SOFIC GROUPS

MSc Thesis

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Budapest, 2014
Acknowledgments

I would like to thank my advisor, Miklós Abért, for introducing me to this subject. I am grateful for the many discussions we had, for his helpful suggestions, and for his patience with me.
I would also like to thank Bandi Szabó for his immense help with the proof of Theorem 2.2.
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Introduction

The idea of sofic groups (originally: initially subamenable groups) was introduced by Gromov (8) in 1999, as a common generalization of residually finite and amenable groups. The name ‘sofic’, from the Hebrew word meaning ‘finite’, was coined by Weiss (11).

Sofic groups are of considerable interest because several important general conjectures of group theory were shown to be true for them. Among these, we discuss Gottschalk’s surjunctivity conjecture, that was proved by Gromov in [8].

Possibly the biggest open question in the subject is whether all countable groups are sofic. So far we know that the class of sofic groups is closed under taking subgroups, direct products, inverse limits, direct limits, free products, extensions by amenable groups (5), and amalgamation over amenable subgroups (6, 10).

The goal of this thesis is to summarize some results about sofic groups and to clarify a proof of Elek and Szabó about the amalgamated products of sofic groups over amenable subgroups. The original proof in [6] contains an error and is incomplete. An alternate proof, using von Neumann algebras, appears in [10].

The structure is as follows: In the first chapter, we define sofic groups, give some useful and interesting characterizations for them, and review some basic examples. The second chapter contains the theorems about the class of sofic groups. Finally, in the third chapter we present a proof of the surjunctivity conjecture for sofic groups.
Chapter 1

Definitions and basic examples

Soficity has many different definitions and characterizations in several papers. In this chapter we introduce some of them.

1.1 Definition

Our definition of sofic groups follows [5].

For a finite set \( A \) let \( \text{Map}(A) \) denote the monoid of self-maps of \( A \) acting on the right. We write \( a \cdot f \) for \( f(a) \) and multiplication in \( \text{Map}(A) \) works as usual: \( a \cdot fg = (a \cdot f) \cdot g = g(f(a)) \). Let \( \varepsilon \in (0, 1) \) be a real number, then we say that two elements \( e, f \in \text{Map}(A) \) are \( \varepsilon \)-similar, or \( e \sim_\varepsilon f \), if the number of points \( a \in A \) with \( a \cdot e \neq a \cdot f \) is at most \( \varepsilon |A| \). We say that \( e, f \) are \((1 - \varepsilon)\)-different if they are not \((1 - \varepsilon)\)-similar.

Let \( G \) be a group, \( \varepsilon \in (0, 1) \) a real number and \( F \subseteq G \) a finite subset. An \((F, \varepsilon)\)-quasi-action of \( G \) on a finite set \( A \) is a function \( \phi : G \to \text{Map}(A) \) with the following properties:

(a) For any two elements \( e, f \in F \) the map \( \phi(ef) \) is \( \varepsilon \)-similar to \( \phi(e)\phi(f) \).

(b) \( \phi(1) \) is \( \varepsilon \)-similar to the identity map of \( A \).

(c) For each \( e \in F \setminus \{1\} \) the map \( \phi(e) \) is \((1 - \varepsilon)\)-different from the identity map of \( A \).

**Definition 1.1.** The group \( G \) is sofic if for all \( \varepsilon \in (0, 1) \) and all finite subsets \( F \subseteq G \) there exists an \((F, \varepsilon)\)-quasi-action of \( G \).

We can ask for more than this:

**Lemma 1.2.** If a group \( G \) is sofic then for each \( F \subseteq G \) and for each \( \varepsilon \in (0, 1) \) there is an \((F, \varepsilon)\)-quasi-action \( \phi \) of \( G \) on a finite set \( A \) satisfying the following extra conditions:
(b') \( \phi(1) \) is the identity map of \( A \), for each \( 1 \neq g \in G \) the map \( \phi(g) \) is a fixpoint free bijection and \( \phi(g^{-1}) = \phi(g)^{-1} \).

(c') For different elements \( e, f \in F \cup \{1\} \) the map \( \phi(e) \) is \( (1 - \varepsilon) \)-different from \( \phi(f) \).

It is clear from the definition that a non-finitely generated group is sofic if and only if all of its finitely generated subgroups are sofic. Because of this, from now on we will mostly consider finitely generated groups.

### 1.1.1 Two characterizations of soficity

We will use the following two characterizations of sofic groups:

1. For the definition used by Weiss in [11], we need the notion of Cayley graphs. Let \( G \) be a finitely generated group, and \( S \subseteq G \) a fixed finite, symmetric (i.e. \( S = S^{-1} \)) generating set. The *Cayley graph* of \( G \) is a directed graph \( \text{Cay}(G, S) \), whose edges are labeled by the elements of \( S \): the set of vertices equals \( G \), and the edges with label \( s \in S \) are the pairs \( (g, sg) \) for all \( g \in G \). Let \( B_r(1) \) denote the \( r \)-ball around \( 1 \in \text{Cay}(G, S) \) (it is an edge-colored graph, and also a finite subset in \( G \)).

**Proposition 1.3.** A finitely generated group \( G \) is sofic if there exists a finite, symmetric generating set \( S \), such that for each \( \delta > 0 \) and each \( r \in \mathbb{N} \) there is a finite directed graph \((V, E)\) edge-labeled by \( S \), and a subset \( V_0 \subseteq V \) with the following properties:

1. For each point \( v \in V_0 \) there is a function \( \psi_v : B_r(1) \rightarrow V \) which is an isomorphism (of labeled graphs) between \( B_r(1) \subseteq \text{Cay}(G, S) \) and the \( r \)-ball around \( v \) in \( V \).
2. \( |V_0| \geq (1 - \delta)|V| \).

We call this finite graph \( V \) an \( (r, \delta) \)-approximation of the Cayley graph.

Moreover, if \( G \) is sofic, then all finite, symmetric generating sets have this property.

The above proposition, i.e. the equivalence of the two definitions was proven in [3].

2. Let \( G \) be a finitely generated group: \( G = \langle S \rangle \), where \( S = S^{-1} \) and \( |S| < \infty \). Let \( F_S \) denote the free group generated by \( S \), and \( f : F_S \rightarrow G \) the factor map.
Proposition 1.4. A finitely generated group $G$ is sofic if there is a finite, symmetric generating set $S$ such that for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$, a homomorphism $\phi : F_S \to S_n$ and a subset $A \subseteq [n]$ ($[n]$ denotes the set of integers from 1 to $n$, $S_n$ acts on $[n]$), $|A| > (1-\varepsilon)n$, such that if $w \in F_S$ and the length of $w$ is at most $k$ then for every $x \in A$, $x\phi(w) = x$ if and only if $f(w) = 1$ in $G$. In this case we call this $\phi$ a $(k, \varepsilon)$-quasi-action.

If $G$ is sofic then this is true for every generating set $S$.

Proof. First suppose we have a set $S$ with this property. Let $\varepsilon \in (0, 1)$ and $F \subset G$, $|F| < \infty$ be arbitrary. We can assume that 1 $\in F$. Since $S$ is a generating set, we can write every element of $F$ as a product of elements of $S$. Consider a shortest such word for each element in $F \cdot F$, and take the maximum of the length of these words, where $F \cdot F = \{ef \mid e, f \in F\}$. Since $F$ is a finite set, this is a finite number, let us denote it by $t$.

We can find a $(k, \varepsilon)$-quasi-action of $G$, where $k = 2t + 1$. Call this quasi-action $\psi : F_S \to S_n$. Let $A \subseteq [n]$ be the set described in the proposition. We define $\phi : F \cdot F \to \text{Map}([n])$ the following way. For $e \in F \cdot F$ and $x \in A$, then we can find $w \in F_S$ such that $f(w) = e$ and the length of $w$ is at most $t$. Define

$$x\phi(e) = x\psi(w).$$

This is well-defined because if $w' \in F_S$ is another word that has length at most $t$ and $f(w') = e$, then $w'w^{-1}$ has length at most $k$ and $f(w'w^{-1}) = 1$ hence $x\psi(w'w^{-1}) = x$, i.e. $x\psi(w') = x\psi(w^{-1})^{-1} = x\psi(w)$.

This way we defined $\phi(e)$ on the elements of $A$, on the rest of the elements we can define it to be the identity. For $g \notin F \cdot F$ let $\phi(g)$ be the identity. It is easy to see that this $\phi : G \to \text{Map}([n])$ is an $(F, \varepsilon)$-quasi-action of $G$.

For the other direction let $G$ be a finitely generated sofic group and $S$ a finite, symmetric generating set. We would like to find a $(k, \varepsilon)$-quasi-action for $k \mathbb{N}$ and $\varepsilon > 0$.

Let $W \subset F_S$ be the set of words of length at most $k$, and let $f(W) = F \subseteq G$. This $F$ is clearly finite, denote the size of $F$ by $m$. Let

$$\delta = \frac{\varepsilon}{m^2 + m},$$

and let $\phi : G \to \text{Map}(A)$ be an $(F, \delta)$-quasi-action of $G$.

Then for each $e_1, e_2 \in F$ there is a subset $A_{e_1,e_2} \subseteq A$, $|A_{e_1,e_2}| > (1-\delta)|A|$ on which $\phi(e_1e_2) = \phi(e_1)\phi(e_2)$. Similarly, for every $e \in F \setminus \{1\}$ there exists

\begin{align*}
\phi(e) & \in A_{e_1,e_2} \\
\text{and} \quad \phi(e_1e_2) & \in A_{e_1,e_2}.
\end{align*}
A_e such that \(|A_e| > (1 - \delta)|A|\) and \(\phi(e)\) has no fixpoints in \(A_e\). Let \(A'\) be the intersection of all these \(A_{e_1,e_2}\)'s and \(A_e\)'s. Clearly

\[|A'| > |A| - \sum_{e_1,e_2 \in F} |A \setminus A_{e_1,e_2}| - \sum_{e \in F \setminus \{1\}} |A \setminus A_e| \geq (1 - m^2\delta - m\delta)|A| = (1 - \varepsilon)|A|.

On this intersection the above statements hold simultaneously for all elements of \(F\).

For \(s \in S \subset F_S\) let us define \(\psi(s)|_{A'} = \phi(f(s))|_{A'}\) and extend it to \(A\) as a bijection. It is possible since by the definition of \(A'\) \(\phi(e)\) is injective on \(A'\) for \(e \in F\). Let \(|A| = n\) and identify \(A\) with \([n]\). This way we defined \(\psi : S \to S_n\). One can easily check that this \(\psi\) is a \((k, \varepsilon)\)-quasi-action.

\[\square\]

### 1.1.2 Other approaches

3. For two permutations \(\sigma, \tau \in S_n\) their normalized Hamming distance \(\text{dist}(\sigma, \tau)\) is defined to be the number of points not fixed by \(\sigma^{-1}\tau\), divided by \(n\). Clearly we can reformulate the definition of sofic groups using this concept:

A group \(G\) is sofic if for all \(\varepsilon \in (0, 1)\) and every finite subset \(F \subset G\) there is some \(n \in \mathbb{N}\) and a map \(\phi : G \to S_n\) such that

- for every \(g \in F \setminus \{e\}\), \(\text{dist}(\phi(g), e) > 1 - \varepsilon\), where \(e\) is the identity element of \(G\),
- for all \(g, h \in F\), \(\text{dist}(\phi(g^{-1}h), \phi(g)^{-1}\phi(h)) < \varepsilon\).

The following is another useful characterization: Given positive integers \(n(k)\) we denote by \(N\) the normal subgroup of \(\prod_{k=1}^{\infty} S_{n(k)}\) consisting of all sequences \((\sigma_k)_{k=1}^{\infty}\) such that \(\lim_{k \to \infty} \text{dist}(\sigma_k, \text{id}_{n(k)}) = 0\), where \(\text{id}_{n(k)}\) is the identity element of the permutation group \(S_{n(k)}\).

**Proposition 1.5** (From [2]). Let \(G\) be a finitely generated group. Then \(G\) is sofic if and only if for some sequence of integers \(n(k)\) there is a group homomorphism

\[\psi : G \to \left(\prod_{k=1}^{\infty} S_{n(k)}\right) / N,\]

given by \(\psi(g) = [(\psi_k(g))_{k=1}^{\infty}]\) for some maps \(\psi_k : G \to S_{n(k)}\) such that

\[\lim_{k \to \infty} \text{dist}(\psi_k(g), \text{id}_{n(k)}) = 1\]

for all nontrivial elements \(g \in G\).
4. Let $G$ be a finitely generated group, and let $Sub(G)$ denote the set of subgroups endowed with the Chabauty topology, which is just the product topology on the set of subsets of $G$ restricted to the set of subgroups. This is a compact topological space, $G$ acts on it by conjugation, and this action is by homeomorphisms.

An invariant random subgroup (IRS) of $G$ is a random subgroup of $G$ whose distribution is a $G$-invariant probability measure on $Sub(G)$. For example if $N \triangleleft G$ then the Dirac measure $\delta_N$ is an IRS. If $H \leq G$, $|G : H| < \infty$, then $H$ has finitely many conjugates, let $\mu_H$ denote the uniform probability measure on these conjugates. This is also an IRS.

