On admissible edges

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Abstract

Let $G = (V + s, E)$ be a 2-edge-connected graph. A pair of edges $rs, st$ is called admissible if splitting off these edges (replacing $rs$ and $st$ by $rt$) preserves the local edge connectivities between all pairs of vertices in $V$.

First we generalize Mader’s result [2] by showing that if $d(s) \geq 4$ then there exists an edge that belongs to at least $\lfloor \frac{d(s)}{3} \rfloor$ admissible pairs. An infinite family of graphs shows that this is best possible.

Second we generalize Frank’s result [1] by characterizing when an edge belongs to no admissible pairs. It provides a new proof for Mader’s theorem.

1 Introduction

In this note $G = (V + s, E)$ is always a 2-edge-connected graph. The operation splitting off is defined as usually: two edges $rs$ and $st$ are replaced by $rt$. A pair of edges $rs, st$ is called admissible if splitting off these edges preserves the local edge connectivities between all pairs of vertices in $V$. We say that an edge incident to $s$ is admissible if it belong to an admissible pair, otherwise it is called non-admissible. Mader proved in [2] that if $d(s) \neq 3$ then there exists an admissible edge. Here we shall strengthen this result by showing in Theorem 3.1 that if $d(s) \geq 4$ then there exists an edge that belongs to at least $\lfloor \frac{d(s)}{3} \rfloor$ admissible pairs. The proof follows the line of [1]. We shall also present an infinite family of graphs showing that our result is best possible. Mader’s theorem [2] implies that at most three edges are non-admissible. Frank showed in [1] that in fact there is at most one non-admissible edge. We shall refine this result by giving in Theorem 4.1 the structure of the graph if it contains a non-admissible edge. The proof technic developed for Theorem 4.1 provides a new proof for Mader’s theorem.

2 Preliminaries

Recall that $G = (V + s, E)$ is a 2-edge-connected graph. $\Gamma(s)$ denotes the neighbours of $s$. For a set $T \subset V$, $G/X$ denotes the graph obtained from $G$ by contracting $T$

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into one vertex. \( d(X, Y) \) (resp. \( \overline{d}(X, Y) \)) denotes the number of edges between \( X - Y \) and \( Y - X \) (resp. \( X \cap Y \) and \( V + s = (X \cup Y) \)), \( d(X) = d(X, V + s - X) \). The **local edge-connectivity** between two vertices \( x \) and \( y \) is defined by \( \lambda(x, y) = \min\{d(X) : x \in X, y \notin X \} \). Let \( R(X) := \max\{\lambda(x, y) : x \in X, y \in V - X \} \) and \( h(X) := d(X) - R(X) \). Then, for \( X, Y \subseteq V, (1), (2), (3), (4) \) and at least one of (5) and (6) and hence at least one of (7) and (8) hold:

\[
\begin{align*}
h(X) &\geq 0, \\
h(X) &= h(V - X) + 2d(s, X) - d(s), \\
d(X) + d(Y) &= d(X \cap Y) + d(X \cup Y) + 2d(X, Y), \\
d(X) + d(Y) &= d(X - Y) + d(Y - X) + 2\overline{d}(X, Y), \\
R(X) + R(Y) &\leq R(X \cap Y) + R(X \cup Y), \\
R(X) + R(Y) &\leq R(X - Y) + R(Y - X), \\
h(X) + h(Y) &\geq h(X \cap Y) + h(X \cup Y) + 2d(X, Y), \\
h(X) + h(Y) &\geq h(X - Y) + h(Y - X) + 2\overline{d}(X, Y).
\end{align*}
\]

A set \( X \) is called **tight** (resp. **dangerous**) if \( h(X) = 0 \) (resp. \( h(X) \leq 1 \)).

The following three claims can be found in [1].

**Claim 2.1.** \( \{su, sv\} \) is admissible if and only if no dangerous set contains \( u, v \). \( \square \)

**Claim 2.2.** Let \( t \in \Gamma(s) \) of minimum degree. If \( t \in M, h(M) \leq 1 \) and \( |\Gamma(s) \cap M| \geq 2 \), then \( R(M - t) \geq R(M) \). \( \square \)

**Claim 2.3.** For a tight set \( T, \{su, sv\} \) is admissible in \( G \) if and only if it is admissible in \( G/T \). \( \square \)

**Claim 2.4.** If \( M \) is a dangerous set, then (a) \( d(s, M) \leq \frac{d(s) + 1}{2} \) (where equality holds only if \( V - M \) is tight) and (b) \( G[M] \) is connected.

