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Self-similar solutions of Navier-Stokes type equations

Master's Thesis

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INTRODUCTION

"La France entière a vu le désastre, au cœur de Paris, du premier pont suspendu que voulut élever un ingénieur, membre de l'Académie des sciences, triste chute qui fut causée par des fautes que ni le constructeur du canal de Briare, sous Henri IV, ni le moine qui a bâti le Pont-Royal, n'eussent faites, et que l'Administration consola en appelant cet ingénieur au Conseil général."

Honoré de Balzac, *Le curé de village* [1]

The proper systematical study of fluid motions has attained a history of over 200 years. One of the main reasons why it serves as a peculiar point of interest is the fact that the governing processes and the associated properties can be discovered not only in the straightforward cases of river flows or the evolution of oceanic currents but also in various other geophysical phenomena (for example in the study of landslides or even with earthquakes) or even in larger scale processes starting from the atmospheric motions up until the the evolution of nebulas. Despite the long history, the indefinite amount of effort, and the vast amount of interested fields in the subject, mathematically speaking it is still poorly understood.

The first mathematically relevant advancements on the study of fluid mehanics were made by Leonhard Euler during the late 1750's when, by applying Newton's second law of motion, he described the motion of a fluid with various equations. The hypotheses made by Euler were rather idealistic, furthermore the equations failed to capture some of the basic governing notions of the motion itself (the friction for

[1] On the infamous incident of Claude-Louis Navier's suspension bridge.
A rough translation could be given as follows:
All France knew of the disaster which happened in the heart of Paris to the first suspension bridge built by an engineer, a member of the Academy of Sciences; a melancholy collapse caused by blunders such as none of the ancient engineers – the man who cut the canal at Briare in Henry IV's time, or the monk who built the Pont Royal – would have made; but our administration consoled its engineer for his blunder by making him a member of the Council-general.

instance), but still the idea of introducing an elaborate system of 'partial differential equations' to express knowledge about the fluid motion proved to be a major breakthrough.

In 1822, Claude-Louis Navier, a young French engineer and former student of the École Nationale des Ponts et Chaussées, deeply inspired by theoretical mathematics (especially the work of Joseph Fourier, who became his doctoral advisor in mathematics as well), published a memoir [19] on his observations and findings related to fluid mechanics. In this report he described more or less adequately the notions of fluid motion in the inhomogeneous case, which could be considered as the starting point of mathematical analysis on the subject. His fondness of mathematics made him revolutionise the theory of elasticity as well as certain aspects of the engineering studies, the latter led to a somewhat blind respect and recognition among his fellows (this could be the main reason why the infamous incident of the collapse of his "innovative suspension bridge" did not manage to ruin his career).

Navier's model still contained some slighty abrupt hypotheses on the molecular level which required slight corrections. In the early 1840's George Stokes published a series of articles related to various cases of fluid motions, which led to the publishment of his study on internal friction of fluids in motion in 1845 [23]. This paper contained the proper derivation of the equations of motion defining a moving fluid (in the compressible case as well), this system of partial equations is what we nowadays know as the Navier-Stokes equations, named after the two main contributors of the formulation.

One hundred and seventy years have passed since the birth of these equations and some fundamental questions still remain unresolved. Undoubtedly, the quest to solve the Navier-Stokes equations (among other challenging partial differential equations) has led to some remarkable improvements in the general study of (hyperbolic) partial differential equations and in the theory of abstract functional analysis as well. This seemingly unassailable problem has certainly earned its right to be considered as one of the seven Millennium Problems of the Clay Mathematical Institute. As Fefferman points out in his short summary of the problem [7], the problematic question is the uniqueness and the regularity of the solutions.

In the current report we shall somewhat remotely address the question of regularity and some associated phenomena. Chapter 1 of this master's thesis is dedicated to the proper setting of the equation. We will present the Navier-Stokes equations in three different forms and discuss what one could understand as a solution to it. After that we shall spend some time recalling some useful Fourier analysis and functional analysis tools for the proper handling of the equations. Last but not least, we will point out some interesting properties of the system (like homogeneity) as well as that of the associate function spaces, which shall serve as a starting point for chapters 3 and 4 as well.

Chapter 2 will be more of an overview on some results related to the existence and the uniqueness of the solutions, since these topics are not the main interests of the present work. Our guide for this chapter will be Marco Cannone's *Harmonic analysis tools for solving the incompressible Navier-Stokes equations* [2]. This well-organised chapter, as part of the third volume of the Handbook of Mathematical Fluid Mechanics, is an excellent supplement of the present report, it provides a deeper insight on the various topics we will only vaguely mention, moreover it contains an astonishing list of references on many topics concerning the problem.

The third and fourth chapter will be dedicated to the main objective of this work, the self-similar solutions associated to the Navier-Stokes equations, introduced by Jean Leray in his pioneering work from 1933 [15]. In these chapters, after presenting the notion of these special type of solutions and explaining the background of them, we shall present some remarkable propositions, theorems and corollaries from the ever expanding study on the subject.

CHAPTER 1

PRELIMINARIES

"... l'analyse mathématique est aussi étendue que la nature elle-même; elle définit tous les rapports sensibles, mesure les temps, les espaces, les forces, les températures; cette science difficile se forme avec lenteur, mais elle conserve tous les principes qu'elle a une fois acquis; elle s'accroit et s'affermit sans cesse au milieu de tant de variations et d'erreurs de l'esprit humain."

Jean Baptiste Joseph Fourier [8] [2]

As the title suggests, this chapter will be dedicated to the proper setting of the equations: in terms of the equation itself, in terms of the correct definition of its solution, and in terms of the function spaces which are worth noting when one has to work with these equations. None of the statements and theorems will be proved here since most of them are elementary and well-known from advanced functional analysis studies (although in some cases references will be given for the interested reader).

Most of the time we are going to work in $\mathbb{R}^3$ due to the obvious physical relevance but some general propositions (especially the ones concerning the function spaces) remain true in higher (or lower) dimensions as well. Moreover, throughout our studies we will handle only the simpler incompressible variant of the equations. Since even the incompressible case for the equations is as hard as it can get, it is not considered to be a huge simplification; usually the well-known general methods and tricks allow us to handle the more difficult compressible case in a similar fashion.

[2] *Mathematical analysis is as extensive as nature itself; it defines all perceptible relations, measures times, spaces, forces, temperatures; this difficult science is formed slowly, but it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind.*

Alexander Freeman (the official English translator)

1.1 The Navier-Stokes equations

In the present master's thesis we shall make some generally accepted assumptions on the physical background of the Navier-Stokes equations in order to be able to arrive to the general and famous form of them, the Cauchy problem for the Navier-Stokes equations describing the time evolution of velocity and pressure in an incompressible viscous fluid.

Throughout the study we are going to denote the time and space dependent velocity as $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ and the pressure as $p(t,x)$. Since we are only interested in the incompressible case, we may assume that the density ϱ (which is constant) is equal to 1. In the domain of $\mathbb{R}^3$ with the presence of an external force $\phi(t,x)$ the incompressible Navier-Stokes equations have the form

$$\begin{aligned} \partial_t v - \nu \Delta v &= -(v \cdot \nabla) v - \nabla p + \phi, \\ \nabla \cdot v &= 0, \\ v(0,x) &= v_0(x) \qquad x \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where ν denotes the viscosity coefficient of the fluid (a positive constant) which can also be supposed to be equal to 1 due to the scaling invariance described in section 1.5.

In appendix A, a more or less formal derivation of the equations can be found starting from the purely physical origin of viscous fluid movements, and all the way to the Cauchy problem associated to the incompressible case (naturally with a fair amount of physical assumptions as well). In the next section we will detail what can one consider as a solution of these equations in general.

A usual assumption considering the equations is that the external force arises from a potential $U(t,x)$, that is the external force can be described as a divergence

$$\phi = \nabla \cdot U.$$

During the later chapters, we will usually completely ignore this force term in the equations (that is we will suppose that $\phi = 0$).

A final remark on the basic system is that the nonlinear term $(v \cdot \nabla)v$ is equal to $\nabla \cdot (v \otimes v)$ because of the divergence-free property $\nabla \cdot v = 0$ (Leibnitz rule).

Before introducing the integral equation associated to (1.1) we shall introduce an intermediate form of the equations which can be obtained simply by applying the well-known Leray-Hopf operator $\mathbb{P}$ on the equations thus arriving to the form

$$\begin{aligned} \partial_t v - \mathbb{P}\Delta v &= -\mathbb{P}\nabla \cdot (v \otimes v) + \mathbb{P}\phi, \\ v(0,x) &= v_0(x) \qquad x \in \mathbb{R}^3. \end{aligned} \tag{1.2}$$

Definition 1.1. *The Leray-Hopf operator* $\mathbb{P}$ *can be defined with the use of Fourier transform as follows*

$$\widehat{(\mathbb{P}v)}_j(\xi) = \sum_{k=1}^{3} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \hat{v}_k(\xi), \quad j = 1, 2, 3. \tag{1.3}$$

This operator can also be defined with the use of the so called Riesz transform[3], named after Frigyes Riesz. The operator $\mathbb{P}$ is in fact the orthogonal projection operator onto the subspace of divergence-free vector field. Moreover, based on the definition one may deduce that it is a pseudo-differential operator of degree zero.

Due to the fact that it is an orthogonal projection onto the kernel of the divergence operator, the term ∇p disappeared from the new equation. It can be recovered in order to return to the original equations (1.1) by assuring that the incompressibility condition $\nabla \cdot v = 0$ is satisfied.

As mentioned before this new equation is usually not used by itself, instead it is the key step in obtaining the integral equation associated to the incompressible Navier-Stokes equations. In order to present its integral form, we shall introduce a semigroup of operators (which is similar to the semigroup associated to the heat equation)

$$\bar{S}(t) = \exp(t\mathbb{P}\Delta). \tag{1.4}$$

With this semigroup at our disposal, it is but a straightforward procedure to derive the following integral equation from (1.2):

$$v(t,x) = \bar{S}(t)v_0(x) - \int_0^t \bar{S}(t-s)\mathbb{P}\nabla\cdot(v\otimes v)(s,x)\,ds + \int_0^t \bar{S}(t-s)\mathbb{P}\nabla\cdot U(s,x)\,ds. \tag{1.5}$$

As usual (it will be detailed in the following section), a solution of the integral equation (1.5) is not necessarily a solution of the original equations. These type of solutions have a huge historical importance, furthermore they proved to be crucial in existence results for the Navier-Stokes equations, hence it is more than worth mentioning.

[3] If we denote $D_j = -i\frac{\partial}{\partial x_j}$ for $j = 1, 2, 3$ (and of course $i^2 = -1$), then the Riesz transforms are

$$R_j = D_j(-\Delta)^{-\frac{1}{2}},$$

where the operator $(-\Delta)^{-\frac{1}{2}}$ can also be defined with the use of Fourier transform, that is $\widehat{(-\Delta)^{-\frac{1}{2}}v}(\xi) = |\xi|\hat{v}(\xi)$. With the Riesz transforms in our hand and by denoting $R = (R_1, R_2, R_3)$ we may deduce that the Leray-Hopf operator is

$$\mathbb{P} = \mathrm{Id} - R \otimes R.$$

A good property of the aforementioned semigroup is that since we only consider the whole space $\mathbb{R}^3$, the semigroup $\bar{S}(t)$ reduces to the famous heat semigroup $\exp(t\Delta)$ which allows an easier analysis of the equation. It will play a significant role in the existence results mentioned in chapter 2. Further remarks on the topic can be found in appendix B.

1.2 Notion of the solution

Our attention in the analysis of the equations will be focused on strongly continuous functions of $t \in [0, T)$ with values in the Banach space $X = X\Big(\mathbb{R}^3\Big)$ of vector distributions, that is $v \in \mathcal{C}\Big([0, T); X\Big)$. The Banach space X will be further specified in the following sections, but in some cases it won't play a significant role. Furthermore, depending on whether T will be finite ($T < \infty$) or infinite ($T = \infty$) we will obtain local or global (in time) solutions respectively.

Due to the extensive use of Fourier transforms and inspired by the generic sense of classical ordinary differential equations we shall assume that the a solution belongs to $\mathcal{S}'$, the space of tempered distributions. Moreover we shall always assume that the Banach space X has the property of $X \hookrightarrow L^2_{loc}$ in order to be able to give (at least) a distributional meaning to the nonlinear term $\nabla \cdot (v \otimes v)$.

Roughly speaking, two main types of solutions exist:

- *strong(er) solutions*, associated to either (1.5) or (1.1) and for which existence and uniqueness results are known;
- *weak(er) solutions*, usually associated to the badic equation (1.1) and for which only existence results are known.

In reality, one can distinguish many different types of solutions, the difference between these definitions lie in general in the class of functions they are associated to. In the following pages we will list four different types of solutions.

Definition 1.2. ***(Classical solution; by Hadamard)*** *A classical solution* $(v(t,x), p(t,x))$ *of the Navier-Stokes equations is a pair of functions* $v : t \to v(t)$ *and* $p : t \to p(t)$ *satisfying the system (1.1), for which all the terms appearing in the equations are continuous functions of their arguments, that is the functions verify the following properties*

- $v(t,x) \in \mathcal{C}\Big([0,T); E\Big) \cap \mathcal{C}^1\Big([0,T); F\Big)$,
- $E \hookrightarrow F$ *(as a continuous embedding),*

- *$v \in E \Rightarrow \Delta v \in F$ (continuous operator),*
- *$\nabla \cdot (v \otimes v) \in F$ (continuous operator),*

where E and F are two Banach spaces of distributions.

For example we may impose the Sobolev space $E = \dot{H}^s(\mathbb{R}^3)$ for $s > \frac{3}{2}$ (in which case E has the structure of an algebra with the usual product of functions). In this case we shall choose $F = \dot{H}^{s-2}(\mathbb{R}^3)$ to ensure that $\Delta v \in F$ if $v \in E$, moreover that $\nabla \cdot (v \otimes v) \in \dot{H}^{s-1}(\mathbb{R}^3) \hookrightarrow F$.

In general it is very difficult to ensure the existence of classical solutions, unless we are looking for exact solutions (and thus it usually involves the disappearance of the nonlinear term) or we impose some (heavily) restrictive conditions on the initial data.

To overcome the flaws of the classical solution, Leray gave sense to a ”weaker” type of solutions, which finally led to a considerable advancement in the field of existence results. In order to define the notion of weak solutions to the Navier-Stokes system, we shall introduce the notation $\mathbb{P}X$ for a Banach space X, which is exactly the subspace of X characterized by the divergence-free condition (in the literature these spaces are often called solenoidal function spaces). Furthermore, in what follows we shall refer to the function $v(t, \cdot)$ considered only in the space variable as $v(t)$ for short.

Definition 1.3. ***(Weak solution; by Leray and Hopf)*** *A weak solution $v(t, x)$ of the Navier-Stokes equations verifies the following conditions:*

- *$v(t, x) \in L^\infty\Big([0, T); \mathbb{P}L^2(\mathbb{R}^3)\Big) \cap L^2\Big([0, T); \mathbb{P}\dot{H}^1(\mathbb{R}^3)\Big)$,*
- *It verifies the following modified integral equation*

$$\begin{aligned} \int_0^T \Big(-\langle v(s), \partial_t \varphi(s)\rangle + \langle \nabla v(s), \nabla \varphi(s)\rangle + \Big\langle (v(s)\cdot\nabla)v(s), \varphi(s)\Big\rangle\Big)\, ds \\ = \Big\langle v_0, \varphi(0)\Big\rangle + \int_0^T \langle \phi(s), \varphi(s)\rangle\, ds \end{aligned} \tag{1.6}$$

  for any $\varphi \in \mathcal{C}_0^\infty\Big([0, T); \mathbb{P}\mathcal{C}_0^\infty(\mathbb{R}^3)\Big)$. The symbol $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product (in the space variable).

- *It verifies the following energy inequality*

$$\frac{1}{2}||v(t)||_2^2 + \int_0^t ||\nabla v(s)||_2^2\, ds \leq \frac{1}{2}||v(0)||_2^2 + \int_0^t \langle \phi(s), v(s)\rangle\, ds, \quad T > t > 0. \tag{1.7}$$

Definition 1.4. *A solution of the Navier-Stokes equations is called turbulent if it is a weak solution, moreover it satisfies the energy inequality for all intervals $(t_0, t_1) \subset (0, T)$, except possibly for a set of Lebesgue mesure zero.*

The notion of turbulent solutions was introduced by Leray in [15] and it has a huge importance when one wishes to consider self-similar solutions, as we plan to do in chapters 3 and 4. Another useful variant of the weak solution is the so called strong solution:

Definition 1.5. ***(Strong solution)*** *A strong solution $v(t,x)$ of the Navier-Stokes equations is a weak solution that verifies the original equations (1.1) as well, moreover it satisfies the following condition*

$$v(t,x) \in \mathcal{C}^\infty\Big([0,T); \mathcal{C}^\infty(\mathbb{R}^3)\Big).$$

Definition 1.6. ***(Strong solution, weaker form)*** *A strong solution $v(t,x)$ of the Navier-Stokes equations could also be considered as a weak solution that verifies the original equations (1.1), moreover it satisfies the following condition*

$$v(t,x) \in \mathcal{C}\Big([0,T); \dot{H}^2(\mathbb{R}^3)\Big) \cap L^2\Big([0,T); \dot{H}^3(\mathbb{R}^3)\Big).$$

We shall remark that many other strong and weak solutions exist in the literature, the idea is generally the same though: a strong solution is based on a weak solution satisfying the original system (1.1) and belongs to a smaller function space than the one associated to the weak solution itself. Our study will be based on Leray's basic definition of a weak solution and also strong solutions arising from it.