Recall that a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of probability measures weak* converges to $\mu$ if for every real valued continuous function $f$ we have
\[
\int f \, d\mu_n \to \int f \, d\mu.
\]

Proposition 1.6 (From [1]). Let $G$ be a finitely generated group, $G \cong F/N$, where $F$ is a free group, $N \triangleleft F$. Then $G$ is sofic if and only if there exists $H_n \leq F$ for every $n \in \mathbb{N}$ such that $|F : H_n| < \infty$ and $\mu_{H_n} \rightharpoonup \delta_N$.

1.2 Some examples

Let $\Gamma$ be a finitely generated group, $S \subseteq \Gamma$ a finite, symmetric generating set. Consider the Cayley graph $\text{Cay}(\Gamma, S)$. For a subgraph $X$ of this graph let $\partial X$ denote the boundary of $X$, i.e., the set of vertices that have at least one neighbor in the complement of $X$.

A Følner sequence in the Cayley graph is a sequence of spanned subgraphs $\mathcal{F} = \{F_1, F_2, \ldots\}$ such that
\[
\bigcup_{n=1}^\infty F_n = \text{Cay}(\Gamma, S),
\]
\[
\lim_{n \to \infty} \frac{|\partial F_n|}{|F_n|} = 0.
\]

Definition 1.7. A finitely generated group $\Gamma$ is amenable if one of its Cayley graphs admits a Følner sequence. Note that this property does not depend on the generating set.

Proposition 1.8. Let $\Gamma$ be a finitely generated amenable group. Then $\Gamma$ is sofic.

Proof. We choose a finite, symmetric generating set $S$ and a Følner sequence in $\text{Cay}(\Gamma, S)$, $\mathcal{F} = \{F_1, F_2, \ldots\}$. Let us fix $\delta$ and $r$. We would like to find a finite graph described in Proposition 1.3.
If \(|S| = 2d\), then the Cayley graph is \(d\)-regular. We know that the \(F_n\)'s have small boundary for large enough \(n\). Let
\[
F_n' = \text{Cay}(\Gamma, S) \setminus N_r(\text{Cay}(\Gamma, S) \setminus F_n),
\]
where \(N_r(X)\) denotes the \(r\)-neighborhood of a subgraph \(X\). This means that we have omitted those points from \(F_n\) that are close to its complement. All the remaining points have the property that their \(r\)-neighborhood in \(F_n\) is isomorphic to an \(r\)-ball in the Cayley graph. Clearly
\[
|F_n'| = |F_n| - |F_n \cap N_r(\text{Cay}(\Gamma, S) \setminus F_n)| > |F_n| - |N_{r-1}(\partial F_n)| \geq |F_n| - |\partial F_n|d^{r-1}.
\]
So
\[
|F_n'| > \left(1 - d^{r-1} \frac{|\partial F_n|}{|F_n|}\right) |F_n|.
\]
This means if we choose \(n\) such that \(\frac{|\partial F_n|}{|F_n|} < \delta^{-1}d^{r-1}\), then
\[
|F_n'| > (1 - \delta)|F_n|,
\]
which is exactly what we wanted. \(\square\)

**Remark 1.9.** Generally we say that a group \(\Gamma\) is amenable is for every \(\varepsilon > 0\) and for each finite set \(F \subseteq \Gamma\) there is a finite subset \(A \subseteq \Gamma\) such that \(F\) does not move \(A\) too much, i.e.
\[
|Ag \setminus A| < \varepsilon|A| \quad \text{for every} \quad g \in F.
\]
Notice that for finitely generated groups, this is equivalent to the definition we used above.
If \(G\) is not necessarily finitely generated, then \(G\) is amenable if and only is all of its finitely generated subgroups are amenable. It follows that every amenable group is sofic.

There is another large class of groups for which it is easy to see that they are sofic: residually finite groups. Recall that a group \(G\) is **residually finite** if for every \(1 \neq g \in G\) there is a normal subgroup \(N \triangleleft G\), \(|G : N| < \infty\) such that \(g \notin N\). This is equivalent to
\[
\bigcap_{N \triangleleft G, |G : N| < \infty} N = \{1\}.
\]

**Proposition 1.10.** Residually finite groups are sofic.

**Proof.** Let \(G\) be residually finite and \(F \subseteq G\) an arbitrary finite set. This means we can find a normal subgroup \(N \triangleleft G\), \(|G : N| < \infty\) such that the factor map
π : G → G/N is injective on $F \cdot F \cup F \cup \{1\}$. Indeed, for each pair $g, h \in F \cdot F \cup F \cup \{1\}$ there is $N_{g,h}$ such that $g^{-1}h \notin N_{g,h}$. Let $N$ be the intersection of these.

Now let $A = G/N$, which is a finite set and let $\phi(g)$ be the right multiplication on $G/N$ by $\pi(g)$. This way clearly $\phi : G \to \text{Map}(A)$ and conditions (a)-(c) in Definition 1.1 hold for any $\varepsilon \in (0, 1)$. \qed
Chapter 2

The class of sofic groups

In this chapter we prove the following two theorems about the class of sofic groups.

**Theorem 2.1** (Elek-Szabó, [5]). *The class of sofic groups is closed under the following constructions:*

1. subgroups, direct products, inverse limits, direct limits;
2. free products
3. certain extensions: if $N \triangleleft G$, $N$ is sofic and $G/N$ is amenable then $G$ is also sofic.

**Theorem 2.2** (Elek-Szabó, Păunescu, [6] and [10]). *Free products of sofic groups amalgamated over arbitrary amenable subgroups are sofic.*

**Proof of part 1. of Theorem 2.1 by [5].** Suppose that $G$ is sofic and $H \leq G$. Let $F \subseteq H$ finite and $\varepsilon \in (0, 1)$. Then of course $F \subseteq G$ as well, and an $(F, \varepsilon)$-quasi-action of $G$ is an $(F, \varepsilon)$-quasi-action of $H$ too. So $H$ is sofic.

Next let $G_1$ and $G_2$ be sofic groups. We would like to prove that $G_1 \times G_2$ is sofic as well. Let $\varepsilon \in (0, 1)$, $F \subseteq G_1 \times G_2$ be a finite subset, then we can choose finite sets $F_i \subseteq G_i$ ($i = 1, 2$) such that $F \subseteq F_1 \times F_2$. Since $G_1$ and $G_2$ are sofic, there exist $(F_i, \varepsilon_i^2)$-quasi-actions $\phi_i : G_i \to \text{Map}(A_i)$. Let $A = A_1 \times A_2$, and define $\phi : G_1 \times G_2 \to \text{Map}(A)$ the following way:

$$(a_1, a_2) \cdot \phi(g_1, g_2) = (a_1 \cdot \phi_1(g_1), a_2 \cdot \phi_2(g_2)).$$

It is easy to see that $\phi$ is an $(F, \varepsilon)$-quasi-action of $G_1 \times G_2$. Iterating this argument gives us that finite direct products of sofic groups are also sofic.

Now let $G = \prod_{i \in I} G_i$ where $\{G_i\}_{i \in I}$ are sofic groups. For a finite subset $F \subseteq G$ there exists a finite $J \subseteq I$ such that the natural projection $\pi : G \to \prod_{i \in J} G_i$ is injective on
For $F \cup \{1\}$. Hence all $(\pi(F), \varepsilon)$-quasi-action of $\prod_{i \in I} G_i$ gives us an $(F, \varepsilon)$-quasi-action of $G$. This is just a finite direct product, and for that we already proved the statement. An inverse limit of groups is by definition a subgroup of their direct product, so that part of the theorem follows from the previous ones.

Assume we have a directed system of sofic groups $\{G_i\}_{i \in I}$, let $G = \lim_{i \in I} G_i$. Fix $\varepsilon \in (0, 1)$ and a finite set $F \subseteq G$. Then there is an index $i$ and a finite subset $F_i \subseteq G_i$ such that the natural homomorphism $\sigma_i : G_i \to G$ is a bijection $F_i \to F$.

Denote by $\widehat{G}$ the image of $G_i$ in $G$. Choose a map $s : \widehat{G} \to G_i$ that has the property that $\sigma_i(s(g)) = g$ for all $g \in \widehat{G}$. Let $\phi_i : G_i \to \text{Map}(A)$ be an $(F_i, \varepsilon)$-quasi-action of $G_i$. Then define $\phi : G \to \text{Map}(A)$ as follows. For $a \in A$ let

$$a \cdot \phi(g) = \begin{cases} a \cdot \phi_i(s(g)), & \text{if } g \in \widehat{G} \\ a, & \text{if } g \in G \setminus \widehat{G}. \end{cases}$$

This is clearly an $(F, \varepsilon)$-quasi-action of $G$, hence $G$ is sofic. \hfill \Box

### 2.1 Free products

In this section we present a proof for part 2. of Theorem 2.1 that uses a different method as the one in [5]. We will use Proposition 1.4, the second characterization of sofic groups mentioned in the first chapter.

Let $G = \langle S \rangle$ and $H = \langle S' \rangle$ be finitely generated sofic groups, $f : F_S \to G$ and $f' : F_{S'} \to H$ the factor maps. If $\phi : F_S \to S_n$ and $\psi : F_{S'} \to S_n$ are $(k, \varepsilon)$-quasi-actions of $G$ and $H$ respectively, then for $\sigma \in S_n$ let us define $\widehat{\phi}_\sigma : F_{S \cup S'} \to S_n$ in the following way: for $s \in S$ $\widehat{\phi}_\sigma(s) = \phi(s)$ and for $s \in S'$ $\widehat{\phi}_\sigma(s) = \sigma^{-1}\psi(s)\sigma$, and extend this to a homomorphism from $F_{S \cup S'}$ (which agrees with $\phi$ on $F_S$ and with $\psi$ on $F_{S'}$).

**Proposition 2.3.** For arbitrary $0 < \delta$ and $k \in \mathbb{N}$, we can choose $\varepsilon$ and $n_0$ such that if $n > n_0$ and $\phi : F_S \to S_n$ and $\psi : F_{S'} \to S_n$ are as above, then the following holds.

There exists $\sigma \in S_n$ such that $\widehat{\phi}_\sigma$ is a $(k, \delta)$-quasi-action of $G \ast H$.

**Proof.** Let $\hat{f}$ denote the factor map from $F_{S \cup S'}$ to $G \ast H$ and let us define the set of good points in $[n]$ for a word $w \in F_{S \cup S'}$:

$$A_{\sigma,w} = \{x \in [n] : x\widehat{\phi}_\sigma(w) = x \text{ if and only if } \hat{f}(w) = 1\}.$$ 

The set good points for all words of length at most $k$ is denoted by

$$A_\sigma = \bigcap_{l(w) \leq k} A_{\sigma,w}.$$
Our goal is to choose the parameters such that $|A_\sigma| > (1 - \delta)n$.

We choose $\sigma$ from $S_n$ uniformly randomly. If the above holds in expected value, i.e. $\mathbb{E}(|A_\sigma|) > (1 - \delta)n$, then we can find a suitable $\sigma$. An estimate on the expected value is

$$\mathbb{E}(|A_\sigma|) = n - \mathbb{E}([n] \setminus A_\sigma) \geq n - \sum_{l(w) \leq k} \mathbb{E}([n] \setminus A_{\sigma,w}) = n - \sum_{l(w) \leq k} (n - \mathbb{E}(|A_{\sigma,w}|)).$$

This means that we have to find a lower bound for $\mathbb{E}(|A_{\sigma,w}|)$. For the identity element $A_{\sigma,1} = [n]$ for every $\sigma$, so any lower bound works here. Let us fix $w \in F_{S_\sigma S'}$, where $w \neq 1$ and has length at most $k$. The word $w$ has the following form: $w = g_1h_1g_2h_2 \ldots g_lh_l$, where $g_j \in F_S$ and $h_j \in F_{S'}$ for each $j$, and for $j \neq 1$ $g_j \neq 1$ and $j \neq l$ $h_j \neq 1$.

There are two cases: The first one is that no $f(g_i)$ or $f'(h_i)$ equals the identity element of $G$ or $H$ (except when $g_1$ or $h_l$ is the empty word in $F_{S_\sigma S'}$). The second is that this doesn’t hold, which means that $w$ is not one of the shortest words in $\hat{f}^{-1}(\hat{f}(w))$.

1. In this case $A_{\sigma,w}$ is the set of non-fixpoints of $\hat{\phi}_\sigma(w)$. Assume that $g_1 \neq 1$ and $h_l \neq 1$.

$$\hat{\phi}_\sigma(w) = \hat{\phi}_\sigma(g_1h_1g_2h_2 \ldots g_lh_l) = \phi(g_1)\sigma^{-1}\psi(h_1)\sigma\phi(g_2)\sigma^{-1}\psi(h_2)\sigma \ldots \sigma^{-1}\psi(h_l)\sigma$$

We call the sequence of points $x$, $x\phi(g_1)$, $x\phi(g_1)\sigma^{-1}$, $x\phi(g_1)\sigma^{-1}\psi(h_1)$, $x\phi(g_1)^{-1}\sigma\psi(h_1)\sigma$, $x\phi(g_1)^{-1}\psi(h_1)\sigma\phi(g_2)$, $\ldots$, $x\phi(g_1)^{-1}\psi(h_l)\sigma$ the trajectory of $x$. Let

$$B_{\sigma,w} = \{ x \in [n] : \text{the trajectory of } x \text{ consists of different points in } [n] \}.$$

Clearly $B_{\sigma,w} \subseteq A_{\sigma,w}$, so $\mathbb{E}(|A_{\sigma,w}|) \geq \mathbb{E}(|B_{\sigma,w}|)$.

$$\mathbb{E}(|B_{\sigma,w}|) = \sum_{x \in [n]} \mathbb{P}(x \in B_{\sigma,w})$$

If no $f(g_i)$ or $f'(h_i)$ is the identity element of $G$ or $H$, then clearly no $\phi(g_i)$ or $\psi(h_i)$ is the identity in $S_n$, because $\phi$ and $\psi$ are $(k, \varepsilon)$-quasi-actions and the length of $g_i$ and $h_i$ is at most $k$. So $B_{\sigma,w} \neq \emptyset$. We will find a lower bound for $\mathbb{P}(x \in B_{\sigma,w})$.

