in (1.1), (1.2) and (4.6), (4.7) all satisfy the following conditions.

Assumptions 4.1 *(i) The domain $\Omega \subset \mathbf{R}^N$ is C^2 diffeomorphic to a convex set.*

(ii) The functions $A : \Omega \times \mathbf{R}^N \to \mathbf{R}^{N\times N}$ and $q : [0, T] \times \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ are measurable and bounded w.r. to the variable $x \in \Omega$ and C^1 in all other variables.

(iii) The matrices $A(x)$ are symmetric and their eigenvalues λ satisfy

$$0 < \mu_1 \le \lambda \le \mu_2 < +\infty$$

with constants μ_1, μ_2 independent form x.

(iv) Let $2 \le p$ (if $N = 2$) or $2 \le p \le \frac{2N}{N-2}$ (if $N > 2$). There exist constants $c_i', c_i, d_i \ge 0$ and $2 \le p_i \le p$ $(i = 1, 2)$ such that for any $x \in \Omega$ (or $x \in \partial\Omega$, respectively) and $\xi \in \mathbf{R}$,

$$-c_1' \le \partial_\xi q(x, \xi) \le c_1 + c_2|\xi|^{p_1-2}, \quad -c_2' \le \partial_\xi s(x, \xi) \le d_1 + d_2|\xi|^{p_2-2}.$$

(vi) For all fixed $t \in [0, T]$ $g(t, x) \in L^2(\Omega)$, and $\gamma(x) \in H_0^1(\Omega)$.

The analogous assumptions are also fulfilled by the functions of the system (1.2) (indexed by j). However we will only use constant matrices A, we state these assumptions in general.

In the first case $X = L^2(\Omega)$, in the second is the product of k Hilbert space, i.e. $X = \big(L^2(\Omega), \ldots, L^2(\Omega)\big)$.

4.3.2 The quasi-Newton method for semilinear PDEs

With these Assumptions 4.1 the quasi-Newton method described in Subsection 4.2.1 is applicable, see [9]. The algorithmic form of it for a fixed finite element subspace $V_h \subset V$ looks as follows:

$$\left\{\begin{array}{ll}
\text{(a)} & u_0^{k+1} := u^k; \\
 & \qquad \text{for a fixed } n \in \mathbf{N}, \text{ if } u_n^{k+1} \in V_h \text{ is known then} \\
\text{(b)} & p_n \in V_h \text{ is the solution of the problem:} \\
 & \displaystyle\int_\Omega \big(\nabla p_n \nabla v + w_n^{(i)}(x) p_n v\big) = \\
 & \displaystyle\quad = -\int_\Omega \Big(\nabla u_n^{k+1} \nabla v + q(u_n^{k+1})v + \frac{u_n^{k+1}}{\tau} v - \frac{u^k}{\tau} v - gv\Big) \qquad (\forall v \in V_h); \\
\text{(c)} & u_{n+1}^{k+1} := u_n^{k+1} + \dfrac{2\tau_n}{M_n + m_n}\, p_n \\
\text{(d)} & u^{k+1} := u_{n+1}^{k+1} \quad \text{if the relative residual error is sufficiently small,}
\end{array}\right. \tag{4.8}$$

where $\tau_n \in (0,1]$ is a damping parameter and $w_n^{(i)}$ is a step function, e.g. for $i = 2$ it is defined by:

$$w_n^{(2)}(x) := \begin{cases} \frac{3}{4}\max_\Omega y_n + \frac{1}{4}\min_\Omega y_n, & \text{if } y_n(x) \geq \frac{1}{2}\big(\max_\Omega y_n + \min_\Omega y_n\big), \\ \frac{1}{4}\max_\Omega y_n + \frac{3}{4}\min_\Omega y_n, & \text{if } y_n(x) < \frac{1}{2}\big(\max_\Omega y_n + \min_\Omega y_n\big). \end{cases}$$

using the notation $y_n(x) = q'(u_n^{k+1}(x))$. For other values of i the function w_n is defined similarly.

4.4 Numerical experiments

We have run our experiments for various semilinear parabolic PDE problems. All of the equations were solved on $[0,T] \times I$ where I is some nice interval, usually $[0,1]$.

