On constructive characterizations of \((k, l)\)-sparse graphs

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December 2003
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Abstract

In this paper we study constructive characterizations of graphs satisfying tree-connectivity requirements. The main result is the following: if \(k\) and \(l\) are positive integers and \(l \leq \frac{k}{2}\), then a necessary and sufficient condition is proved for a node being the last node of a construction in a graph having at most \(k|X| - (k + l)\) induced edges in every subset \(X\) of nodes.

Keywords: sparse graph, constructive characterization

1 Constructive characterizations

A constructive characterization of a graph property is meant to be a building procedure consisting of some simple operations so that the graphs obtained from some specified initial graph by these operations are precisely those having the property. For example, a graph is connected if and only if it can be obtained from a node by the operation: add a new edge connecting an existing node with either an existing node or a new one. Another well-known result is the so-called ear-decomposition of 2-connected graphs.

A graph is said to be \(k\)-edge-connected if the deletion of at most \(k - 1\) edges results in a connected graph. From now on, adding an edge means adding a new edge connecting two existing nodes. This new edge can be parallel to existing ones, but it cannot be a loop unless otherwise stated. In 1976 Lovász \[1\] proved the following result.

Theorem 1.1. An undirected graph \(G = (V, E)\) is 2\(k\)-edge-connected if and only if \(G\) can be obtained from a single node by the following two operations:

(i) add a new edge (possibly a loop),

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(ii) add a new node \( z \), subdivide \( k \) existing edges by new nodes, and identify the \( k \) subdividing nodes with \( z \).

Operation (ii) is called pinching \( k \) edges.

Similar constructive characterizations for \( 2k + 1 \)-edge-connectivity were given by Mader. A directed counterpart of the previous results is also due to Mader [11]. This kind of characterizations can be very useful. For example, Lovász used his result to derive Nash-Williams' theorem [12] on \( k \)-edge-connected orientations of graphs, while Mader used his result to derive Edmonds' theorem [2] on disjoint arborescences.

\( k \)-edge-connectivity is the common way to formulate one's intuitive feeling for high 'edge-connection' of an undirected graph but there may be other possibilities, as well.

An undirected graph is called \( k \)-tree-connected if it contains \( k \) edge-disjoint spanning trees. The following constructive characterization of \( k \)-tree-connected graphs was given by Frank in [3] by observing that a combination of a theorem of Mader and a theorem of Tutte gives rise to the following. (For a direct proof, see Tay [14].)

**Theorem 1.2.** An undirected graph \( G = (V, E) \) is \( k \)-tree-connected if and only if \( G \) can be built from a single node by the following two operations:

(i) add a new edge,

(ii) add a new node \( z \) and \( k \) new edges ending at \( z \),

(iii) pinch \( i \) \((1 \leq i \leq k − 1)\) existing edges with a new node \( z \), and add \( k − i \) new edges connecting \( z \) with existing nodes.

Which constructive characterization can be considered to be good. Jüttner [8] gave the following building procedure for graphs having a Hamiltonian cycle. Beginning from \( K_3 \) use the following two operations: adding a new edge between two existing nodes and subdividing an edge incident to a node of degree 2 by a new node. It is clear that this procedure builds up a graph \( G \) if and only if \( G \) has a Hamiltonian cycle.

Why do not we think that this is a good constructive characterization? We did not accept the characterization of \( k \)-tree-connected graphs by taking immediately \( k \) edge-disjoint trees because it does not take the nodes one by one. Here it is satisfied. The main problem of this characterization here is that it cannot be checked for a graph in polynomial time if it can be obtained this way or not.

Nash-Williams [13] proved the following theorem concerning coverings by trees. For a graph \( G = (V, E) \), \( \gamma_G(X) \) denotes the number of edges of \( G \) with both end-nodes in \( X \subseteq V \).

**Theorem 1.3 (Nash-Williams).** A graph \( G = (V, E) \) is the union of \( k \) edge-disjoint forests if and only if \( \gamma_G(X) \leq k|X| − k \) for all nonempty \( X \subseteq V \).

In [3] two variants of the notion of \( k \)-tree-connectivity were considered. A graph \( G \) (with at least 2 nodes) is called nearly \( k \)-tree-connected if \( G \) is not \( k \)-tree-connected but adding any new edge to \( G \) results in a \( k \)-tree-connected graph. Let \( K_{2^{k-1}} \) denote the graph on two nodes with \( k − 1 \) parallel edges. (Based on the work of Henneberg...
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and Laman [3], Tay and Whiteley [16] gave the proof of the following theorem in the special case of \( k = 2 \).

**Theorem 1.4.** An undirected graph \( G = (V, E) \) is nearly \( k \)-tree-connected if and only if \( G \) can be built from \( K_2^{k-1} \) by applying the following operations:

\((O1')\) add a new node \( z \) and \( k \) new edges ending at \( z \) so that no \( k \) parallel edges can arise,

\((O2')\) choose a subset \( F \) of \( i \) existing edges \( (1 \leq i \leq k-1) \), pinch the elements of \( F \) with a new node \( z \), and add \( k - i \) new edges connecting \( z \) with other nodes so that there are no \( k \) parallel edges in the resulting graph.