If $x$ is a fixpoint of $\phi(g_1) = \hat{\phi}_\sigma(g_1)$ then this probability is zero, but this happens at most in $\varepsilon n$ cases, because $\phi$ is a $(k, \varepsilon)$-quasi-action. If $x\phi(g_1) = x_1 \neq x$, then let $\phi(g_i) = \alpha_{2i-1}$ and $\psi(h_i) = \alpha_{2i}$, so we can write

$$\hat{\phi}_\sigma(w) = \alpha_1\sigma^{-1}\alpha_2\sigma\alpha_3\sigma^{-1}\alpha_4 \ldots \alpha_{2l}\sigma.$$
Let \( x_0 = x \), and for \( 1 \leq t \leq l \) let 
\[
x_{4t-3} = x_{4t-3} \alpha_1^{-1} \ldots \alpha_{2t-1}^{-1}, \quad x_{4t-1} = x_{4t-1} \alpha_1^{-1} \ldots \alpha_{2t}^{-1}, \quad x_{4t-2} = x_{4t-2} \alpha_1 \ldots \alpha_{2t-1}^{-1}, \quad x_{4t} = x_{4t} \alpha_1 \ldots \alpha_{2t} \sigma.
\]
This sequence is the trajectory of \( x \). Let \( E_m \) denote the event that \( x_0, x_1, \ldots, x_{2m-1} \) are all different.

We estimate \( \mathbb{P}(E_{m+1}|E_m) \) as follows. Suppose \( E_m \) happens, let \( \sigma' \) denote \( \sigma \) if \( m \) is even and \( \sigma^{-1} \) if \( m \) is odd (since the distribution of \( \sigma^{-1} \) is the same as of \( \sigma \), the calculations are the same in both cases). Then 
\[
x_{2m} = x_{2m-1} \sigma' \quad \text{and} \quad x_{2m+1} = x_{2m-1} \sigma' \alpha_{m+1}.
\]
We choose \( \sigma' \) uniformly randomly, so the images of any tuple of different points are independent. In our case so far we know the image of \( m - 1 \) points, which implies that \( x_{2m-1} \sigma' \) is uniformly distributed among the other \( n - (m - 1) \) points. However, there are some bad choices when \( E_{m+1} \) doesn’t hold.

1. If \( x_{2m} = x_i \) for \( 0 \leq i \leq 2m - 1 \).
2. When \( x_{2m} \) is a fixpoint of \( \alpha_{m+1} \), hence \( x_{2m+1} = x_{2m} \). There are at most \( \varepsilon n \) such points. (Because \( \alpha_{m+1} \neq \text{id} \).)
3. If \( x_{2m+1} = x_i \) for \( 0 \leq i \leq 2m - 1 \), so \( x_i \alpha_m^{-1} \) are also bad points.

Summing these up we get that there are at most \( 4m + \varepsilon n \) bad points. In the case when \( m + 1 = 2l \) there are no bad points of the second and third kind, but if we subtract those as well, we still get a lower bound. So the probability we are interested in is
\[
\mathbb{P}(E_{m+1}|E_m) \geq \frac{n - (m - 1) - (4m + \varepsilon n)}{n - (m - 1)} \geq \frac{n - 5m + \varepsilon n}{n} \geq \frac{(1 - \varepsilon)n - 10l}{n}.
\]
The last inequality holds because \( m \leq 2l \).

Since \( E_{m+1} \subseteq E_m \),
\[
\mathbb{P}(E_{m+1}|E_m) = \frac{\mathbb{P}(E_{m+1})}{\mathbb{P}(E_m)}.
\]
Using this we get a lower bound on \( E_{2l} \):
\[
\mathbb{P}(E_{2l}) = \mathbb{P}(E_1) \prod_{i=1}^{2l-1} \mathbb{P}(E_{i+1}|E_i) \geq (1 - \varepsilon) \prod_{i=1}^{2l-1} \frac{(1 - \varepsilon)n - 10l}{n} =
\]
\[
= (1 - \varepsilon) \left( \frac{(1 - \varepsilon)n - 10l}{n} \right)^{2l-1},
\]
where we used the fact that \( E_1 \) is the event that \( x \neq x_1 \). But \( E_{2l} \) is the event that the trajectory of \( x \) consists of different points, so
\[
\mathbb{P}(x \in B_{\sigma,w}) \geq (1 - \varepsilon) \left( \frac{(1 - \varepsilon)n - 10l}{n} \right)^{2l-1} \geq (1 - \varepsilon) \left( \frac{(1 - \varepsilon)n - 5k - 10}{n} \right)^{k+1}.
\]
The last inequality holds because the length of \( w \) is at most \( k \) so \( k \geq 2l - 2 \).

So in this case we have

\[
\mathbb{E}(|A_{\sigma,w}|) \geq \mathbb{E}(|B_{\sigma,w}|) \sum_{x \in [n]} \mathbb{P}(x \in B_{\sigma,w}) \geq n(1 - \varepsilon) \left( \frac{(1 - \varepsilon)n - 5k - 10}{n} \right)^{k+1}.
\]

In the case when \( g_1 = 1 \) or \( h_l = 1 \), the calculations work similarly and we get slightly higher lower bounds for the probability.

2. Suppose \( w \) is of the second type. If for some \( i \), \( f(g_i) = 1 \) in \( G \), then \( w' = g_1h_1 \ldots h_{i-1}h_i g_{i+1} \ldots g_l h_l \), a shorter word in \( F_{S \cup S'} \), goes to the same element of \( G * H \). We can say even more: \( \phi \) is a quasi-action and the length of \( g_i \) is at most \( k \), \( f(g_i) = 1 \), so \( \phi(g_i) = \text{id} \in S_\sigma \). Hence \( \hat{\phi}_\sigma(w') = \hat{\phi}_\sigma(w) \). This is also true when \( f'(h_i) = 1 \) in \( H \) for some \( i \), we can delete \( h_i \) as well.

Iterate this step until we get a word \( \hat{w} \) which has the form \((\hat{g}_1)\hat{h}_1\hat{g}_2 \ldots \hat{g}_l(\hat{h}_l)\), where none of the \( \phi(\hat{g}_i) 's and \psi(\hat{h}_i) 's are equal to the identity. We can get here in finitely many steps since the length of the word decreases in each step.

In other words, this \( \hat{w} \) belongs to the first case. So we already have a lower bound for \( \mathbb{E}(|A_{\sigma,\hat{w}}|) \). As we saw, these steps do not change the \( \hat{\phi}_\sigma \)-image of our word, so \( \hat{\phi}_\sigma(w) = \hat{\phi}_\sigma(\hat{w}) \). This means that the same lower bound works for \( \mathbb{E}(|A_{\sigma,w}|) \) too.

We showed that the above holds for each \( w \) of length at most \( k \). From this we get

\[
\mathbb{E}(|A_\sigma|) \geq n - \sum_{l(w) \leq k} (n - \mathbb{E}(|A_{\sigma,w}|)) \geq n - \sum_{l(w) \leq k} (n - n(1 - \varepsilon) \left( \frac{(1 - \varepsilon)n - 5k - 10}{n} \right)^{k+1}) 
\]

\[
\mathbb{E}(|A_\sigma|) \geq n \left( 1 - \sum_{l(w) \leq k} \left( 1 - (1 - \varepsilon) \left( \frac{(1 - \varepsilon)n - 5k - 10}{n} \right)^{k+1} \right) \right).
\]

Let \( r \) denote the number of words in \( F_{S \cup S'} \) that have length at most \( k \). We could calculate precisely but all that matters is that \( r \) is some finite number, that depends only on \( |S \cup S'| \) and \( k \). Let \( \varepsilon < \delta/r \).

Let

\[
(1 - \varepsilon) \left( \frac{(1 - \varepsilon)n - 5k - 10}{n} \right)^{k+1} = a_{n,\varepsilon}.
\]

As \( n \) goes to infinity, \( a_{n,\varepsilon} \to (1 - \varepsilon)^{k+2} \). Choose \( \varepsilon \) such that \( (1 - \varepsilon)^{k+2} > 1 - \varepsilon'/2 \).

This way there exists \( n_0 \) such that if \( n > n_0 \), then \( a_{n,\varepsilon} > 1 - \varepsilon' \). So for \( n > n_0 \) we
have
\[ E(|A_\sigma|) \geq n \left( 1 - \sum_{l(w) \leq k} (1 - a_{n,\varepsilon}) \right) > n \left( 1 - \sum_{l(w) \leq k} \varepsilon' \right) = n(1 - r\varepsilon') > n(1 - \delta). \]

The proof of the proposition is complete. \(\square\)

**Theorem 2.4.** If \(G\) and \(H\) are sofic groups then their free product \(G \ast H\) is also a sofic group.

**Proof.** First suppose that \(G\) and \(H\) are finitely generated.

Fix some \(\delta > 0\) and \(k \in \mathbb{N}\). We want to find a \((k, \delta)\)-quasi-action of \(G \ast H\) using the method described in the previous proposition. According to Proposition 2.3 we can choose \(\varepsilon > 0\) such that for each large enough \(n\) there exists a good \(\sigma \in S_n\). The only things missing are a \((k, \varepsilon)\)-quasi-action \(\phi\) of \(G\) and a \((k, \varepsilon)\)-quasi-action \(\psi\) of \(H\) on \(n\) points for some \(n > n_0\).

Since \(G\) and \(H\) are sofic, there are \((k, \varepsilon)\)-quasi-actions for them on \(n_1\) and \(n_2\) points respectively.

Notice that if we have a quasi-action of a group on \(n\) points, then we have one on any multiple of \(n\): if we have \(mn\) points, divide it to \(m\) classes of \(n\) elements, and we use the original action on every class.

So there are \((k, \varepsilon)\)-quasi-actions of \(G\) and \(H\) on \(n_1n_2m\) points for every \(m \in \mathbb{N}\). Choose \(m\) such that \(n = n_1n_2m > n_0\). For this \(n\) there exist the needed quasi-actions so we can find a good \(\sigma\) and a \((k, \delta)\)-quasi-action of \(G \ast H\) on \(n\) points.

If \(G\) and \(H\) are not necessarily finitely generated, then the free product \(G \ast H\) is the direct limit of \(\{G_\alpha \ast H_\alpha\}\), where \(G_\alpha\) and \(H_\alpha\) are finitely generated groups. Since the direct limit of sofic groups is sofic, \(G \ast H\) is a sofic group. \(\square\)

### 2.2 Extensions

**Proposition 2.5.** Let \(N\) be a sofic group, and let \(N \triangleleft G\) such that \(G/N\) is amenable. Then \(G\) is sofic.

**Remark 2.6.** Before we begin the proof, let us get a picture of what we would like to do. Consider the finitely generated case, when \(N = \langle X \rangle\), \(G = \langle X \cup Y \rangle\) such that \(G/N = \langle \pi(Y) \rangle\) where \(\pi : G \to G/N\) is the factor map. Fix the numbers \(r \in \mathbb{N}\) and \(\delta > 0\).

Because of the soficity of \(N\) we can choose an approximation of \(\text{Cay}(N, X)\), call it \(B\). Since \(G/N\) is amenable, we can find a subgraph \(A \subseteq \text{Cay}(G/N, \pi(Y))\) with small boundary.

We construct a graph in the following way:
The vertex set equals $V(A) \times V(B)$.

Into each copy of $V(B)$ we can draw the graph $B$, these will be the $X$-colored edges.

Now let $\{a_1\} \times B$ and $\{a_2\} \times B$ be two copies of this $B$ such that $a_1a_2$ is a $y$-labeled edge for some $y \in Y$. First draw an $y$-edge between $(a_1, b)$ and $(a_2, b)$ for some $b \in B$. This one edge determines the other $y$-colored edges between $\{a_1\} \times B$ and $\{a_2\} \times B$. Denote by $b \cdot g$ the endpoint of the path corresponding to $g \in N$ starting from $b$. Then draw a $y$-edge between $(a_1, b \cdot g)$ and $(a_2, b \cdot y^{-1}g)$.

The previously described strategy works for the edges of a spanning tree of $A$. After that if we take an edge $a_1a_2$ then this gives us a cycle. This means that we have to be careful when drawing the first edge between $\{a_1\} \times B$ and $\{a_2\} \times B$, because that cycle in $\text{Cay}(G/N, \pi(Y))$ might not be a cycle in $\text{Cay}(G, X \cup Y)$. However, it does us a well-defined element in $N$. This determines where we should draw the first edge, and after this, the procedure is the same.

This method gives us a finite graph edge-labeled by $X \cup Y$. It turns out that we can choose these approximations to be so close to the corresponding Cayley graphs, so that this final graph that we constructed is an $(r, \delta)$-approximation.

What we described was the main idea of the proof, but in this ‘graph-picture’ the calculations would be much more complicated. So we present a combinatorial proof.