Last but not least, we are going to present the so called mild solution to the Navier-Stokes equations. Since it is based on the integral equation (1.5), the methods applied in treating it differ somewhat from the tools used in handling classical or weak solutions. The importance of mild solutions lies in the fact that it utilises methods akin to the fixed point algorithm (which is stable and constructive) instead of relying on energy estimates.

Definition 1.7. ***(Mild solution; by Masahito Yoshida)*** *A mild solution $v(t,x)$ of the Navier-Stokes equations satisfies the integral equation (1.5) and is such that*

$$v(t,x) \in \mathcal{C}\Big([0,T); \mathbb{P}X\Big),$$

where X is a Banach space of distributions on which the heat semigroup $\{\exp(t\Delta); t \geq 0\}$ is strongly continuous and the integrals in (1.5) are well defined in the sense of Bochner: the integrals exist because the integrals of the norms in X of the respecitve terms exist.

1.3 Some useful aspects of the Fourier analysis

It is noticeable that we have already made an extensive use of various aspects of Fourier analysis in the previous sections just by introducing the proper setting of the Navier-Stokes system: the reformulation of the equations relies heavily on operators based on elementary Fourier transforms and the notion of the solution often required the use of Sobolev spaces (and will require the introduction of even more general spaces, like Besov spaces and Triebel-Lizorkin spaces).

So one can see that it is of greater significance to provide a relatively thorough overview on the most important aspects of Fourier analysis, which we shall do in the following pages.

1.3.1 Fourier transform

Definition 1.8. *For an $f : \mathbb{R}^d \to \mathbb{C}$ function in $L^1(\mathbb{R}^d)$, its Fourier transform is the function $\mathcal{F}f$ (or $\hat{f}$ for short) defined by*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx \quad \forall \xi \in \mathbb{R}^d. \tag{1.8}$$

Proposition 1.1. ***(Inversion formula)*** *For a function $f \in L^1(\mathbb{R}^d)$, if $\hat{f} \in L^1(\mathbb{R}^d)$ holds as well, then we have that*

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi \quad \forall x \in \mathbb{R}^d. \tag{1.9}$$

We shall remark that depending on what is the exact use of the Fourier transform, a couple of other slightly different definitions exist, most of them involve the replacement of the constant $(2\pi)^d$ (for example if one would like to emphasize an isomorphism with the linear operator $\mathcal{F}$ he would prefer to adjust the 'constant multiplier' in the definition of $\mathcal{F}$).

Working with the Fourier transform in the Lebesgue framework can be a bit too restrictive, so we shall consider a much more useful framework for it which will allow us to introduce a large class of distributions over $\mathbb{R}^d$ to whom the notion of Fourier transform can be extended.

Definition 1.9. *The Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ is the set of smooth functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that* $\forall k \in \mathbb{N}$

$$||f||_{k,\mathcal{S}} = \sup_{\substack{x\in\mathbb{R}^d,\\ \alpha\in\mathbb{N}^d, |\alpha|\le k}} \Big(1+|x|\Big)^k |\partial^\alpha f(x)| < \infty.$$

Proposition 1.2. *The operator of Fourier transform $\mathcal{F}$ maps continuously from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ in the sense of Fréchet, that is $\forall k \in \mathbb{N}$ $\exists j \in \mathbb{N}, \exists c_k$ constant such that*

$$||\mathcal{F}\varphi||_{k,\mathcal{S}} \le c_k ||\varphi||_{j,\mathcal{S}} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Theorem 1.1. ***(Fourier-Plancherel)*** *The Fourier transform can be extended to an automorphism of $L^2(\mathbb{R}^d)$ with the inverse defined in the inversion formula. Moreover, for any functions f and g from $L^2(\mathbb{R}^d)$ (or more restrictively from $\mathcal{S}(\mathbb{R}^d)$) we have the following Parseval formula:*

$$\int_{\mathbb{R}^d} f(x)\overline{g}(x)\,dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(x)\overline{\hat{g}}(x)\,dx.$$

For introducing the more general version of the Fourier transform, we shall first recall the dual of the Schwartz space, that is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$:

Definition 1.10. *A tempered distribution over $\mathbb{R}^d$ is a continuous linear functional over $\mathcal{S}(\mathbb{R}^d)$, in other words: by $u \in \mathcal{S}'(\mathbb{R}^d)$ we mean that $\exists k \in \mathbb{N}$ and $\exists c_k \ge 0$ such that*

$$\Big|\langle u, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}\Big| = |u(\varphi)| \le c_k ||\varphi||_{k,\mathcal{S}(\mathbb{R}^d)}.$$

Remark 1.1. *Since we have that $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, we can easily deduce that a tempered distribution is indeed a distribution.*

Based on proposition 1.2 we may infer the following result, which provides the proper definition of the Fourier transform for tempered distributions (in general this statement is true for any $\mathcal{A}$ continuous linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$).

Proposition 1.3. *For $\forall u \in \mathcal{S}'(\mathbb{R}^d)$ and $\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$ let us consider the map ${}^t\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ defined by*

$$\langle {}^t\mathcal{F}u, \varphi\rangle = \langle u, \mathcal{F}\varphi\rangle. \tag{1.10}$$

Then the operator ${}^t\mathcal{F}$ is well-defined, linear and continuous over $\mathcal{S}'(\mathbb{R}^d)$.

Remark 1.2. *In the special case of $u \in L^1(\mathbb{R}^d)$, we have that ${}^t\mathcal{F}u = \mathcal{F}u$ by a simple application of Fubini's theorem. Then, by density argument, we have defined the Fourier transform $\mathcal{F}$ for any $u \in \mathcal{S}'(\mathbb{R}^d)$.*

To end this short part of revision, we shall list some of the most important formulae involving the Fourier transform:

Proposition 1.4. *For any $u, v \in \mathcal{S}'(\mathbb{R}^d)$, any $\lambda > 0$, any $\alpha \in \mathbb{N}^d$, and any $a, \omega \in \mathbb{R}^d$ we have*

- $\mathcal{F}(\partial^\alpha u) = (i\xi)^\alpha \mathcal{F}u$;

- $\mathcal{F}(x^\alpha u) = (i\partial)^\alpha \mathcal{F}u$;

- $\mathcal{F}(\tau_a u)(\xi) = e^{-ia\cdot\xi}\mathcal{F}u(\xi)$, *where* $\tau_a u(x) = u(x-a)$ *denotes a translation;*

- $\Big(\tau_\omega \mathcal{F}u\Big)(\xi) = \mathcal{F}\Big(e^{i\omega\cdot x}u\Big)(\xi)$;

- $\mathcal{F}(u(\lambda\cdot)) = \lambda^{-d}\mathcal{F}u\left(\frac{\cdot}{\lambda}\right)$;

- $\mathcal{F}(u * v) = \mathcal{F}u \cdot \mathcal{F}v$, *where* $*$ *denotes the convolution operator.*

1.3.2 Sobolev spaces

For the rest of this section, we shall recall the definition and some elementary properties of the Sobolev spaces, one of the most common (and naturally arising) function spaces when one has to work with problems related to mathematical physics (such as obviously the Navier-Stokes system).

Definition 1.11. ***(Nonhomogeneous Sobolev space)*** *Let s be a real number. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ is said to belong to the (nonhomogeneous) Sobolev space $H^s(\mathbb{R}^d)$ if*

- $|\hat{u}|^2$ *is Lebesgue measurable,*

- *we have that*
$$||u||_{H^s} = \left(\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} < \infty.$$

Remark 1.3. *In the special case of $s \in \mathbb{N}$ we have that*

$$H^s = \Big\{u \in L^2(\mathbb{R}^d) \text{ such that } \partial^\alpha u \in L^2(\mathbb{R}^d)\ \forall \alpha \in \mathbb{N}^d, |\alpha| \le s\Big\}.$$

Proposition 1.5. *The Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert-space with the following inner product*
$$(u|v)_{H^s} = \int_{\mathbb{R}^d}\Big(1+|\xi|^2\Big)^s\,\hat{u}(\xi)\overline{\hat{v}}(\xi)\,d\xi.$$

Since the nonhomogeneous Sobolev spaces are all Hilbert spaces (independently of the index s), we may obtain various other useful and "unrestricted" properties.

Proposition 1.6. *For $s < t$ arbitrary real numbers, we have that*

1. *the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of test functions and the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ are dense in $H^s(\mathbb{R}^d)$.*
2. *the $H^s \hookrightarrow H^t$ continuous embedding holds.*

3. for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the map $u \mapsto \varphi u$ is continuous on $H^s(\mathbb{R}^d)$.

Proposition 1.7. ***(Interpolation, nonhomogeneous case)*** *For any $s_1, s_2 \in \mathbb{R}$ and $\theta \in [0,1]$ if $u \in H^{s_1}(\mathbb{R}^d) \cap H^{s_2}(\mathbb{R}^d)$ and if we denote $s = \theta s_1 + (1-\theta)s_2$ then we have that $u \in H^s(\mathbb{R}^d)$, moreover*

$$||u||_{H^s} \leq ||u||_{H^{s_1}}^{\theta} ||u||_{H^{s_2}}^{1-\theta}.$$

Proposition 1.8. ***(Duality, nonhomogeneous case)*** *Let $s \in \mathbb{R}$ and $u \in H^{-s}(\mathbb{R}^d)$. Then the functional L_u defined on $\mathcal{S}(\mathbb{R}^d)$ by*

$$L_u(\varphi) = \langle u, \varphi \rangle_{\mathcal{S}' \times \mathcal{S}}$$

can be extended to a continuous linear functional $\langle u, \cdot \rangle_{H^{-s} \times H^s}$ on $H^s(\mathbb{R}^d)$. Reciprocally if $L \in \left(H^s(\mathbb{R}^d)\right)'$ then there exists a unique $u \in H^{-s}(\mathbb{R}^d)$ for which $\forall \varphi \in H^s(\mathbb{R}^d)$ we have that $\langle u, \varphi \rangle_{H^{-s} \times H^s} = L(\varphi)$. Furthermore, the map $u \mapsto (2\pi)^d \langle u, \cdot \rangle_{H^{-s} \times H^s}$ is an isometric isomorphism from $H^{-s}(\mathbb{R}^d)$ to $\left(H^s(\mathbb{R}^d)\right)'$.

We shall also provide some of the more well-known embedding theorems related to the Sobolev spaces:

Theorem 1.2. ***(Sobolev embeddings, nonhomogeneous case)***

1. *If $0 \leq s < \frac{d}{2}$ then $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for every $p \in [2, p_c]$, where the critical exponent is defined by*
$$-\frac{d}{p_c} = -\frac{d}{2} + s. \tag{1.11}$$
2. *If $s = \frac{d}{2}$ then $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for every $p \in [2, \infty)$.*
3. *If $s > \frac{d}{2}$ then $H^s(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^1(\mathbb{R}^d)$, where $\mathcal{F}L^1(\mathbb{R}^d)$ denotes the space of distributions such that their Fourier transform is in $L^1(\mathbb{R}^d)$.*

Theorem 1.3. ***(Compact "embedding")*** *For $0 < s < \frac{d}{2}$ and $2 \leq p < p_c$ (with the p_c critical exponent defined by (1.11)) we have that the map $\mathrm{Id} : H^s(\mathbb{R}^d) \to L^p_{loc}(\mathbb{R}^d)$ is compact, that is $\mathrm{Id} : H^s(\mathbb{R}^d) \to L^p(B(0,R))$ is compact for $\forall R > 0$.*

More precisely, if $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^d)$ then $\exists (u_{\varphi(n)})_{n\in\mathbb{N}}$ subsequence such that $\exists u \in H^s(\mathbb{R}^d)$,

1. *$u_{\varphi(n)} \rightharpoonup u$ in $H^s(\mathbb{R}^d)$,*
2. *$||u||_{H^s} \leq \liminf_{n\to\infty} ||u_{\varphi(n)}||_{H^s}$,*
3. *$u_{\varphi(n)} \to u$ (strong convergence) in $L^p(B(0,R))$ $\forall R > 0$.*

Remark 1.4. *The theorem is false if $p = p_c$ (the famous profile decomposition theorem). The theorem is also false if we omit the localisation of the Lebesgue space!*

These are just some of the more remarkable properties of the nonhomogeneous Sobolev spaces. Despite all these handy facts one major drawback of them is that they are not invariant under any type of scaling (which is one of the most important properties of the Navier-Stokes equations, and many other equations motivated by physics). Thankfully the homogeneous counterpart of the Sobolev spaces has the scaling invariance property, although we have to sacrifice some of the more useful properties of the nonhomogeneous Sobolev spaces.

Definition 1.12. ***(Homogeneous Sobolev space)*** *Let s be a real number. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ is said to belong to the (nonhomogeneous) Sobolev space $\dot{H}^s(\mathbb{R}^d)$ if*

- *$\hat{u}$ belongs to $L^1_{loc}(\mathbb{R}^d)$,*
- *we have that*

$$||u||_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^2 s |\hat{u}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} < \infty.$$

The quantity $|| \cdot ||_{\dot{H}^s}$ is (in general) only a seminorm, furthermore unlike the nonhomogeneous Sobolev spaces, these spaces can't be compared for the inclusion with each other. Nevertheless we have that

Proposition 1.9. *For $s \in \mathbb{R}$ we have that*

1. *if $s > 0$ then $||u||_{\dot{H}^s} \leq ||u||_{H^s}$,*
2. *if $s = 0$ then $\dot{H}^0(\mathbb{R}^d) = H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$,*
3. *if $s < 0$ then $||u||_{H^s} \leq ||u||_{\dot{H}^s}$.*

Another relatively 'sad' fact is the following statement:

Proposition 1.10. *The space $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert-space if and only if $s < \frac{d}{2}$.*

Proposition 1.11. *The space $\dot{H}^s(\mathbb{R}^d)$ can be defined as the completion of $\mathcal{S}(\mathbb{R}^d)$ for $|| \cdot ||_{\dot{H}^s}$ only in the case of $|s| \leq \frac{d}{2}$. This can be slightly generalized, that is the space $\dot{H}^s(\mathbb{R}^d)$ can be defined as the completion of $\mathcal{S}_0(\mathbb{R}^d)$ for $|| \cdot ||_{\dot{H}^s}$ if $s < \frac{d}{2}$ with*

$$\mathcal{S}_0(\mathbb{R}^d) = \left\{u \in \mathcal{S}(\mathbb{R}^d);\ 0 \notin \operatorname{supp} \hat{u}\right\}. \tag{1.12}$$

The good news is that the interpolation inequality is still not affected by the value of s, so we still have that

Proposition 1.12. ***(Interpolation, homogeneous case)*** *For any $s_1, s_2 \in \mathbb{R}$ and $\theta \in [0, 1]$ if $u \in \dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d)$ and if we denote $s = \theta s_1 + (1 - \theta)s_2$ then we have that $u \in \dot{H}^s(\mathbb{R}^d)$, moreover*

$$||u||_{\dot{H}^s} \leq ||u||^{\theta}_{\dot{H}^{s_1}} ||u||^{1-\theta}_{\dot{H}^{s_2}}.$$

The bad news is that the duality theorem and the embeding theorems are all under the "influence" of the index s, so we are forced to modify them.

Proposition 1.13. ***(Duality, homogeneous case)*** *If $|s| < \frac{d}{2}$, then the bilinear functional*

$$\mathcal{B} : \begin{cases} \mathcal{S}_0(\mathbb{R}^d) \times \mathcal{S}_0(\mathbb{R}^d) \to \mathbb{C} \\ (\phi, \varphi) \mapsto \int_{\mathbb{R}^d} \phi(x)\varphi(x)\, dx \end{cases}$$

can be extended to a continuous bilinear functional on $\dot{H}^{-s}(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$. Moreover, if L is a continuous linear functional on $\dot{H}^s(\mathbb{R}^d)$, a unique tempered distribution u esists in $\dot{H}^{-s}(\mathbb{R}^d)$ such that

$$\mathcal{B}(u, \varphi) = \langle L, \varphi \rangle \quad \forall \varphi \in \dot{H}^s(\mathbb{R}^d);$$
$$||L||_{(\dot{H}^s)'} = ||u||_{\dot{H}^{-s}}.$$

Theorem 1.4. ***(Sobolev embeddings, homogeneous case)*** *For $0 \leq s < \frac{d}{2}$ and p_c given by (1.11) as before, we have that $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^{p_c}(\mathbb{R}^d)$.*

1.4 Functional setting of the equations

From a historical point of view, the systematic use of harmonic analysis tools paved the way for important discoveries for the Navier-Stokes system: the existence of a global solution for "highly oscillating data", the uniqueness of this solution, and its asymptotic behaviour via the existence of self-similar solutions, the latter of which will serve as one of the main goals of this work.

So, partly as a straighforward continuation of the previous section, we shall progress further into the field of harmonic analysis, revising some of the quite recent results and theories, providing a supplement to the various useful tools in our hands in order to start properly handling the Navier-Stokes equations.