The experiments were carried out in the following way:

- We applied semidiscretization, with implicit Euler method (denoted by (IE)).
- The space discretization was done by FEM discretization with Courant elements up to order $p = 7$. For simplicity, but not necessarily, we used uniform meshes.
- We carried out element-by-element assembly, i.e. a reference element was used. The numerical integrations were done in order $2p$ with Simpson's quadrature, and in order $5p$ for the integrals containing the nonlinearity q.
- The quasi-Newton method was damped.
- The stopping criterion for the auxiliary elliptic problem was $\dfrac{\|F_h(u_n) - b_h\|_h}{\|F_h(u_0) - b_h\|_h} < 10^{-11}$; we always displayed the relative residual errors.
- The code was written in MATLAB and the auxiliary problems were solved using the built-in solver of MATLAB: `\mldivide`.

4.4.1 Bistable problem

Consider the single equation of the bistable problem (a special case of (1.3), see more in [5, p. 4-7.] or [1]):

$$\begin{cases} \partial_t - \varepsilon\Delta u = u - u^3 & ((t,x) \in [0,+\infty) \times [0,1]) \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 & (t \in [0,+\infty)) \\ u(0,x) = u_0(x) & (x \in [0,1]). \end{cases}$$

In the case when ε is sufficiently small the solution $u \equiv 1$ and $u \equiv -1$ are both stable equilibrium solutions, and there is no other. All solutions, except unstable equilibria, converge to one of these solutions, however the convergence is extremely slow.

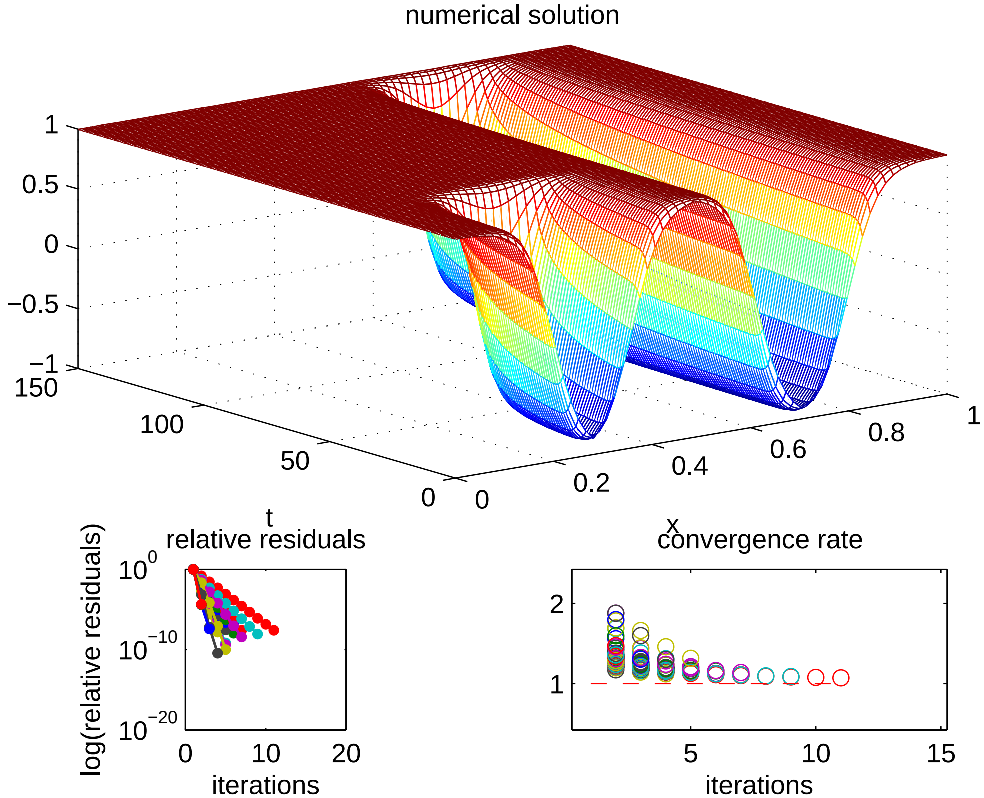
Our initial data is as follows:

$$u_0(x) := \begin{cases} \mathrm{th}\big((0.2 - x)/(2\sqrt{\varepsilon})\big), & 0 \leq x < 0.28 \\ \mathrm{th}\big((x - 0.36)/(2\sqrt{\varepsilon})\big), & 0.28 \leq x < 0.4865 \\ \mathrm{th}\big((0.613 - x)/(2\sqrt{\varepsilon})\big), & 0.4865 \leq x < 0.7065 \\ \mathrm{th}\big((x - 0.8)/(2\sqrt{\varepsilon})\big), & 0.7065 \leq x \leq 1. \end{cases}$$

We note here that for $\varepsilon = 0.0009$ the function u_0 is very close to a metastable state.