**Proof of Proposition 2.5.** Let $F \subseteq G$ be a finite set and $\varepsilon \in (0, 1)$. We aim to find an $(F, \varepsilon)$-quasi-action of $G$.

Denote by $\pi : G \to G/N$ the factor map, and choose a section of this homomorphism: $\sigma : G/N \to G$, such that $\pi(\sigma(h)) = h$ for each $h \in G/N$. This is equivalent to saying that $g\sigma(\pi(g))^{-1} \in N$ for all $g \in G$.

By the amenability of $G/N$ we can find $\bar{A} \subseteq G/N$ such that for each $h \in \pi(F)$ we have

$$|\bar{A}h \setminus \bar{A}| < \frac{\varepsilon}{3}|\bar{A}|.$$ 

Let $A = \sigma(\bar{A})$ (so $\pi(A) = \bar{A}$) and $H = (A \cdot F \cdot A^{-1}) \cap N \subseteq N$. Clearly $|H| < \infty$ so we can find an $(H, \varepsilon/3)$-quasi-action of $N$, $\psi : N \to B$.

Now we define a map $\phi : G \to \text{Map}(A \times B)$ as follows. Note that this is the point where we use the idea described before the proof,

$$(a, b) \cdot \phi(g) = \begin{cases} \left(\sigma(\pi(ag)), b \cdot \psi(ag\sigma(\pi(ag))^{-1})\right), & \text{if } \pi(ag) \in \pi(A) \\ (a, b) & \text{otherwise.} \end{cases}$$
This definition makes sense because \( ag\sigma(\pi(ag))^{-1} \in N \). We will prove that this \( \phi \) is an \((F, \varepsilon)\)-quasi-action of \( G \) on the set \( A \times B \). Let us check conditions (a)-(c) of Definition 1.1.

(b) If \( g = 1 \) then \( \pi(a1) \in \bar{A} \), so

\[
(a, b) \cdot \phi(1) = (\sigma(\pi(a)), b \cdot \psi(a\sigma(\pi(a))^{-1})) = (a, b \cdot \psi(1)),
\]

here \( \psi(1) \) is \( \varepsilon/3 \)-similar to the identity map of \( B \), hence \( \phi(1) \) is \( \varepsilon/3 \)-similar to the identity map of \( A \times B \).

(c) Let \( e \in F \setminus \{1\} \). For this \( e \) there are at most \( \varepsilon|A|/3 \) elements \( a \in A \) such that \( \pi(\epsilon a) = \pi(a)\pi(e) \notin \bar{A} \).

If \( e \notin N \) then \( \pi(e) \neq 1 \) in \( N \). Suppose that \( \pi(\epsilon a) = \pi(a)\pi(e) \in \bar{A} \) for some \( a \in A \). Then obviously \( \pi(\epsilon a) \neq \pi(a) \), hence \( \sigma(\pi(\epsilon a)) \neq \sigma(\pi(a)) = a \). So in this case, for any \( b \in B \), \( (a, b) \) cannot be a fixpoint of \( \phi(e) \), i.e., \( \phi(e) \) is \((1-\varepsilon/3)\)-different from the identity map of \( A \times B \).

Now consider the case when \( e \in N \). Then \( \pi(e) = 1 \). So \( \pi(\epsilon a) = \pi(a) \in \bar{A} \) for every \( a \in A \), and

\[
(a, b) \cdot \phi(e) = (\sigma(\pi(a)), b \cdot \psi(ae\sigma(\pi(\epsilon a))^{-1})) = (a, b \cdot \psi(ae^{-1})).
\]

Here \( aea^{-1} \in H \) and \( aea^{-1} \neq 1 \), this means that \( \psi(aea^{-1}) \) is \((1-\varepsilon/3)\)-different from the identity map of \( B \), hence \( \phi(e) \) is \((1-\varepsilon/3)\)-different from the identity map of \( A \times B \).

(a) Let \( e, f \in F \). As before, there are at most \( \varepsilon|A|/3 \) elements of \( A \) where \( \pi(\epsilon a) \notin \bar{A} \), and the same amount for \( f \). This means that altogether we have at most \( 2\varepsilon|A|/3 \) ‘wrong’ elements in \( A \).

Now consider those \( a \in A \) when \( \pi(\epsilon a) \in \bar{A} \) and \( \pi(\epsilon f) \in \bar{A} \). In this case

\[
(a, b) \cdot \phi(e) \cdot \phi(f) = (\sigma(\pi(\epsilon a)), b \cdot \psi(ae\sigma(\pi(\epsilon a))^{-1})) \cdot \phi(f) =
\]

\[
= (\sigma(\sigma(\pi(\epsilon a)f)), b \cdot \psi(ae\sigma(\pi(\epsilon a))^{-1}) \cdot \psi(\sigma(\pi(\epsilon a)f)\sigma(\pi(\epsilon a)f))^{-1})).
\]

Here

\[
\sigma(\pi(\pi(\epsilon a)f)) = \sigma(\pi(\pi(\epsilon a)))\pi(f) =
\]

\[
= \sigma(\pi(\epsilon a)\pi(f)) = \sigma(\pi(\epsilon af)),
\]

So using the notations \( g_1 = ae\sigma(\pi(\epsilon a))^{-1} \) and \( g_2 = \sigma(\pi(\epsilon a))f\sigma(\pi(\epsilon af))^{-1} \), we get

\[
(a, b) \cdot \phi(e) \cdot \phi(f) =
\]
\[
(\sigma(\pi(aef)), b \cdot \psi(aef) \sigma(\pi(aef))^{-1}) \cdot \psi(\sigma(\pi(aef))f \sigma(\pi(aef))^{-1})) = \\
(\sigma(\pi(aef)), b \cdot \psi(g_1) \cdot \psi(g_2)).
\]

On the other hand, we have
\[
(a,b) \cdot \phi(ef) = (\sigma(\pi(aef)), b \cdot \psi(aef) \sigma(\pi(aef))^{-1})) = (\sigma(\pi(aef)), b \cdot \psi(g_1g_2)).
\]

As we can see, the first coordinate is the same, and the second coordinates are equal too if \(b \cdot \psi(g_1) \cdot \psi(g_2) = b \cdot \psi(g_1g_2).\) Since \(g_1, g_2 \in H\) and \(\psi\) is an \((H, \varepsilon/3)\)-quasi-action, this is true for at least \((1 - \varepsilon/3)|B|\) values of \(b.\)

We got that there are at most \(2\varepsilon|A|/3\) wrong elements in \(A\) and at most \(\varepsilon|B|/3\) wrong elements in \(B\) for each good \(a \in A.\) For all the other pairs \((a, b)\) we have
\[
(a,b) \cdot \phi(e) \cdot \phi(f) = (a,b) \cdot \phi(ef).
\]

So \(\phi(e)\phi(f)\) is \(\varepsilon\)-similar to \(\phi(ef).\)

We proved that \(\phi\) is an \((F, \varepsilon)\)-quasi-action, hence \(G\) is a sofic group. \(\square\)

### 2.3 Amalgamated products over an amenable subgroup

**Definition 2.7.** Let \(G\) be a finitely generated group with a symmetric generating set \(S.\) Let \(B\) be a finite graph, such that each directed edge of \(B\) is labeled by an element of \(S.\) We say that \(B\) is an \(r\)-approximation of the Cayley graph of \(G, Cay(G, S)\) if there exists a subset \(W \subseteq V(B)\) such that

- \(|W| > (1 - \frac{1}{r}) |V(B)|\) and
- if \(p \in W\) then the \(r\)-neighborhood of \(p\) is rooted isomorphic to the \(r\)-neighborhood of a vertex of the Cayley-graph (as an edge labeled graph).

In other words, \(B\) is an \(r\)-approximation if it is an \((r, \frac{1}{r})\)-approximation of the Cayley graph.

**Proposition 2.8.** The group \(G\) is sofic if and only if for any \(r \geq 1\) there exists an \(r\)-approximation of \(Cay(G, S)\) by a finite graph.

**Proof.** If \(G\) is sofic, then using Proposition \[1.3\] for \(\delta = \frac{1}{r}\) and \(r\) provides us such a finite graph.

For the other direction choose \(R \geq \max\{r, \frac{1}{3}\},\) this way an \(R\)-approximation is good for \(r\) and \(\delta\) in Proposition \[1.3\] proving that the group \(G\) is sofic. \(\square\)
Definition 2.9. Let $E(A)$ denote the set of edges in a colored graph $A$. We say that two colored graphs $A$ and $B$ are $r$-isomorphic for some $r > 0$ if there are subgraphs $A' \subseteq A$ and $B' \subseteq B$ such that

$$|E(A')| \geq \left(1 - \frac{1}{r}\right)|E(A)|, \quad |E(B')| \geq \left(1 - \frac{1}{r}\right)|E(B)|$$

and $A'$ is isomorphic to $B'$ (as colored graphs). The isomorphism between $A'$ and $B'$ is called an $r$-isomorphism.

Lemma 2.10. Let $G$ be a sofic group, and $S$ a finite, symmetric generating set of $G$. Then for each integer $r > 0$ there exists an integer $R_{\text{app}}(r)$ which has the following property. Whenever a finite graph $B$ is an $R_{\text{app}}(r)$-approximation of $\text{Cay}(G,S)$ and $A$ is $R_{\text{app}}(r)$-isomorphic to $B$, then $A$ is an $r$-approximation of $\text{Cay}(G,S)$.

Proof. Suppose that for some $R > 0$ $A$ and $B$ are $R$-isomorphic and $B$ is an $R$-approximation of the Cayley graph. Then there are $B' \subseteq B$ and $A' \subseteq A$ subgraphs such that $B' \cong A'$ and $|B'| > \left(1 - \frac{1}{R}\right)|B|$, $|A'| > \left(1 - \frac{1}{R}\right)|A|$. There is also $\hat{B} \subseteq B$ which is small: $|\hat{B}| < \frac{1}{R}|B|$, and each point which is in $B \setminus \hat{B}$ has a good $R$-neighborhood in $B$.

As before, for a subgraph $X$ let $N_r(X)$ denote the $r$-neighborhood of $X$. Let

$$A'' = A \setminus N_r(A \setminus A') \subset A'.$$

Let $d = |S|$, so our graphs are $d$-regular. Since $|A \setminus A'| < |A|/R$, clearly $|N_r(A \setminus A')| < d^r|A|/R$. So

$$|A''| > \left(1 - \frac{d^r}{R}\right)|A|.$$

Let $\psi : B' \to A'$ be an isomorphism. Let $\hat{A} = \psi(\hat{B} \cap B')$, the $\psi$-image of the bad points in $B$. Consider the subgraph $A'' \setminus \hat{A}$. Its $r$-neighborhood is still isomorphic to a subgraph of $B$ and its $\psi^{-1}$-image in $B$ is disjoint from $\hat{B}$. This means that the $r$-ball around any point of $A'' \setminus \hat{A}$ is rooted isomorphic to an $r$-ball in the Cayley graph.

We also need a lower bound for the size of this subgraph. Clearly

$$|\hat{A}| \leq |\hat{B}| < \frac{1}{R}|B| < \frac{1}{R}\left(1 - \frac{1}{R}\right) = \frac{1}{R-1}|B'| = \frac{1}{R-1}|A'| \leq \frac{1}{R-1}|A|.$$

Hence

$$|A'' \setminus \hat{A}| > \left(1 - \frac{d^r}{R} - \frac{1}{R-1}\right)|A|.$$

We can choose $R = R_{\text{app}}(r)$ such that

$$\frac{d^r}{R} + \frac{1}{R} < \frac{1}{r},$$

so the proof of the lemma is complete.
Definition 2.11. Let $\mathcal{T} = (T_1, T_2, \ldots, T_m)$ be a finite sequence of colored graphs. Their linear combination with coefficient vector $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, denoted by $\alpha \cdot \mathcal{T}$, is the disjoint union of $\alpha_1$ copies of $T_1$, $\alpha_2$ copies of $T_2$, $\ldots$, $\alpha_m$ copies of $T_m$. As a special case, if $m = 1$ then we talk about the integer multiples of $T_1$.

Remark 2.12. Let $\Gamma$ be a finitely generated amenable group, $S \subseteq \Gamma$ a finite symmetric generating set. Then there exists a Følner sequence in the Cayley graph $\text{Cay}(\Gamma, S)$, let us denote it by $\mathcal{F} = \{F_1, F_2, \ldots\}$. It is easy to see that for all $r > 0$, all but finitely many of the $F_n$ are $r$-approximations of $G$.

2.3.1 Quasi-tilings

Let us recall the notion of quasi-tilings from [9]. Let $X$ be a finite set and $\{A_i\}_{i=1}^n$ be finite subsets of $X$. We say that $\{A_1, A_2, \ldots, A_n\}$ are $\varepsilon$-disjoint if there exist subsets $\bar{A}_i \subseteq A_i$ such that

- For any $1 \leq i \leq n$, $|\bar{A}_i|/|A_i| \geq 1 - \varepsilon$.
- If $i \neq j$ then $\bar{A}_i \cap \bar{A}_j = \emptyset$.