Actually, we proved this result in a slightly more general form. We proved the following conjecture in case \( l = 1 \). Let \( k, l \) be two integers such that \( k \geq 2 \) and \( \frac{k}{2} \geq l \geq 0 \). A graph \( G = (V, E) \) is said to be \((k, l)\)-sparse if \( \gamma_G(X) \leq k|X|-(k+l) \) for all \( X \subseteq V, |X| \geq 2 \). (By convention the graph with one single node is \((k, l)\)-sparse.)

**Conjecture 1.5.** Let \( 1 \leq l < \frac{k+2}{3} \). An undirected graph \( G = (V, E) \) is \((k, l)\)-sparse if and only if \( G \) can be built from a single node by applying the following operations:

\((P1)\) add a new node \( z \) and at most \( k \) new edges ending at \( z \) so that no \( k-l+1 \) parallel edges can arise.

\((P2)\) Choose a subset \( F \) of \( i \) existing edges \( (1 \leq i \leq k-1) \), pinch the elements of \( F \) with a new node \( z \), and add \( k - i \) new edges connecting \( z \) with other nodes so that there are no \( k-l+1 \) parallel edges in the resulting graph.

(If \( l = 0 \) is allowed, then Theorem 1.2 is also a special case which has been already verified.) By the fundamental Theorem 1.3 of Nash-Williams, a graph is \((k, l)\)-sparse if and only if the edge-set can be covered by \( k \) spanning trees after adding \( l \) new edges arbitrarily.

We call a graph highly \( k \)-tree-connected if the deletion of any existing edge leaves a \( k \)-tree-connected graph. Frank and Király [11] gave a constructive characterization (among others) for highly 2-tree-connected graphs. In [3] this was extended for arbitrary \( k \geq 2 \).

We mention a recent result of Berg and Jordán [12] who proved a conjecture of Connelly. A 2-connected undirected graph \( G = (V, E) \) is a generic circuit if \( |E| = 2|V| - 2 \) and \( \gamma_G(X) \leq 2|X| - 3 \) for all \( 2 \leq |X| \leq |V| - 1 \).

**Theorem 1.6.** An undirected graph \( G = (V, E) \) is a generic circuit if and only if \( G \) can be built up from \( K_4 \) by the following operation:

- subdivide an edge \( uv \) by a new node \( z \) and add an edge \( zw \) so that \( w \neq u, v \).

These graphs have a role in rigidity theory. We also remark that Whiteley in [17] provided some rigidity property of nearly \( k \)-tree-connected graphs.

Jackson and Jordán considers sparse graphs in connection with rigidity properties in [7]. In [15] Tay proved for inductive reasons that a node of degree at most \( 2k - 1 \)
either can be “split off”, or “reduced” to obtain a smaller nearly $k$-tree-connected graph. Theorem [4] says that there always is a node which can be “split off”.

We have the following theorem which follows easily from the definition of $(k,l)$-sparse graphs.

**Theorem 1.7.** Let $1 \leq l \leq \frac{k}{2}$. If an undirected graph $G = (V,E)$ can be built up from a single node by applying the operations (P1) and (P2), then $G$ is $(k,l)$-sparse.

Inspired by the previous constructive characterizations we would conjecture that the reverse of the above theorem is also true for all $k$ and $l$ satisfying $\frac{k}{2} \geq l$. But as we will show in Section 4, this is not true if $l \geq \frac{k+2}{3}$. We believe that Conjecture 1.5 will be proved soon.

## 2 Splittings for $(k,l)$-sparse graphs

In the definition of $(k,l)$-sparse graphs why do not we allow bigger $l$ values? The answer is that, if $\frac{k}{2} < l$ and $|E| = 3k - (k + l) = 2k - l$, then there is no graph on 3 nodes satisfying $\gamma_G(X) \leq k|X| - (k + l)$ for all $X \subseteq V, |X| \geq 2$. Indeed, if there was one $G = (V,E)$, then $|E| \leq 3(k - l)$ since an edge may have multiplicity at most $k - l$. Since $2k - l > 3k - 3l$, we get a contradiction.

With the same reasoning the following can be proved.

**Lemma 2.1.** There is no graph on $m \geq 3$ nodes with $|E| = km - (k + l)$ satisfying $\gamma_G(X) \leq k|X| - (k + l)$ for all $X \subseteq V, |X| \geq 2$ if $\frac{m-1}{m+1}k < l$.

**Proof.** Since $|E| \leq \frac{m(m-1)}{2}(k - l)$ by the maximal multiplicity of an edge, we have $km - (k + l) = |E| \leq \frac{m(m-1)}{2}(k - l)$. But

$$km - (k + l) - \frac{m(m-1)}{2}(k - l) =$$

$$\frac{(m^2 - m - 2)l - (m^2 - 3m + 2)k}{2} = \frac{(m - 2)((m + 1)l - (m - 1)k)}{2} >$$

$$\frac{1}{2} \left( (m + 1) \frac{m - 1}{m + 1} k - (m - 1)k \right) = 0,$$

a contradiction. \hfill \Box

That is why we study here only the case of $l \leq \frac{k}{2}$.

