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The stable roommates problem with choice functions

Tamás Fleiner

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Abstract

The stable marriage theorem of Gale and Shapley states that for n men and n women there always exists a stable marriage scheme, that is, a set of marriages such that no man and woman exists that mutually prefer one another to their partners. The stable marriage theorem was generalized in two directions: the stable roommates problem is the "one-sided" version, where any two agents on the market can form a partnership. The generalization by Kelso and Crawford is in the "two-sided" model, but on one side of the market agents have a so-called substitutable choice function, and stability is interpreted in a natural way. It turned out that even if both sides of the market have these substitutable choice functions, there still exists a stable assignment. This latter version contains the "many-to-many" model where up to a personal quota, polygamy is allowed for both men and women in the two-sided market.

The goal of this work is to solve the stable partnership problem, a generalization of the one-sided model with substitutable choice functions. We do not quite reach that: besides substitutability, we also need the increasing property for the result. Luckily, choice functions in well-known stable matching theorems comply with this property. The main result is a generalization of Irving's algorithm, that is the first efficient method to solve the stable roommates problem. This general algorithm allows us to deduce a generalization of Tan's result on characterizing the existence of a stable matching and to prove a generalization of the so-called splitting property of many-to-many stable matchings. We show that our algorithm is linear-time in some sense and indicate that for general (i.e. not necessary increasing) substitutable choice functions the stable partnership problem is NP-complete.

Keywords: Stable marriage problem, stable roommates problem, Irving's algorithm, choice function

1 Introduction

Gale and Shapley [8] introduced their famous marriage model almost a half century ago. The model consists of n men and n women such that each person has a linear preference order on the members of the opposite gender. The marriage theorem states, that for each such model there exists a stable marriage scheme, that is, a set of disjoint couples in such a way that no man and woman mutually prefer one another to their partners. It turned out that variants of the model are useful in Game Theory, Economics, Graph Theory, Complexity

*Budapest University of Technology and Economics, Department of Computer Science and Information Theory, Magyar tudósok körútja 2. H-1117, Budapest, Hungary. Research is supported by OTKA grants K69027 and K60802, the MTA-ELTE Egerváry Research Group (EGRES) and the János Bolyai research fellowship of the Hungarian Academy of Sciences. e-mail: `fleiner@cs.bme.hu`.

Theory, Combinatorial Optimization and the Theory of Algorithms. Also, stable matchings have a rich structure, and this also led to further generalizations. There are two natural directions of these generalizations. For the first one, we drop the two-sided property of the market. This way we get the stable roommates problem: any two agents may have a relationship, the solution of the problem is more difficult, and a stable matching does not always exist. Irving [10] was the first who gave an efficient algorithm that finds a stable matching in a given model if it exists. Since then, many different approaches are known for the same problem. (Cf. Tan [16], Tan and Hsueh [17], Subramanian [15], Feder [5] and others. See also the book of Roth and Sotomayor [12] and of Gusfield and Irving [9] for further details.)

The other direction for a possible generalization is that we allow that an agent may participate in several relationships, so instead of a matching, we look for a certain subgraph of the underlying graph with degree prescriptions. This is done by keeping linear preference orders, but introducing quotas for the agents (this model is present already in the original paper of Gale and Shapley), or in a much more general way, by choice functions. This is what was initiated by Crawford and Knoer [4], and continued by Kelso and Crawford [11]. This choice function model is closely related to the lattice theoretical fixed point theorem of Knaster and Tarski [18] as it was pointed out by Fleiner [6]. It is worth mentioning that the nonbipartite stable roommates problem has to do with an other fixed point theorem: as Aharoni and Fleiner [1] observed, it is a special case of the well known game theoretical Scarf's lemma [13]. (Scarf's lemma can be regarded as a discrete version of Kakutani's fixed point theorem, which is a generalization of the topological fixed point theorem of Brouwer.)