1.4.1 The Littlewood-Paley decomposition

We shall start with a quite recent localisation technique, the dyadic decomposition of John E. Littlewood and Raymond Paley. Based on proposition 1.4, we have that a derivative counts more or less as a multiplication by the variable (that is, a dilatation of some kind) in the Fourier space, so it provides a good motivation to how one should start handling the derivatives' norms for example with partial differential equations. This property is even more tangible if we only consider functions whose Fourier transform is supported in a ball or annulus. And this is where the Littlewood-Paley decomposition will come in handy since it will provide a relatively easy device which allows us to split a function into a sum of well localised functions. Furthermore, providing a proper tool for a reasonable localisation is always useful with these types of problems, because often it is easier to handle a problem locally than globally.

At the base of the theory lies an even more elementary observation, more exactly a pair of observations attributed to Sergei N. Bernstein, that describe explicitely the good behaviour of a derivative in the L^p norms.

Proposition 1.14. ***(Direct Bernstein inequality)*** *Let* $R > 0$*. Let us suppose that* $\operatorname{supp} \hat{u} \subset B(0, \lambda R)$ *for some* $\lambda > 0$*, then there exists a constant* C *such that for every* $1 \leq p \leq q \leq \infty$ *and* $\forall k \in \mathbb{N}$

$$\left|\left|D^k u\right|\right|_{L^q} \leq C \lambda^{k + \frac{d}{p} - \frac{d}{q}} ||u||_{L^p}, \tag{1.13}$$

where D^k *denotes a derivation of order* k*.*

Proposition 1.15. ***(Indirect Bernstein inequality)*** *Let* $0 < R_1 < R_2$*. Let us suppose that* $\operatorname{supp} \hat{u} \subset \left\{\xi \in \mathbb{R}^d; R_1 \lambda \leq |\xi| \leq R_2 \lambda\right\}$ *for some* $\lambda > 0$*, then there exists a constant* C' *such that for every* $p \in [1, \infty]$ *and* $\forall k \in \mathbb{N}$

$$(C')^{-1} \lambda^k ||u||_{L^p} \leq \left|\left|D^k u\right|\right|_{L^p} \leq C' \lambda^k ||u||_{L^p}. \tag{1.14}$$

As part of the fundaments of the Littlewood-Paley decomposition we should also mention the so called dyadic unit partition, a variant of which is the following proposition.

Proposition 1.16. ***(Dyadic unit decomposition)*** *Let* $C = \left\{\xi \in \mathbb{R}^d; \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ *(an annulus), then there exist two smooth radial* $\mathbb{R}^d \to [0,1]$ *functions* χ *and* φ *such that* $\operatorname{supp} \chi \subset B\left(0, \frac{4}{3}\right)$ *and* $\operatorname{supp} \varphi \subset C$*, moreover the following properties hold:*

1. *Nonhomogeneous partition:* $\forall \xi \in \mathbb{R}^d$

$$1 = \chi(\xi) + \sum_{j \geq 0} \varphi\left(2^{-j}\xi\right);$$

2. *Homogeneous partition:* $\forall \xi \in \mathbb{R}^d \backslash \{0\}$

$$1 = \sum_{j \in \mathbb{Z}} \varphi\left(2^{-j}\xi\right);$$

3. *For any* $j, j' \in \mathbb{Z}$ *such that* $|j - j'| \geq 2$ *we have that* $\operatorname{supp} \varphi\left(2^{-j}\cdot\right) \cap \operatorname{supp} \varphi(2^{-j'}\cdot) = \emptyset$*, furthermore for any* $k \in \mathbb{N}$*,* $k \geq 1$ *we have that* $\operatorname{supp} \chi \cap \operatorname{supp} \varphi\left(2^{-k}\cdot\right) = \emptyset$*.*

4. Nonhomogeneous L^2 norm estimate: $\forall \xi \in \mathbb{R}^d$

$$\frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2\left(2^{-j}\cdot\right) \leq 1.$$

5. Homogeneous L^2 norm estimate: $\forall \xi \in \mathbb{R}^d \backslash \{0\}$

$$\frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2\left(2^{-j}\cdot\right) \leq 1.$$

Let us remark that other, considerably different unit partition functions can be constructed as well (with differing hypotheses, that is) but we shall stick to the aforementioned version, mainly because of the smoothness condition that it provides (even though it leads to another drawback which will be mentioned shortly). We may also suppose that the two functions defined in the previous proposition verify the following additional equality (without loss of generality)

$$\varphi(\xi) = \chi\left(\frac{\xi}{2}\right) - \chi(\xi) \quad \forall \xi \in \mathbb{R}^d.$$

Definition 1.13. ***(Nonhomogeneous truncation/cut-off operators)*** *Let us introduce the following operators for a $u \in L^p(\mathbb{R}^d)$ function:*

- *For $j \geq 0$ let us denote by Δ_j the following operation:*

$$\Delta_j u \left(= \varphi\left(2^{-j}D\right)u\right) = \mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\mathcal{F}u\right) = 2^{jd}h\left(2^j\cdot\right) * u \text{ with } h = \mathcal{F}^{-1}\varphi.$$

- *For $j = -1$ we shall consider Δ_{-1} defined by*

$$\Delta_{-1}u\left(= \chi(D)u\right) = \mathcal{F}^{-1}\left(\chi\mathcal{F}u\right) = \check{h} * u \text{ with } \check{h} = \mathcal{F}^{-1}\chi.$$

- *And for the sake of completeness we shall pose $\Delta_j = 0$ for $j \leq -2$.*

- *We may also define the low frequency operators, denoted by S_j $\forall j \in \mathbb{Z}$, and defined by*

$$S_j u\left(= \chi\left(2^{-j}D\right)u\right) = \mathcal{F}^{-1}\left(\chi\left(2^{-j}\cdot\right)\mathcal{F}u\right) = 2^{jd}\check{h}\left(2^j\cdot\right) * u.$$

Definition 1.14. ***(Homogeneous truncation/cut-off operators)*** *Let us introduce the following operators for a $u \in L^p(\mathbb{R}^d)$ function:*

- *For $j \in \mathbb{Z}$ let us denote by $\dot{\Delta}_j$ the following operation:*

$$\dot{\Delta}_j u\left(= \varphi\left(2^{-j}D\right)u\right) = \mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\mathcal{F}u\right) = 2^{jd}h\left(2^j\cdot\right) * u.$$

- *We may also define the corresponding low frequency operators, denoted by $\dot{S}_j$ $\forall j \in \mathbb{Z}$, defined by*

$$\dot{S}_j u \left(= \chi\left(2^{-j} D\right) u\right) = \mathcal{F}^{-1}\left(\chi\left(2^{-j}\cdot\right)\mathcal{F}u\right) = 2^{jd}\check{h}\left(2^{j}\cdot\right) * u.$$

Remark 1.5. *The operators Δ_j, S_j, $\dot{\Delta}_j$ and $\dot{S}_j$ are well defined, continuous on $L^p(\mathbb{R}^d)$ and their norm does not depend on j itself! Moreover these definitions can be trivially extended to the case where $u \in \mathcal{S}'(\mathbb{R}^d)$ with similar properties.*

Lemma 1.1. *The operators Δ_j and $\dot{\Delta}_j$ are not L^2 orthogonal projectors, however they still verify the following quasi-orthogonality property*

$$\Delta_j \Delta_k = \dot{\Delta}_j \dot{\Delta}_k = 0 \text{ if } |j-k| > 1.$$

With these operators at our disposal we are now able to state the main theorem concerning the Litlewood-Paley decomposition.

Theorem 1.5. ***(Littlewood-Paley decomposition)*** *Let us consider a $u \in \mathcal{S}'(\mathbb{R}^d)$ tempered distribution.*

1. *Nonhomogeneous case: without any restriction the following equality holds*

$$u = \sum_{j \geq -1} \Delta_j u. \tag{1.15}$$

2. *Homogeneous case: the below equality always holds modulo polynomials*

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u. \tag{1.16}$$

The meaning of this equality is somewhat subtle, so we shall reformulate it with the following restriction: the above mentioned equality is true for any $u \in \mathcal{S}_h'(\mathbb{R}^d)$, where

$$\mathcal{S}_h'(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d);\ \lim_{j \to -\infty} \left\| \dot{S}_j u \right\|_{L^\infty} = 0 \right\}.$$

1.4.2 Besov spaces

Now that we have provided a somewhat brief overview on the topic of the Littlewood-Paley decomposition, we might as well take another step into harmonic analysis and introduce (or possibly revise) some of the more fundamental notions of the theory surrounding the Besov spaces. The structure of this part is going to be similar to the section handling the Sobolev spaces, since (for obvious reasons) we will be able to distinguish two different types of Besov spaces: nonhomogeneous and homogeneous.

The general remarks on the comparison between the homogeneous and nonhomogeneous version of the Sobolev spaces still remain true in the case of Besov spaces: the nonhomogeneous Besov spaces have slightly better properties but again they are missing the scaling invariance property. Also it is worth mentioning that now the differences between these two types is mostly based on the fact that the decomposition with the homogeneous truncation operators only works for the subspace $\mathcal{S}_h'(\mathbb{R}^d)$, thus resulting in relatively minor differences.

Definition 1.15. ***(Nonhomogeneous Besov space)*** *For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, $1 \leq r \leq \infty$ we set*

$$||u||_{B^s_{p,r}} = \left(\sum_{j \geq -1} 2^{jrs} ||\Delta_j u||^r_{L^p} \right)^{\frac{1}{r}} \tag{1.17}$$

and define the nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ as the subset of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $||u||_{B^s_{p,r}} < \infty$.

In the following theorems we shall list the classical and important properties of the nonhomogeneous Besov spaces.

Theorem 1.6. ***(Elementary properties, nonhomogeneous case)*** *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$*

1. *the Besov space $B^s_{p,r}(\mathbb{R}^d)$ is complete;*
2. *(Fatou property) if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions from $B^s_{p,r}(\mathbb{R}^d)$ that converges in $\mathcal{S}'(\mathbb{R}^d)$ to $u \in \mathcal{S}'(\mathbb{R}^d)$, then $u \in B^s_{p,r}(\mathbb{R}^d)$, moreover $||u||_{B^s_{p,r}} \leq$ $\leq C \liminf_{n\to\infty} ||u_n||_{B^s_{p,r}}$ with a suitable constant C;*
3. *(Duality) if u is in $\mathcal{S}'(\mathbb{R}^d)$ then we have that*

$$||u||_{B^s_{p,r}} \leq C \sup_{\varphi} \langle u, \varphi \rangle$$

   where the supremum is taken over those φ in $\mathcal{S}(\mathbb{R}^d) \cap B^{-s}_{p',r'}(\mathbb{R}^d)$ such that $||\varphi||_{B^{-s}_{p',r'}} \leq 1$. Here p' and r' denote the coefficients satisfying the equalities

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{r} + \frac{1}{r'} = 1;$$

4. *(Interpolation) for $s_1 \neq s_2$, $\theta \in (0,1)$, and $1 \leq p \leq \infty$ we have*

$$||u||_{B^{\theta s_1 + (1-\theta)s_2}_{p,1}} \leq C' ||u||^{\theta}_{B^{s_1}_{p,\infty}} ||u||^{1-\theta}_{B^{s_2}_{p,\infty}};$$

5. *The (partial) derivative operator ∂_k maps $B^s_{p,r}(\mathbb{R}^d)$ in $B^{s-1}_{p,r}(\mathbb{R}^d)$.*

Theorem 1.7. ***(Besov embeddings, nonhomogeneous case)***

1. *For $s \in \mathbb{R}$, $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$ we have*

$$B^s_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow B^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}(\mathbb{R}^d). \tag{1.18}$$

2. *For $s' < s$ and any $1 \le p, r_1, r_2 \le \infty$, the embedding of $B^s_{p,r_1}(\mathbb{R}^d)$ in $B^{s'}_{p,r_2}(\mathbb{R}^d)$ is locally compact. In particular, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $u \mapsto \varphi u$ is compact from $B^s_{p_1,r_1}(\mathbb{R}^d)$ to $B^{s'-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}(\mathbb{R}^d)$.*
3. *The Besov space $B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ is continuously embedded in the set $\mathcal{C}_b(\mathbb{R}^d)$ of bounded continuous functions. If $p < \infty$ it is also embedded in the set $\mathcal{C}_0(\mathbb{R}^d)$ of continuous functions goint to 0 at infinity.*

As already mentioned before, the homogeneous counterpart of the Besov spaces has similar properties except for the homogeneity property stated below. One of the more frequently used function spaces is going to be a member of this family during the upcoming chapters.

Definition 1.16. ***(Homogeneous Besov space)*** *For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$ we set*

$$||u||_{\dot{B}^s_{p,r}} = \left|\left| 2^{js} ||\dot{\Delta}_j u||_{L^p(\mathbb{R}^d)} \right|\right|_{l^r(\mathbb{Z})} \tag{1.19}$$

and define the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ as the subset of tempered distributions $u \in \mathcal{S}'_h(\mathbb{R}^d)$ such that $||u||_{\dot{B}^s_{p,r}} < \infty$.

Remark 1.6. *With easy calculations one can prove that $\dot{B}^s_{2,2}(\mathbb{R}^d) = \dot{H}^s(\mathbb{R}^d)$ (and the same result holds in the nonhomogeneous case as well).*

Theorem 1.8. ***(Elementary properties, homogeneous case)*** *For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$*

1. *the Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is complete if and only if $s < \frac{d}{p}$ or $\Big(s \le \frac{d}{p}$ and $r = 1\Big)$;*
2. *(Fatou property) if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions from $\dot{B}^s_{p,r}(\mathbb{R}^d)$ that converges in $\mathcal{S}'(\mathbb{R}^d)$ to $u \in \mathcal{S}'_h(\mathbb{R}^d)$, then $u \in \dot{B}^s_{p,r}(\mathbb{R}^d)$, moreover $||u||_{\dot{B}^s_{p,r}} \le$ $\le C \liminf_{n\to\infty} ||u_n||_{\dot{B}^s_{p,r}}$ with a suitable constant C;*
3. *(Duality) if u is in $\mathcal{S}'_h(\mathbb{R}^d)$ then we have that*

$$||u||_{\dot{B}^s_{p,r}} \le C \sup_{\varphi} \langle u, \varphi \rangle$$

where the supremum is taken over those φ in $\mathcal{S}(\mathbb{R}^d) \cap \dot{B}^{-s}_{p',r'}(\mathbb{R}^d)$ such that $||\varphi||_{\dot{B}^{-s}_{p',r'}} \le 1$;

4. *(Interpolation) for* $s_1 \neq s_2$, $\theta \in (0,1)$, *and* $1 \leq p \leq \infty$ *we have*

$$||u||_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq C'||u||^{\theta}_{\dot{B}^{s_1}_{p,\infty}}||u||^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}};$$

5. *The (partial) derivative operator* ∂_k *maps* $\dot{B}^s_{p,r}(\mathbb{R}^d)$ *in* $\dot{B}^{s-1}_{p,r}(\mathbb{R}^d)$.

Theorem 1.9. ***(Besov embeddings, homogeneous case)***

1. *For any* $p \in [1,\infty]$ *we have the following chain of continuous embeddings:*

$$\dot{B}^0_{p,\min(p,2)}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow \dot{B}^0_{p,max(p,2)}(\mathbb{R}^d).$$

2. *For* $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ *and* $1 \leq r_1 \leq r_2 \leq \infty$ *we have*

$$\dot{B}^s_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}(\mathbb{R}^d). \tag{1.20}$$

3. *The Besov space* $\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ *is continuously embedded in the set* $\mathcal{C}_b(\mathbb{R}^d)$. *If* $p < \infty$ *it is also embedded in the set* $\mathcal{C}_0(\mathbb{R}^d)$.

Proposition 1.17. ***(Homogeneity property)*** *For any* $s \in \mathbb{R}$ *and any* $1 \leq p \leq \infty$, $1 \leq r \leq \infty$ *there exists a constant* C *such that for all positive* λ *and* $u \in \dot{B}^s_{p,r}(\mathbb{R}^d)$, *we have*

$$C^{-1}\lambda^{s-\frac{d}{p}}||u||_{\dot{B}^s_{p,r}} \leq ||u(\lambda\cdot)||_{\dot{B}^s_{p,r}} \leq C\lambda^{s-\frac{d}{p}}||u||_{\dot{B}^s_{p,r}}. \tag{1.21}$$

Apart from the scaling invariance of the homogeneous Besov spaces, the versatility of the definition provides additional beneficial tools for us. In order to state the alternative definitions, let us first denote by $S(t)$ the heat semigroup (again without the Leray-Hopf operator $\mathbb{P}$). Additionally let us denote by Θ the function for which $\hat{\Theta}(\xi) = |\xi|\exp(-|\xi|^2)$. Now, if we define $\Theta_t = t^{-3}\Theta\left(\frac{\cdot}{t}\right)$ then we have the following:

Proposition 1.18. *For* $s < 0$ *and* $1 \leq p,r \leq \infty$ *the following four quantities are equivalent and can all be used to express the Besov norm for* $u \in \dot{B}^s_{p,r}(\mathbb{R}^3)$.

$$\left(\sum_{j\in\mathbb{Z}}\left(2^{sj}||\dot{\Delta}_j u||_{L^p}\right)^r\right)^{\frac{1}{r}},$$

$$\left(\sum_{j\in\mathbb{Z}}\left(2^{sj}||\dot{S}_j u||_{L^p}\right)^r\right)^{\frac{1}{r}},$$

$$\left(\int_0^\infty t^{-1}\left(t^{-\frac{s}{2}}||S(t)u||_{L^p}\right)^r dt\right)^{\frac{1}{r}},$$

$$\left(\int_0^\infty t^{-1}\left(t^{-s}||\Theta_t u||_{L^p}\right)^r dt\right)^{\frac{1}{r}}.$$

Finally, we shall turn our attention to a seemingly different question: given u and v belonging to two Besov spaces, what can we say about their product uv? A formal answer to this question was given by Jean-Michel Bony in 1987, and even though his method has been greatly improved since, the following so called Bony's paraproduct rule still bears his name.