We have solved numerically the bistable problem on $[0, 150] \times [0, 1]$ with the initial condition above. The results can be seen in Figure 4.1, the very slow convergence to one of the stable solutions can be nicely seen.

Figure 4.1: Solution of the bistable problem



The analytical solutions of the bistable problems are not known, but we were able to reproduce the well known behavior of this problem.

The plots in the bottom of Figure 4.1 show the superlinear convergence of the iteration when solving the auxiliary problems. The left diagram shows the relative residual errors on a logarithmic scale, the right one shows the estimated convergence rate on each step.

4.4.2 Superconductivity of liquids

Consider the system of two equations describing superconductivity of liquids :

$$\partial_t u - \varepsilon \Delta u = (1 - |u|^2)u + g$$

or in the form:

$$\begin{cases} \partial_t u_1 - \varepsilon_1 \Delta u_1 = \left(1 - \sqrt{u_1^2 + u_2^2}\right) u_1 + g_1 \\ \partial_t u_2 - \varepsilon_2 \Delta u_2 = \left(1 - \sqrt{u_1^2 + u_2^2}\right) u_2 + g_2, \end{cases}$$

which is also a special case of (1.1). For the original problem detailed in e.g. [5, p. 2.] the functions g_1, g_2 are both zero. To show some convergence results we set them so that the solution of the equation is

$$\begin{aligned} u_1(t,x) &= e^{-t} \sin(k_1 \pi x) \qquad (k_1 \in \mathbf{R}^+) \\ u_2(t,x) &= e^{-t} \sin(k_2 \pi x) \qquad (k_2 \in \mathbf{R}^+). \end{aligned}$$

We are mainly interested in the convergence in time. For $k_1 = 1$, $k_2 = 3$ and $\varepsilon_1 = \varepsilon_2 = 1$ the absolute errors can be seen in Table 4.1 below. The first order convergence in time can be nicely seen.

Table 4.1: Errors for the superconductivity problem

		$\lVert \mathbf{u}_h - \mathbf{u}^* \rVert$			
t	τ^{-1}	$h^{-1} = 2^3$	$h^{-1} = 2^4$	$h^{-1} = 2^5$	$h^{-1} = 2^6$
0.25	4	0.0118164	0.0084477	0.0075186	0.0072806
	8	0.0097014	0.0055349	0.0043869	0.0040930
	16	0.0084827	0.0038333	0.0025536	0.0022261
0.5	4	0.0116118	0.0089549	0.0082256	0.0080390
	8	0.0080694	0.0051503	0.0043496	0.0041448
	16	0.0061635	0.0031542	0.0023286	0.0021178
1	4	0.0071782	0.0062540	0.0060033	0.0059393
	8	0.0040499	0.0031775	0.0029404	0.0028799
	16	0.0025401	0.0017057	0.0014787	0.0014207
2	4	0.0024642	0.0023823	0.0023611	0.0023557
	8	0.0012037	0.0011405	0.0011248	0.0011207
	16	0.0006191	0.0005644	0.0005507	0.0005471
4	4	0.0003275	0.0003235	0.0003225	0.0003223
	8	0.0001559	0.0001536	0.0001531	0.0001530
	16	0.0000764	0.0000750	0.0000746	0.0000746

4.4.3 Time discretization with higher order methods

High order one-step numerical methods for ordinary differential equations are well known. One of the most widespread methods are Runge-Kutta methods.