**Proof.** (a) By (1) and (2). (b) If \( \emptyset \neq X \subset M \), then \( -1 \geq h(M) - 2 \geq h(X) + h(M - X) - 2d(X, M - X) \geq -2d(X, M - X) \) that is there is at least one edge between \( X \) and \( M - X \). \( \square \)

**Lemma 2.5.** Let \( st \in E \) and \( \mathcal{M} \) be a minimal collection of dangerous sets in \( V \) such that \( t \in M_i \) for all \( M_i \in \mathcal{M} \) and \( d(s, \bigcup \mathcal{M}) \geq \frac{d(s) + 1}{2} \). Suppose that \( |\mathcal{M}| \geq 3 \) and every tight set is a singleton.

\[
\text{Then for } M_i, M_j \in \mathcal{M}, (a) \ (8) \text{ does not apply, (b) } M_i \cap M_j \text{ is tight, so by (9),} \\
M_i \cap M_j = t.
\]

**Proof.** (a) \( 1 \geq h(M_i), 1 \geq h(M_j) \) thus if (8) applied, then \( h(M_i - M_j) = 0 \) (so by (9), \( M_i - M_j = s \)) and \( d(M_i, M_j) = 1 \). Let \( M_k \in \mathcal{M} - M_i - M_j \). Then, by Claim 2.4(b), \( 1 \leq d(M_i \cap M_j, M_k - M_i \cap M_j) \leq \overline{d}(M_i, M_j) - d(M_i \cap M_j, s) \leq 1 - 1 = 0 \), contradiction.

(b) By Claim 2.5(a), (7) applies for \( M_i \) and \( M_j \). Then, since \( 1 \geq h(M_i), 1 \geq h(M_j) \), and by the minimality of \( \mathcal{M} \), \( h(M_i \cup M_j) \geq 2 \), we have \( h(M_i \cap M_j) = 0 \). \( \square \)
3 A $\left\lfloor \frac{d(s)}{3} \right\rfloor$-admissible edge

Theorem 3.1. If $d(s) \geq 4$, then there is an edge $sr$ belonging to at least $\left\lfloor \frac{d(s)}{3} \right\rfloor$ admissible pairs.

Proof. Induction on $|V|$. By Claim 2.3 we may assume that (9) is satisfied. Let $t$ be a minimum degree neighbour of $s$. Suppose $t$ belongs to less than $\left\lfloor \frac{d(s)}{3} \right\rfloor$ admissible pairs. Then, by Claim 2.1, there is a minimal collection $\mathcal{M}$ of dangerous sets in $V$ such that $t \in M_i$ for all $M_i \in \mathcal{M}$ and (8) $d(s, \bigcup \mathcal{M}) \geq d(s) - \left\lfloor \frac{d(s)}{3} \right\rfloor + 1 = \left\lfloor \frac{2d(s)}{3} \right\rfloor + 1$. By Claim 2.4(a), $|\mathcal{M}| \geq 2$. Let $M_1, M_2 \in \mathcal{M}$.

Claim 3.2. $\mathcal{M} = \{M_1, M_2\}$.

Proof. By Claim 2.2, $R(M_1-t) \geq R(M_1)$ and $R(M_2-t) \geq R(M_2)$. Suppose $|\mathcal{M}| \geq 3$. Then, by Lemma 2.5(b), $M_1 \cap M_2 = t$, thus $M_1$ and $M_2$ satisfy (8), a contradiction by Lemma 2.5(a).

Claim 3.3. (a) $M_1 - M_2 = r_1$, $M_2 - M_1 = r_2$, (b) $d(M_1 \cap M_2, s) = 1$, $d(s, r_1) + d(s, r_2) \geq \left\lfloor \frac{2d(s)}{3} \right\rfloor$.