Proposition 1.19. *We have the following formal equality for any u and v distributions:*

$$uv = T_u v + T_v u + R(u,v), \tag{1.22}$$

with

$$\begin{aligned} T_u v &= \sum_{j\in\mathbb{Z}} S_{j-1} u \Delta_j v \quad \text{(paraproduct term)} \\ R(u,v) &= \sum_{j\in\mathbb{Z}} \sum_{|j'-j|\leq 1} \Delta_j u \Delta_{j'} v \quad \text{(remainder term)} \end{aligned}$$

Remark 1.7. *A similar decomposition holds for the homogeneous case with the restriction $u, v \in \mathcal{S}'_h(\mathbb{R}^d)$. In this case however, the existence of the remainder term is still not guaranteed.*

This decomposition has the advantage that it separates the product into three separate terms, two of them have considerably different properties thus allowing the use of different methods on each term. In the following proposition we shall list some of the most common continuity properties for Bony's paraproduct rule.

Proposition 1.20. *For any $s \in \mathbb{R}$, and any $1 \leq p, r \leq \infty$ we have*

- $T : L^\infty(\mathbb{R}^d) \times B^s_{p,r}(\mathbb{R}^d) \to B^s_{p,r}(\mathbb{R}^d)$ *continuous and bilinear,*
- $T : B^{-t}_{\infty,\infty}(\mathbb{R}^d) \times B^s_{p,r}(\mathbb{R}^d) \to B^{s-t}_{p,r}(\mathbb{R}^d)$ *continuous and bilinear for any $t > 0$.*

For any $s_1, s_2 \in \mathbb{R}$, and any $1 \leq p_1, p_2, r_1, r_2 \leq \infty$ we have

- *for the case of $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$ then $R : B^{s_1}_{p_1,r_1}(\mathbb{R}^d) \times B^{s_2}_{p_2,r_2}(\mathbb{R}^d) \to B^{s_1+s_2}_{p,r}(\mathbb{R}^d)$ continuous and bilinear,*
- *for the case of $s_1 + s_2 = 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, and $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$ then $R : B^{s_1}_{p_1,r_1}(\mathbb{R}^d) \times B^{s_2}_{p_2,r_2}(\mathbb{R}^d) \to B^{0}_{p,\infty}(\mathbb{R}^d)$ continuous and bilinear.*

(Similar results hold for the homogeneous case.)

A straightforward application of this proposition leads to the following corollary.

Corollary 1.1. *Let $s > 0$ and $1 \leq p, r \leq \infty$. For any $u, v \in \left(L^\infty(\mathbb{R}^d) \cap B^s_{p,r}(\mathbb{R}^d)\right)$ there exists a constant C (depending only on d, p and s) such that*

$$||uv||_{B^s_{p,r}} \leq C\left(||u||_{L^\infty}||v||_{B^s_{p,r}} + ||v||_{L^\infty}||u||_{B^s_{p,r}}\right)$$

(Similar results hold for the homogeneous case.)

Remark 1.8. *Consequently, since $B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ is embedded in $L^\infty(\mathbb{R}^d)$, we deduce that whenever $p < \infty$, the Besov space $B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ is an algebra. (Same result holds for the homogeneous case.) Furthermore we may also conclude that any Besov space $B^s_{p,r}(\mathbb{R}^d)$ with $s > \frac{d}{p}$ is stable for the product.*

1.4.3 Other useful function spaces

A not that well-known function space is the so called Triebel-Lizorkin space. Esentially it could be regarded as the "twin brother" of the Besov spaces, since the definition is so similar and has similar properties as well, still it is a bit harder to work with them when dealing with exact computations.

Definition 1.17. ***(Triebel-Lizorkin space, nonhomogeneous version)*** *Let $s \in \mathbb{R}$, $1 \leq p < \infty$, and $1 \leq r \leq \infty$. Then a tempered distribution u belongs to the nonhomogeneous Triebel-Lizorkin space $F^s_{p,r}(\mathbb{R}^d)$ if and only if*

$$\left|\left|\left(\sum_{j\geq -1}(2^s j|\Delta_j u|)^r\right)^{\frac{1}{r}}\right|\right|_{L^p(\mathbb{R}^d)} < \infty.$$

Definition 1.18. ***(Triebel-Lizorkin space, homogeneous version)*** *Let $s \in \mathbb{R}$, $1 \leq p < \infty$, and $1 \leq r \leq \infty$. Then a tempered distribution u belongs to the nonhomogeneous Triebel-Lizorkin space $\dot{F}^s_{p,r}(\mathbb{R}^d)$ if and only if $u \in \mathcal{S}'_h(\mathbb{R}^d)$ and*

$$\left|\left|\left(\sum_{j\in\mathbb{Z}}\left(2^s j|\dot{\Delta}_j u|\right)^r\right)^{\frac{1}{r}}\right|\right|_{L^p(\mathbb{R}^d)} < \infty.$$

For the Triebel-Lizorkin spaces we have the same interpolation inequality and the same embedding theorems as for the corresponding Besov spaces. Moreover, parallel to the homogeneous Besov spaces in $\mathbb{R}^3$ the homogeneous Triebel-Lizorkin spaces also have somewhat useful equivalent definitions:

Proposition 1.21. *For $s < 0$, $1 \leq p < \infty$, and $1 \leq r \leq \infty$ the following four quantities are equivalent and can all be used to express the Triebel-Lizorkin norm* *for* $u \in \dot{F}^s_{p,r}(\mathbb{R}^3)$.

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(2^{sj} |\dot{\Delta}_j u| \right)^r \right)^{\frac{1}{r}} \right\|_{L^p},$$
$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(2^{sj} |\dot{S}_j u| \right)^r \right)^{\frac{1}{r}} \right\|_{L^p},$$
$$\left\| \left(\int_0^\infty t^{-1} \left(t^{-\frac{s}{2}} |S(t)u| \right)^r \, dt \right)^{\frac{1}{r}} \right\|_{L^p},$$
$$\left\| \left(\int_0^\infty t^{-1} \left(t^{-s} |\Theta_t u| \right)^r \, dt \right)^{\frac{1}{r}} \right\|_{L^p}.$$

Remark 1.9. *Like with the Besov spaces, the Triebel-Lizorkin spaces are also the generalisations of the Lebesgue and Sobolev spaces, that is: it can be shown that for* $1 < p < \infty$, $L^p(\mathbb{R}^d) = \dot{F}^0_{p,2}(\mathbb{R}^d)$ *and* $\dot{H}^s(\mathbb{R}^d) = \dot{F}^s_{2,2}(\mathbb{R}^d)$ $(s \in \mathbb{R})$*.*

Another interesting function space, which could be regarded as a certain generalisation of a special case of the Triebel-Lizorkin spaces for $p = \infty$ (for $s = 0$ and $r = 2$) as well, is the so called Bounded Mean Oscillation (BMO) space.

Definition 1.19. *The* BMO *norm of a function* f *is defined by the following quantity*

$$||f||_{BMO} = \sup_{B\in\mathcal{B}} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^2 \, dx \right)^{\frac{1}{2}},$$

where $\mathcal{B}$ *stands for the set of Euclidean balls,* $\mu(B)$ *is the volume of the ball* B *and* f_B *denotes the average of the function* f *over the ball* B*, say*

$$f_B = \frac{1}{\mu(B)} \int_B f(x) \, dx.$$

Remark 1.10. $L^\infty(\mathbb{R}^d) \hookrightarrow BMO$ *but these two spaces are different.*

Definition 1.20. *The space* BMO^{-1} *(or* ∇BMO*) is defined by the tempered distributions* u *such that there exists a vector function* $g = (g_1, \ldots, g_d)$ *belonging to* $(BMO)^d$ *such that* $f = \nabla \cdot g$*. The norm of* BMO^{-1} *is defined by*

$$||u||_{BMO^{-1}} = \inf_{g\in BMO^d} \sum_{j=1}^d ||g_j||_{BMO}.$$

The BMO and BMO^{-1} function spaces play an important role in the handling of the so called critical spaces. They also have a considerable amount of importance in the establishment of advanced asymptotic results as well as in certain types of weak regularity criteria.

For the time being we shall mention one more family of function spaces, the Lorentz spaces, which can be considered as a not entirely obvious generalisation of the Lebesgue spaces.

Definition 1.21. ***(Lorentz spaces)*** *Let $1 \le p, q \le \infty$, then a function f belongs to the Lorentz space $L^{(p,q)}(\mathbb{R}^d)$ if and only if we have that*

$$||f||_{L^{(p,q)}} = \left(\frac{q}{p} \int_0^\infty t^{-1} \left(t^{\frac{1}{p}} f^*(t) \right)^q \, dt \right)^{\frac{1}{q}} < \infty,$$

where f^* is the decreasing rearrangement of f:

$$f^*(t) = \inf\{s \ge 0; |\{|f| > s\}| \le t\}, \quad t \ge 0.$$

(For $q = \infty$, we pose $||f||_{L^{(p,\infty)}} = \sup\limits_{t>0} \left(t^{\frac{1}{p}} f^*(t) \right)$.)

Remark 1.11. *In general, the quantity $||.||_{L^{(p,q)}}$ is not a norm.*

Remark 1.12. *If $p = q$ then $L^{(p,p)}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$.*

1.5 Homogeneity of the equations

A remarkable property of the Navier-Stokes equations is their homogeneity to certain types of rescaling. This invariant structure not only allows us to disregard the viscosity coefficient ν in the equation but it provides a solid foothold for the notion of critical spaces introduced later in this section. Also it provides a somewhat vague motivation for the self-similar solutions introduced in later chapters.

Let us consider $v(t, x)$ and $p(t, x)$ functions that solve the system (1.1) (with the external force ϕ equaling zero) in all $(0, \infty) \times \mathbb{R}^3$. One can easily see that in this case, if we consider a $\lambda > 0$ constant (it could be negative as well) and by regarding the functions $v(\lambda t, \lambda x)$ and $p(\lambda t, \lambda x)$, these new functions solve a slightly different Navier-Stokes system (trivial application of the chain rule). These new functions are solutions of a Navier-Stokes system where ν is replaced by $\lambda\nu$. This allows us to make the assumption of $\nu = 1$, as we have already done in the very beginning of the chapter (for an arbitrary positive ν we may find a $\lambda > 0$ such that $\lambda\nu = 1$).

The truly remarkable scaling invariance lies in a slightly different pair of modified functions. Let us consider the following rescaled functions:

$$v_\lambda(t, x) = \lambda v\Big(\lambda^2 t, \lambda x\Big), \qquad p_\lambda(t, x) = \lambda^2 p\Big(\lambda^2 t, \lambda x\Big).$$

It is really easy to see that the new functions v_λ and p_λ solve exactly the same system (1.1) (with the external force ϕ equaling zero)! This fundamental and intrinsic propery of the Navier-Stokes equations has many different and interesting consequences. A far-reaching one will be that it provides the very basis of the notion of self-similar solutions which will be the center of our interests during the second half of the present paper.

Another important consequence is that it provides a reason to consider the so-called critical spaces to the system as well as subcritical and supercritical spaces. We shall only give a proper definition for the first type of function spaces, mainly because of the ambiguity that lies within the latter two. That is, there is no consensus in the literature on when exactly should we call a function space supercritical and when subcritical since it heavily depends on the intents of the person who is using them.

Definition 1.22. *A translation invariant Banach space of tempered distributions X is called a critical space for the Navier-Stokes equations if its norm is invariant under the action of the scaling $f(x) \to \lambda f(\lambda x)$ for any $\lambda > 0$. Equivalently we require from a critical space X the following two properties*

1. *the continuous embedding $X \hookrightarrow \mathcal{S}'$,*
2. *for any $f \in X$*

$$||f(\cdot)||_X = ||\lambda f(\lambda \cdot - x_0)||_X \quad \forall \lambda > 0, \forall x_0 \in \mathbb{R}^3.$$

Making use of the invariance properties of some well-known or earlier presented function spaces, we are able to give a couple of different examples to critical spaces for the Navier-Stokes equations (obviously in all cases we are still talking about $\mathbb{R}^3$ for the space variable):

- the Lebesgue space L^3,
- the Lorentz spaces $L^{(3,q)}$ $(1 \leq q \leq \infty)$,
- the Sobolev space $\dot{H}^{1/2}$,
- the Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,r}$ $(1 \leq p < \infty, 1 \leq r \leq \infty)$,
- the Triebel-Lizorkin spaces $\dot{F}^{\frac{3}{p}-1}_{p,r}$ $(1 \leq p < \infty, 1 \leq r \leq \infty)$.

We shall end this chapter by stating a relatively interesting fact about critical spaces.

Proposition 1.22. *If X is a critical space, then X is continuously embedded in the Besov space $\dot{B}^{-1}_{\infty,\infty}$.*

Since this proposition is of lesser importance considering our main goals, we shall not prove it now. The proof of it is basically the application of the definitions as well as making use of the fact that the function $\exp\left(-|x|^2/4\right)$ is in $\mathcal{S}$. A detailed proof can be found in [2] as well as a thorough description of nearly a dozen different function spaces which bear some significance in the theory of the Navier-Stokes equations.

CHAPTER 2

SOME EXISTENCE AND UNIQUENESS RESULTS

"... it is usually through paradoxes that mathematical work has the greatest influence on physics. In terms of existence and uniqueness theory, this means that the most important thing to discover is what is not true. When one proves the Navier-Stokes equations have solutions, the physicist yawns. If one can prove these solutions are not unique (say), he opens his eyes instead of his mouth."

Marvin Shinbrot [22]

Due to the steady advancement in functional analysis and harmonic analysis during the last century, nowadays an overwhelming number of different existence and uniqueness theorems are at our disposal considering the Navier-Stokes equations. Each and every one of them imposes various restrictions on the initial data, the type of solution searched for or the function spaces related to the solutions and the initial data (among other restricting factors as well).

One could fill hundreds and hundreds of pages with the detailing of these theorems and the lemmas and propositions, as well as the diverse techniques used in their proofs [1], which is clearly out of the scope of the present master's thesis (moreover it would lead relatively far from our original humble goals). In the following pages we shall list some of the more well-known or historically more important theorems and related mathematical settings. The proofs will be omitted (references will be given pointing to a preferably detailed proof) and we will try to focus our attention on those who have some relevance considering the theory of self-similar solutions.

2.1 Existence theorems

In this section we are going to present some existence results for the Navier-Stokes system. We shall start with the fundamental results concerning weak solutions. After that we will present a relatively basic method which is generally useful if one wants to work with the integral form (1.5) of the equations. Moreover it has some historical significance, especially in the theory of mild solutions.

2.1.1 Weak solutions

The basic theorem for the existence of abstract solutions for the Navier-Stokes system is attributed to Leray [16]. In his series of articles from 1933-34, he presented the notion of weak solutions as well as proved their local and global existence in various settings. Nowadays, a couple of different proofs are known, some of the more elementary ones (like the proof of Eberhard Hopf from 1951) can be found in standard literature as well.

Theorem 2.1. ***(Leray)*** *Let* $v_0 \in L^2(\mathbb{R}^3)$ *with* $\nabla \cdot v_0 = 0$*. Then there exists a* $v(t,x)$ *weak solution of the Navier-Stokes system (1.1) (with the external force* ϕ *equaling zero) satisfying the energy inequality*

$$||v(t)||_{L^2}^2 + 2\int_0^t ||\nabla v(s)||_{L^2}^2\, ds \leq ||v_0||_{L^2}^2$$

for almost every $t \geq 0$*. Moreover* $||v(t) - v_0||_{L^2} \to 0$ *if* $t \to 0$*.*

There are many different results considering the existence of strong solutions based on Leray's definitions. Some of the more general ones are attributed to Giga ([11]) and are based on the semigroup theory associated to $\bar{S}(t)$ (for further details, see appendix B). We are also going to present some recent existence results related to the so-called forward self-similar solutions in chapter 4.

2.1.2 Mild solutions

We recall that, by the definition of mild solution, we are interested in finding a solution to the integral equation (1.5). The right hand side of the equation contains three different terms, each of them can be treated separately.

The linear term containing the initial data

$$S(t)v_0 = \exp(t\Delta)v_0 \tag{2.1}$$

is behaving well in all translation invariant Banach spaces, since due to the well regularised heat semigroup component, one may apply the generalised Young inequality to obtain the following result.

Lemma 2.1. *Let X be a Banach space whose norm is translation invariant. For any $T > 0$ and any $v_0 \in X$, we have*

$$\sup_{0<t<T} ||S(t)v_0||_X = ||v_0||_X.$$

As a reminder, we shall remark (again) that due to the fact that we are treating the whole space $\mathbb{R}^3$, we were able to omit the Leray-Hopf operator $\mathbb{P}$ from this term and use purely the heat semigroup.