In the present work, we move into both directions. In Section 2, we describe our one-sided (nonbipartite) model and define the stability concept by choice functions. By a generalization of Irving's algorithm, we solve the defined stable partnership problem in Section 3 for so called increasing choice functions. Furthermore, the same way as Tan [16] did, we show that a certain half-integral solution always exists, and such a solution is either a generalized stable matching or it is an immediate proof for the nonexistence of it. The reader who finds it difficult to follow all the details may focus only on the solution of the stable partnership problem and ignore the existence of half-integral solutions. It is fairly easy to reduce the algorithm to the integral stable partnership problem, and proofs become much easier. If the choice functions of the model are given by an oracle then the algorithm works in polynomial time, and Section 4 gives a detailed analysis. We also show there that for general choice functions the stable partnership problem is NP-complete. Section 5 contains a structural result on stable partnerships: we generalize a result of Fleiner [7] (that was independently found by Teo et al. [14]) on the splitting property of stable matchings. We conclude in Section 6 with raising the question of a possible generalization of the well-known Scarf's lemma.

2 Preliminaries

Let $G = (V, E)$ be a finite graph. For a subset E' of E and vertex v of G let us denote by $E'(v)$ the set of edges of E' that are incident with v. Let C_v denote the *choice function of vertex* v, i.e. function $C_v : 2^{E(v)} \to 2^{E(v)}$ maps any set X of edges incident with v to a subset of X that v chooses from X. We shall assume that choice functions we handle are *substitutable*, that is, if $x \neq y$ and $x \in C_v(X)$, then $x \in C_v(X \setminus \{y\})$ holds. This roughly means that if agent v would select some option x from set X of available options, then v

would still select option x even if some other options are not available any more. For choice function C_v, let us define function $\overline{C}_v$ of ignored options by $\overline{C}_v(X) := X \setminus C_v(X)$. It is useful to see a connection between the above two notions.

Theorem 2.1. *A choice function C on a finite groundset is substitutable if and only if $\overline{C}$ is monotone, that is, if $Y \subseteq X$ implies that $\overline{C}(Y) \subseteq \overline{C}(X)$.*

In [6] choice functions C with the property that $\overline{C}$ is monotone are called *comonotone*. So by theorem 2.1, a choice function on a finite ground set is comonotone if and only if it is substitutable. Note that on infinite ground sets, comonotonicity of a choice function is a stronger property than substitutability.

Proof. For finite groundsets, substitutability is equivalent with the property that for any $Y \subseteq X$ we have $Y \cap C(X) \subseteq C(Y)$. Monotonicity of $\overline{C}$ is equivalent with the property that for any $Y \subseteq X$, $C(X)$ is disjoint from $\overline{C}(Y)$, which is clearly equivalent with the first property. □

We say that a subset S of E is a *stable partnership* if for any vertex v $C_v(S(v)) = S(v)$ (this property is often called *individual rationality*) and no *blocking edge* $e = uv \notin S$ exists such that both $e \in C_u(S(u) \cup \{e\})$ and $e \in C_v(S(v) \cup \{e\})$ holds. The *stable partnership problem* is given by a graph $G = (V, E)$, choice functions C_v for each vertex, and our task is to decide whether a stable partnership exists, and if yes, we have to find one.

An example of a substitutable choice function C_v on ground-set $E(v)$ is a *linear choice function*: we have a linear order on $E(v)$ and $C_v(X)$ is the minimal element of X according to this linear order. If for all vertices v of G, choice function C_v is linear then a stable partnership is simply a stable matching. Another possible choice function is when for each vertex v, we have a "quota" $b(v)$, and $C_v(X)$ is the $b(v)$ smallest element of X according to the linear order. For these linear choice functions with quotas, a stable partnership is nothing but the well-studied "many-to-many" stable matching (or stable b-matching). An even more general substitutable choice function can be defined with matroids: let $\mathcal{M}$ be a matroid on $E(v)$ and fix a linear order on $E(v)$ as well. Let $C_v(F)$ be the output of the greedy algorithm on F, that is, we scan the elements of F in the given linear order, and scanned element e is selected into $C_v(F)$, if e together with the previously selected elements is independent in $\mathcal{M}$.

Fleiner [6] generalized a result of Kelso and Crawford [11] by showing the following theorem.