In graph $G$ splitting off a pair $zu$ and $zv$ of edges for distinct $u$ and $v$ means that we delete these two edges and add a new edge $uv$ (maybe parallel to the other existing edges) to $G$. After applying this operation, $uv$ is called a split edge. A splitting off in a $(k,l)$-sparse graph $G$ is admissible if the resulting graph on node set $V - z$ is $(k,l)$-sparse.
**Definition 2.2.** Let $b_G$ denote the following function for any $X \subseteq V, |X| \geq 2$

$$b_G(X) := k|X| - (k + l) - \gamma_G(X).$$

By this definition a graph $G = (V, E)$ is $(k, l)$-sparse if and only if $b_G(X) \geq 0$ for all subsets $X \subseteq V, |X| \geq 2$. If $b_G(X) = 0$ and $X \neq V$, then $X$ is said to be a $G$-tight set. Furthermore $G$ is a union of $k$ edge-disjoint spanning trees after adding arbitrary $l$ edges if and only if $G$ is $(k, l)$-sparse and $b_G(V) = 0$. We will abbreviate $b_G$ by $b$.

**Observation 2.3.** Splitting off $zu$ and $zv$ at node $z$ is not admissible if and only if there exists a tight subset in $V - z$ containing $u$ and $v$.

We say that splitting off $j$ disjoint pairs of edges $(1 \leq j \leq k - 1)$ at node $z$ is admissible if it consists of admissible splittings. Obviously the order of the pairs in a splitting sequence is irrelevant. The length of a splitting sequence $S$ is the number of its pairs and it is denoted by $|S|$. $G_S$ denotes the graph obtained after applying the splitting sequence $S$.

An admissible splitting sequence at node $z$ of length $d_G(z) - k$ (which number is denoted by $i$) is called a full splitting for $d_G(z) \geq k + 1$. For the sake of convenience, at a node $z$ with degree at most $k$ the inverse of operation (P1) (that is, the deletion of $z$ and all of its adjacent edges) is also called a full splitting. The main result of this chapter is an necessary and sufficient condition of a node admitting a full splitting. We hope that it will lead to a proof of Conjecture [L5] just like in the special case of $l = 1$.

Note that $b_G(X)$ is an upper bound for the number of split edges induced by $X \subseteq V - z$ provided by an admissible sequence of splittings at some node $z$.

The next four claims are about $(k, l)$-sparse graphs. ($d_G(X, Y)$ is defined to be the number of edges between the node-sets $X$ and $Y$.)

**Claim 2.4.** If $X, Y \subseteq V$ and $|X \cap Y| \geq 2$, then

$$b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d(X, Y).$$

**Proof.** $b(X) + b(Y) = k|X| - (k + l) - \gamma_G(X) + k|Y| - (k + l) - \gamma_G(Y) = k(|X| + |Y|) - 2(k + l) - (\gamma_G(X \cap Y) + \gamma_G(X \cup Y) - d_G(X, Y)) = k|X \cap Y| - (k + l) - \gamma_G(X \cap Y) + k|X \cup Y| - (k + l) - \gamma_G(X \cup Y) + d_G(X, Y) = b(X \cap Y) + b(X \cup Y) + d(X, Y). \square$

**Claim 2.5.** If $X, Y \subseteq V$ and $|X \cap Y| = 1$, then

$$b(X) + b(Y) = b(X \cup Y) - l + d(X, Y).$$

**Proof.** $b(X) + b(Y) = k|X| - (k + l) - \gamma_G(X) + k|Y| - (k + l) - \gamma_G(Y) = k(|X| + |Y| - 1) - (k + l) - l - (\gamma_G(X) + \gamma_G(Y)) = k|X \cup Y| - (k + l) - l - (\gamma_G(X \cup Y) - d_G(X, Y)) = b(X \cup Y) - l + d(X, Y). \square$

**Claim 2.6.** If $X_1, X_2, X_3 \subseteq V$ and $|X_j \cap X_m| = 1$ for $1 \leq j < m \leq 3$ and $|X_1 \cap X_2 \cap X_3| = 0$, then

$$b(\bigcup_{j=1}^{3} X_j) \leq \sum_{j=1}^{3} b(X_j) - k + 2l.$$
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Proof. \(b(\bigcup_{j=1}^{3} X_j) = k|\bigcup_{j=1}^{3} X_j| - (k + l) - \gamma_G(\bigcup_{j=1}^{3} X_j) \leq k(\sum_{j=1}^{3} |X_j| - 3) - (k + l) - \sum_{j=1}^{3} \gamma_G(X_j) = \sum_{j=1}^{3} (|X_j| - (k + l) - \gamma_G(X_j)) - k + 2l = \sum_{j=1}^{3} b(X_j) - k + 2l. \ □

Remark. Especially, all of \(X_1, X_2, X_3\) cannot be tight at the same time for \(k \geq 2l + 1\). If \(k = 2l\) and \(X_1, X_2, X_3\) are tight sets, then \(\bigcup_{j=1}^{3} X_j\) is also tight.

Claim 2.7. Let \(z \in V\) and \(X \subset V - z\) be a maximal tight set containing the distinct nodes \(c_1, c_2\). Let \(d\) be a node in \(V - X - z\). If there is a tight set in \(V - z\) containing \(c_1\) and \(d\), then there is no tight set in \(V - z\) containing \(c_2\) and \(d\).

Proof. According to Claim 2.4, \(P \cap X = \{c_1\}\) since \(X\) is maximal. By Claims 2.4 and 2.6, we obtain that there is no tight set containing \(c_2\) and \(d\).

Let \(G\) be a \((k, l)\)-sparse graph. Since \(\sum_{v \in V} d_G(v) = 2|E| \leq 2k|V| - 2(k + l) < 2k|V|\), it follows that there is a node \(z\) of \(G\) with \(d_G(z) \leq 2k - 1\).