The most probelmatic part of the equation (1.5) is the middle term expressing the nonlinearity of the equation, arising from the bilinear operator

$$B(v,u)(t) = -\int_0^t \exp((t-s)\Delta)\mathbb{P}\nabla \cdot (v \otimes u)(s)\, ds. \tag{2.2}$$

Finally, we have the term involving the external force (if present), which can be treated by the linear operator

$$L(V)(t) = \int_0^t \exp((t-s)\Delta)\mathbb{P}\nabla \cdot V(s)\, ds. \tag{2.3}$$

As we have already mentioned in chapter 1, the main idea in searching for mild solutions is the application of the Banach fixed point theorem. More exactly, one may obtain the following result based on the fixed point theorem.

Lemma 2.2. *Let X be an abstract Banach space, $L : X \to X$ a linear operator such that for a given $\lambda \in (0,1)$ and for any $x \in X$,*

$$||L(x)||_X \leq \lambda ||x||_X$$

and $B : X \times X \to X$ a bilinear operator such that for any $x_1, x_2 \in X$,

$$||B(x_1,x_2)||_X \leq \eta ||x_1||_X ||x_2||_X,$$

then, for any $y \in X$ such that

$$4\eta ||y||_X < (1-\lambda)^2,$$

the equation $x = y + B(x,x) + L(x)$ has a solution $x \in X$. In particular, the solution is such that

$$||x||_X \leq \frac{2||y||_X}{1-\lambda},$$

moreover the solution is unique if we also have that

$$||x||_X < \frac{1-\lambda}{2\eta}.$$

It is apparent that this lemma provides the means to find a (unique) mild solution in a suitable Banach space X given the norm of the initial data is sufficiently small. So, to be able to deduce actual theorems on the existence of mild solutions to the Navier-Stokes equations, we have to find good X Banach spaces where we are able to handle the integral terms (2.2) and (2.3). For this we shall introduce yet another type of function spaces called adapted spaces:

Definition 2.1. ***(Adapted space)*** *The function space X is adapted to the Navier-Stokes equations if the following conditions hold:*

1. *X is a Banach space with the following continuous embeddings*

$$\mathcal{S}(\mathbb{R}^3) \hookrightarrow X \hookrightarrow \mathcal{S}'(\mathbb{R}^3).$$

2. *The norm of X is translation invariant.*
3. *The product of two tempered distributions from X is still a tempered distribution, that is $\forall f, g \in X$ we have that $fg \in \mathcal{S}'(\mathbb{R}^3)$.*
4. *There exists a series of real numbers $\eta_j > 0$ ($j \in \mathbb{Z}$) such that*

$$\sum_{j=-\infty}^{\infty} 2^{-|j|}\eta_j < \infty,$$

and

$$\forall j \in \mathbb{Z}, \forall f, g \in X; \; \left|\left|\dot{\Delta}_j(fg)\right|\right|_X \leq \eta_j ||f||_X ||g||_X.$$

Let us introduce the notation $\dot{B}_X = X \cap \mathcal{S}'_h(\mathbb{R}^3)$, that is the space of tempered distributions belonging to the adapted space X for which the homogeneous Littlewood-Paley decomposition also holds. With these at our disposal, we are finally able to state the main existence theorem for mild solutions.

Theorem 2.2. *Let X be an adapted space to the Navier-Stokes equations. Then for any initial value $v_0 \in X$ such that $\nabla \cdot v_0 = 0$, there exists a $T = T(||v_0||_X) > 0$ and a local mild solution $v(t,x) \in \mathcal{C}\Big([0,T); X\Big)$ such that $v(0,x) = v_0(x)$. Furthermore, we have that $v(t) - S(t)v_0$ belongs to the space $\mathcal{C}\Big([0,T); \dot{B}_X\Big)$ and*

$$\lim_{t\to 0} \sup_{||v_0||\leq 1} ||v(t) - S(t)v_0||_X = 0$$

The proof of this theorem and the aforementioned lemmas can be found in [1]. Nevertheless, to be more accurate, we shall name some of the well-known function spaces belonging to the class of adapted spaces, thus having an existence result for mild solutions.

For Lebesgue spaces, again by the application of the Young inequality, one may deduce the following property.

Lemma 2.3. *The Lebesgue space $L^p(\mathbb{R}^3)$ is adapted to the Navier-Stokes equations if and only if $p > 3$.*

Thus, as a particular case of theorem 2.2, we have

Theorem 2.3. *Let $p > 3$ be fixed. Then for any initial value $v_0 \in L^p(\mathbb{R}^3)$ such that $\nabla \cdot v_0 = 0$, there exists a $T = T(||v_0||_{L^p}) > 0$ and a local mild solution $v(t,x)$ in $\mathcal{C}\Big([0,T); L^p(\mathbb{R}^3)\Big)$ such that $v(0,x) = v_0(x)$. Furthermore, we have that $v(t) - S(t)v_0$ belongs to the space*

$$\mathcal{C}\Big([0,T); \dot{B}^{-1}_{p,\infty}(\mathbb{R}^3)\Big) \cap \mathcal{C}\Big([0,T); \dot{B}^{s}_{p,\infty}(\mathbb{R}^3)\Big)$$

with $s = s(p) = 1 - \frac{3}{p} > 0$.

For Sobolev spaces, one may deduce the following results.

Lemma 2.4. *The Sobolev space $\dot{H}^s(\mathbb{R}^3)$ is adapted to the Navier-Stokes equations if and only if $s > \frac{1}{2}$.*

Theorem 2.4. *Let $s > \frac{1}{2}$ be fixed. Then for any initial value $v_0 \in \dot{H}^s(\mathbb{R}^3)$ such that $\nabla \cdot v_0 = 0$, there exists a $T = T(||v_0||_{\dot{H}^s}) > 0$ and a local mild solution $v(t,x)$ in $\mathcal{C}\Big([0,T); \dot{H}^s(\mathbb{R}^3)\Big)$ such that $v(0,x) = v_0(x)$. Furthermore, we have that $v(t) - S(t)v_0$ belongs to the space*

- $\mathcal{C}\Big([0,T); \dot{H}^{2s-\frac{1}{2}-\varepsilon}(\mathbb{R}^3)\Big)$, *if $\frac{1}{2} < s \leq \frac{3}{2}$, and for any $\varepsilon > 0$;*
- $\mathcal{C}\Big([0,T); \dot{H}^{s+1-\varepsilon}(\mathbb{R}^3)\Big)$, *if $s > \frac{3}{2}$, and for any $\varepsilon > 0$.*

Let us remark that these two families of function spaces are often called super-critical due to the fact that the respective critical spaces serve as a "strict boundary" for them ($p = 3$ for the Lebesgue spaces and $s = \frac{1}{2}$ for the Sobolev spaces). As a consequence, for critical spaces the above mentioned algorithm does not provide any results, in order to handle them, slight adjustments had to be made. But before recalling what is known for the critical spaces, let us mention two lemmas which allow the introduction of existence theorems for the Besov and Triebel-Lizorkin spaces.

Lemma 2.5. *The Besov space $\dot{B}^s_{p,r}(\mathbb{R})^3$ is adapted in the following five cases:*

- *$s > \frac{3}{p}$, $1 \leq p \leq \infty$, $1 \leq r \leq \infty$;*

- $s = \frac{3}{p}$, $1 \le p \le \infty$, $r = 1$;
- $\frac{3}{p} - 1 < s \le \frac{3}{p}$, $2 \le p \le 3$, $1 \le r \le \infty$;
- $0 < s \le \frac{3}{p}$, $3 < p < \infty$, $1 \le r \le \infty$;
- $s = 0$, $3 < p \le \infty$, $1 \le r \le 2$.

Lemma 2.6. *The Triebel-Lizorkin space $\dot{F}^s_{p,r}(\mathbb{R})^3$ is adapted in the following six cases:*

- $s > \frac{3}{p}$, $1 < p < \infty$, $1 < r < \infty$;
- $s > \frac{3}{p}$, $1 \le p < r \le \infty$;
- $s > \frac{3}{2}\left(\frac{1}{p} + \frac{1}{r}\right)$, $1 \le p \le r < \infty$;
- $\frac{3}{p} - 1 < s \le \frac{3}{p}$, $2 \le p \le 3$, $1 \le r \le \infty$;
- $0 < s \le \frac{3}{p}$, $3 < p < \infty$, $1 \le r \le \infty$;
- $s = 0$, $3 < p \le \infty$, $1 \le r \le 2$.

In order to treat the critical spaces as well, we have to return to the origin of the previous algorithm for finding mild solutions, that is to the fixed point theorem. We should remark that the Banach fixed point theorem in a given function space is only a sufficient condition to ensure the existence of a solution in that given space, so we shall consider a slightly different strategy instead.

In what follows we shall only mention the case of $\mathcal{N} = \mathcal{C}\Big([0,T); L^3(\mathbb{R}^3)\Big)$, the "natural function space" associated to the equation (1.5). The sketch of the method presented here could be extended to more general function spaces as it is done for example in [10]. A sufficient condition leading to the existence of a mild solution in $\mathcal{N}$ is to find a function space $\mathcal{F}$ such that:

- the elements of the banach space $\mathcal{F}$ are functions $v(t,x)$ with $0 < t < T$ and $x \in \mathbb{R}^3$;
- the bilinear term $B(v,u)(t)$ defined by (2.2) is continuous as $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$;
- the bilinear term $B(v,u)(t)$ is continuous as $\mathcal{F} \times \mathcal{F} \to \mathcal{N}$ too;
- if $v_0 \in L^3(\mathbb{R}^3)$ then $S(t)v_0 \in \mathcal{F}$.

Here, the second and fourth properties ensure that there exists a mild solution $v(t,x) \in \mathcal{F}$ by the fixed point algorithm in $\mathcal{F}$! Thanks to the third condition, this solution will belong to $\mathcal{N}$ as well. The difficulty here is to construct a suitable function space $\mathcal{F}$ with these properties.

Since the 1980's, a couple of different ways were presented to fill in the blank gap left by the unknown function space $\mathcal{F}$ (Weissler's space from 1981, Kato's space from 1984, Giga's space from 1986, Calderón's space from 1990, etc.), for a more or less complete list of these spaces, as well as the corresponding proofs, the reader is referred to [2]. As a clear consequence to any of these theorems we shall state the relevant existence theorem in $L^3(\mathbb{R}^3)$.

Theorem 2.5. *For any $v_0 \in L^3(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$, there exists a $T = T(v_0)$ such that the Navier-Stokes equations have a local solution in $\mathcal{C}\Big([0,T); L^3(\mathbb{R}^3)\Big)$. Moreover there exists a $\delta > 0$ such that if $||v_0||_{L^3} < \delta$, then the solution is global, i.e. we can take $T = \infty$.*

The fact that with small initial data we have a global existence theorem is more or less unique to the critical spaces associated to the Navier-Stokes system and it serves as a rather intriguing addition to the aforementioned local theorems. We won't go into details concerning the different function spaces with which one may obtain this theorem, still we would like to highlight two theorems that will have additional consequences in what follows.

Theorem 2.6. ***(Kato)*** *Let $3 < p < \infty$ and $s = 1 - \frac{3}{p}$ be fixed. There exists a constant $\delta_p > 0$ such that for any initial data $v_0 \in L^3(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$ in the sense of distributions such that*

$$\sup_{0<t<T} t^{\frac{s}{2}} ||S(t)v_0||_{L^p} < \delta_p, \tag{2.4}$$

then there exists a mild solution $v(t,x)$ to the Navier-Stokes equations belonging to $\mathcal{N}$, which tends strongly to v_0 as time goes to zero. Moreover, the smallness condition (2.4) holds for arbitrary $v_0 \in L^3(\mathbb{R}^3)$ provided we consider $T(v_0)$ small enough, and remains true for $T = \infty$ provided the norm of v_0 in the Besov space $\dot{B}^{-s}_{p,\infty}(\mathbb{R}^3)$ is smaller than δ_p.

Remark 2.1. *We shall recall that a simple application of the embedding theorems leads to the property*

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}^{-s}_{p,\infty}(\mathbb{R}^3) \quad \forall q \in (3, \infty].$$

Theorem 2.7. ***(Giga)*** *Let $3 < p < 9$ and $s = 1 - \frac{3}{p}$ be fixed. There exists a constant $\delta_p > 0$ such that for any initial data $v_0 \in L^3(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$ in the sense of distributions such that*

$$\left(\int_0^T ||S(t)v_0||^{\frac{2}{s}}_{L^p} \, dt \right)^{\frac{s}{2}} < \delta_p, \tag{2.5}$$

then there exists a mild solution $v(t,x)$ to the Navier-Stokes equations belonging to $\mathcal{N}$, which tends strongly to v_0 as time goes to zero. Moreover, the smallness condition (2.5) holds for arbitrary $v_0 \in L^3(\mathbb{R}^3)$ provided we consider $T(v_0)$ small enough, and remains true for $T = \infty$ provided the norm of v_0 in the Besov space $\dot{B}^{-s}_{p,\frac{2}{s}}(\mathbb{R}^3)$ is smaller than δ_p. Finally, the "fluctuation" function $v(t,x) - S(t)v_0(x)$ belongs to the function space

$$\mathcal{C}\Big([0,T); \dot{B}^0_{3,2}(\mathbb{R}^3)\Big) \cap L^2\Big((0,T); L^\infty(\mathbb{R}^3)\Big).$$

2.2 Uniqueness theorems

This section is dedicated to a selected assortment of uniqueness theorems. We shall emphasize that (as it was already pointed out in the introduction) the true problems involving the Navier-Stokes system lie in the uniqueness and smoothness properties of it, thus giving rise to the fact that there are considerably less propositions in this field. It also implies that the techniques (more exactly the attempts) and observations involved here differ somewhat from most of the more common methods for proving unicity.

2.2.1 Case of weak solutions

As we have already mentioned in the introduction, the true problems for the Navier-Stokes system start at the question of the unicity of the solutions. For the most general case, that is for weak solutions, no general theorem is known to the public. The problem lies basically in the nonlinear part of the equations, more precisely, in order to be able to properly handle the higher order term (for example with energy inequalities) one has to require supplementary regularity assumptions on the solution. These type of hypotheses are often referred to as *"weak=strong"* type properties.

To illustrate it, the basic method for proving unicity would be to consider two solutions for the same system with the same initial values, subtract one from the other and apply the energy inequalities to estimate the L^2 norms, since by definition of the weak solution, we are to apply estimations for the natural function spaces involved. A simple application of Green's formula would yield to an inequality on the L^2 norm of the difference (usually with the use of Gronwall's lemma). If we were to apply these simple steps, we would first bump into the problem of how exactly should one estimate an integral term involving the nonlinearity of the equation. There are various methods to handle this but so far all of them end up with another problem: the final estimation on the norms would involve the fact that at least one of the two solutions has to have additional smoothness properties as opposed to the definition of weak solution. For an illustration and proper calculations, the reader is referred to [2].

Nevertheless we shall present some of the more well-known unicity results for the weak solutions of the Navier-Stokes system.

Proposition 2.1. *Let v_1 and v_2 be two solutions of the same Navier-Stokes equations (1.1) (with the external force ϕ equaling zero) with the same initial value $v_1(0) = v_2(0)$, furthermore with the same behaviour at infinity, that is $(v_1 - v_2)(t, x)$ tends to 0 if $|x| \to \infty$ for any $0 < t < T$ in their common interval of existence $[0, T)$. If in addition, we have that the quantity $\int_0^t ||\nabla v_2(s)||_{L^\infty}\, ds$ or the quantity $\int_0^t ||\nabla v_2(s)||_{L^2}^4\, ds$ is bounded, then $v_1 = v_2$.*

A relatively complete list of the various unicity theorems for weak solutions can be found in [13], here we shall only name two more theorems which have the most historical and mathematical significance for our cause.

Theorem 2.8. ***(Unicity of Serrin)*** *Suppose that v is a weak solution of the Navier-Stokes system on $[0, T)$ satisfying the energy inequality*

$$||v(t)||_{L^2}^2 + 2\int_0^t ||\nabla v(s)||_{L^2}^2\, ds \leq ||v_0||_{L^2}^2$$

for $0 \leq t \leq T$ (by definition). If

$$\int_0^T ||v(t)||_{L^p}^r\, st$$

is finite for some p and r such that $\frac{2}{r} + \frac{3}{p} \leq 1$ and $p > 3$, then such a solution v on $[0, T)$ is unique!

Theorem 2.9. ***(Unicity of Sohr and von Wahl)*** *Let v_1 and v_2 be two weak solutions of the Navier-Stokes equations (1.1) (with the external force ϕ equaling zero) on $[0, T) \times \mathbb{R}^3$ such that they have the same initial value v_0 at $t = 0$. If in addition we have that v_1 belongs to the function space $\mathcal{C}\Big([0, T); L^3(\mathbb{R}^3)\Big)$ then $v_1 = v_2$.*

2.2.2 Case of mild solutions

Sadly, the uniqueness theorems for the weak solutions do not apply in general to the mild solutions in $\mathcal{C}\Big([0, T); L^p(\mathbb{R}^3)\Big)$ $(p > 3)$ as well, since in the latter case the initial data only belongs to $L^p(\mathbb{R}^3)$ and usually not in $L^2(\mathbb{R}^3)$. (With bounded domains, we would be able to infer such conclusions, but currently we are only treating the equations in the whole space $\mathbb{R}^3$.)