Theorem 2.2 (Fleiner [6])**.** *If $G = (V, E)$ is a finite bipartite graph and for each vertex v, choice function C_v on $E(v)$ is substitutable, then there always exists a stable partnership.* □

Choice functions in the above examples satisfy the following additional property. Choice function C_v is *increasing* if $Y \subseteq X$ implies that $|C_v(Y)| \leq |C_v(X)|$. This roughly means that if extra choices are added, then we select at least as many options as we would have picked without the extra ones. For a choice function C_v, we say that subset X of E *v-dominates* element x if $x \in \overline{C}_v(X(v) \cup \{x\})$. If it causes no ambiguity, then instead of v-domination we may speak about *domination*. For a choice function C_v, let $D_v(X)$ denote the set of elements (v-)dominated by X. Clearly, $\overline{C}_v(X) \subseteq D_v(X)$ and $C_v(X) = X \setminus D_v(X)$ holds for any subset X of $E(v)$.

Lemma 2.3. *If choice function C_v is substitutable then dominance function D_v is monotone.*

If choice function C_v is substitutable and increasing and $Y \subseteq D_v(X)$ then $C_v(X \cup Y) = C_v(X)$, thus $D_v(X \cup Y) = D_v(X)$.

Proof. Assume first that $A \subseteq B$ and $a \in D_v(A)$. Monotonicity of $\overline{C}_v$ implies that $a \in \overline{C}_v(A \cup \{a\}) \subseteq \overline{C}_v(B \cup \{a\})$, so $a \in D_v(B)$, hence D_v is monotone. (Actually, the implication is true in the other direction, as well. If D_v is monotone then $X \mapsto X \cap D_v(X) = \overline{C}_v(X)$ is also also monotone, hence C_v is substitutable by Lemma 2.1.)

For the second part, let $y \in Y \cup \overline{C}_v(X)$ be an arbitrary element dominated by X. So $y \in \overline{C}_v(X \cup \{y\}) \subseteq \overline{C}_v(X \cup Y)$, where the second relation follows from the monotonicity of $\overline{C}_v$. Hence $Y \cup \overline{C}_v(X) \subseteq \overline{C}_v(X \cup Y)$, that is, $C_v(X \cup Y) = (X \cup Y) \setminus \overline{C}_v(X \cup Y) \subseteq (X \cup Y) \setminus (\overline{C}_v(X) \cup Y) \subseteq X \setminus \overline{C}_v(X) = C_v(X)$. As $X \subseteq X \cup Y$, the increasing property of C_v implies that $|C_v(X)| \leq |C_v(X \cup Y)|$, so $C_v(X) = C_v(X \cup Y)$ follows. □

With this notion of dominance, we can reformulate the notion of a stable partnership.

Theorem 2.4. *If $G = (V, E)$ is a finite graph and for each vertex v, choice function C_v on $E(v)$ is substitutable then S is a stable partnership if and only if $E \setminus S = \bigcup_{v \in V} D_v(S(v))$, that is, if S dominates exactly $E \setminus S$.*

Proof. If S is a stable partnership then by individual rationality, no edge of S is dominated by S. As no blocking edge exists, each edge outside S is dominated by S. On the other hand, if $E \setminus S = \bigcup_{v \in V} D_v(S \cap E(v))$ then no edge of S is dominated by S, thus S is individually rational. As each edge outside S is dominated, no blocking edge exists. □

Let C_v be a choice function and let $X \subseteq E(v)$. For an edge x in $C_v(X)$ the *X-replacement* of x according to C_v is the set $R = C_v(X \setminus \{x\}) \setminus C_v(X)$. Roughly speaking, if option x is not available any more, then from X we select the options of R instead of option x.

Lemma 2.5. *If C_v is an increasing and substitutable choice function on $E(v)$, $X \subseteq E(v)$ and $x \in C_v(X)$, then the X-replacement R of x contains at most one element.*

Proof. We have $C_v(X) \setminus \{x\} \subseteq C_v(X \setminus \{x\})$ by substitutability, so $C_v(X \setminus \{x\}) = C_v(X) \cup R \setminus \{x\}$. From the increasing property of C_v, we get $|C_v(X)| \geq |C_v(X \setminus \{x\})| = |C_v(X) \cup R \setminus \{x\}| = |C_v(X)| + |R| - 1$, and the lemma follows. □