Claim 2.8. Let \(G = (V, E)\) be a \((k, l)\)-sparse graph. \(d_G(u, v) \leq k - l\) for any two nodes \(u, v\).

Proof. By the definition of \((k, l)\)-sparse graphs, \(\gamma_G(\{u, v\}) \leq k|\{u, v\}| - (k + l) = k - l\) for set \(\{u, v\}\).

3 Full splittings in \((k, l)\)-sparse graphs

In this section we derive a necessary and sufficient condition for an arbitrary specified node to admit a full splitting.

Let \(k \geq 2\) and \(0 < l < \frac{k}{2}\). Let \(G\) be a \((k, l)\)-sparse graph. Consider a node \(z\) with degree at most \(2k - 1\) for which there is no full splitting. If \(d_G(z) \leq k\), then the deletion of \(z\) and its adjacent edges results in a \((k, l)\)-sparse graph, hence \(d_G(z) \geq k + 1\).

Assume that a longest admissible splitting sequence \(S\) at \(z\) is not full. Since \(z\) does not admit a full splitting, \(|S| < i := d_G(z) - k\).

Let \(N_D(w)\) denote the set of the neighbours of a node \(w\) in graph \(D\).

Claim 3.1. If \(|N_{G^*_S}(z)| \geq 2\), then there exists a maximal \(G_S\)-tight subset \(P_{\text{max}}\) of \(V - z\) including \(N_{G^*_S}(z)\).

Proof. Let \(za\) and \(zb\) denote two non-parallel edges. Since \((za, zb)\) is not an admissible splitting off, there is a \(G_S\)-tight set \(X \subseteq V - z\) containing \(a\) and \(b\). According to Claim 2.4, there is a maximal tight set \(P \subseteq V - z\) containing \(a\) and \(b\).

If there is another neighbour \(c\) of \(z\) which is not in \(P\), then there is a tight set \(Y \subseteq V - z\) containing \(a\) and \(c\), since \((za, zc)\) is not an admissible splitting off. Since \(P\) is maximal, \(Y \cap P = \{a\}\). By Claim 2.7, \((zb, zc)\) is an admissible splitting off, a contradiction, that is, \(P\) contains all the neighbours of \(z\). \(\ □\)

Claim 3.2. If \(|N_{G^*_S}(z)| \geq 2\), then there exists a split edge which is disjoint from the nodes of \(P_{\text{max}}\).
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Proof. Since there is no admissible splitting off at \(z\) in \(G_S\), according to Claim 2.1, there exists \(P_{\text{max}} \subseteq V - z\). Let \(j,h,m\) denote the number of split edges with exactly, respectively, 2, 1, 0 end-node in \(P_{\text{max}}\). \(j + h + m = |S| < i\) since \(S\) is not full.

\[
k|P_{\text{max}} + z| - (k + l) \geq \gamma_G(P_{\text{max}} + z) = \gamma_{G_S}(P_{\text{max}}) + j + h + d_{G_S}(z, P_{\text{max}})
\]

\[
= \gamma_{G_S}(P_{\text{max}}) + j + h + (k + i - 2(j + h + m))
\]

\[
= \gamma_{G_S}(P_{\text{max}}) + k + (i - (j + h + m) - m > k|P_{\text{max}}| - (k + l) + k - m
\]

\[= k|P_{\text{max}} + z| - (k + l) - m,
\]

which implies \(m > 0\). \(\square\)

Claim 3.3. If \(|N_{G_S}(z)| \geq 2\), then \(|N_{G_S}(z)| = 2\). There is a neighbour \(s\) of \(z\) for which \(d_{G_S}(z,s) = 1\).

Proof. First assume that \(|N_{G_S}(z)| \geq 3\). Let \(a_1,a_2,a_3\) denote three of these nodes. By Claim 2.2 there is a split edge \(uv\) disjoint from \(P_{\text{max}}\). Let \(J = \{1,2,3\}\).

By Claim 2.7, \(S - (zu,zv) \cup (zu, za_j)\) is an admissible splitting sequence for at least two elements \(j\) of \(J\). The same is true for \(S - (zu,zv) \cup (zv, za_j)\). Hence we may assume that \(S - (zu,zv) \cup (zu, za_1)\) and \(S - (zu,zv) \cup (zv, za_2)\) are both admissible splitting sequences. We claim that \(S' := S - (zu,zv) \cup (zu, za_1) \cup (zv, za_2)\) is an admissible splitting sequence. If not, then there is a tight set \(Y\) in \(G_S - z\) containing \(u,v,a_1,a_2\). Then, according to Claim 2.4, \(P_{\text{max}} \cup Y\) is a tight set in \(G_S - z\) contradicting the maximality of \(P_{\text{max}}\). The length of \(S'\) is greater than the length of \(S\), a contradiction.

Now assume that \(|N_{G_S}(z)| = 2\). Let \(s\) and \(t\) be the two neighbours of \(z\) and assume that \(d_{G_S}(z,s) \geq 2\) and \(d_{G_S}(z,t) \geq 2\). By Claim 2.2 there is a split edge \(uv\) disjoint from \(P_{\text{max}}\). According to Claim 2.7 \(S - (zu,zv) \cup (zu, zt)\) or \(S - (zu,zv) \cup (zu, zs)\) is an admissible splitting sequence. This also holds for \(zv\) instead of \(zu\).