Still, the uniqueness results for the existence theorems mentioned in section 2.1.2 can be easily deduced by the usual technique of considering two solutions of the same integral equation for the Navier-Stokes system with the same initial data and subtracting one of them from the other. The L^p norm estimate of the difference of the two solutions follows from a straighforward application of the Young inequality (the same estimation works for more general adapted function spaces as well)!

Let us recall that lemma 2.2 already involves a uniqueness statement too, for relatively small initial data that is. Thus we may be able to state for example the following basic uniqueness results for mild solutions in the supercritical case:

Theorem 2.10. *Let $3 < p \leq \infty$ be fixed. For any $v_0 \in L^p(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$ and any $T > 0$, there exists at most a mild solution in $\mathcal{C}\Big([0,T); L^p(\mathbb{R}^3)\Big)$ to the Navier-Stokes equations.*

Theorem 2.11. *Let $\frac{1}{2} < s$ be fixed. For any $v_0 \in \dot{H}^s(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$ and any $T > 0$, there exists at most a mild solution in $\mathcal{C}\Big([0,T); \dot{H}^s(\mathbb{R}^3)\Big)$ to the Navier-Stokes equations.*

Similar to the existence results, for the critical spaces we are required to use the alternative approach of introducing the auxiliary function space $\mathcal{F}$ in order to properly handle the uniqueness. But compared to the existence results, it was no sooner than in 1997 that the first complete proof was presented for the simpler case of the Lebesgue space $L^3(\mathbb{R}^3)$ by Giulia Furioli et al. [9]. The proof relies heavily on Lorentz space and Besov space based auxiliary spaces.

Theorem 2.12. *For any $v_0 \in L^3(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$, and any $T > 0$, there exists at most a mild solution $v(t,x)$ to the Navier-Stokes equations such that $v(t,x)$ in $\mathcal{C}\Big([0,T); L^3(\mathbb{R}^3)\Big)$.*

CHAPTER 3

BACKWARD SELF-SIMILAR SOLUTIONS

"Si j'avais réussi à construire des solutions des équations de Navier qui deviennent irrégulières, j'aurais le droit d'affirmer qu'il existe effectivement des solutions turbulentes ne se réduisent pas, tout simplement, à des solutions régulières. Même si cette proposition était fausse, la notion de solution turbulente, qui n'aurait dès lors plus à jouer aucun rôle dans l'étude des liquides visqueux, ne perdrait pas son intérêt: il doit bien se présenter des problèmes de Physique mathématique pour lesquels les causes physiques de régularité ne suffisent pas à justifier les hypothèses de régularité faites lors de la mise en équation."

Jean Leray [16] [4]

During the next two chapters, we shall introduce and examine the so-called self-similar solutions for the Navier-Stokes equations. We will only treat them in the simplified case where no external force is present, that is we are going to work with the equations (1.1) with $\phi \equiv 0$, so with

$$\begin{aligned} \partial_t v(t,x) - \nu \Delta v(t,x) &= -(v(t,x)\cdot\nabla)v(t,x) - \nabla p(t,x), \\ \nabla\cdot v(t,x) &= 0, \\ v(0,x) &= v_0(x). \end{aligned} \tag{3.1}$$

More exaclty, partially inspired by the invariance property of the equations de-

[4] *If I had succeeded in constructing a solution to the Navier equations that becomes irregular, I would have the right to claim that turbulent solutions, which do not simply reduce to regular ones, exist. Even if this proposition were wrong, the notion of turbulent solution, that for the study of viscous fluids would not play any role anymore, would not lose interest: there have to exist some problems in Mathematical physics such that the physical causes of regularity are not sufficient to justify the hypotheses introduced when the equations are derived.*

tailed in section 1.5, we would like to examine the following type of solutions:

$$v(t,x) = \lambda(t) V\Big(\lambda(t)x\Big), \qquad\qquad p(t,x) = \lambda^2(t) P\Big(\lambda(t)x\Big), \tag{3.2}$$

$\lambda(t)$ being a function of time, $P(x)$ a function of x and $V(x)$ a divergence-free vector field. The functions V and P are also called the profiles (or profile functions) of the velocity field and the pressure (respectively). The definition of these solutions is based on the homogeneity of the Navier-Stokes equations and implies a basic scaling invariance in the space variables.

This chapter will be dedicated to the analysis of theso-called backward self-similar solutions.

Definition 3.1. ***(Backward self-similar solution)*** *A backward self-similar solution of the Navier-Stokes equations (3.1) is of the form (3.2), where* $\lambda(t) = \frac{1}{\sqrt{2a(T-t)}}$ *for* $a > 0$*,* $T > 0$ *and* $t < T$*.*

A straighforward computation (a consecutive application of the chain rule) leads to the following simple lemma:

Lemma 3.1. *For a backward self-similar solution, the profile functions* $V(x)$ *and* $P(x)$ *solve the system*

$$\begin{aligned} -\nu\Delta V + aV + a(x\cdot\nabla)V + (V\cdot\nabla)V + \nabla P = 0, \\ \nabla\cdot V = 0. \end{aligned} \tag{3.3}$$

An apparent remark about the backward self-similar solutions is that the notion of mild solution does not exactly have any obvious meaning for them, since they are not defined for the integral equation (at least for now), thus the emphasis will be placed on the weak and strong solutions arising from these special type of solutions.

The notion of backward self-similar solutions was introduced by Leray for the sake of being able to study the possible appearing singularities of the Navier-Stokes system. Indeed, if such a nontrivial solution would exist, it would possess a singularity at $t = T$, since $\lim_{t\nearrow T} ||\nabla v(t)||_{L^2} = \infty$. The main reason why Leray started looking for self-similar solutions with $\lambda(t)$ as in the definition was that he managed to prove (see [15]) that a turbulent solution at T blows up like $\frac{1}{\sqrt{2a(T-t)}}$ when t tends to T.

A solution to this "problem" was proposed by Nečas et al. in 1996 [20], where they established a mathematical proof for the nonexistence of such solutions, hence closing of the search for backward self-similar solutions and blowing up for solutions with similar invariant structures in general. An excellent summary on the subject can be found in [14].

3.1 Nonexistence for slow growth solutions

The basic theorem concerning the nonexistence of nontrivial backward self-similar solutions that are sufficiently decreasing at infinity is based on the following variant of the maximum principle.

Proposition 3.1. ***(Maximum principle)*** *Let* Π, b_1, b_2 *and* b_3 *be functions with real variables such that* $\Pi \in \mathcal{C}^2(\mathbb{R}^3)$ *and* $b_j \in \mathcal{C}(\mathbb{R}^3)$ *with* $\lim_{|x|\to\infty} \frac{b_j(x)}{|x|} = 0$ *for* $j = 1, 2, 3$. *Furthermore, let us fix* $a > 0$, *and denote* $b(x) = (b_1(x), b_2(x), b_3(x))$. *If we also have that*

$$\Delta\Pi(x) - ((b(x) + ax) \cdot \nabla)\,\Pi(x) \geq 0 \quad \forall x \in \mathbb{R}^3 \;\text{ and }\; \lim_{|x|\to\infty} \frac{|\Pi(x)|}{|x|^2} = 0,$$

then Π *is constant.*

Proof: Let us denote by $\mathcal{L}$ the following differential operator:

$$\mathcal{L}f(x) = \Delta f(x) - ((b(x) + ax) \cdot \nabla)\, f(x). \tag{3.4}$$

By the hypothesis of the proposition, we have that $\mathcal{L}\Pi \geq 0$. We shall prove this proposition by contradiction.

Let us suppose that Π is not constant, that is there exist two points in the space (x_0 and x_1) such that $\Pi(x_0) < \Pi(x_1)$ (without loss of generality). Then take two strictly positive real numbers, R_0 and R_1, such that $R_0 < |x_0 - x_1| < R_1$, moreover we impose that

- R_1 is big enough to guarantee that $aR_1^2 > 6$ and that for any x such that $|x - x_0| \geq R_1$ we have

$$|b(x) + ax_0| \leq a\frac{|x - x_0|}{2};$$

- R_0 is small enough to guarantee that for any x such that $|x - x_0| \leq R_0$ we have

$$\Pi(x) \leq \frac{\Pi(x_0) + \Pi(x_1)}{2}.$$

The first condition is due to the asymptotic condition requested from b, the second condition can be verified by continuity.

Let us consider the following function:

$$F(x) = \Pi(x) + \alpha\left(e^{-\beta|x-x_0|^2} - \gamma|x - x_0|^2\right), \tag{3.5}$$

where the coefficients α, β and γ are strictly positive real numbers defined by the following ruleset:

- First of all, we will define β. We want to set it sufficiently large in order to have that $\mathcal{L}\varphi_\beta(x) > 0$ for any x such that $|x - x_0| \geq R_0$, where the function φ_β is defined by
$$\varphi_\beta(x) = e^{-\beta|x-x_0|^2}.$$
This choice is always possible, since we have that
$$\begin{aligned}\mathcal{L}\varphi_\beta(x) &= \Delta\varphi_\beta(x) - ((b(x) + ax) \cdot \nabla)\, \varphi_\beta(x) \\ &= \Delta e^{-\beta|x-x_0|^2} - ((b(x) + ax) \cdot \nabla)\, e^{-\beta|x-x_0|^2} \\ &= e^{-\beta|x-x_0|^2} \left(4\beta^2|x - x_0|^2 - 6\beta - (b(x) + ax) \cdot \left(-2\beta e^{-\beta|x-x_0|^2}\right)(x - x_0)\right) \\ &= \beta e^{-\beta|x-x_0|^2} \Big((4\beta + 2a)|x - x_0|^2 - 6 + 2(b(x) + ax_0) \cdot (x - x_0)\Big)\,,\end{aligned}$$
and because of the fact that for $|x - x_0| \geq R_0$ we may define a constant C such that $|b(x)| \leq C(R_0)|x - x_0|$, it is enough to choose $\beta > 0$ satisfying
$$4\beta + 2a > 2C(R_0) + 6R_0^{-2} + a|x_0|R_0^{-1}.$$

- Then we will choose an appropriate γ sufficiently small such that for the function $\psi(x) = |x - x_0|^2$ the following inequality holds:
$$\gamma \cdot \left(\sup_{R_0 \leq |x-x_0| \leq R_1} |\mathcal{L}\psi(x)|\right) < \inf_{R_0 \leq |x-x_0| \leq R_1} \mathcal{L}\varphi_\beta.$$
This can be simply guaranteed since both φ_β and ψ are smooth functions, thus $|\mathcal{L}\psi|$ and $\mathcal{L}\varphi_\beta$ are continuous, so the infimum and supremum of them on an annulus are both finite (and nonnegative as well).

- Finally we set the value of α such that we have the following inequality
$$\alpha \cdot \left(\sup_{|x-x_0| \leq R_1} |\varphi_\beta(x) - \gamma\psi(x)|\right) < \frac{\Pi(x_1) - \pi(x_0)}{4}.$$
This can also be simply guaranteed because on the right hand side, we have a positive real number, moreover α is multiplied by a finite nonnegative number due to the choice of γ.

Since the function $\Pi(x)$ is of class $\mathcal{C}^2$, $F(x)$ is also of class $\mathcal{C}^2$. Furthermore, from its definition (and the hypothesis on Π's asymptotic behaviour) we may infer that $F(x) \sim -\alpha\gamma|x|^2$ when $|x| \to \infty$, thus F attains its maximum at a finite point x_2. In this point we have $\nabla F(x_2) = 0$ and $\Delta F(x_2) \leq 0$ (the trace of its

Hessian, or equivalently the sum of the eigenvalues can't be positive), so we have that $\mathcal{L}F(x_2) \leq 0$.

1. $|x_2 - x_0| \leq R_0$ is not possible: for $|x - x_0| \leq R_0$ we have that

$$\begin{aligned} V(x) &\leq \Pi(x) + \alpha \sup_{|x-x_0|\leq R_0} |\varphi_\beta(x) - \gamma\psi(x)| \\ &\leq \frac{\Pi(x_0)+\Pi(x_1)}{2} + \frac{\Pi(x_1)-\pi(x_0)}{4} \leq \Pi(x_0) + \frac{3}{4}(\Pi(x_1) - \Pi(x_0)) \\ &< \Pi(x_0) + (\Pi(x_1) - \Pi(x_0)) + \alpha(\varphi_\beta(x) - \gamma\psi(x)) = V(x_1) \end{aligned}$$

   and $x_1 \neq x_2$ in this case!

2. $R_0 \leq |x_2 - x_0| \leq R_1$ is not possible: for $R_0 \leq |x - x_0| \leq R_1$ we have that $\gamma|\mathcal{L}\psi(x)| < \mathcal{L}\varphi_\beta(x)$ and $\mathcal{L}\Pi(x) \geq 0$, so $\mathcal{L}V(x) > 0$.
3. $R_1 \leq |x_2 - x_0|$ is not possible as well: for $|x - x_0| \geq R_1$ we have $\mathcal{L}\varphi_\beta(x) > 0$, $\mathcal{L}\Pi(x) \geq 0$ and

$$\begin{aligned} \mathcal{L}\psi(x) &= \Delta|x-x_0|^2 - ((b(x)+ax)\cdot\nabla)\,|x-x_0|^2 = 6 - 2(b(x)+ax)\cdot(x-x_0) \\ &= 6 - 2(b(x)+ax_0)\cdot(x-x_0) - 2a|x-x_0|^2 \leq 6 - a|x-x_0|^2 < 0 \end{aligned}$$

   by the hypotheses made on R_1, so we obtain that $\mathcal{L}V(x) > 0$ again.

So none of these cases are possible which is clearly a contradiction, Π must be constant. □

Theorem 3.1. ***(Slowly increasing backward self-similar solutions)*** *Let us consider $v(t,x)$ and $p(t,x)$ backward self-similar solution for the equations (3.1) with $\nu = 1$. That is, the corresponding profile functions $V(x)$ and $P(x)$ solve the following system (for $a > 0$):*

$$\begin{aligned} \Delta V &= aV + a(x\cdot\nabla)V + (V\cdot\nabla)V + \nabla P, \\ \nabla\cdot V &= 0. \end{aligned} \tag{3.6}$$

Let us suppose that $V \in \mathcal{C}^3(\mathbb{R}^3)$ and $P \in \mathcal{C}^2(\mathbb{R}^3)$. If V and P verify the asymptotic equations

$$\lim_{|x|\to\infty} \frac{|V(x)|}{|x|} = 0; \text{ and } \lim_{|x|\to\infty} \frac{|P(x)|}{|x|^2} = 0$$

then V is a constant function.

Proof: We shall start by introducing the function Π and the differential operator $\mathcal{L}$

$$\begin{aligned} \Pi(x) &= \frac{1}{2}|V(x)|^2 + P(x) + ax\cdot V(x), \\ \mathcal{L}f(x) &= \Delta f(x) - ((V(x)+ax)\cdot\nabla)\,f(x). \end{aligned}$$

First of all, we would like to reason why $\mathcal{L}\Pi \geq 0$ holds. We will start by applying the divergence operator on the first equation of (3.6). Due to the divergence-free property $\nabla \cdot V(x) = 0$, we also have that $\nabla \cdot \Delta V(x) = \Delta(\nabla \cdot V(x)) = 0$. Furthermore,

$$\nabla \cdot ((x \cdot \nabla)V)(x) = \nabla \cdot V(x) + (x \cdot \nabla)\nabla \cdot V(x) = 0.$$

So the equation reduces to the following identity

$$\begin{aligned}\Delta P(x) = \nabla \cdot \nabla P(x) &= -\nabla \cdot \Big((V(x) \cdot \nabla)V(x)\Big) \\ &= -\sum_{j=1}^{3}\sum_{k=1}^{3} \partial_j \partial_k \Big(V_j(x)V_k(x)\Big) = -\sum_{j=1}^{3}\sum_{k=1}^{3} \partial_j V_k(x) \partial_k V_j(x).\end{aligned}$$

Moreover we have that

$$\Delta(x \cdot V(x)) = 2\nabla \cdot V(x) + x \cdot \Delta V(x) = x \cdot \Delta V(x),$$

as well as

$$\Delta\left(\frac{1}{2}|V(x)|^2\right) = V(x) \cdot \Delta V(x) + |\nabla \otimes V(x)|^2.$$

To sum it up, we have that

$$\Delta \Pi(x) = |\nabla \otimes V(x)|^2 - \sum_{j=1}^{3}\sum_{k=1}^{3} \partial_j V_k(x) \partial_k V_j(x) + (V(x) + ax) \cdot \Delta V(x).$$

On the other hand, we may also deduce that

$$\begin{aligned}((V(x) + ax) \cdot \nabla)\, \Pi(x) &= (V(x) + ax) \cdot (\nabla P(x) + aV(x) + ((V(x) + ax) \cdot \nabla)V(x)) \\ &= (V(x) + ax) \cdot \Delta V(x),\end{aligned}$$

so we have that

$$\mathcal{L}\Pi(x) = |\nabla \otimes V(x)|^2 - \sum_{j=1}^{3}\sum_{k=1}^{3} \partial_j V_k(x) \partial_k V_j(x) = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} |\partial_j V_k(x) - \partial_k V_j(x)|^2 \geq 0. \tag{3.7}$$

With this at our disposal, we will be able to apply the maximum principle from before. Due to the hypotheses on the asymptotic behaviour of $V(x)$ and $P(x)$, the asymptotic property required from $\Pi(x)$ is also verified, thus the principle can be applied. This means that Π is constant, thus we have that $\mathcal{L}\Pi(x) = 0$. So we have equality in the inequality (3.7), that means that specially we obtained that $\partial_j V_k(x) = \partial_k V_j(x)$ for $j, k = 1, 2, 3$. Since we also have that $V(x)$ is a divergence-free field, then we may deduce that $V(x)$ is a conservative field, more exactly $V(x) = \nabla F(x)$ for a harmonic function F!