Let $G = (V, E)$ be a graph and for each vertex v, let C_v be a choice function on $E(v)$. Let S be a subset of E and fix disjoint subsets $S_1, S_2, \ldots, S_l$ of S such that each S_i is an odd cycle with a fixed orientation. (Note that cycle S_i is not necessarily a circuit: it can traverse the same vertex several times.) Let $S_i^+(v)$ and $S_i^-(v)$ denote the set of edges of S_i that leave and enter vertex v, respectively. Define edge set $S^+(v) := S(v) \setminus \bigcup_{i=1}^{l} S_i^-(v)$ as the unoriented edges of $S(v)$ together with the arcs of the S_i's that leave v. Similarly let $S^-(v) := S(v) \setminus \bigcup_{i=1}^{l} S_i^+(v)$, denote the set unoriented edges of $S(v)$ and all arcs of the S_i's that enter v. We say that $(S, S_1, \ldots, S_l)$ is a *stable half-partnership* if

1. for each $v \in V$, $C_v(S(v)) = S^+(v)$. Moreover,
2. If arcs $e \in S_i^-(v)$ and $f \in S_i^+(v)$ are consecutive on S_i then e is the $S(v)$-replacement of f according to C_v.
3. $E \setminus S = \bigcup_{v \in V} D_v(S^-(v))$.

(The name of the notion corresponds to the fact that a stable half-partnership can be regarded as a stable fractional partnership, where edges of the S_i's have weight $\frac{1}{2}$, and all other edges of S have weight 1.) A consequence of the definition is that if $(S, S_1, \ldots, S_l)$ is a stable half-partnership and $e = uv$ is an edge then exactly one of the following three possibilities holds. Either e is an unoriented edge of S (that does not belong to any of the S_i's), or e is an edge of some S_i, and hence if $e = S_i^-(v)$ then e is dominated by $S_i^+(v)$, or $e \notin S$ and hence e is dominated by $S^-(u)$ at some vertex u of e. If $(S, S_1, \ldots, S_l)$ has properties 1. and 2. and edge $e = uv \notin S$ is not dominated (i.e. $e = uv$ and $e \notin D_u(S^-(u)) \cup D_v(S^-(v))$) then we say that e is *blocking* $(S, S_1, \ldots, S_l)$. Observe that if (S) is a stable half-partnership (that is, no oriented odd cycles are present) if and only if S is a stable partnership. Now we can state our main result.

Theorem 2.6. *If $G = (V, E)$ is a finite graph and for each vertex v, choice function C_v on $E(v)$ is increasing and substitutable then there exists a stable half-partnership. Moreover, if $(S, S_1, \ldots, S_l)$ and $(S', S'_1, \ldots, S'_m)$ are stable half-partnerships, then $l = m$ and sets of oriented cycles $\{S_1, \ldots, S_l\}$ and $\{S'_1, \ldots, S'_m\}$ are identical.*

Corollary 2.7. *If $(S, S_1, \ldots, S_l)$ is a stable half-partnership then either $l = 0$ and S is a stable partnership, or no stable partnership exists whatsoever.* □

Corollary 2.7 shows that to solve the stable partnership problem in case of increasing substitutable choice functions, it is enough to find a stable half-partnership. Note that Theorem 2.6 is a generalization of Tan's result [16] on stable half-matchings (or on "stable partitions" in Tan's terminology).

3 Proof of the main result

To prove our main result, we follow Tan's method. Tan extended Irving's algorithm such that it finds a stable half-matching, and, with the help of the algorithm, he proved the unicity of the oriented odd cycles. Here, instead of linear orders, we work with increasing substitutable choice functions. To handle this situation, we shall generalize Irving's algorithm to our setting. Irving's algorithm works in such a way that it keeps on deleting edges such that no new stable matching is created after a deletion, and, if there was a stable matching before a deletion, there should be one after it, as well. Irving's algorithm terminates if the actual graph is a matching, which, by the deletion rules is a stable matching for the original problem. Similarly, our algorithm will delete edges in such a way that after a deletion no new stable half-partnership can be created. Moreover, if there was a stable half-partnership $(S, S_1, \ldots, S_l)$ before some deletion, then we cannot delete an edge of any of the S_i's and at least one stable half-partnership has to survive the deletion. If our algorithm terminates then we are left with a graph G' such that edge set $E(G')$ of G' is a stable half partnership of G', hence it is a stable half partnership of G, as well. Our algorithm has different deletion rules, and there is a priority of them. The algorithm always takes a highest priority step that can be made.