Hence at least one of the following splitting sequences is admissible: \(S - (zu,zv) \cup (zu, zt)\cup(zv, zt), S - (zu,zv) \cup (zu, zt)\cup(zv, zs), S - (zu,zv) \cup (zu, zs)\cup(zv, zt), S - (zu,zv) \cup (zu, zt)\cup(zv, zs)\), a contradiction. \(\square\)

Now we prove that if \(d_G(z)\) is at most \(k + l\), then a full splitting always exists at \(z\).

Proposition 3.4. Let \(G\) be a \((k,l)\)-sparse graph. If \(z \in V\) has degree at most \(k + l\), then there exists a full splitting at \(z\).

Proof. If \(d_G(z)\) is at most \(k\), then if we delete \(z\) with its adjacent edges, then we obviously get a \((k)\)-sparse graph, that is, \(z\) admits a full splitting.

We claim that there always exists a full splitting at a node \(z\) with degree \(k+i\) where \(1 \leq i \leq l\). There is no \(G\)-tight set \(X \subseteq V - z\) which contains all the neighbours of \(z\) because, if there was one, then \(b_G(X + z) = b_G(X) + k - d_G(z) \leq 0 + k - (k + 1) < 0\) which contradicts that \(G\) is \((k,l)\)-sparse. Since there are no edges with multiplicity greater than \(k - l\), the neighbour-set of \(z\) in \(G\) has at least two elements, so by Observation 2.3 there is an admissible splitting off at \(z\). Hence the longest admissible splitting sequence at \(z\) has length at least 1.
Let $S$ be a longest admissible splitting sequence at $z$. If $|S| \geq i$, then we are done. If $h := |S| < i$, then $d_{G_S}(z) \geq d_G(z) - 2(i - 1) = k + i - 2i + 2 = k + 2 - i + 2 \geq k - l + 2$. Hence by Claim 3.3, $|N_{G_S}(z)| \geq 3$ or $|N_{G_S}(z)| = 2$ and both neighbours are joined to $z$ by at least two edges. By Claim 3.3, $S$ is not longest, a contradiction. \hfill $\Box$

Let $i := d_G(z) - k$ (here $2 \leq i \leq k - 1$). Call a node $z$ small if $k + l + 1 \leq d_G(z) \leq 2k - 1$.

Theorem 3.5. A small node $z$ of $G$ does not admit a full splitting if and only if $z$ has a neighbour $t$ and there is a family $P_z$ of subsets of $V - z$ with at least two elements such that:

$$X \cap Y = \{t\} \text{ for } X, Y \in P_z,$$

$$\sum_{X \in P_z} b(X) < d_G(z,t) - (k - i) - d_G(z,V - z - \cup P_z),$$

where $\cup P_z$ denotes $\bigcup_{X \in P_z} X$.

Proof. Suppose first that $t$ and $P_z$ satisfy (*) and (**) and let $S$ be an admissible splitting sequence. The number of split edges incident to $t$ with other end-nodes outside of $\cup P_z$ is at most $d_G(z, V - z - \cup P_z)$. The number of split edges incident to $t$ with their other end-nodes in $\cup P_z$ is at most $\sum_{X \in \cup P_z} b(X)$. In a full splitting we would have at least $d_G(z,t) - (k - i)$ split edges incident to $t$ which implies by (**) that $S$ is not full.

To see the other direction, let $S$ be a longest admissible splitting sequence at $z$ for which the following pair is lexicographically maximal: $(|N_{G_S}(z)|, |P_{\max}|)$ where $P_{\max}$ denotes a maximal tight set in $G_S$ which includes $N_{G_S}(z)$ but does not contain $z$. If there is no such a tight set, then let $P_{\max} := \emptyset$. Since $z$ does not admit a full splitting, $|S| < i$. From now on $G_S$-tight is abbreviated by tight.

By Claim 3.3 there are only the following two Cases. An edge not incident to $t$ is called $t$-disjoint.

CASE 1. $|N_{G_S}(z)| = 2$ and $z$ has a neighbour $s$ for which $d_{G_S}(z,s) = 1$.

Let $u \in V - t - s$ be an arbitrary node for which there is a $t$-disjoint split edge $uv$. There is a tight set $X \subseteq V - z$ containing $u$ and $t$, otherwise $S' := S - (zu,zv) \cup (zu,zt)$ is an other longest admissible splitting sequence for which if $v \neq s$, then $|N_{G_{S'}}(z)| = 3$, if $v = s$ and $d_{G_{S'}}(z,t) \geq 3$, then $d_{G_{S'}}(z,t) \geq d_{G_{S'}}(z,s) \geq 2$, which is a contradiction by Claim 3.3. If $v = s$ and $d_{G_{S'}}(z,t) = 2$ and $d_{G_{S'}}(z,s) = 1$, then by Claim 3.3 there is a split edge $ab$ which is disjont from $P_{\max} \cup \{u\}$. Since $S^* := S - (za,zb) - (zu,zs) \cup (za,zs) \cup (zb,zs) \cup (zu,zt)$ is not admissible, we have a tight set in $G_S$ containing $a, b, t, s, u$ contradicting the maximal choice of $P_{\max}$ by Claim 2.3 (it also contradicts that there is no tight set containing $t$ and $u$). (By the previous cases and Claim 2.3, there is no tight set containing $(a$ or $b)$ and $s$.)