Moreover, by the definiton of F, we also have that the coordinate functions $V_1(x)$, $V_2(x)$ and $V_3(x)$ are also harmonic. Putting this together with the fact that $V(x)$ is of $o(|x|)$ at infinity, that is, we have a harmonic tempered distribution, thus a harmonic polynome of $o(|x|)$, so $V(x)$ is necessarily constant! $\square$

3.2 More general cases

Theorem 3.1 by itself is not that useful, because it expects the profile functions to be extremely regular, moreover the asymptotic conditions imposed are also quite restrictive. Still this theorem was the fundamental step in order to gain proofs for the nonexistence of backward self-similar solutions. Nowadays we know many different enhancements of this theorem from which we shall name some of the more well-known ones, attributed to the two group of authors who achieved the greatest progressions in the field: Josef Málek, Jindřich Nečas et al. and Tai-Peng Tsai et al.

Both of these groups had the same general approach to the field but the slightly different techniques led to varying results. Basically, in order to obtain a more general theorem one has to establish appropriate asymptotic estimates at infinity as well as one needs to incorporate some regularity results known (or obtained during the course of the work). Some of the most famous regularity results for the solutions of the Navier-Stokes equations are attributed to James Serrin, who revolutionised the field by introducing a rather intricate boothstrapping algorithm for different chains of function space embeddings.

Remark 3.1. *Perhaps the most famous regularity theorem of Serrin treats the case of a solution $v(t,x) \in L^\infty([0,T);\mathbb{R}^3)$ and states that in this case, one may deduce that for any $0 < T_0 < T_1 < T$ and any $k \in \mathbb{N}$ we have that $v(t,x)$ belongs to the space $L^\infty\Big([T_0, T-1); \dot{B}^k_{\infty,\infty}(\mathbb{R}^3)\Big)$ and as a far reaching consequence we may obtain that $v(t,x)$ is in fact of class $\mathcal{C}^\infty$.*

3.2.1 The results of Nečas

Historically speaking, the first complete proof on the nonexistence in a relatively general setting was posed by Nečas et al. in 1996 ([20]). Later, the same group made some adjustments on their theorem with some minor alterations and generalisations. Although one of their fundamental steps was a similar proposition to the one established in the previous section, their main focus was on the modified (1.2) equations and its solutions in order to be able to apply some smoothness properties appearing with the Leray-Hopf operator in the equation.

Theorem 3.2. ***(Nečas)*** *If $v(t,x)$ is a backward self-similar solution to the equations (3.1) on $(0,T) \times \mathbb{R}^3$ with $V(x) \in L^3(\mathbb{R}^3)$ then $v(t,x) = 0$.*

In some of their more recent works, they have also provided some quite remarkable generalisations involving the general Sobolev spaces $\dot{H}^k_{loc}(\mathbb{R}^3)$ (mainly the nonexistence result for a solution belonging to the space $\dot{H}^1_{loc}(\mathbb{R}^3)$) [18].

3.2.2 The results of Tsai

More or less at the same time, another research crew emerged with results for the nonexistence of backward self-similar solutions. Following a series of articles, in 1998 Tsai published the collection of his results on the topic [25]. The methods involved in his proofs were similar to his "predecessor's" and the proof presented for the nonexistence of slowly increasing backward self-similar solutions is more or less his original proof. But the introduction of special types of energy estimates allowed him to get even further in the field thus we have the following results.

Theorem 3.3. ***(Tsai's first theorem)*** *If $v(t,x)$ is a backward self-similar solution to the equations (3.1) with $V(x) \in \mathcal{C}_0(\mathbb{R}^3)$ then $v(t,x) = 0$.*

Remark 3.2. *As a consequence of this theorem, one may also prove Nečas' original theorem!*

Theorem 3.4. *A weak backward self-similar solution for the equations (3.1) belonging to $L^p(\mathbb{R}^3)$ for some $3 < p \leq \infty$ must necessarily be constant (and hence identically zero for $p < \infty$).*

Theorem 3.5. ***(Tsai's second theorem)*** *If $v(t,x)$ is a backward self-similar solution to the Navier-Stokes equation (3.1) on $(0,T) \times \mathbb{R}^3$ and for a $T_0 < T$ and an $R_0 > 0$ we have the following energy estimates*

$$\sup_{T_0<t<T} \int_{|x|<R_0} |v(t,x)|^2 \, dx < \infty, \tag{3.8}$$

$$\int_{T_0}^{T} \int_{|x|<R_0} |\nabla \otimes v(t,x)|^2 \, dx \, dt < \infty, \tag{3.9}$$

then $v(t,x) = 0$.

CHAPTER 4

FORWARD SELF-SIMILAR SOLUTIONS

"... un écoulement fluide initialement régulier reste régulier durant un certain intervalle de temps; ensuite il se poursuit indéfiniment; mais reste-t-il régulier et bien déterminé? On ignore la réponse à cette double question. Elle fut posée il y a soixante ans dans un cas extrêmement particulier. Alors H. Lebesgue, consulté, déclara: "Ne consacrez pas trop de temps à une question aussi rebelle. Faites autre chose!" "

Jean Leray [17] [5]

Compared to the backward self-similar solutions, their counterparts, the forward self-similar solutions are in a much favorable situation, as we will see in the following couple of pages.

Definition 4.1. ***(Forward self-similar solution)*** *A forward self-similar solution of the Navier-Stokes equations (3.1) is of the form (3.2), where* $\lambda(t) = \frac{1}{\sqrt{2a(T+t)}}$ *for* $a > 0$*,* $T > 0$ *and* $t > -T$*.*

A straighforward computation (consecutive application of the chain rule) leads to the following simple lemma:

Lemma 4.1. *For a forward self-similar solution, the profile functions* $V(x)$ *and* $P(x)$ *solve the system*

$$\begin{aligned} -\nu\Delta V - aV - a(x\cdot\nabla)V + (V\cdot\nabla)V + \nabla P = 0,\\ \nabla\cdot V = 0. \end{aligned} \tag{4.1}$$

[5] *... an initially regular fluid motion remains so during a certain time interval; then it goes on indefinitely; but does it remain regular and well-determined? We ignore the answer to this double question. It was addressed sixty years ago in an extremely particular case. At that time H. Lebesgue, asked on the matter, declared: "Don't spend too much time on such a tenacious question. Do something different!"*

The fact that the forward self-similar solutions are in a much better position comes from the various existence and uniqueness theorems concerning them. In fact, some very general existence theorems where also proved by searching (and finding) these special type of solutions! A less clear use of them is their connection with the asymptotic behaviour of the solutions to the Navier-Stokes system (for instance their "possible" connection with attractor sets) which makes it highly beneficial to study them in order to deeper understand the asymptotic properties of the equations as well.

We shall remark that in general obtaining a self-similar solution implies that we were originally given a -1 homogeneous initial function which is especially beneficial since it belongs to certain critical spaces by definition. Finally we shall point out that the particular case of $a = \frac{1}{2}$ and $T = 0$, that is $\lambda(t) = (t)^{-\frac{1}{2}}$ arises somewhat naturally from the scaling property: since we know that the function $v_\lambda(t,x) = \lambda(\lambda^2 t, \lambda x)$ solves the same equations as $v(t,x)$, we may pose $\lambda^2 t = 1$ (for each $t > 0$ we have such a λ), so we obtain solutions of the form

$$v(t,x) = \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right). \tag{4.2}$$

4.1 Mild self-similar solutions

The main idea in this section is that we would like to apply the more or less general existence and uniqueness theorems presented in chapter 3. For small initial data we can assure global existence of a mild solution, say $v(t,x)$, moreover if we require the initial data to be -1 homogeneous (as it should be due to the fact that we are searching for self-similar solutions), then by the uniqueness theorem, we have that for any $\lambda > 0$, the rescaled function $v_\lambda(t,x)$ is a mild solution, thus for any $\lambda > 0$ $v_\lambda(t,x)$ is the same solution as $v(t,x)$ which implies that the solution is self-similar!

Hence, requiring only that the initial data should be compatible with a self-similar solution (that is, the -1 homogeneity), the uniqueness theorem guarantees us that the existing global solution will be self-similar. Naturally it will be forward self-similar solution due to the nature of the definition itself. So from the global existence theorems 2.6 we automatically obtain existence theorems for forward self-similar solutions as well.

Theorem 4.1. *Let $3 < p < \infty$ and $s = 1 - \frac{3}{p}$ be fixed. There exists a constant $\delta_p > 0$ such that for any initial data $v_0 \in \dot{B}^{-s}_{p,\infty}(\mathbb{R}^3)$, homogeneous of degree -1, $\nabla \cdot v_0 = 0$ in the sense of distributions, and such that $||v_0||_{\dot{B}^{-s}_{p,\infty}} < \delta_p$ then there exists a global mild forward self-similar solution $v(t,x)$ to the Navier-Stokes equations such that it has the form of (4.2) where $V(x)$ is a divergence-free function belonging to $\dot{B}^{-s}_{p,\infty}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$.*

Remark 4.1. *For forward self-similar solutions one might go even further than the general theory on global existence for mild solutions. Not only can the above mentioned theorem be proved for the critical case* $\dot{B}^0_{3,\infty}(\mathbb{R}^3)$ *but the same result also holds for* $p = 2$*, that is for* $\dot{B}^{\frac{1}{2}}_{2,\infty}(\mathbb{R}^3)$ *too. [1]*

Another interesting generalisation involves the replacement of the Besov spaces with the Lorentz spaces. General existence and uniqueness results are relatively scarce with these "new" function spaces but for forward self-similar solutions, we can state nearly the same result.

Theorem 4.2. ***(Barazza, Meyer)*** *There exists a constant* $\delta > 0$ *such that for any initial data* $v_0 \in L^{(3,\infty)}(\mathbb{R}^3)$*, homogeneous of degree* -1*,* $\nabla \cdot v_0 = 0$ *in the sense of distributions, and such that* $||v_0||_{L^{(3,\infty)}} < \delta$ *then there exists a global mild forward self-similar solution* $v(t,x)$ *to the Navier-Stokes equations such that it has the form of (4.2) where* $V(x)$ *is a divergence-free function belonging to* $L^{(3,\infty)}(\mathbb{R}^3)$*.*

4.2 Strong self-similar solutions

In a quite recent article, Šverák presented a rather new approach in proving existence results for generalised weak solutions (the so-called Leray solutions) with the use of forward self-similar solutions [12]. They introduced a new set of regularity criteria and established some rather intriguing results on the existence of weak solutions (which, due to the regularity properties, instantly became extremely strong solutions as well)!

Definition 4.2. ***(Leray solution)*** *A vector field* $v \in L^2_{loc}([0,\infty) \times \mathbb{R}^3)$ *is called a Leray solution to the Navier-Stokes equations with initial data* v_0*, where* $v_0 \in L^2_{loc}(\mathbb{R}^2)$*,* $\nabla \cdot v_0 = 0$ *and* $\sup\limits_{x_0 \in \mathbb{R}^3} \int\limits_{B(x_0,1)} |v_0|^2(x)\,dx < \infty$*, if it satisfies:*

1. *For any* $R < \infty$ *we have that*

$$\sup_{0 \le t \le R^2} \sup_{x_0 \in \mathbb{R}^3} \int\limits_{B(x_0,R)} \frac{|v|^2}{2}(t,x)\,dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int\limits_{B(x_0,R)} |\nabla v|^2(t,x)\,dx\,dt < \infty$$

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int\limits_{B(x_0,R)} |v|^2(t,x)\,dx\,dt = 0.$$

2. *For a distribution* p *in* $(0,\infty) \times \mathbb{R}^3$*,* (v,p) *satisfies the equations (3.1) (for simplicity with* $\nu = 1$*) in the sense of distributions.*
3. *For any compact set* $K \subset \mathbb{R}^3$ *we have that*

$$\lim_{t \to 0+} ||v(t,\cdot) - v_0(\cdot)||_{L^2(K)} = 0.$$

4. *$v(t,x)$ is suitable in the sense of Caffarelli-Kohn-Nirenberg, that is, we have the following local energy estimate for any smooth $\varphi \geq 0$ with $\operatorname{supp}\varphi \subset (0,\infty)\times\mathbb{R}^3$*

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla v|^2 \varphi(t,x)\,dx\,dt \leq \\ \leq \int_0^\infty \int_{\mathbb{R}^3} \left(\frac{|v|^2}{2}(\partial_t \varphi + \Delta\varphi)(t,x) + \frac{|v|^2}{2} v\cdot\nabla\varphi(t,x) + pv\cdot\nabla\varphi(t,x) \right) dx\,dt. \tag{4.3}$$

The set of all Leray solutions starting from v_0 will be denoted by $\mathcal{N}(v_0)$. This definition was inspired by the preliminary work of Leray from 1933 [15], and aims to generalise some aspects of the traditional weak solution, like replacing the energy estimate with a local version, or localising the function spaces whenever it is possible (in a suitable manner, hence the additional hypotheses).

Remark 4.2. *Our definition of weak solution (which is sometimes referred to as Leray-Hopf solution as well) is in particular a Leray solution.*

Due to the nature of the local energy inequality, a relatively elaborate chain of apriori estimates and weak-regularity criteria can be established for the set $\mathcal{N}(v_0)$ which was done by Šverák et al. All these could be counted as basic tools and preliminaries for the main results presented in the following paragraphs.

Proposition 4.1. ***(Apriori estimate for forward self-similar Leray solutions)*** *Let v_0 be divergence free and -1 homogeneous from the space $L^2_{loc}(\mathbb{R}^3)$ with the additional assumption that $v_0|_{\partial B(0,1)} \in \mathcal{C}^\infty(\partial B(0,1))$. Let us consider $v \in \mathcal{N}(v_0)$ a scale-invariant solution. Then the corresponding profile function $V(x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}^3)$ and we have that*

$$|D^\alpha (V(x) - S(1)v_0(x))| \leq \frac{C(\alpha, v_0)}{(1+|x|)^{3+|\alpha|}}, \quad \forall |\alpha| \geq 0, \alpha \in \mathbb{N}^3, \tag{4.4}$$

where D^α is an arbitrary differential operator of order α.

Theorem 4.3. *Let $v_0 \in \mathcal{C}^\infty(\mathbb{R}^3\backslash\{0\})$ divergence-free and -1 homogeneous. Then there exists $v \in \mathcal{N}(v_0)$ scale-invariant solution for the system (3.1). Moreover we have that $v \in \mathcal{C}^\infty((0,\infty)\times\mathbb{R}^3)$ and its profile function satisfies the apriori estimate.*

Theorem 4.4. *Let $v_0 \in \dot{B}^s_{\infty,\infty}(\mathbb{R}^3\backslash\{0\})$ (with $s \in (0,1)$) a -1 homogeneous and divergence-free function. Then there exists $v \in \mathcal{N}(v_0)$ scale-invariant solution for the system (3.1). Moreover we have that its profile function is smooth and satisfies the apriori estimate.*

4.3 Further applications of forward self-similar solutions

Forward self-similar solutions not only provide an excellent tool for searching for existence results in (close to) critical spaces, they have many additional possible applications as well. A rather clear one is that they provide a solid basis for the construction of exact solutions since their special properties allow us to only work with the space variable, thus rendering the equations considerably simpler in some cases.

Apart from existence results they provide a useful tool for examining regularity properties (or even regularity issues) in certain function spaces or for certain types of solutions. Furthermore they have varying applications in the theory of asymptotic behaviour for the solutions of the Navier-Stokes system. In the following section we shall present a selected few theorems and remarks from these fields.

4.3.1 A singularity result

Initially, in the search for singularities for the Navier-Stokes equations, the backward self-similar solutions were a much better candidate. However, as we have seen in chapter 3, such solutions do not exist in general, so in reality they won't help in this question. On the contrary, forward self-similar solutions can provide singularity results. Indeed, for a self-similar solution of the form (4.2), we may also observe that at time $t \to 0$ such a solution may develop a singularity of the type $\sim \frac{1}{|x|}$.

An interesting result by Tian and Xin [24] is the following:

Theorem 4.5. *All the solutions $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ and $p(t,x)$ to the system (3.1) (with $\nu = 1$) in the pointwise sense, which are steady, symmetric about the x_1-axis, homogeneous of order -1 and regular except for 0 are given by the following explicit formulae:*

$$\begin{aligned} v_1(x) &= 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \\ v_2(x) &= 2\frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \\ v_3(x) &= 2\frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \\ p(x) &= 4\frac{cx_1 - |x|}{|x|(c|x| - x_1)^2}, \end{aligned} \tag{4.5}$$

where c is an arbitrary constant such that $|c| > 1$.

We shall remark that in the generalised (or distributional) sense, this explicit solution corresponds to the very singular $\phi(x) = \big(b(c)\delta_0(x), 0, 0\big)$ external force, which should explain to some extent the reason behind the singularity.

Remark 4.3. *The pointwise solution presented in the above theorem is not globally of finite energy but it belongs to* $L^2_{loc}(\mathbb{R}^3)$.