To start the algorithm we need some definitions. We say that the *first choices* of v are the edges of $C_v(E(v))$. These are the best possible options for agent v. If edge $e = vw$ is a first choice of v then we call arc $e = vw$ a *1-arc*. Note that if vw is a 1-arc then it is possible that wv is also a 1-arc. Let A denote the set of 1-arcs. For a vertex v let $A^+(v)$ and $A^-(v)$ stand for the set of 1-arcs that are oriented away from v and towards v, respectively.

The **1st priority (proposal) step** is that we find and orient all 1-arcs.

As the problem did not change (we did not delete anything), the set of stable half-partnerships is the same as it was before the orientation. After we found all 1-arcs, we execute the

2nd priority (rejection) step: If $D_v(A^-(v)) \neq \emptyset$ for some vertex v then we delete $D_v(A^-(v))$.

Lemma 3.1. *The set of stable half-partnerships does not change by a 2nd priority step.*

Proof. Assume that $(S, S_1, \ldots, S_l)$ is a stable half-partnership after the deletion. This means that for each 1-arc $f = uv$ of $A^-(v)$ either f belongs to $S^-(v)$, or, as f cannot be dominated by $S^-(u)$ at u, f is dominated by $S^-(v)$ at v. Lemma 2.3 implies that $D_v(A^-(v)) = D_v(C_v(A^-(v))) \subseteq D_v(S^-(v) \cup C_v(A^-(v))) = D_v(S^-(v))$, hence no deleted edge can block $(S, S_1, \ldots, S_l)$. This means that no new stable half-partnership can emerge after a 2nd priority deletion.

Assume now that $(S, S_1, \ldots, S_l)$ is a stable half-partnership before the deletion and some edge $e \in D_v(A^-(v))$ belongs to S. Similarly to the previous argument, this means that for each 1-arc $f = uv$ of $C_v(A^-(v))$ either f belongs to $S^-(v)$, or (as f cannot be dominated by $S^-(u)$ at u) f is dominated by $S^-(v)$ at v. Lemma 2.3 implies that $e \in D_v(A^-(v)) = C_v(D_v(A^-(v))) \subseteq D_v((S^-(v)) \cup C_v(A^-(v)) = D_v(S^-(v))$, so e cannot belong to $S(v)$, a contradiction. $\square$

Later we need the following lemma.

Lemma 3.2. *If no* 1*st and* 2*nd priority steps can be made then* $|A^+(v)| = |A^-(v)|$ *for each vertex* v.

Proof. By the increasing property of C_v, we have $|A^-(v)| = |C_v(A^-(v))| \leq |C_v(E(v))| = |A^+(v)|$. So each vertex v has at least as many outgoing 1-arcs as the number of 1-arcs entering v. As both the total number of outgoing 1-arcs and the total number of ingoing 1-arcs is exactly $|A|$, the previous inequality must be an equality for each vertex v. $\square$

If no more 1st and 2nd priority steps can be made, we check for replacements. For a 1-arc $e = uv$, let $e^r = uw$ denote the $E(u)$-replacement of e according to C_u. (It might happen that e^r does not exist.)

3rd priority (replacement) step: For each 1-arc $e = uv$, find $E(u)$-replacement e^r.

As we do not delete anything in a 3rd priority step, the set of stable partnerships does not change by this step. Next we study replacements of 1-arcs.

Lemma 3.3. *Assume that no 1st and 2nd priority steps can be made and that* 1*-arc* $e = uv$ *is not bidirected, that is,* vu *is not a* 1*-arc. Then the* $E(u)$*-replacement* e^r *of* e *is a unique edge of* $E(u)$*.*

Proof. By Lemma 3.2 and the increasing property of C_u, we have $|A^+(u)| = |A^-(u)| = |C_u(A^-(u))| \leq |C_u(E(u) \setminus \{e\})| \leq |C_u(E(u))| = |A^+(u)|$, hence there is equality throughout. In particular, we see that $|C_u(E(u) \setminus \{e\})| = |A^+(u)|$. By substitutability of C_u, we have $A^+(u) \setminus \{e\} = C_u(E(u)) \setminus \{e\} \subseteq C_u(E(u) \setminus \{e\})$. This means that the $E(u)$-replacement of e (that is, $C_u(E(u) \setminus \{e\}) \setminus A^+(u)$) is a unique edge of $E(u)$. $\square$

