Two counterexamples on completely independent spanning trees

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Abstract

For each \( k \geq 2 \), we construct a \( k \)-connected graph which does not contain two completely independent spanning trees. This disproves a conjecture of Hasunuma. Furthermore, we also give an example for a 3-connected maximal plane graph not containing two completely independent spanning trees.

1 Introduction

The notion of completely independent spanning trees was introduced by Hasunuma [1] inspired by former investigations on independent trees. Both concepts have applications to fault-tolerant broadcasting problems in interconnection networks.

Let \( G \) be a graph. Two paths \( P_1 \) and \( P_2 \) between vertices \( x \) and \( y \) in \( G \) are said to be \textit{openly disjoint} if they are vertex-disjoint apart from their end vertices. Let \( T_1, T_2, \ldots, T_k \) be spanning trees of \( G \). \( T_1, T_2, \ldots, T_k \) are \textit{completely independent} if for any two vertices \( u, v \) of \( G \), the paths between \( u \) and \( v \) in \( T_1, T_2, \ldots, T_k \) are pairwise openly disjoint.

By this time two positive results are known about the existence of completely independent spanning trees, both are due to Hasunuma:

\textbf{Theorem 1.} [1] Let \( L(G) \) be a \( k \)-connected line digraph. Then there are \( k \) completely independent spanning trees in the underlying undirected graph of \( L(G) \).

\textbf{Theorem 2.} [2] There are two completely independent spanning trees in any 4-connected maximal plane graph.

If we replace ‘openly disjoint’ with ‘edge-disjoint’ in the definition of completely independent trees, we obtain an equivalent definition of edge-disjoint spanning trees. On edge-disjoint spanning trees, the celebrated theorem of Nash-Williams states that there are \( k \) edge-disjoint spanning trees in any \( 2k \)-edge-connected graph. This theorem and the results above led Hasunuma to pose the following analogous conjecture:

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Conjecture 1. There are $k$ completely independent spanning trees in any $2k$-connected graph.

In this note we show that Conjecture 1 proves to be wrong, in fact we construct a $k$-connected graph for any $k$ which does not contain even two completely independent spanning trees. As this shows that there is no direct relation between the existence of completely independent spanning trees and connectivity, and considering the fact that the existence of two completely independent spanning trees implicates only 2-connectivity we may ask whether in Theorem 2 the condition of being 4-connected is necessary. However, our second construction will show that in that case the 4-connectivity cannot be weakened: there exists a 3-connected maximal plane graph which does not contain two completely independent spanning trees.

2 Preparations

Hasunuma also gave the following clear and useful characterization for completely independent spanning trees:

**Theorem 3.** Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph $G$. Then $T_1, T_2, \ldots, T_k$ are completely independent if and only if $T_1, T_2, \ldots, T_k$ are edge-disjoint and for any vertex $v$ of $G$, there is at most one spanning tree $T_i$ such that $v$ is not a leaf in $T_i$.

We call a vertex of a tree an inner vertex if it is not a leaf. The characterization says that every vertex of the graph is an inner vertex of at most one of the trees. In other words, the vertices of the graph can be coloured with $k$ colours in such a way that if a vertex is an inner vertex of tree $T_i$ then its colour is $i$. (The vertices which are leaves in each tree can be coloured arbitrarily.)

**Observation 1.** If $T_1$ and $T_2$ are completely independent spanning trees, then $T_1$ is not a star, i.e. it has more than one inner vertex.

**Proof:** If it had a sole inner vertex $v$, then all edges of $G$ incident to $v$ should belong to $T_1$ and thus $v$ would not be reachable in $T_2$, a contradiction.

**Observation 2.** If $T_1$ and $T_2$ are completely independent spanning trees in $G$ and $v \in V(G)$ is an arbitrary vertex, then $v$ has a neighbour, which is an inner vertex in $T_1$.

**Proof:** If $v$ is an inner vertex in $T_1$, then it has a neighbour which is also an inner vertex as $T_1$ is not a star by Observation 1. If $v$ is not an inner vertex in $T_1$ then it is a leaf and thus has a neighbour which is an inner vertex (we may suppose that $|V(T_1)| \geq 3$).
3 Counterexamples

**Proposition 1.** For any \( k \geq 2 \), there exists a \( k \)-connected graph that does not contain two completely independent spanning trees.