Let $P_u$ be such a tight set containing minimal number of $t$-disjoint split edges which is inclusion-wise maximal. Similarly, there is a tight set $X \subseteq V - z$ containing $s$ and $t$, otherwise $S \cup (zs,zt)$ is a longer admissible splitting sequence than $S$. Let $P_s$ be such
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Let \(P_z := \{X \subseteq V - z : \exists u \in V \text{ incident to a } t\text{-disjoint split edge such that } X = P_u \text{ or } X = P_s \}\). For nodes \(u \neq v\), \(P_u\) can be equal to \(P_v\), but there is only one copy of them in \(P_z\). Now we prove some essential properties of \(P_z\).

**Figure 1:** A set-system \(P_z\).

**Proposition 3.6.** There is no \(t\)-disjoint split edge in any member \(X\) of \(P_z\).

**Proof.** First let us assume that \(X = P_s\). Let us suppose indirectly that there is a \(t\)-disjoint split edge \(ab\) in \(P_s\). \(S' := S - (za,zb) \cup (zt,zs)\) is an admissible splitting sequence with three remaining neighbours of \(z\) in \(G_{S'}\), which is a contradiction by Claim 3.3.

Now let us assume \(X = P_u\) and \(u \neq s\). By the definition of \(P_u\) we have a \(t\)-disjoint split edge \(uv\). Let us suppose indirectly that there is a \(t\)-disjoint split edge \(ab\) in \(P_u\). We may suppose that \(b \neq u\).

If \(v \neq s\), then \(v \notin P_u\) (if \(v \in P_u\), then \(S - (zu,zv) \cup (zt,zu)\) is an admissible splitting sequence with the same length but with one more remaining neighbour of \(z\)). \(P_e \cap P_u = \{t\}\) according to Claim 2.4. \(S - (za,zb) - (zu,zv) \cup (zt,zu) \cup (zv,za)\) is an other longest splitting sequence with one more remaining neighbour of \(z\), so it cannot be admissible, that is, there is a set \(Y \subseteq V - z\) containing \(a,u,v,t\), which is tight in \(G_S\). \(Y\) does not contain \(b\), hence the tight set \(Y \cap P_u\) contains a smaller number of split edges than \(P_u\), a contradiction. If \(v = s\) and \(v \notin P_u\), then the proof is the same.

Suppose that \(v = s\) and \(v \in P_u\). Let us consider a split edge \(cd\) which is disjoint from \(P_{max}\) and hence from \(P_u\) (such an edge exists according to Claim 3.2). By the previous paragraph tight sets \(P_c\) and \(P_d\) do not contain \(t\)-disjoint split edges. According to Claim 2.4, \(P_c \cap P_{max} = \{t\}\).

According to Claim 2.4, \(S' := S - (zc,zd) \cup (zc,zs)\) is an admissible splitting sequence. For \(S'' := S' - (zu,zv) \cup (zt,zu)\), the cardinality of \(N_{G_{S''}}(z) = \{t,s,d\}\) is 3, hence \(S''\) cannot be admissible, that is, there is a tight set \(Y \subseteq V - z\) containing \(c,s,u,t\) in \(G_{S''}\). \(Y \cup P_{max}\) (in \(G_{S''}\)) contradicts the choice of \(S\) by the maximality of \(P_{max}\). \(\square\)

Now it follows that \((**\rangle\) holds for \(P_z\).
Claim 3.7. Let $X,Y$ be two distinct members of $\mathcal{P}_z$. $X \cap Y = \{t\}$.

Proof. Let us suppose $X = P_u$ and $Y = P_v$ for some $u,v \in V$. By Proposition 3.6, $P_u \not\subseteq P_v$. If $|P_u \cap P_v| \geq 2$, then by Claim 2.4, $d_G(P_u,P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any $t$-disjoint split edge, it contradicts the maximal choice of $P_u$. □

Hence (*) holds for $\mathcal{P}_z$.

CASE 2. $|N_G(z)| = 1$. Let $t$ denote the only neighbour of $z$ in $G_S$.

Claim 3.8. There exists a $t$-disjoint split edge.

Proof. Let $l$ and $m$ be the number of split edges incident to, respectively, not incident to $t$. Since $\mathcal{S}$ is not full, $l + m = |\mathcal{S}| < i$. In the original graph $G$ by Claim 2.8

$$k - 1 \geq d_G(z,t) = d_G(z) - l - 2m = k + i - l - 2m = k + (i - l - m) - m > k - m,$$

which implies that $m > 1$. □

Since $\mathcal{S}$ is not a full splitting: $d_G(z) \geq k + i - 2(i - 1) = k - i + 2 \geq 3$. Now we define $\mathcal{P}_z$. Let $u \in V - t$ be an arbitrary node for which there is a $t$-disjoint split edge $uv$. There is a tight set $X \subseteq V - z$ containing $u$ and $t$, otherwise $S' := S - (zu,zv) \cup (zu,zt)$ is an other longest admissible splitting sequence for which $|N_G(z)| = 2$, which contradicts the choice of $\mathcal{S}$. Let $P_u$ be such a tight set containing minimal number of $t$-disjoint split edges which is inclusion-wise maximal. Let $\mathcal{P}_z := \{X \subseteq V - z : \exists u \in V$ incident to a $t$-disjoint split edge such that $X = P_u\}$. (The only difference to Case 1. is that there is no set $P_u$ here.)

Proposition 3.9. There is no $t$-disjoint split edge in an arbitrary element of $\mathcal{P}_z$.