We made some experiments with maybe the most universal R-K method: the trapezoid method ((T)), also known as the the Crank-Nicolson method ($(C\text{-}N)$), which is of second order. The formulation of this method can be derived based on the latter name, i.e. $(T) = (C\text{-}N) = \big((IE) + (EE)\big)/2$ in the sense:

$$(EE) \qquad \frac{u^{k+1} - u^k}{\tau} - \Delta u^k + q^k(u^k) = g^k$$

$$(IE) \qquad \frac{u^{k+1} - u^k}{\tau} - \Delta u^{k+1} + q^{k+1}(u^{k+1}) = g^{k+1},$$

by summing up we have:

$$2\frac{u^{k+1} - u^k}{\tau} - \Delta u^{k+1} + q^{k+1}(u^{k+1}) - \Delta u^k + q^k(u^k) = g^{k+1} + g^k,$$

which yields the auxiliary elliptic problems:

$$-\Delta u^{k+1} + q^{k+1}(u^{k+1}) + \frac{2}{\tau}u^{k+1} = \Delta u^k - q^k(u^k) + \frac{2}{\tau}u^k + g^{k+1} + g^k.$$

We also implemented this method with the same conditions as before.

In Table 4.2 we compare (IE) and (T) on problem (3.4) where $q(\xi) = |\xi|^3\xi$ and g is chosen so that the solution of the problem is $e^{-t}\sin(k\pi x)$.

Table 4.2: Compare of (IE) and (T) for $k = 1$.

		$\|\mathbf{u}_h - \mathbf{u}^*\|$					
		Implicit Euler			Trapezoid method		
t	τ^{-1}	$h^{-1} = 2^4$	$h^{-1} = 2^5$	$h^{-1} = 2^6$	$h^{-1} = 2^4$	$h^{-1} = 2^5$	$h^{-1} = 2^6$
0.25	1	0.2211992	0.2211992	0.2211992	0.2211992	0.2211992	0.2211992
	2	0.0073580	0.0070489	0.0069715	0.0004847	0.0002373	0.0004184
	3	0.0043382	0.0039663	0.0038732	0.0005159	0.0000593	0.0000551
0.5	1	0.0146955	0.0144679	0.0144110	0.0011770	0.0019806	0.0021825
	2	0.0079571	0.0076921	0.0076258	0.0002825	0.0002317	0.0002864
	3	0.0042469	0.0039677	0.0038980	0.0000314	0.0000101	0.0000582
1	1	0.0117001	0.0115536	0.0115171	0.0005704	0.0004214	0.0003834
	2	0.0056645	0.0055279	0.0054936	0.0001013	0.0001758	0.0002007
	3	0.0028156	0.0026879	0.0026556	0.0000771	0.0000139	0.0000427

The contrast between the first order convergence of the implicit Euler method and the second order convergence of the trapezoidal method can be seen.

Summary

Semilinear parabolic problems can be treated by techniques of semigroup theory. We introduced and applied this theory for partial differential equations. Unlike the usual generation techniques (Hille-Yoshide and Lumer-Philips) we used a theorem which is more special and more typical in this field, whose main assumption was the compactness of the inverse operator [10].

We were given existence and uniqueness theorems for a large class of semilinear parabolic partial differential equations and systems, where the nonlinearity is locally Lipschitz continuous, as we were used to also in the theory of ordinary differential equations (Theorem 3.5 and 3.6).

We have introduced the technique of time discretization and coupled with the numerical solution of the auxiliary elliptic equations. Based on [12] we chose quasi-Newton method to solve them (Section 4.2).

The discussed methods were implemented, we could solve equations and even systems numerically in the class of our investigation. Numerical experiments have been carried out for various problems, hence the convergence rates of the time discretizations are numerically reinforced by our results.

Magyar nyelvű összefoglaló

Félcsoportelmélet segítségével hatokényan kezelhetőek az általunk tárgyalt szemilineáris parabolikus problémák. Ezt az elméletet röviden bemutattuk és alkalmaztuk parciális differenciálegyenletek esetében. A szokásos generálási tételek (Hille-Yoshide and Lumer-Philips) helyett mi a területre jellemzőbb, speciálisabb tételt használtunk melynek, fő feltétele az operátor inverzének kompaktsága [10].

Létezési és egyértelműségi tételeket adtunk szemilineáris parabolikus parciális differenciálegyenletekre és -rendszerekre, abban az esetben, ha a nemlinearitás lokálisan Lipschites (3.5 és 3.6 Tétel).

Részleteztük az általunk használt időbeli diszkretizáció módszerét, és az adódó elliptikus feladatok numerikus megoldási módszereit. Választásunk [12] alapján a kvázi-Newton módszerre esett (4.2 rész).

A tárgyalt numerikus módszereket megvalósítottuk, majd különböző feladatokat és rendszereket oldottunk meg velük, így numerikusan alátámasztva a konvergenciarendeket.

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