**Proof:** We will give a construction that results in the desired graphs. Let \( k \) be fixed. As a starting point, let \( H \) be a complete graph on \( 2k - 1 \) vertices. For all \( k \)-tuples of vertices of \( H \) let us add a new vertex to the graph and connect it to the \( k \) vertices of the tuple. Let us denote the obtained graph by \( G_k \). (So \( G_k \) has \( 2k - 1 + \binom{2k-1}{k} \) vertices.) It is clear that \( G_k \) is \( k \)-connected.

We show that there are no two completely independent spanning trees in \( G_k \). Suppose that \( T_1 \) and \( T_2 \) are such trees. By applying Observation 2 for the vertices outside of \( H \) we obtain that choosing any \( k \) vertices from \( H \) at least one of them is an inner vertex of \( T_1 \). It means that out of the vertices of \( H \) at most \( k - 1 \) can be a leaf in \( T_1 \). Analogously in \( T_2 \), so at least one vertex of \( H \) should be an inner vertex in both trees \( T_1 \) and \( T_2 \), a contradiction. \( \square \)

**Proposition 2.** There exists a 3-connected maximal plane graph which does not contain two completely independent spanning trees.

**Proof:** We say that a cycle of a plane graph is a cut cycle if the graph has at least one additional vertex inside the cycle and at least one outside of it. Suppose that we have a plane graph that contains two completely independent spanning trees. Colour their inner vertices blue and red, respectively (due to the characterisation every vertex gets at most one colour), and then colour the remaining uncoloured vertices arbitrarily. Consider a cut cycle. As both trees contain a path from the outside of the cycle to its inside, we obtain that the cycle contains at least one blue and at least one red vertex. The basic idea of the counterexample is to show that there exists an appropriate plane graph whose cut cycles cannot be coloured in such a way.

![Figure 1](image)

Figure 1: Solid vertices and edges form \( G \), the additional vertices and edges of \( G' \) are dashed.
First observe that if we have an arbitrary 3-connected plane graph $G$, then we can construct a 3-connected maximal plane graph $G'$ in which $G$ is contained as a subgraph and every cycle of $G$ is a cut cycle in $G'$. Indeed, let us add a new vertex for every face of $G$ (including the exterior face), drawing them inside the corresponding face and connect them to every vertex of the face (see Figure 1). The graph thus obtained is clearly 3-connected and maximal. So it suffices to show that there exists a 3-connected plane graph whose vertices cannot be coloured with two colours without constructing a monochromatic cycle. Our counterexample is based on a construction of Tutte and uses the fact that it is a maximal plane graph.

From now on we use ideas from [3]. Suppose that $G$ is a 3-connected maximal plane graph where vertices can be coloured this way. Then, according to the assumptions the 3-length cycles surrounding the faces are not monochromatic, so one of the colours occurs exactly twice on them and the other one exactly once. Let $F$ be the set of edges whose two end-vertices have the same colour. Then $F$ contains exactly one edge from each face, which means that in $G^*$ (the dual graph of $G$) the set of the duals of the edges in $F$ (let us denote it with $F^*$) forms a perfect matching. Moreover, as there is no monochromatic cycle in $G$, i.e. there is no cut in $G^*$ whose all edges are in $F^*$, $G^* - F^*$ is connected. As $G^*$ is 3-regular, $G^* - F^*$ is 2-regular and thus, as being connected, is a Hamiltonian cycle. In summary, we obtained that if there is a required colouring in $G$ then there is a Hamiltonian cycle in $G^*$.

As the dual of a 3-connected plane graph is 3-connected, it suffices to find a 3-connected 3-regular plane graph which is not Hamiltonian. Tait conjectured that all such graphs are Hamiltonian but this statement was disproved by Tutte in 1946 (see e.g. [4]). The counterexample of Tutte constructed for this purpose is the so-called Tutte graph (see Figure 2). Taking its dual and adding the new vertices and edges to the obtained graph in the way written above we get a 3-connected maximal plane graph which does not contain two completely independent spanning trees. □
References