Lemma 3.4. *Assume that no 1st and 2nd priority step can be executed and* $e = uv$ *is a* 1*-arc such that* vu *is not a* 1*-arc. If* $e^r = uw$ *is the* $E(u)$*-replacement of* e*, then* $D_w(\{e^r\} \cup A^-(w))$ *contains exactly one* 1*-arc* $e^r_r = xw$*. Moreover,* 1*-arc* $e^r_r = xw$ *has the property that its inverse* wx *is not a* 1*-arc.*

Proof. By the 2nd priority step $e^r \notin D_w(A^-(w))$, hence $e^r \in C_w(\{e^r\} \cup A^-(w))$. The increasing property of C_w gives that $|A^-(w)| = |C_w(A^-(w))| \leq |C_w(\{e^r\} \cup A^-(w))| \leq |C_w(E(w))| = |A^+(w)| = |A^-(w)|$, where the last equality is due to Lemma 3.2. So we have equality throughout, i.e. $|A^-(w)| = |C_w(\{e^r_r\} \cup A^-(w))|$, so e^r has a unique $\{e^r\} \cup A^-(w)$-replacement $e^r_r = xw$. Clearly, if wx was a 1-arc then $e^r_r \in C_w(\{e^r_r\} \cup A^-(w))$ holds, a contradiction. □

Assume now that no 1st, 2nd and 3rd priority steps are possible and consider the following two cases.

Case 1. All 1-arcs are bidirected, that is, if $e = uv$ is a 1-arc, then its opposite vu is also a 1-arc. In other words, $A^+(v) = A^-(v) = A(v) = E(v)$ for each vertex v. This means that the edge set of our graph is a stable partnership, the algorithm terminates and outputs (S).

Case 2. There exists a 1-arc $e = uv$ that is not bidirected. So there is an $E(u)$-replacement e^r of e. Edge e^r_r in Lemma 3.4 is another 1-arc that is not bidirected. Following the alternating sequence of nonbidirected 1-arcs and their replacements, we shall find a sequence $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r, e_{m+1} = e_1)$ in such a way that $(e_i)^r_r = e_{i+1}$ for $i = 1, 2, \ldots, m$ and edges $e_1, e_2, \ldots, e_m$ are different 1-arcs, none of them is bidirected. After Irving, we call such an alternating sequence $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$ of 1-arcs and edges a *rotation*.

Lemma 3.5. *Assume that $(S, S_1, \ldots, S_l)$ is a stable half-partnership in graph G and no 1st, 2nd and 3rd priority step is possible. If $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$ is a rotation and $e_i = xv \in S(v)$ then $e_{i-1} = uw \in S^+(u)$ follows, where addition is meant modulo m.*

In particular, if $e_i \in S$ then $\{e_1, e_2, \ldots, e_m\} \subseteq S$.

Proof. First we show that $e^r_{i-1} = uv \notin S^-(v) \cup D_v(S^-(v))$. Indirectly, assume $e^r_{i-1} = uv \in S^-(v) \cup D_v(S^-(v))$. If $f = zv \in A^-(v)$ is a 1-arc then f (being a first choice) cannot be dominated at z, so $f \in S^-(v) \cup D_v(S^-(v))$ follows, that is,

$$A^-(v) \subseteq S^-(v) \cup D_v(S^-(v)) \ . \tag{1}$$

By Lemma 2.3,

$$e_i = (e_{i-1})^r_r \in D_v(A^-(v) \cup \{(e_{i-1})^r\}) \subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup \{(e_{i-1})^r\}) = D_v(S^-(v))$$

so $e_i \notin S(v)$, a contradiction. Thus $e^r_{i-1} = uv \notin S^-(v)$, hence $e^r_{i-1} = uv \notin S^+(u)$ and $e^r_{i-1} \notin D_v(S^-(v))$, hence $e^r_{i-1} \in D_u(S^+(u))$. As e^r_{i-1} is the $E(u)$-replacement of first choice e_{i-1}, it follows that $e_{i-1} \in S^+(u)$, as Lemma 3.5 claims. □

Lemma 3.6. *Assume that no 1st, 2nd and 3rd priority step is possible and that $(S, S_1, \ldots, S_l)$ is a stable half-partnership. If $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$ is a rotation then sets $\{e_1, e_2, \ldots, e_m\}$ and $\{e^r_1, e^r_2, \ldots, e^r_m\}$ are either disjoint or identical.*

If $\{e_1, e_2, \ldots, e_m\} = \{e^r_1, e^r_2, \ldots, e^r_m\}$ then m is odd and $\{e_1, e_2, \ldots, e_m\}$ is one of the S_j's.