Proof. By (2) and (1), $h(M_1 \cup M_2) \geq 2d(s, M_1 \cup M_2) - d(s) \geq 2(\left\lfloor \frac{2d(s)}{3} \right\rfloor + 1) - d(s) \geq 3$, so (7) does not apply and hence (8) applies for $M_1$ and $M_2$. Then $h(M_1 - M_2) = 0 = h(M_2 - M_1)$, so by (9), $M_1 - M_2 = r_1$ and $M_2 - M_1 = r_2$; and $d(M_1 \cap M_2, s) = 1$. By Claim 3.2 and (8), $d(s, r_1) + d(s, r_2) = d(s, M_1 \cup M_2) - d(s, M_1 \cap M_2) \geq \left\lfloor \frac{2d(s)}{3} \right\rfloor + 1 - 1 \geq \left\lfloor \frac{2d(s)}{3} \right\rfloor$.

Claim 3.4. Let $e_i$ be any edge connecting $s$ and $r_i$ for $1 \leq i \leq 2$. Then $\{e_1, e_2\}$ is admissible.

Proof. Otherwise, by Claim 2.1, there is a dangerous set $X$ with $r_1, r_2 \in X$, and then, by (2), (1), Claim 3.3(b) and $d(s) \geq 4, 1 \geq h(X) \geq 2d(s, X) - d(s) \geq 2\left\lfloor \frac{2d(s)}{3} \right\rfloor - d(s) \geq 2$, contradiction.

By Claim 3.3(b), wlog. $d(s, r_1) \geq \left\lfloor \frac{d(s)}{3} \right\rfloor$. Then, by Claims 3.4, $e_2$ belongs to at least $\left\lfloor \frac{d(s)}{3} \right\rfloor$ admissible pairs.

Example: There exists an infinite class of graphs in which each edge incident to $s$ belongs to exactly $\left\lfloor \frac{d(s)}{3} \right\rfloor$ admissible pairs. See Figure 1.

4 A non-admissible edge

Theorem 4.1. An edge $st$ belongs to no admissible pair if and only if $d(s)$ is odd and there exist two disjoint tight sets $C_1, C_2 \subseteq V - t$ such that $d(s, C_1) = d(s, C_2) = \frac{d(s) - 1}{2}$. Moreover, if $d(s) \neq 3$, then for every $c_1 \in C_1 \cap \Gamma(s), c_2 \in C_2 \cap \Gamma(s)$, $\{sc_1, sc_2\}$ is an admissible pair.
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Proof. if: Suppose \( d(s) \) is odd and there exist two disjoint tight sets \( C_i \subseteq V - t \) such that \( d(s, C_i) = \frac{d(s)-1}{2} \). Then, by \( (2) \), \( V - C_i \) is dangerous so, by Claim 2.1, \( st \) belongs to no admissible pair.

only if: Induction on \( |V| \).

Lemma 4.2. We may assume that \( (9) \) is satisfied.

Proof. if \( T \) was a tight set with \( |T| > 1 \), then let \( G' := G/T \). By Claim 2.3 \( st \) belongs to no admissible pair in \( G' \) and \( |V(G')| < |V| \), hence, by induction, \( d(s) \) is odd and there exist two disjoint tight sets in \( G' \) \( C_1, C_2 \subseteq V(G') - t \) such that \( d(s, C_1) = d(s, C_2) = \frac{d(s)-1}{2} \) and if \( d(s) \neq 3 \) then for every \( c_1 \in C_1 \cap \Gamma(s), c_2 \in C_2 \cap \Gamma(s), \{s_{c_1}, s_{c_2}\} \) is an admissible pair in \( G' \). Then, by Claim 2.3 and since \( C_1 \) and \( C_2 \) are also tight in \( G \), we are done.

By Claim 2.1, there is a minimal collection \( M \neq \emptyset \) of dangerous sets in \( V \) such that for every \( r_i \in \Gamma(s) - t \) there exists \( M_i \in M \) containing \( t \) and \( r_i \). \( |M| \geq 2 \), by Claim 2.4(a) and \( d(s) \geq 2 \).

Lemma 4.3. If \( M = \{M_1, M_2\} \) then \( C_1 := M_1 - M_2 \) and \( C_2 := M_2 - M_1 \) satisfy the statement of the Theorem.