Remark 4.4. *An extensive analysis on the system shows that the function* $b(c)$ *is decreasing from* ∞ *to* 0 *on the interval* $(1, \infty)$, *hence it is possible to choose an appropriate* c *with which we may guarantee the existence and uniqueness of this solution by fixed point algorithm methods!*

For further facts about how this solution implies exactly the loss of regularity for large initial data, the reader should consult with [3] and [5].

4.3.2 Asymptotic properties

Asymptotic studies involving self-similar solutions are not unheard of. In fact, such connetions were known for a long time in the analysis of nonlinear heat equations for instance, that is: solutions are asymptotically close to self-similar solutions. A relatively thorough study on the topic was presented by Planchon in [21].

The main idea is that for a global solution $v(t, x)$ to the Navier-Stokes equations we would like to somehow define the limit $\lim_{\lambda\to\infty} v_\lambda(t, x)$ with $v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x)$. Formally speaking, if we denote this limit by $u(t, x)$ then it is easy to see that this 'function' is scale-invariant, furthermore one would expect that it solves the same system (3.1), since $\forall\lambda > 0$ $v_\lambda(t, x)$ was a solution to it.

Definition 4.3. *We say that* $v(t, x)$ *"converges in* L^p *norm" to a function* $U(x)$ *if and only if one of the two equivalent conditions is satisfied:*

1. *For all compact intervals* $[a, b] \subset (0, \infty)$

$$v_\lambda(t, x) \xrightarrow{L^p(dx)} \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right) \quad \text{as} \quad \lambda \to \infty,$$

   uniformly for $t \in [a, b]$.

2. *For* $t \to \infty$

$$\sqrt{t} v\left(t, \sqrt{t} x\right) \xrightarrow{L^p(dx)} U(x).$$

One of the main results involving asymptotic behaviours of certaion solutions is the following:

Theorem 4.6. *Let us fix $3 < p < \infty$ and pose $s = 1 - \frac{3}{p}$. Let $v(t,x)$ be a mild solution to (3.1) such that*

$$\sup_{t \geq 0} \left\| \sqrt{t} v \left(t, \sqrt{t} x\right) \right\|_{L^p} < \infty$$

and that $v(t,x)$ converges weakly to $v_0(x)$ when $t \to 0$. If v "converges in L^p norm" to $U(x)$, then the initial data $v_0(x)$ belongs to $\dot{B}^{-s}_{p,\infty}(\mathbb{R}^3)$, the function $u(t,x) = = \frac{1}{\sqrt{t}} U \left(\frac{x}{\sqrt{t}} \right)$ is a forward self-similar solution to (3.1) and $S(t)v_0$ "converges in L^p norm" to $S(1)u_0(x)$ where $u_0(x)$ is the initial data of the self-similar solution.

APPENDIX

"There are many fascinating problems and conjectures about the behavior of solutions of the Euler and Navier-Stokes equations. Since we don't even know whether these solutions exist, our understanding is at a very primitive level. Standard methods from PDE appear inadequate to settle the problem. Instead, we probably need some deep, new ideas."

Charles L. Fefferman [7]

APPENDIX A

DERIVATION OF THE EQUATIONS

The aim of this chapter is to provide a brief but considerably formal derivation of the incompressible Navier-Stokes equations presented in chapter 1 (system (1.1)), starting from purely physical layout. Since the equations are relatively simplified there will be a lot of physical assumptions on the examined fluid (gas or liquid).

We shall emphasize that, for the sake of being able to apply Fourier analysis methods and for simplification reasons as well, the fluid domain is the whole space $\mathbb{R}^3$ (thus boundary effects are neglected). In the Eulerian description, one can associate to every material point x in $\mathbb{R}^3$ at time $t \in \mathbb{R}$ the following physical quantities:

- the velocity field $\vec{v} = \vec{v}(t, x) \in \mathbb{R}^3$,
- the density $\varrho = \varrho(t, x) \in \mathbb{R}_+$,
- the internal energy $e = e(t, x) \in \mathbb{R}$,
- the entropy $s = s(t, x) \in \mathbb{R}$,
- the pressure $p = p(t, x) \in \mathbb{R}$.

There are various other relevant quantities as well, like for example the momentum ($\varrho\vec{v}$) or the temperature ($T = T(t, x)$), we only tried to name the most relevant ones for the establishment of the equations.

The equations governing the motion of the fluid arise from the several conservation laws of mechanics and thermodinamics. In order to be able to properly handle them mathematically, we shall introduce the notion of the flow of the velocity field $\vec{v}$, since the mathematical interpretation of the conservation laws will be based on the trivial fact that certain physical quantities are conserved along the particle trajectories.

Definition A.1. *The flow ψ of $\vec{v}$ is the solution of the following ordinary differential equation*

$$\frac{d}{dt}\psi(t,x) = \vec{v}(t,\psi(t,x)), \quad \psi(0,x) = x$$

(the point x being treated as a parameter here).

For a domain $\Omega \subset \mathbb{R}^3$ let us denote $\Omega_t = \psi_t(\Omega)$, where $\psi_t(x) = \psi(t,x)$. The lemma below will be a useful method to rewrite the soon to be presented conservation laws in a more appropriate fashion (in the case of $\mathcal{C}^1$ functions, it is easy to prove as well).

Lemma A.1. *Let Ω be an open subdomain of $\mathbb{R}^3$, ψ be the flow of $\vec{v}$ with $\Omega_t = \psi_t(\Omega)$. Let b be a scalar function. Then we have*

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} b(t,x)\,dx &= \int_{\Omega_t} \Big(\partial_t b(t,x) + \nabla\cdot(b(t,x)\vec{v}(t,x))\Big)\,dx \\ &= \int_{\Omega_t} \partial_t b(t,x)\,dx + \int_{\partial\Omega_t} \Big(b(t,x)\vec{v}(t,x)\cdot\vec{n}\Big)\,d\Sigma.\end{aligned}$$

(Here $\vec{n}$ denotes the unit outer normal vector at the boundary.)

Now we may start treating the various preserved physical quantities for our viscous fluid motion. We shall consider an arbitrary Ω open subdomain of $\mathbb{R}^3$ for the descriptions.

Mass conservation: we are working with a 'closed fluid', that is there is no production or loss of mass inside any part of the fluid during the time evolution of the movement. The mass of the fluid in Ω at time t can be expressed by

$$M_\Omega(t) = \int_\Omega \varrho(t,x)\,dx.$$

Since the mass is conserved, we may deduce that

$$\frac{d}{dt}M_{\Omega_t}(t) = \frac{d}{dt}\int_{\Omega_t} \varrho(t,x) = 0.$$

So by the formal application of lemma A.1 we obtain the equation for the **mass balance**:

$$\partial_t\varrho + \nabla\cdot(\varrho\vec{v}) = 0. \tag{A.1}$$

Momentum conservation: Newton's second law states that all the forces acting on the part of the fluid in Ω are to be equal to the force arising from the change of momentum. The momentum can be written in the following integral form

$$\vec{P}_\Omega(t) = \int_\Omega (\varrho\vec{u})(t,x)\,dx.$$

The change of the momentum equals the sum of all the forces acting on the body, that is the external force $\vec{\phi}$ (gravity for instance) and the surface forces at the boundary of the domain. This second force can be represented by the second order stress tensor σ (a matrix), which will be specified later, based on further physical assumptions. So we obtain

$$\frac{d}{dt}\vec{P}_{\Omega_t}(t) = \frac{d}{dt}\int_{\Omega_t}(\varrho\vec{v})(t,x)\,dx = \int_{\Omega_t}(\varrho\vec{\phi})(t,x)\,dx + \int_{\partial\Omega_t}(\sigma\cdot\vec{n})(t,x)\,d\Sigma,$$

and by the lemma we have the equation for the **momentum balance**:

$$\partial_t(\varrho\vec{v}) + \nabla\cdot(\varrho\vec{v}\otimes\vec{v}) = \varrho\vec{\phi} + \nabla\cdot\sigma. \tag{A.2}$$

Energy conservation: as a consequence of the first law of thermodynamics, we also have that the total energy, expressed by the quantity

$$E_\Omega(t) = \int_\Omega\left(\frac{1}{2}\varrho|\vec{v}|^2 + \varrho e\right)(t,x)\,dx$$

is also conserved, hence

$$\begin{aligned}\frac{d}{dt}E_{\Omega_t}(t) &= \frac{d}{dt}\int_{\Omega_t}\varrho(t,x)\left(e + \frac{|\vec{v}|^2}{2}\right)(t,x)\,dx\\ &= \int_{\Omega_t}(\varrho\vec{\phi}\cdot\vec{v})(t,x)\,dx + \int_{\partial\Omega_t}\Big((\sigma\cdot\vec{n})\cdot\vec{v}\Big)(t,x)\,d\Sigma - \int_{\partial\Omega_t}(\vec{q}\cdot\vec{v})(t,x)\,d\Sigma,\end{aligned}$$

where the first two integrals on the right hand side represent the energy arising from the acting forces, and the third integral stands for the heat loss along the boundary, with $\vec{q}$ being the heat flux vector. By applying the lemma again, we arrive at the following equation

$$\partial_t\left(\varrho\left(e + \frac{|\vec{v}|^2}{2}\right)\right) + \nabla\cdot\left(\varrho\left(e + \frac{|\vec{v}|^2}{2}\right)\vec{v}\right) = \varrho\vec{\phi}\cdot\vec{v} + \nabla\cdot(\sigma\cdot\vec{v}) - \nabla\cdot\vec{q}.$$

However, by multiplying (A.2) by $\vec{v}$ and introducing the kinetic energy by unit volume $E_k = \frac{1}{2}\varrho|\vec{v}|^2$, with the application of the Leibnitz rule we obtain the following equation as well:

$$\partial_t E_k + \nabla\cdot(E_k\vec{v}) = \varrho\vec{\phi}\cdot\vec{v} + (\nabla\cdot\sigma)\cdot\vec{v}.$$

Finally by subtracting the two previous equations from each other we obtain the simplified equation describing the **energy balance**:

$$\partial_t(\varrho e) + \nabla\cdot(\varrho e\vec{v}) = \nabla\cdot(\sigma\cdot\vec{v}) - (\nabla\cdot\sigma)\cdot\vec{v} - \nabla\cdot\vec{q}. \tag{A.3}$$

Entropy balance: the second law of thermodynamics states bluntly that the entropy can only increase, so for the entropy in the domain Ω,

$$S_\Omega(t) = \int_\Omega (\varrho s)(t,x)\,dx$$

should satisfy the following inequality

$$\frac{d}{dt} S_{\Omega_t} = \frac{d}{dt}\int_{\Omega_t} (\varrho s)(t,x)\,dx \geq -\int_{\Omega_t} \left(\frac{\vec{q}\cdot\vec{n}}{T}\right) d\Sigma.$$

Thus we are able to explicit the **entropy inequality**:

$$\partial_t(\varrho s) + \nabla\cdot(\varrho s\vec{v}) \geq -\nabla\cdot\left(\frac{\vec{q}}{T}\right). \tag{A.4}$$

Up until now we have only made little to no physical assumptions on the fluid, meaning that equations (A.1), (A.2), (A.3) and (A.4) are valid for a rather general fluid. In what follows we shall assume that the *fluid is Newtonian*, which implies the following three conditions:

- the fluid is isotropic, the physical quantities depend only on (t,x);
- the tensor σ is a linear function of the Jacobian $D\vec{v}$, invariant under rigid transformations;
- the angular momentum is conserved, based on which we may deduce a more exact form for the stress tensor:

  $$\sigma = \tau - p\,\mathrm{Id} \text{ with } \tau = \lambda(\nabla\cdot\vec{v})\,\mathrm{Id} + 2\mu D(\vec{v}),$$

  where $\tau = \tau(t,x)$ is called the viscous stress tensor, the real numbers λ and μ are the viscosity coefficients and $D(\vec{v}) = \frac{1}{2}(D\vec{v} +^t D\vec{v})$ is the deformation tensor.

By the Gibbs-Duhem relation, which states that

$$T\,ds = de + p\,d\left(\frac{1}{\varrho}\right)$$

a straightforward calculation (basically just the multiple application of the Leibnitz rule) yields

$$\partial_t(\varrho s) + \nabla\cdot(\varrho s\vec{v}) = \frac{\tau : D(\vec{v})}{T} - \nabla\cdot\left(\frac{\vec{q}}{T}\right) - \frac{\nabla T\cdot\vec{q}}{T^2},$$

where ":" denotes the Frobenius inner product (component-wise inner product).

Using this equation with the entropy inequality (A.4) leads us to the following simplified form of it:

$$\tau : D(\vec{v}) - \frac{\nabla T \cdot \vec{q}}{T} \geq 0.$$

By the definition of τ and making use of the so-called Fourier law: $\vec{q} = -k\nabla T$ (with k the thermal conductivity coefficient) we obtain that the entropy inequality is verified if and only if

$$\lambda(\nabla \cdot \vec{v})^2 + 2\mu \operatorname{Tr}(D(\vec{u}))^2 + k\frac{|\nabla T|^2}{T} \geq 0,$$

thus we have the following constraints on the coefficients:

$$k \geq 0, \quad \mu \geq 0, \quad \text{and} \quad 2\mu + 3\lambda \geq 0. \tag{A.5}$$

So we can safely conclude that the entropy inequality did not play much of a role in describing the movement (we shall remind the reader that we assumed that these coefficients do not depend on other variables of the system, or even they were supposed to be constants).

One final adjustment shall be the two state equations interrelating p, ϱ, e and T. These equations can be written as follows under the assumption of having the properties of a perfect gas:

$$e = C_v T \quad \text{and} \quad p = R\varrho T, \tag{A.6}$$

where C_V and R are given positive constants. By introducing the adiabatic constant $\gamma = \frac{C_p}{C_V}$ where $C_p = C_V + R$ constant, we may deduce that $p = (\gamma - 1)\varrho e$. Therefore the equation describing the energy balance obtains the following form

$$\partial_t p + \vec{v} \cdot \nabla p + \gamma p \nabla \cdot \vec{v} - (\gamma - 1)\nabla \cdot (k\nabla T) = (\gamma - 1)\left(\varrho\vec{\phi} + 2\mu D(\vec{u}) : D(\vec{u}) + \lambda(\nabla \cdot \vec{u})^2\right). \tag{A.7}$$

With the additional assumptions of p depending only on the density ϱ, and λ and μ not depending on neither T nor ϱ, we can see that this modified energy equation became uncoupled from the mass and momentum equations (A.1) and (A.2). So, since the energy equation defines p, if ϱ and $\vec{u}$ are already known, only the mass and momentum equations remain for us to be solved:

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho \vec{v}) = 0 \\ \partial_t(\varrho \vec{v}) + \nabla \cdot (\varrho \vec{v} \otimes \vec{v}) - \mu \Delta \vec{v} - (\mu + \lambda)\nabla(\nabla \cdot \vec{v}) + \nabla p = \varrho \vec{\phi}. \end{cases} \tag{A.8}$$

And by posing $\varrho \equiv 1$ we can deduce the desired (1.1) system.

APPENDIX B

REMARKS ON THE HEAT SEMIGROUP

As the name suggests, the heat semigroup arises from the exact solution of the homogeneous initial value problem associated to the heat equation

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= 0, \\ u(0,x) &= u_0(x). \end{aligned} \tag{B.1}$$

It is a well-known result (and a relatively simple calculation using Fourier transforms), that the solution of this problem can be expressed in the following form

$$u(t,x) = S(t)u_0(x),$$

where the operator $S(t)$ is a convolution-type operator of the form (for the case of being in $\mathbb{R}^3$)

$$S(t) = \exp(t\Delta) = \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \exp\left(-\frac{|x|^2}{4t}\right) * \tag{B.2}$$

(here naturally $*$ denotes the convolution operator).

This $S(t)$ operator is also called the heat semigroup and as the name suggests, it is indeed a semigroup of operator (for $t > 0$). This semigroup is usually the simplest example of operator semigroups and is considered to be a 'fundamental piece' in operator semigroup theory.

A bit more complicated example introduced in chapter 1 when deriving the modified equation of the Navier-Stokes system (1.5) was the heat semigroup "composed" with the Leray-Hopf projection operator, which was denoted by $\bar{S}(t)$, that is

$$\bar{S}(t) = \exp t\mathbb{P}\Delta.$$

Proposition B.1. *The semigroup $\bar{S}(t)$ is a bounded holomorphic semigroup in $X^p(\mathbb{R}^3)$, where $X^p(\mathbb{R}^3)$ is the space of functions from $L^p(\mathbb{R}^3)$ with divergence equaling* *zero. That is, for every nonnegative integer m, the estimate*

$$||(-\mathbb{P}\Delta)^m \exp t\mathbb{P}\Delta f||_{L^p} \leq Ct^{-m}||f||_{L^p}$$

is valid with a constant C for any $f \in X^p$ and $t > 0$

Conceptually speaking, the homolorphicity is referring to the fact that the solution $\bar{S}(t)u_0$ is holomorphic in time. This property often remains true for similar semigroups arising from parabolic equations thus the notion of holomorphic semigroup is sometimes referred to as parabolic semigroup too. For a proof of this proposition and some generalisations of this property, as well as further connections to some strong solution existence results for the integral equation associated to the Navier-Stokes system, the reader is invited to consult [11].

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