On the other hand, if $\{H_j\}_{j=1}^m$ are finite subsets of $X$, we say that they $\alpha$-cover $X$ if

$$\frac{|\bigcup_{j=1}^m H_j|}{|X|} \geq \alpha.$$ 

Definition 2.13. Let $\Gamma$ be a finitely generated amenable group with a symmetric generating set $S$ and let $1 \in F_1 \subseteq F_2 \subseteq \ldots$, $\cup_{n=1}^\infty F_n = \Gamma$ be a Følner-exhaustion. Let $B_r(x)$ denote the $r$-neighborhood of the vertex $x$ in a graph. Let $B$ be a finite graph edge-labeled by the elements of $S$, and let $L$ be a natural number. Let

$$Q^B_L = \{x \in B : B_L(x) \simeq B_L(1) \subset \text{Cay}(\Gamma, S) \text{ as edge-labeled graphs}\}.$$ 

Let $\{F_{n_1}, F_{n_2}, \ldots, F_{n_s}\}$ be a finite collection of the Følner sets above such that for any $1 \leq i \leq s$, $F_{n_i} \subset B_{L/2}(1)$. Then for any $x \in Q^B_L$ and $1 \leq i \leq s$, $T_x(F_{n_i})$ is the image of $F_{n_i}$ under the unique colored isomorphism $B_L(1) \rightarrow B_L(x)$ mapping 1 to $x$. We call such a subset a tile of type $F_{n_i}$ and say that $x$ is the center of $T_x(F_{n_i})$. A system of tiles $\varepsilon$-quasi-tile $V(B)$ if they are $\varepsilon$-disjoint and form an $(1 - \varepsilon)$-cover.

Proposition 2.14 (Theorem 2 from [3]). For any $\varepsilon > 0$, $n > 0$, $\delta > 0$ and a finite collection $\{F_{n_1}, F_{n_2}, \ldots, F_{n_s}\} \subseteq B_{L}(1)$ of Følner sets such that $n_i > n$ and if

$$\frac{|Q^B_L|}{|V(B)|} > 1 - \delta$$

then $V(B)$ can be $\varepsilon$-quasi-tiled by tiles of the form $T_x(F_{n_i})$, $x \in Q^B_L$, $1 \leq i \leq s$. 
Definition 2.15. If $\mathcal{T} = \{F_1, \ldots, F_m\}$ is a sequence of Følner sets and $\alpha \in \mathbb{N}^m$ a nonzero lattice vector, then we say that a finite graph can be $\varepsilon$-quasi-tiled by $\alpha \cdot \mathcal{T}$ if it can be $\varepsilon$-quasi-tiled by the union of $\alpha_1$ tiles of type $T_1$, $\alpha_2$ tiles of type $T_2$, $\ldots$, $\alpha_m$ tiles of type $T_m$.

Suppose there exists some $\beta \in \mathbb{N}^m$ such that $\beta_i \leq \alpha_i$ for every $1 \leq i \leq m$, $|V(\beta \cdot \mathcal{T})| < \varepsilon|V(\alpha \cdot \mathcal{T})|$ and a finite graph $B$ can be $\varepsilon$-quasi-tiled by $(\alpha - \beta) \cdot \mathcal{T}$. In this case we say that $\alpha \cdot \mathcal{T}$ almost $\varepsilon$-quasi-tiles $B$.

Proposition 2.16. If a finite graph $B$ is almost $1/2r$-quasi-tiled by $\{A_1, A_2, \ldots, A_n\}$, then $B$ is $r$-isomorphic to the disjoint union $\bigcup_{i=1}^n A_i$.

Proof. We can assume that for $m \leq n \{A_1, \ldots, A_m\}$ $\frac{1}{2r}$-quasi-tiles $B$, and

$$\sum_{i=1}^m |A_i| > \left(1 - \frac{1}{2r}\right)\sum_{i=1}^n |A_i|.$$

Let $\bar{A}_i \subseteq A_i$ for $1 \leq i \leq m$ such that $|\bar{A}_i| > (1 - 1/2r)|A_i|$ and $\bar{A}_i \cap \bar{A}_j = \emptyset$ if $i \neq j$. So we have a map $\varphi : \bigcup_{i=1}^m \bar{A}_i \to B$ that is injective. Here

$$\sum_{i=1}^m |\bar{A}_i| > \left(1 - \frac{1}{2r}\right)\sum_{i=1}^m |A_i| > \left(1 - \frac{1}{2r}\right)^2 \sum_{i=1}^n |A_i| > \left(1 - \frac{1}{r}\right)\sum_{i=1}^n |A_i|,$$

and

$$\sum_{i=1}^m |\bar{A}_i| > \left(1 - \frac{1}{2r}\right)\sum_{i=1}^m |A_i| > \left(1 - \frac{1}{2r}\right)^2 |B| > \left(1 - \frac{1}{r}\right)|B|.$$ 

So they are indeed $r$-isomorphic. \qed

The following is a variation of Lemma 2.5 from [6].

Definition 2.17. For a vector $0 \neq \alpha \in \mathbb{R}^m$ we define the unit vector $\sigma(\alpha) = \alpha/\|\alpha\|$ where $\|\alpha\|$ denotes the (usual) length of $\alpha$. Also, for an arbitrary $\alpha = (\alpha_1, \ldots, \alpha_m)$ let $[\alpha] = ([\alpha_1], \ldots, [\alpha_m]) \in \mathbb{Z}^m$.

Lemma 2.18. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ be a finite sequence of colored (non-empty) graphs. For each $\varepsilon \in (0, 1)$ there is an integer $M(\varepsilon) > 0$ (also depending on $\mathcal{T}$) such that whenever $\alpha, \beta \in \mathbb{N}^m$ are nonzero lattice vectors with $\|\beta\| \geq M(\varepsilon)\|\alpha\|$ and $\|\sigma(\alpha) - \sigma(\beta)\| \leq 1/M(\varepsilon)$ then the graph $\beta \cdot \mathcal{T}$ can be almost $\varepsilon$-quasi-tiled by $(t\alpha) \cdot \mathcal{T}$ for some integer $t$.

Proof. Let $v$ be the maximum number of vertices in $T_i$, and let $t$ be the largest integer such that $\|t\alpha\| \leq \|\beta\|$. Clearly $M(\varepsilon) \leq t$ and $\|\alpha\| \geq 1$, hence

$$\|\beta - t\alpha\| \leq \|\alpha\| + t\|\sigma(\beta) - \sigma(\alpha)\| \leq \|\alpha\| + t\frac{1}{M(\varepsilon)} \leq \frac{2}{M(\varepsilon)}\|\alpha\|t,$$
Our problem is that these \( t \)’s. The plan is to omit those which intersect each other and prove that we still have enough \( B^i \)'s. Since our graph is \( d \)-regular, if \( D \) is a subgraph then \( |N_\epsilon(D)| \leq d\epsilon |D| \). Now let

\[
A'_i = A_i \setminus N_\epsilon(A_i \setminus \bar{A}_i).
\]

Moreover, suppose that for \( 1 \leq i \leq n \), \( A_i \) is almost \( \varepsilon_1 \)-quasi-tiled by \( \{ B^i_1, B^i_2, \ldots, B^i_{k_i} \} \) where \( B^i_j \subset A_i \), that is, we can omit some of them such that we get an \( \varepsilon_1 \)-quasi-tiling.

Let \( B = \{ B^j_i : 1 \leq i \leq n, 1 \leq j \leq k_i \} \) and \( v = \max\{|B^j_i| : 1 \leq i \leq n, 1 \leq j \leq k_i\} \).

Suppose that \( d^\epsilon \varepsilon_2 + \varepsilon_2 + 4\varepsilon_1 < 1 \).

In this case \( B \) almost \((d^\epsilon \varepsilon_2 + \varepsilon_2 + 4\varepsilon_1)\)-quasi-tiles \( B \).

Proof. We can assume that \( A_i \) is \( \varepsilon_1 \)-quasi-tiled by \( \{ B^i_1, B^i_2, \ldots, B^i_{l_i} \} \) for some \( l_i \leq k_i \).

Let us keep in mind that this implies \( \sum_{j=1}^{l_i} |B^j_i| > (1 - \varepsilon_1) \sum_{j=1}^{k_i} |B^j_i| \).

According to the definition of \( \varepsilon \)-disjointness, there are \( \bar{A}_i \subset A_i \) and \( \bar{B}^j_i \subset B^j_i \) subgraphs such that

- \( |\bar{A}_i| > (1 - \varepsilon_2)|A_i| \) for \( 1 \leq i \leq n \);

- \( \bar{A}_i \cap \bar{A}_{i'} = \emptyset \) if \( i \neq i' \);

- \( |\bar{B}^j_i| > (1 - \varepsilon_1)|B^j_i| \) for \( 1 \leq i \leq n, 1 \leq j \leq l_i \);

- \( \bar{B}^j_i \cap \bar{B}^{j'}_i = \emptyset \) if \( j \neq j' \).

Our problem is that these \( \bar{B}^j_i \)'s are not disjoint for different \( i \)'s. The plan is to omit those which intersect each other and prove that we still have enough \( B^i_j \)'s.
Clearly $A'_i \subseteq A_i$. The following calculations come from the previous observations:
\[
|A_i \setminus \bar{A}_i| < \varepsilon_2 |A_i|; \\
|N_v(A_i \setminus \bar{A}_i)| < d^v \varepsilon_2 |A_i|; \\
|A'_i| > |A_i| - d^v \varepsilon_2 |A_i| = (1 - d^v \varepsilon_2) |A_i|.
\]
It is also clear that if for some $j$, $A'_i \cup B^i_j \neq \emptyset$ then $B^i_j \cup (A_i \setminus \bar{A}_i) = \emptyset$ (because $v \geq |V(B^i_j)|$). Let us omit those $B^i_j$’s that do not intersect their $A'_i$. We can assume that we are left with $\{B^i_j : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ for some $m_i \leq l_i$. This way for $i \neq i'$ we have
\[
B^i_j \subseteq \bar{A}_i, \quad B'^i_{j'} \subseteq \bar{A}_{i'},
\]
so $B^i_j \cap B'^i_{j'} = \emptyset$.

This means that our new system of tiles $\{B^i_j : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ is $\varepsilon_1$-disjoint.

We need to count two more things: How many vertices of $B$ do they cover and how many vertices did we omit from $\mathcal{B} = \{B^i_j : 1 \leq i \leq n, 1 \leq j \leq k_i\}$?

1. It is easier to count the vertices covered by the $\bar{B}^i_j$’s because they are disjoint. First consider just one of the $A_i$’s. Here we have $\{B^i_1, B^i_2, \ldots, B^i_{m_i}\}$. We know that
\[
\sum_{j=1}^{l_i} |B^i_j| \geq (1 - \varepsilon_1) |A_i|,
\]
and since $|\bar{B}^i_j| > (1 - \varepsilon_1) |B^i_j|$, we have
\[
\sum_{j=1}^{l_i} |\bar{B}^i_j| > (1 - \varepsilon_1)^2 |A_i| > (1 - 2\varepsilon_1) |A_i|.
\]
So there are at most $2\varepsilon |A_i|$ points not covered by $\bigcup_{j=1}^{l_i} \bar{B}^i_j$ in $A_i$, consequently there are at most the same number of points not covered by the same set in $A'_i$. If $\bar{B}^i_j$ intersects $A'_i$ then $j \leq m_i$. So every vertex in $A'_i$ which is in $\bigcup_{j=1}^{l_i} \bar{B}^i_j$ is contained in $\bigcup_{j=1}^{m_i} \bar{B}^i_j$ as well. Hence
\[
\left| \bigcup_{j=1}^{m_i} \bar{B}^i_j \cap A'_i \right| > |A'_i| - 2\varepsilon |A_i|,
\]
\[
\left| \bigcup_{j=1}^{m_i} \bar{B}^i_j \right| > (1 - d^v \varepsilon_2 - 2\varepsilon_1) |A_i|.
\]
From this, clearly
\[
\left| \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \bar{B}^i_j \right| > (1 - d^v \varepsilon_2 - 2\varepsilon_1) \left| \bigcup_{i=1}^{n} A_i \right| > (1 - d^v \varepsilon_2 - 2\varepsilon_1)(1 - \varepsilon_2) |B|,
\]
so \[ \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} B_i^j > (1 - d^r \varepsilon_2 - 2 \varepsilon_1)(1 - \varepsilon_2)|B| > (1 - d^r \varepsilon_2 - \varepsilon_2 - 2 \varepsilon_1)|B|. \]

2. We saw that for each \( i \) we have
\[ \left| \bigcup_{j=1}^{m_i} \tilde{B}_j^i \right| > (1 - d^r \varepsilon_2 - 2 \varepsilon_1)|A_i|. \]
Hence
\[ \sum_{j=1}^{m_i} |B_j^i| > (1 - d^r \varepsilon_2 - 2 \varepsilon_1)|A_i|. \]
Since we know that \( \sum_{j=1}^{l_i} |B_j^i| < \left| \bigcup_{j=1}^{l_i} \tilde{B}_j^i \right|/(1 - \varepsilon_1) \leq |A_i|/(1 - \varepsilon_1) \), we can estimate
\[ \frac{\sum_{j=1}^{m_i} |B_j^i|}{\sum_{j=1}^{l_i} |B_j^i|} > \frac{(1 - d^r \varepsilon_2 - 2 \varepsilon_1)|A_i|}{\sum_{j=1}^{l_i} |B_j^i|} = (1 - \varepsilon_1)(1 - d^r \varepsilon_2 - 2 \varepsilon_1) > 1 - d^r \varepsilon_2 - 3 \varepsilon_1. \]
From the choice of \( l_i \) we know that
\[ \sum_{j=1}^{l_i} |B_j^i| > (1 - \varepsilon_1) \sum_{j=1}^{k_i} |B_j^i|, \]
so
\[ \frac{\sum_{j=1}^{m_i} |B_j^i|}{\sum_{j=1}^{k_i} |B_j^i|} = \frac{\sum_{j=1}^{m_i} |B_j^i| \sum_{j=1}^{l_i} |B_j^i|}{\sum_{j=1}^{k_i} |B_j^i| \sum_{j=1}^{k_i} |B_j^i|} > (1 - d^r \varepsilon_2 - 3 \varepsilon_1)(1 - \varepsilon_1) > 1 - d^r \varepsilon_2 - 4 \varepsilon_1. \]
We can see that these \( B_j^i \)'s form an \((1 - d^r \varepsilon_2 - \varepsilon_2 - 2 \varepsilon_1)\)-cover of \( B \) and they contain \((1 - d^r \varepsilon_2 - 4 \varepsilon_1)\) times all the vertices of \( B \). This means that \( B \) almost \((d^r \varepsilon_2 + \varepsilon_2 + 4 \varepsilon_1)\)-quasi-tile \( B \).