Proof. \( C_1 \cap C_2 = \emptyset \), \( t \in M_1 \cap M_2 \) so \( C_1, C_2 \subseteq V - t \). By Claim 2.4(a), \( 2 \frac{d(s)+1}{2} \geq d(s, M_1) + d(s, M_2) = d(s) + d(s, M_1 \cap M_2) \geq d(s) + 1 \), so \( d(s) \) is odd, \( d(s, M_i) = \frac{d(s)+1}{2} \) and \( d(s, M_1 \cap M_2) = 1 \), that is \( d(s, C_i) = \frac{d(s)-1}{2} \). By Claim 2.4(a), \( V - M_i \) is tight so by \( (9) \), \( C_j \subseteq V - M_i = r_j \subseteq M_j - M_i = C_j \), hence \( C_j \) is tight. Suppose indirect that for \( c_1 \subseteq C_1 \cap \Gamma(s), c_2 \subseteq C_2 \cap \Gamma(s), \{s_{c_1}, s_{c_2}\} \) is not an admissible pair. Then there exists a dangerous set \( X \) containing \( c_1 \) and \( c_2 \). By \( c_i = c_i \) and by Claim 2.4(a), \( 2 \frac{d(s)-1}{2} \leq d(s, X) \leq \frac{d(s)+1}{2} \), that is \( d(s) \leq 3 \), contradiction.

We suppose from now on that \( |M| \geq 3 \). By Lemma 2.5(b), for all \( M_i, M_j \in M, \) \( M_i - M_j = M_i - t \).

Claim 4.4. If $R(M_1) = \lambda(a, b)$, $a \in M_1, b \in V - \bigcup M$, then for some $M_k \in M - M_1, R(M_k - t) > d(t)$.

Proof: $\sum_{M_j \in M - M_1} d(M_j) + d(M_1) \geq d(\bigcup M \cup s) + d(\bigcup M - t, s) + (|M| - 1)d(t) + 1 \geq d(M_1) - 1 + |M| + (|M| - 1)d(t) + 1 = (|M| - 1)(d(t) + 1) + d(M_1) + 1$ so there exists $M_k \in M - M_1$ with $d(M_k) > d(t) + 1$. Since $M_k$ is dangerous, it follows that $R(M_k) \geq d(M_k) - 1 \geq d(t) + 1$, that is $R(M_k - t) > d(t)$.

Claim 4.5. There exists $M_i \in M$ for which $R(M_i - t) \geq d(t)$.

Proof: By Lemma 2.5(b), $R(t) = d(t)$ thus $Y := \{y \in V - t : d(t) = \lambda(t, y)\}$ $\neq \emptyset$. If $y \in M_i \cap Y$ for some $M_i \in M$, then $R(M_i - t) \geq \lambda(t, y) = d(t)$. Thus we suppose that $Y \subseteq V - \bigcup M$. Let $y \in Y$. Then $R(M_i) = d(t) = \lambda(t, y)$ and Claim 4.4 provides the statement.

Claim 4.6. If $M_j \in M - M_i$, then $R(M_j - t) < R(M_j) = d(t)$.

Proof: Suppose $R(M_j - t) \geq R(M_j)$. By Claim 4.5, $R(M_i - t) \geq R(M_i)$. So (8) applies for $M_i$ and $M_j$, contradicting Lemma 2.5(a). $R(M_j - t) < R(M_j)$ and $R(t) = d(t)$ implies $R(M_j) = d(t)$.

Claim 4.7. If $R(M_i) = \lambda(a, b)$, $a \in M_i, b \in V - M_i$, then $b \in V - \bigcup M$.

Proof: Suppose indirect $b \in M_j \in M$. Then, $R(M_j - t) \geq \lambda(a, b) = R(M_i)$. By Claims 4.6 and 4.5, $R(M_j) = d(t) \leq R(M_i - t)$. Thus (8) applies for $M_i$ and $M_j$, contradiction by Lemma 2.5(a).

Claim 4.7, Claim 4.4 applied for $M_1 = M_i$ and Claim 4.6 provides a contradiction.

References