Proof. Assume $X = P_u$. By the definition of $P_u$ we have a $t$-disjoint split edge $uv$. Let us suppose indirectly that there is a $t$-disjoint split edge $ab$ in $P_u$. We may suppose that $b \neq u, v \not\subseteq P_u$, otherwise $S - (zu,zv) \cup (zt,zu)$ is an admissible splitting sequence with the same length but with one more remaining neighbour of $z$. $P_v \cap P_u = \{t\}$ according to Claim 2.4. $S - (za,zb) - (zu,zv) \cup (zt,zu) \cup (zv,za)$ is an other longest splitting sequence with one more remaining neighbour of $z$, so it cannot be admissible, that is, there is a set $Y \subseteq V - z$ containing $a,u,v,t$, which is tight in $G_S$. $Y$ does not contain $b$, hence the tight set $Y \cap P_u$ contains a smaller number of split edges than $P_u$, a contradiction. □

Now it follows that (**) holds for $\mathcal{P}_z$.

Claim 3.10. Let $X,Y$ be two distinct members of $\mathcal{P}_z$. $X \cap Y = \{t\}$.

Proof. Let us suppose $X = P_u$ and $Y = P_v$ for some $u,v \in V$. By Proposition 3.6, $P_u \not\subseteq P_v$. If $|P_u \cap P_v| \geq 2$, then by Claim 2.4, $d_G(P_u,P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any $t$-disjoint split edge, it contradicts the maximal choice of $P_u$. □

Hence (*) holds for $\mathcal{P}_z$. 

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We have showed that if a small node \( z \) does not admit a full splitting, then the neighbour \( t \) of \( z \) and set-system \( P_z \) satisfy both (\( * \)) and (\( ** \)).

We state the following easy consequence of Theorem 3.5. The neighbour \( t \) of \( z \) in Theorem 3.5 is called the *blocking* node of \( z \).

**Corollary 3.11.** Let \( z \) be a small node in a \((k,l)\)-sparse graph \( G \). If \( z \) does not admit a full splitting, then the blocking node \( t \) of \( z \) is uniquely determined.

## 4 Counterexamples

In this section we give a \((k,l)\)-sparse graph for any \( k \geq 2 \), \( \frac{k+2}{3} \leq l \leq \frac{k}{2} \) which cannot be obtained by the operations of Theorem 1.7. This is surprising because we managed to prove almost all the ingredients of the proof of the constructive characterization of \((k,1)\)-sparse graphs also for these graphs. We remark that, for the given graph \( G_{(k,l)} = (V_{(k,l)}, E_{(k,l)}) \), \( |V_{(k,l)}| = 15k - 5l + 10 \), which is 60 in the smallest case \((4,2)\) and 85 in case \((6,3)\).

Let us consider \( m := 3k - l + 2 \) copies of the following graph \( G_1 = (V_1, E_1) \) and let the subscripts go from 1 to \( m \). Graph \( G_1 \) has \( |V_1| = 5 \) nodes and \( |E_1| = k|V_1|-(k+l) = 4k-l \) edges. Edges \( a_1d_1, b_1d_1, c_1d_1, z_1d_1 \) have multiplicity \( k-l \), \( b_1z_1, c_1z_1 \) has \( l \), \( a_1b_1 \) has \( l-1 \), \( a_1z_1 \) has 1, and all the other edges multiplicity 0. See Figure 2, the multiplicity of the edges are shown in the figure.

![Graph G1](image)

**Figure 2:** Graph \( G_1 \)

It is easy to see, that \( G_1 \) is \((k,l)\)-sparse since it can be obtained by the operations (i.e. \( z_1, d_1, c_1, b_1, a_1 \)).
Let $G_{(k,l)} = (V_{(k,l)}, E_{(k,l)})$ where $V_{(k,l)} := \bigcup_{j=1}^{m} V_j$, $E_{(k,l)} := \bigcup_{j=1}^{m} E_j \cup E^*$ and $E^* := K_1 \cup K_2 \cup K_3 \cup K_{1,2} \cup K_{3,2} \cup K_{1,3}$, where

\[ K_1 = \{ a_i a_j : 1 \leq i < j \leq k + 1 \} \]

\[ K_2 = \{ c_1 c_j : 2k - l + 3 \leq j \leq 3k - l + 2 \} \cup \{ c_i c_j : 2k - l + 3 \leq i < j \leq 3k - l + 2 \} \]

\[ K_3 = \{ b_i b_j : k + 2 \leq j \leq 2k - l + 2 \} \cup \{ b_i b_j : k + 2 \leq i < j \leq 2k - l + 2 \} \]

\[ K_{1,2} = \{ b_i a_j : 2 \leq i \leq k + 1, k + 2 \leq j \leq 2k - l + 2 \} \]

\[ K_{3,2} = \{ b_i c_j : 2k - l + 3 \leq i \leq 3k - l + 2, k + 2 \leq j \leq 2k - l + 2 \} \]

\[ K_{1,3} = \{ c_i a_j : 2 \leq i \leq k + 1, 2k - l + 3 \leq j \leq 3k - l + 2 \} \]

See Figure 3. We will use the following two facts about $E^*$

- $d_{E^*}(v) \leq k$ for all $v \in V$,
- $d_{G_{(k,l)}}(V_i, V_j) = 1$ for all $1 \leq i < j \leq 3k - l + 2$.