This is a stronger version of Proposition 2.8 from [6]:

**Proposition 2.20.** Let \( \Gamma \) be a finitely generated amenable group, \( S \subset \Gamma \) a finite symmetric generating set and \( F \) a Følner sequence in \( \text{Cay}(\Gamma, S) \). Then for each integer \( r > 0 \) there is an integer \( R_{qt}(r) \) (which depends also on \( F \)) and a finite subsequence \( T \subset F \) of say \( m \) members and a lattice vector \( 0 \neq \alpha \in \mathbb{N}^m \) with the following property. Each member of \( T \) is an \( r \)-approximation of \( \text{Cay}(\Gamma, S) \), and each \( R_{qt}(r) \)-approximation of \( \text{Cay}(\Gamma, S) \) can be almost \( \frac{1}{r} \)-quasi-tiled by an integer multiple of \( \alpha \cdot T \).

**Proof.** By discarding a few elements from \( F \) we may assume that the remaining elements are all \( r \)-approximations of \( \text{Cay}(\Gamma, S) \). We apply Proposition 2.14 with parameters \( \varepsilon_1 \) and \( n = 0 \) (\( \varepsilon_1 \) shall be given later).
We obtain \( \mathcal{T} = \{F_1, F_2, \ldots, F_m\} \subset \mathcal{F} \), \( L \) and \( \delta \). Choose \( R \) such that \( R > \max(L, 1/\delta) \). This way whenever a finite edge-colored graph \( B \) satisfies \(|Q_0^R|/|V(B)| > 1 - \delta \), it is an \( R \)-approximation of the Cayley-graph. So it follows from the proposition that each \( R \)-approximation of \( \text{Cay}(\Gamma, S) \) can be \( \varepsilon_1 \)-quasi-tiled by some linear combination of \( \mathcal{T} \).

Let \( \mathcal{F}' \subset \mathcal{F} \) be those Følner sets that are \( R \)-approximations of \( \text{Cay}(\Gamma, S) \). For each member \( F \in \mathcal{F}' \) we choose a lattice vector \( 0 \neq \beta_F \in \mathbb{N}^m \) such that \( \beta_F \cdot \mathcal{T} \varepsilon_1 \)-quasi-tiles \( F \).

Now \( \{\sigma(\beta_F) : F \in \mathcal{F}'\} \) is a sequence of unit vectors, so it has an accumulation point \( u \in \mathbb{R}^m \). We apply Lemma 2.18 to \( \mathcal{T} \) and \( \varepsilon_1 \), and obtain the bound \( M(\varepsilon_1) = M \). Let us fix a member \( H \in \mathcal{F}' \) such that \( \|\sigma(\beta_H) - u\| < \frac{1}{2M} \) and then let
\[
\mathcal{F}'' = \{F \in \mathcal{F}' : \|\sigma(\beta_F) - u\| < \frac{1}{2M} \text{ and } \|\beta_F\| \geq M\|\beta_H\|\}.
\]

This \( \mathcal{F}'' \) is still a Følner sequence and for each \( F \in \mathcal{F}'' \) we have \( \|\sigma(\beta_F) - \sigma(\beta_H)\| < 1/M \), hence \( \beta_F \cdot \mathcal{T} \) can be almost \( \varepsilon_1 \)-quasi-tiled by \( (t_F \beta_H) \cdot \mathcal{T} \) for some \( t_F \in \mathbb{N} \).

Applying again Proposition 2.14 for \( \mathcal{F}'' \) and \( \varepsilon_2 \) we get a new finite subsequence \( \mathcal{T}' = \{Q_1, Q_2, \ldots, Q_k\} \subset \mathcal{F}'' \) and \( L', \delta' \). Choose \( R' \) such that \( R' > \max(L', \frac{1}{\delta'}) \).

Now let \( X \) be an \( R' \)-approximation of \( \text{Cay}(\Gamma, S) \). Then \( X \) can be \( \varepsilon_2 \)-quasi-tiled by a linear combination of \( \mathcal{T}' \), say \( \gamma \cdot \mathcal{T}' \). Let us examine a tile of type \( Q_i \), it can be \( \varepsilon_1 \)-quasi-tiled by \( \beta_{Q_i} \cdot \mathcal{T} \). Since \( Q_i \in \mathcal{F}'' \), we can choose \( (t_{Q_i} \beta_H - (2t_{Q_i} \|\beta_H\|/M(\varepsilon_1)) \mathcal{T}) \) tiles among \( \beta_{Q_i} \cdot \mathcal{T} \) such that we omitted at most \( \varepsilon_1 \|\beta_{Q_i} \cdot \mathcal{T}\| \) vertices (according to the proof of Lemma 2.18). This means that our new tiles cover at least \( (1 - 2\varepsilon_1) |Q_i| \) vertices. In other words, \( (t_{Q_i} \beta_H) \cdot \mathcal{T} \) almost \( 2\varepsilon_1 \)-quasi-tiles this tile of type \( Q_i \). We can do the same for every tile.

So we have exactly the situation of the previous lemma with \( \gamma \cdot \mathcal{T}' \varepsilon_2 \text{-quasi-tiling } X \) and \( (t_{Q_i} \beta_H) \cdot \mathcal{T} \) almost \( 2\varepsilon_1 \)-quasi-tiling the tiles of type \( Q_i \), \( v = \max\{|F| : F \in \mathcal{T}\} \). Lemma 2.19 tells us that \((\gamma_1 t_{Q_1} + \gamma_2 t_{Q_2} + \cdots + \gamma_k t_{Q_k}) \beta_H \cdot \mathcal{T} \) almost \( (d^v \varepsilon_2 + \varepsilon_2 + 8\varepsilon_1) \)-quasi-tile \( X \). Here \( d = |S|/2 \), since \( S \) is a symmetric generating set.

Choose \( \varepsilon_1 \) to be \( 1/16r \). This determines \( \mathcal{T} \) and consequently \( v \). Then we can choose \( \varepsilon_2 = 1/(2r(d^v + 1)) \). This way
\[
d^v \varepsilon_2 + \varepsilon_2 + 8\varepsilon_1 = \frac{1}{r},
\]
so this integer multiple of \( \beta_H \cdot \mathcal{T} \) almost \( \frac{1}{r} \)-quasi-tiles \( X \).

Remark 2.21. Note that this proposition clearly implies Proposition 2.8 from \cite{[6]}.\]

\[\Box\]
2.3.2 The proof of the theorem

Definition 2.22. Let $\alpha$ and $\beta$ partitions of a finite set $S$. The incidence graph of $\alpha$ and $\beta$ is a bipartite graph, whose two sets of vertices consist of the classes of $\alpha$ and the classes of $\beta$, and the edges are the elements of $S$, each element connects its $\alpha$-class with its $\beta$-class.

Proposition 2.23 (From [5].) For each triple $(a, b, r)$ of integers there is a finite set $S$ with two partitions $\alpha$ and $\beta$ on it such that each $\alpha$-class has $a$ elements, each $\beta$-class has $b$ elements, an $\alpha$-class can meet a $\beta$-class in at most one element, and in the incidence graph of $\alpha$ and $\beta$ each simple cycle is longer than $2r$.

Remark 2.24. Suppose we have $S$ as described in the proposition, and there is a graph $A$ whose vertex set is $S$ and for every edge its endpoints are in the same $\alpha$-class or in the same $\beta$-class. The condition that in the incidence graph there are no short cycles means the following here. If we have a simple cycle in $A$ such that its length is at most $2r$ then it is contained in an $\alpha$-class or in a $\beta$-class.

Definition 2.25. Let $A$ be a colored graph and $Z$ a subset of the colors used in $A$. Then $A|_Z$ denotes the subgraph obtained from $A$ by omitting all edges whose color does not belong to $Z$. Suppose that $C$ is another colored graph and $\phi$ is an $r$-isomorphism between $A|_Z$ and $C|_Z$.

We build another graph called the enhancement of $C$ with $A$ along $\phi$, denoted by $A \Rightarrow C$. We start from $C$ and add new edges to it. Namely, for each edge $a \rightarrow b$ of $A$ whose color does not belong to $Z$ we add a new edge $\phi(a) \rightarrow \phi(b)$ of the same color, provided that $\phi$ is defined at the endpoints $a$ and $b$.

Theorem 2.26. Let $G$ and $H$ be finitely generated sofic groups, $\Gamma \leq G$ a finitely generated amenable subgroup and $\phi : \Gamma \hookrightarrow H$ an injective homomorphism. Then the amalgamated product $G \star_{\phi} H$ is also a sofic group.

Proof. For simplicity we identify $G$ and $H$ with their canonical image in $G \star_{\phi} H$. This identifies $\Gamma$ and $\varphi(\Gamma)$ with their image too. Let $X \subset G$, $Y \subset H$ and $Z \subset \Gamma$ be finite symmetric generating sets such that $Z = X \cap Y$.

Let $r > 0$ be an integer. We aim to find an $r$-approximation of $\text{Cay}(G \star_{\phi} H, X \cup Y)$. Now we define recursively a sequence of numbers $k_i$ for $0 \leq i \leq 2r + 1$. Consider all the words containing letters from $(X \cup Y) \setminus Z$ that have length at most $2r + 1$. Some of these words may belong to the amenable subgroup $\Gamma$ in $G \star_{\phi} H$. If a word $w \in \Gamma$, let us call the $Z$-length of $w$ the length of the shortest word $\hat{w}$ such that $\hat{w}$ consists of letters from $Z$ and $\hat{w} = w$ in $G \star_{\phi} H$. Since we have finitely many such words, the maximum of their $Z$-lengths is a finite number, let us call it $k_0$. 

If we already have $k_0, k_1, \ldots, k_{i-1}$ then we can find $k_i$ as follows. Consider all the words that contain at most $2r + 1 - i$ letters from $(X \cup Y) \setminus Z$ and at most $ik_{i-1}$ letters from $Z$, and consider the $Z$-lengths of those that are in $\Gamma$. Let the maximum of these $Z$-lengths be $k_i$.

Let $d = |X \cup Y| = |X| + |Y| - |Z|$, $K = k_{2r+1} + 2r + 1$, and set

\begin{align*}
  r_0 > K, \\
  r_1 > 8r_0d^{2r+1}, \\
  r_2 > \max\{R^X_{\text{app}}(r_1), R^Y_{\text{app}}(r_1)\},
\end{align*}

where $R^X_{\text{app}}(r_1)$ denotes the number Lemma 2.10 provides for $\text{Cay}(G, X)$ and $r_1$, and $R^Y_{\text{app}}(r_1)$ is what we get in the case of $\text{Cay}(H, Y)$.

We choose a Følner sequence $\mathcal{F}$ in $\text{Cay}(\Gamma, Z)$, we can assume that each $F \in \mathcal{F}$ is connected. Then applying Proposition 2.20 to the number $r_2$ provides a finite $T = \{F_1, \ldots, F_m\} \subset \mathcal{F}$ and a nonzero vector $\alpha \in \mathbb{N}^m$. Let us denote by $D$ the graph $\alpha \cdot T$. Let

\[ r_3 > \max\{R^Z_{\text{qt}}(2r_2), K + |D|\}. \]

Let us start with $r_3$-approximations $A$ resp. $B$ of the Cayley graphs $\text{Cay}(G, X)$ resp. $\text{Cay}(H, Y)$. Then $A|_Z$ and $B|_Z$ are both $r_3$-approximations of $\text{Cay}(\Gamma, Z)$. Since $r_3 > R^Z_{\text{qt}}(2r_2)$, Proposition 2.20 implies that both $A|_Z$ and $B|_Z$ can be almost $1/(2r_2)$-quasi-tiled by some integer multiples of $\alpha \cdot T$, say $a(\alpha \cdot T)$ and $b(\alpha \cdot T)$. In particular this means that $A|_Z$ is $r_2$-isomorphic to $\alpha \cdot D$ and $B|_Z$ is $r_2$-isomorphic to $b \cdot D$.

**Claim 2.27.** Let $N = \sum_{i=1}^m \alpha_i$ and $A' = N \cdot A$, $B' = N \cdot B$. Then there exists an $r_2$-isomorphism $\psi$ from $Na \cdot D$ to $A'$ such that for each copy of $D$ $\psi$ takes all the connected components of this copy to different copies of $A$, and there exists an $r_2$-isomorphism from $Nb \cdot D$ to $B'$ with the same property.

**Proof.** We have an $r_2$-isomorphism $\varphi : a \cdot D \to A$. Let $D_1, \ldots, D_a$ denote the copies of $D$, so $\varphi : \bigcup_{i=1}^a D_i \to A$. We construct an $r_2$-isomorphism from $Na \cdot D$ to $A'$ in the following way. Let us arrange $Na \cdot D$ in a matrix form, let $\{D_{i,j} : 1 \leq i \leq N, 1 \leq j \leq a\}$ denote the copies of $D$.

Now fix some $1 \leq j \leq a$ and look at $D_{i,j}$, which is isomorphic to $\alpha \cdot T$. Hence this graph contains $N$ of the Følner sets, so it has exactly $N$ connected components, let us denote them by $T^1_{ij}, T^2_{ij}, \ldots, T^N_{ij}$. We can assume that $T^1_{ij} \cong T^2_{ij} \cong \ldots \cong T^N_{ij}$ for each $k$. For $1 \leq l \leq N$ let

\[ D^l_j = \{T^l_{kj} : k - i \equiv l \mod N\}. \]
Each $D_j^i$ contains one $T_{k^{i,j}}$ for a fixed $k$, so because of our assumption on the $T_{k^{i,j}}$'s $D_j^i$ is isomorphic to $D$. Clearly $D_j^1, \ldots, D_j^N$ are pairwise disjoint and $\bigcup_{i=1}^N D_j^i = \bigcup_{i=1}^N D_{i,j}$. This way we ensured that the connected components of a $D_{i,j}$ lie in different $D_j^i$'s. Now for a fixed $l$ we know that $\bigcup_{i=1}^a D_j^i$ is isomorphic to $\bigcup_{i=1}^a D_i$, let $\varphi_0$ denote the isomorphism which takes $D_{j}^i$ to $D_j$. Then

$$\varphi \circ \varphi_0 : \bigcup_{j=1}^a D_j^i \to A$$

is an $r_2$-isomorphism. Let us do the same for each $l$, except that the images should be different copies of $A$, say we take $\bigcup_{j=1}^a D_j^i$ to $A_l$. This way we defined

$$\hat{\varphi} : \bigcup_{j=1}^a \bigcup_{i=1}^N D_j^i \to \bigcup_{l=1}^N A_l,$$

it is the same as

$$\hat{\varphi} : \bigcup_{i=1}^a \bigcup_{j=1}^a D_{i,j} \to A'.$$

This $\hat{\varphi}$ has the property that is takes all the connected components of a $D_{i,j}$ to different copies of $A$ and it is indeed an $r_2$-isomorphism.

The other $r_2$-isomorphism from $Nb \cdot D$ to $B'$ can be constructed similarly.

This means that by replacing $A$ by $A'$ we can assume that $A|_Z$ is $r_2$-isomorphic to $a \cdot D$ and this $r_2$ isomorphism has the property that there is no path in $A$ between different connected components of a $D$-copy. Replace $B$ by $B'$, hence $B$ has the same property.

Apply Proposition 2.23 for the triple $(a, b, r + 1)$, we obtain a finite set $S$ and partitions $\alpha$ and $\beta$ of $S$. Define the graph $C_0 = S \times D$, the union of $|S|$ disjoint copies of $D$. Now for an $\alpha$-class $\sigma \subset S$ the subgraph $\sigma \times D$ is exactly the graph $a(\alpha \cdot T)$. So it is $r_2$-isomorphic to $A|_Z$ and we can enhance it with $A$ along the $r_2$-isomorphism we constructed above. Repeating this enhancement for all $\alpha$-classes we obtain a new graph $C_1$ on the same vertex set $(S \times D)$. This graph $C_1$ is clearly $r_2$-isomorphic to $(|S|/a) \cdot A$ and $C_1|_Z$ is still isomorphic to $S \times D$.

Next for each $\beta$-class $\rho \subset S$ we consider the subgraph $C_1^\rho \subset C_1$ spanned by the subset $\rho \times D$. Then $C_1^\rho|_Z$ is $r_2$-isomorphic to $B|_Z$, so we can enhance it with $B$ along the other $r_2$-isomorphism we constructed. Repeating this enhancement for all $\beta$-classes we obtain a new graph $C$ on the same vertex set $S \times D$. This $C|_X$ is $r_2$-isomorphic to $(|S|/a) \cdot A$ and $C|_Y$ is $r_2$-isomorphic to $(|S|/b) \cdot B$.

We would like to prove that this graph $C$ is an $r$-approximation of the group $G*_{\alpha} H$.

We need to find a set of good points: those which have $r$-neighborhood isomorphic to an $r$-ball in the Cayley graph.
Consider the set \( C \) and the points in this set still have the previously described properties.

Each \( \alpha \)-class in \( C \) is \( r_2 \)-isomorphic to \( A \), this means we have \( A_2 \subseteq A \) that is isomorphic to a subgraph of our \( \alpha \)-class. Let us call the union of these images \( C_2^X \), and similarly we get \( C_2^Y \). Clearly

\[
|C_2^X| > \left( 1 - \frac{1}{r_2} \right) |C|, \quad |C_2^Y| > \left( 1 - \frac{1}{r_2} \right) |C|.
\]

Since every \( \alpha \)-class is \( r_2 \)-isomorphic to \( A \) and \( A \) is an \( r_3 \)-approximation of \( \text{Cay}(G, X) \), the \( \alpha \)-classes are \( r_1 \)-approximations of \( \text{Cay}(G, X) \). So for an \( \alpha \)-class \( \sigma \) there is a subset \( A^\sigma_1 \subseteq \sigma \times D \) for which \( |A^\sigma_1| > (1 - 1/r_1)|\sigma \times D| \) and for each point in \( A^\sigma_1 \) the \( r_1 \)-neighborhood of this point looks good. Let us denote the union of these by \( C_1^X \). We get \( C_1^Y \) similarly.

Consider the set

\[
C' = C_3^X \cap C_3^Y \cap C_2^X \cap C_2^Y \cap C_1^X \cap C_1^Y.
\]

Clearly we have

\[
|C'| > \left( 1 - \frac{2}{r_3} - \frac{2}{r_2} - \frac{2}{r_1} \right) |C|,
\]

and the points in this set still have the previously described properties.

Let \( C'' = C \setminus N_{2r_0+1}(C \setminus C') \). This means that the \( 2r_0+1 \)-neighborhood of \( C'' \) is in \( C' \). Clearly

\[
|C \setminus C'| < \frac{8}{r_1} |C|,
\]

and hence

\[
|N_{2r_0+1}(C \setminus C')| < d^{2r_0+1} \frac{8}{r_1} < d^{2r_0+1} \frac{8}{8r_0d^{2r_0+1}} = \frac{1}{r_0} |C|,
\]

\[
|C''| = |C \setminus N_{2r_0+1}(C \setminus C')| > \left( 1 - \frac{1}{r_0} \right) |C|.
\]
Claim 2.28. For each point \( p \in C'' \) the \( r \)-neighborhood of \( p \) in \( C \) is rooted isomorphic to an \( r \)-ball in \( \text{Cay}(G \ast \phi H, X \cup Y) \).

Proof. Let us fix a vertex \( p \in C'' \). We need to show two things. First, if there is a word with letters from \( X \cup Y \) that is equal to the identity in the group and has length at most \( 2r + 1 \), then the corresponding path starting from \( p \) in \( C \) must be a cycle. Secondly, if we have a simple cycle containing \( p \) of length at most \( 2r + 1 \) then the corresponding word must be the identity. These clearly imply our claim.

We will use the following lemma, which can be easily verified by the reader.

Lemma 2.29. Let \( w \) be a word with letters from \( X \cup Y \), and suppose that \( w \) is the identity in \( G \ast \phi H \). Then we can get the empty word from \( w \) using the following operation finitely many times. We choose a subword \( u \) of \( w \) such that \( u \) contains letters only from \( X \) or only from \( Y \), and the group element \( u \) is in the amenable subgroup \( \Gamma \). Then we change it to a word that is equal to \( u \) in the group and contains letters only from \( Z \). (If it is the identity then we can delete it.)

This is the analogue of the statement that in a free product \( G \ast H \) a word \( g_1 h_1 g_2 h_2 \ldots g_k h_k \) can only be simplified if one of the \( g_i \)'s is the identity element in \( G \) or one of the \( h_i \)'s is the identity of \( H \).

Now we can prove the claim.

1. Let \( w \) be a word which has length at most \( 2r + 1 \) and \( w = 1 \) in \( G \ast \phi H \), and let us denote by \( P \) the corresponding path from \( p \). Since the \( 2r_0 + 1 \)-neighborhood of \( p \) is in \( C' \), all points of the path \( P \) are also in \( C' \).

According to the lemma, we have finitely many, say \( M \) operations of the above kind. Let us call \( w_i \) the word we have after the \( i \)th step, so \( w_M \) is the empty word. Suppose \( w \) has \( k \) letters from \( (X \cup Y) \setminus Z \) and \( l \) letters from \( Z \) (\( k + l \leq 2r + 1 \)). In this case clearly \( M \leq k \) since in each step we decrease the number of letters which are not in \( Z \).

Let us denote by \( P_i \) the path in the graph \( C \) starting from \( p \) which corresponds to the word \( w_i \). We will prove the followings by induction on \( i \).

(a) The word \( w_i \) has at most \( k - i \) letters from \( (X \cup Y) \setminus Z \) and at most \( l + ik_i \) letters from \( Z \). So in particular the length of \( w_i \) is less than \( K \).

(b) The path \( P_i \) is contained in \( C'' \).

(c) \( P_i \) has the same endpoint as \( P_{i-1} \).

If \( w_0 = w \), the statements are trivial for \( i = 0 \). Note that (b) follows from (a), because the length of the path is less than \( K \).
Now consider the step when we get \( w_i \) from \( w_{i-1} \). Let \( w_{i-1} = v_1w_2 \) and \( w_i = v_1u'v_2 \). Let the endpoint of the path corresponding to \( v_1 \) be \( q \). Now we know that \( u \) contains at most \( k - (i - 1) \leq 2r + 1 - (i - 1) \) letters from \( (X \cup Y) \setminus Z \) and at most \( l + (i - 1)k_{i-1} \leq ik_{i-1} \) letters from \( Z \), hence its \( Z \)-length is at most \( k_i \) according to the definition of \( k_i \). So the length of \( u' \) is at most \( k_i \), and \( w_i \) contains at most \( k_i + l + (i - 1)k_{i-1} < l + ik_i \) letters from \( Z \).

For the third statement, we know that \( q \in C' \). Assume that \( u \) has letters only from \( X \) (the proof in the other case is the same), then the path corresponding to the word \( u \) and \( u' \) both lie in \( C \mid X \). They are equal in the group, both have length at most \( K \). We also know that the \( r_0 \)-neighborhood of \( q \) in \( C \mid X \) looks like an \( r_0 \)-ball in \( \text{Cay}(G, X) \). So the paths corresponding to \( u \) and \( u' \) must end at the same point. So \( P_{i-1} \) and \( P_i \) have the same endpoint as well.

We know that \( P_M \) is just the empty word, hence its endpoint is \( p \). This means that all \( P_i \)'s end at \( p \), in particular the original path \( P \) ends there too. So \( P \) is a cycle, this is what we wanted to prove.

2. Now let \( P \) be a simple cycle starting and ending at \( p \) which has length at most \( 2r + 1 \). Let \( w \) be the corresponding word in the group. We would like to prove that this word is equal to the identity in \( G *_{\phi} H \). Assume that \( w \) contains at most \( 2r + 1 - i \) letters from \( (X \cup Y) \setminus Z \), and at most \( i \leq ik_{i-1} \) letters from \( Z \).

Take the map \( C \to S \), where \( \{s\} \times D \) goes to the point \( s \in S \). Consider the image of \( P \) in \( S \). For every edge of \( P_i \), whose color does not belong to \( Z \), we draw an edge between the images of the endpoints. The images of the \( Z \)-edges would be loops, that is why we do not consider them. This is a cycle, though usually it is no longer simple.

But clearly we can find a simple part of this cycle, i.e., there exists \( P' \subseteq P \) such that the endpoints of \( P' \) are in the same copy of \( D \) in \( C \). We can assume that \( p \notin P' \), since there are always at least two such simple parts of a cycle, so we can choose one that does not contain \( p \).

According to Remark 2.24, this \( P' \) is contained in an \( \alpha \)-class or in a \( \beta \)-class. Suppose that it lies in an \( \alpha \)-class, the proof in the other case goes similarly.

Let the corresponding word to \( P' \) be \( u \), this is a subword in \( w \). There are two kinds of \( Y \setminus Z \)-colored edges in \( C \): those that have endpoints in the same \( D \)-copy, and those that have endpoints in different \( \alpha \)-classes. The path \( P' \) clearly cannot contain the one of the latter, and since the image of \( P' \) is simple, it
cannot contain one of the former kind. So \( u \) does not contain elements from \( Y \setminus Z \).

Let \( w = v_1 w_2 \), and let us denote by \( q \) the endpoint of the path corresponding to \( v_1 \). So \( P' \) is a path starting from \( q \) and ending in the same \( D \)-copy, let us call this endpoint \( q' \). Because of the assumption on the \( r_2 \)-isomorphism between \( A \) and the \( \alpha \)-classes, there are no \( X \)-colored paths between different components of this copy of \( D \). Since \( P' \) is an \( X \)-colored path with endpoints in the same copy, its endpoints must be in the same connected component of \( D = \alpha \cdot T \).