![Figure 3: A subgraph of $G_{(k,l)}$](image)

It is clear that $|V_{(k,l)}| = 5m = 5(3k - l + 2) = 15k - 5l + 10$ and $|E_{(k,l)}| = m|E_1| + |E^*| = m(4k - l) + \frac{1}{2} m(3k - l + 1)$. In $G_{(k,l)}$ we have the following degrees for any $1 \leq j \leq m$

\[ d(a_j) = d(b_j) = d(b_j) = 2k, \]

\[ d(d_j) = 4(k-l) \geq 4\frac{k}{2} = 2k, \]

\[ d(z_j) = k + l + 1. \]
Hence the only small nodes are \( z_j \)-s. Since \( \{a_j, d_j\}, \{b_j, d_j\}, \{c_j, d_j\} \) are tight sets, there is no full splitting at \( z_j \), hence graph \( G_{(k,l)} \) cannot be obtained by the operations.

It is remained to see that \( G_{(k,l)} \) is \((k, l)\)-sparse for the given \( k \) and \( l \). We are going to prove that \( b(X) \geq 0 \) for all \( X \subseteq V_{(k,l)} \). It can be shown easily that if \( X \subseteq V_{(k,l)} \) includes at least two nodes of \( V_j \) for some \( j \), then \( b(X) \geq b(X \cup V_j) \). Hence it is enough to prove the condition for subsets \( X \) either including \( V_j \) or having the cardinality of the intersection with it at most 1 for all \( j \).

Let \( n \) denote the number of \( V_j \)'s that are included entirely in \( X \) and \( r \) denote the number of \( V_j \)'s having a one-element intersection with \( X \). \( |X| = 5n + r \), hence we must prove

\[
|E[X]| \leq k|X| - (k + l) = k(5n + r) - (k + l) = 5kn + kr - k - l. \tag{1}
\]

We have

\[
|E[X] - E^*| = n|E_1| = n(4k - l).
\]

\[
|E[X] \cap E^*| \leq \frac{n(n + r - 1) + rk}{2},
\]
since \( d(V_i, V_j) = 1 \) and \( d(a_i, V - V_i) = d(c_i, V - V_i) = k, d(b_i, V - V_i) = k - l + 1 < k \) for all \( i, j \). Hence

\[
|E[X]| = |E[X] - E^*| + |E[X] \cap E^*| \leq n(4k - l) + \frac{n(n + r - 1) + kr}{2}. \tag{2}
\]

We will prove that the difference of the right hand side of (1) and (2) is at least 0, which will finish the proof that \( G \) is \((k, l)\)-sparse. Let us compute, but first multiply by 2,

\[
2(5kn + kr - k - l) - 2 \left( n(4k - l) + \frac{n(n + r - 1) + kr}{2} \right) = \]

\[
(10kn + 2kr - 2k - 2l) - (8kn - 2ln + n^2 + nr - n + kr) = \]

\[
10kn + 2kr - 2k - 2l - 8kn + 2ln - n^2 - nr + n - kr = \]

\[
2kn + kr - 2k - 2l + 2ln - n^2 - nr + n = \]

\[
(n + r)(k - n) + n(k + 2l + 1) - 2(k + l). \tag{3}
\]

If \( 2 \leq n \leq k \), then (3) is obviously at least 0. \( n + r \leq m = 3k - l + 2 \). If \( n > k \), then we continue the computation:

\[
\geq m(k - n) + n(k + 2l + 1) - 2(k + l) =
\]

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\[(3k - l + 2)(k - n) + n(k + 2l + 1) - 2(k + l) =\]

\[(3k - l + 2)k + n(3l - 2k - 1) - 2(k + l) \geq\]

\[\text{since } 3l - 2k - 1 < 0,\]

\[\geq (3k - l + 2)k + (3k - l + 2)(3l - 2k - 1) - 2(k + l) =\]

\[(3k - l + 2)(3l - k - 1) - 2(k + l) =\]

\[(3k - l + 2)(3l - k - 2) + (3k - l + 2) - 2k - 2l =\]

\[(3k - l + 2)(3l - k - 2) + (k - 3l + 2) =\]

\[(3k - l + 1)(3l - k - 2). (4)\]

Since \(l \geq \frac{k+2}{3}\), that is, \(3l \geq k+2\), (4) is at least 0. If \(n = 1\) or 0, \(E[X] \leq k|X| - (k+l)\) can be shown with a much shorter computation. Hence we proved that \(G\) is really \((k,l)\)-sparse.

5 Open problems

The main problem is proving Conjecture 1.5 in the remaining cases. Another important question is finding an appropriate constructive characterization theorem for \((k,l)\)-sparse graphs if \(\frac{k+2}{3} \leq l \leq \frac{k}{2}\). One possibility if the following. If we allow \(i = k\) in (P2), is the reverse of Theorem 1.7 true?

This operation can be allowed in the cases which are already proved, of course, but it is not necessary.

Are the examples of Section 4 the graphs with the smallest number of nodes? We think they are.

Give a constructive characterization for \((k,l)\)-sparse graphs, if \(\frac{k}{2} \leq l \leq k\). We may have to allow operations which glue together bigger graphs and the nodes are not considered one by one.

A graph is said to be \([k,m]\)-sparse, if \(0 \leq m \leq k\) and \(\gamma_G(X) \leq k|X| - m\) for all \(X \subseteq V, |X| \geq 2\). These graphs have not a direct connection to covering by trees but may have a similar construction.

References

References