Since \( p \in C'' \), \( P \subseteq C' \). This means that \( P' \) has an image in \( A \), call this image \( \hat{P}' \), the endpoints \( \hat{q} \) and \( \hat{q}' \). These endpoints are in the same tile of the quasi-tiling, because \( q \) and \( q' \) are in the same component of \( D \).

The size of this tile is at most the size of \( D \), hence we have a path \( Q \) in the tile containing only \( Z \)-edges between \( \hat{q}' \) and \( \hat{q} \), that has length at most \( |D| \). This means that \( \hat{P}' Q \) is a cycle in \( A \).

We also know that \( \hat{q} \) has a good \( r_3 \)-neighborhood in \( A \). We chose \( r_3 \) to be greater than \( |D| + K \), and the length of \( \hat{P}' Q \) is at most \( |D| + K \), so the word corresponding to this path is the identity in the group \( G \).

Let us denote by \( u' \) the corresponding word to \( Q^{-1} \). The previous observation means that \( u' = u \) in \( G \). Since \( Q \) has only edges labeled by \( Z \), \( u' \) contains only letters from \( Z \). Since \( u \) had at most \( i k_{i-1} \) letters from \( Z \) and at most \( 2r + 1 - i \) letters from \( X \setminus Z \), the length of \( u' \) is at most \( k_i \). This \( k_i \) is smaller than \( K < r_0 \), and in \( C \) the \( r_0 \)-ball around \( q \) looks good, so the corresponding path to \( u'u^{-1} \) is a cycle in \( C \) starting from \( q \).

So we can change our word \( w = v_1 w_2 v_2 \) to \( v_1 u' v_2 \). Let \( w_1 \) be this new word, and let \( P_1 \) be the cycle corresponding to \( w_1 \). Then \( w_1 \) is the identity if and only if \( w \) is the identity, so we can continue with \( w_1 \). We also know that \( w_1 \) contains at most \( 2r + 1 - i - 1 \) letters from \( (X \cup Y) \setminus Z \) and at most \( i \cdot k_{i-1} + k_i \leq (i + 1) k_i \) letters from \( Z \).

From now on, we can do the same as before, everything works similarly (in other words, we can use induction). We get the words \( w_2, w_3, \ldots \), each of them having length at most \( K \) and in the group \( G \ast_{\phi} H \) they are all equal. After \( 2r + 1 - i \) steps we have \( w_{2r+1-i} \), that contains only letters from \( Z \), and has length at most \( K \). The corresponding path \( P_{2r+1-i} \) is a cycle in \( C \), hence it is a cycle in \( A \) (or in \( B \)) as well, and this \( p \) has a good \( r_0 \)-neighborhood in \( A \). So \( w_{2r+1-i} = 1 \) in \( \Gamma \), hence it is the identity element in \( G \ast_{\phi} H \).

We finished the proof of the claim.

\( \Box \)
So we found $C'' \subseteq C$, for which

$$|C''| > \left(1 - \frac{1}{r_0}\right)|C| > \left(1 - \frac{1}{r}\right)|C|,$$

and for each point $p \in C''$, the $r$-ball around $p$ in $C$ is rooted isomorphic to an $r$-ball in the Cayley graph of the amalgamated product. This proves that $C$ is an $r$-approximation of the Cayley graph, so the group $G *_{\phi} H$ is sofic.

We still need to prove the statement in the case when our groups are not necessarily finitely generated.

**Theorem 2.30.** Let $G$ and $H$ be sofic groups, $\Gamma \leq G$ an amenable subgroup and $\phi : \Gamma \hookrightarrow H$ an injective homomorphism. Then the amalgamated product $G *_{\phi} H$ is also sofic.

**Proof.** The amalgamated product $G *_{\phi} H$ is the direct limit of amalgamated products $\{G_\alpha *_{\phi} H_\alpha\}$ over amenable subgroups $\Gamma_\alpha$, where these $G_\alpha$, $H_\alpha$ and $\Gamma_\alpha$ are all finitely generated groups. We know that direct limits of sofic groups are sofic, so this proves the theorem. \qed
Chapter 3

Surjunctivity conjecture

The following conjecture is due to Gottschalk [7].

**Conjecture 3.1** (Gottschalk). Let $G$ be a countable group and $X$ a finite set. Consider the compact metrizable space $X^G$ of $X$-valued functions on $G$ equipped with the product topology. Let $A : X^G \to X^G$ be a continuous map that commutes with the natural right $G$-action. Then if $A$ is injective, it is surjective as well.

We present the main ideas of the proof of this conjecture for sofic groups. The following proof was communicated by Miklós Abért.

**Theorem 3.2** (Gromov). Let $\Gamma$ be a finitely generated sofic group, $X$ a finite set and let $A : X^\Gamma \to X^\Gamma$ be a continuous $\Gamma$-equivariant map, and suppose that $A$ is injective. Then $A$ is surjective as well.

We will use the following two easy lemmas. We leave the proofs for the reader.

**Lemma 3.3.** Let $A : X^\Gamma \to X^\Gamma$ be a continuous $\Gamma$-equivariant map. This means that $A$ commutes with the natural right $\Gamma$-action on $X^\Gamma$. Then the following statements hold for $A$.

(a) We define $\hat{A} : X^\Gamma \to X$ such that for $f : \Gamma \to X$, $\hat{A}(f) = f(1)$. Here $1 \in \Gamma$ is the identity element of $\Gamma$. Then $\hat{A}$ determines $A$.

(b) There exists a finite subset $H \subseteq \Gamma$ (which depends on $A$), such that if $f, f' : \Gamma \to X$, $f|_H = f'|_H$ then $\hat{A}(f)(1) = \hat{A}(f')(1)$. In other words, the restriction of $f$ to $H$ determines the value of its $\hat{A}$-image on the identity element.

(c) $A(X^\Gamma) \subseteq X^\Gamma$ is a $\Gamma$-invariant closed subset.

**Lemma 3.4.** Suppose that $A : X^\Gamma \to X^\Gamma$ is not surjective, i.e., $A(X^\Gamma) \neq X^\Gamma$. Then there exists a ‘forbidden pattern’: $T \subseteq \Gamma$, $|T| < \infty$ and $\bar{f} : T \to X$, such that for any $f \in A(X^\Gamma)$, $f|_T \neq \bar{f}$. 
Proof of Theorem 3.2. Let \( \Gamma = \langle S \rangle \) be a sofic group, where \( S \) is a finite symmetric generating set. For \( f \in X^{\Gamma} \) we regard \( f \) as a coloring of \( \Gamma \) by the elements of \( X \).

Part (b) of Lemma 3.3 provides a finite subset \( H \subseteq \Gamma \). Then there exists an integer \( L > 0 \) such that \( H \subseteq B_L(1) \) in \( \text{Cay}(\Gamma, S) \). This means that for each element \( g \in \text{Cay}(\Gamma, S) \) and every \( f \in X^{\Gamma} \) the \( f \)-coloring of the \( L \)-neighborhood of \( g \) determines \( A(f)(g) \), the \( A(f) \)-color of \( g \).

Assume that \( A \) is not surjective. Then Lemma 3.4 gives us \( T \) and \( \tilde{f} : T \to X \). Let \( N_L(T) \) denote the \( L \)-neighborhood of \( T \) in \( \text{Cay}(\Gamma, S) \) and choose \( k \in \mathbb{N} \) such that \( N_L(T) \subseteq B_k(1) \).

Choose \( \{ \varepsilon_n \}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} \varepsilon_n = 0 \), and \( \{ r_n \}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} r_n = \infty \) and \( r_n > k \) for each \( n \in \mathbb{N} \). Since \( \Gamma \) is sofic, for every \( n \) we can find a finite graph \( G_n \) that is an \((r_n, \varepsilon_n)\)-approximation of \( \text{Cay}(\Gamma, S) \).

There are subsets \( V_n \subseteq V(G_n) \) for each \( n \) such that for each \( p \in V_n \), \( B_{r_n}(p) \) is isomorphic to \( B_{r_n}(1) \) in \( \text{Cay}(\Gamma, S) \), and we have

\[
|V_n| > (1 - \varepsilon_n) |V(G_n)|.
\]

We define \( A_n : X^{G_n} \to X^{G_n} \) as follows. Let \( f_n \in X^{G_n} \). If \( p \in V_n \), then \( B_{r_n}(p) \) is isomorphic to \( B_{r_n}(1) \) in \( \text{Cay}(\Gamma, S) \). Since \( r_n > L \), we have \( H \subseteq B_{r_n}(1) \). Then, according to Lemma 3.3, this coloring determines the value of its \( \hat{A} \)-image on the centre of the \( r_n \)-ball. Let this color be \( x \in X \). Let \( A_n(f_n)(p) = x \). For \( p \notin V_n \) let \( A_n(f_n)(p) = x_0 \), where \( x_0 \in X \) is a fixed element.

For every \( p \in V_n \) the \( r_n \)-ball around \( p \) is isomorphic to \( B_{r_n}(1) \). Since \( r_n > k \), this ball contains the \( L \)-neighborhood of the forbidden pattern \( T \). So we have a copy of \( T \) and its \( L \)-neighborhood in \( B_{r_n}(p) \). In the \( A \)-image of a coloring the color of a point depends only on its \( L \)-neighborhood. This is true for \( A_n \) and \( p \in V_n \) too. Since \( \tilde{f} : T \to X \) is a coloring that is not in the image of \( A \), this pattern does not occur in the image of \( A_n \).

Let us denote by \( B_n \subseteq X^{G_n} \) the image of \( A_n \).

Claim 3.5. Since \( B_n \) does not contain the forbidden pattern, there exists \( n_0 \in \mathbb{N} \) and a constant \( c > 1 \), such that for \( n > n_0 \) we have

\[
|X^{G_n}| > c^{|V(G_n)|} |B_n|.
\]

This means that for \( n > n_0 \) we can choose colorings \( f_n \in B_n \subseteq X^{G_n} \), such that \( |A_n^{-1}(f_n)| > c^{|V(G_n)|} \).

For \( n > n_0 \) let \( x_n \in A_n^{-1}(f_n) \). We would like to find \( y_n \in A_n^{-1}(f_n) \) and \( p_n \in V_n \) such that \( x_n(p_n) \neq y_n(p_n) \). Suppose that for some \( n \) there is no such coloring. This means that for each \( x' \in A_n^{-1}(f_n) \), \( x' \) differs from \( x_n \) only in the points of \( V(G_n) \setminus V_n \). Since
we have \(|X|^{V(G_n)\setminus V_n}\) ways to color these points, we can have at most this amount of colorings in \(A_n^{-1}(f_n)\). So
\[
\epsilon|V(G_n)| < |A_n^{-1}(f_n)| < |X|^{V(G_n)\setminus V_n} < |X|^\varepsilon_n|V(G_n)|.
\]
As \(\varepsilon_n \to 0\), there exists \(n_1\) such that for \(n > n_1\), \(|X|^{\varepsilon_n} < c\). So for \(n > n_1\) we have
\[
\epsilon|V(G_n)| > |X|^{\varepsilon_n}|V(G_n)|,
\]
which is exactly the opposite of the above inequality. This implies that for \(n > n_1\) we can find the desired \(y_n\) and \(p_n\).

Recall that a sequence of rooted graphs \(\{(\hat{G}_i, q_i)\}_{i=1}^\infty\) is rooted convergent and converges to the rooted graph \((G, q)\) if for every \(R\) there exist \(i(R)\) such that for \(i > i(R)\) the \(R\)-ball around \(q_i\) in \(\hat{G}_i\) is rooted isomorphic to the \(R\)-ball around \(q\) in \(G\).

Now for \(n > n_1\) consider \((G_n, p_n)\) as a rooted graph. Since \(p_n \in V_n\), i.e., \(p_n\) has a good \(r_n\)-neighborhood in \(G_n\), the rooted limit of \(\{(G_n, p_n)\}\) is clearly \((\text{Cay}(\Gamma, S), 1)\).

We define three sequences of colorings of \(\Gamma\) as follows. For \(n > n_1\), \(B_{r_n}(p_n)\) is rooted isomorphic to \(B_{r_n}(1)\) in \(\text{Cay}(\Gamma, S)\). Using this isomorphism, \(x_n\) gives us a coloring of \(B_{r_n}(1)\). Extend this to the other vertices by \(x_0 \in X\), this way we obtain the coloring \(\bar{x}_n \in X^\Gamma\). We get \(\bar{y}_n\) from \(y_n\), and \(\bar{f}_n\) from \(f_n\) similarly.

Since \(X^\Gamma\) is a compact space, every sequence has a convergent subsequence. So we can find \(\{n_k\}_{k=1}^\infty\), such that \(\{\bar{x}_{n_k}\}\), \(\{\bar{y}_{n_k}\}\) and \(\{\bar{f}_{n_k}\}\) are convergent. Let \(x_{n_k} \to x\) and \(y_{n_k} \to y\) and \(f_{n_k} \to f\). Since \(x_n(p_n) \neq y_n(p_n)\), we get that \(x(1) \neq y(1)\).

**Claim 3.6.** *For these \(x, y, f \in X^\Gamma\), we have \(A(x) = A(y) = f\).*

This finishes the proof, since \(x \neq y\), \(A(x) = A(y)\) contradicts the injectivity of \(A\).
Bibliography