If sets $\{e_1, e_2, \ldots, e_m\}$ and $\{e^r_1, e^r_2, \ldots, e^r_m\}$ are disjoint and $e_i \in S$ then $\{e_1, \ldots, e_m\} \subseteq S \setminus (S_1 \cup \ldots \cup S_l)$. Moreover, $(S', S_1, \ldots, S_l)$ is a stable half-partnership for $S' = S \setminus \{e_1, e_2, \ldots, e_m\} \cup \{e^r_1, e^r_2, \ldots, e^r_m\}$.

Proof. Assume first that sets $\{e_1, e_2, \ldots, e_m\}$ and $\{e^r_1, e^r_2, \ldots, e^r_m\}$ are not disjoint, so $e_i = uv = (e_{i+k})^r$ for some $i \in \{1, 2, \ldots, m\}$ and $1 \leq k < m$, where addition is meant modulo m. If 1-arc e_{i+k} is a first choice of x then $(e_{i+k})^r$ cannot be the first choice of x by the definition of a replacement. This means that 1-arc $e_{i+k} = vw$ is a first choice of v (and not of u). As

$(e_{i+k})^r$ is an $E(v)$-replacement of first choice e_{i+k} of v, it follows that $(e_{i+k})^r \in D_v(F)$ imply $e_{i+k} \in F$. Choose $F = A^-(v) \cup \{(e_{i-1})^r\}$. We know that $(e_{i+k})^r = e_i = (e_{i-1})^r_r \in D_v(F)$ by the definition of $(e_{i-1})^r_r$, hence $e_{i+k} \in A^-(v) \cup \{(e_{i-1})^r\}$. As $e_{i+k} = vw$ is a 1-arc in the rotation, $e_{i+k} \notin A^-(v)$, so $e_{i+k} = (e_{i-1})^r$.

We proved that $e_i = (e_{i+k})^r$ implies $e_{i+k} = (e_{i-1})^r$. The above argument for this latter equality means that $e_{i-1} = (e_{i+k-1})^r$. That is, $e_i = (e_{i+k})^r$ yields $e_{i-1} = (e_{i+k-1})^r$, $e_{i-2} = (e_{i+k-2})^r$, $e_{i-3} = (e_{i+k-3})^r$, and so on. So for any j, we have $e_j = (e_{j+k})^r$. In particular, we see that $\{e_1, e_2, \ldots, e_m\} = \{e_1^r, e_2^r, \ldots, e_m^r\}$.

Another consequence is that $e_{i+k} = (e_{i-1})^r = e_{i-1-k}$, so $i+k \equiv i-1-k (mod\ m)$, that is, $2k+1 \equiv 0 (mod\ m)$. As $1 \le k < m$, we get that $m = 2k+1$, and edges $e_1, e_2, \ldots, e_m$ form a cycle in order $e_1, e_{k+1}, e_m, e_k, e_{m-1}, e_{k-1}, \ldots, e_2, e_{k+2}$. We shall prove that $\{e_1, e_2, \ldots, e_m\} \subseteq S$. If $e_i \in S$ then this follows by Lemma 3.5. Otherwise, $(e_{i+k})^r = e_i \notin S$. This means that $(e_{i+k})^r \in D_v(S^-(v))$, so $e_{i+k} \in S^-(v) \subseteq S(v)$, and $\{e_1, e_2, \ldots, e_m\} \subseteq S$ by Lemma 3.5 again.

By property (1), $A^-(v) \subseteq S^-(v) \cup D_v(S^-(v))$. Hence, by Lemma 2.3, we have

$$\begin{aligned} e_i = (e_{i-1})^r_r &\in D_v(A^-(v) \cup \{(e_{i-1})^r\}) \\ = D_v(A^-(v) \cup \{e_{i+k}\}) &\subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup \{e_{i+k}\}) \\ &\subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup S^+(v)) = D_v(S^+(v)) \ , \end{aligned}$$

thus, $e_i \in S(v) \setminus S^+(v)$. This means that e_i belongs to one of the cycles S_j of stable half-partnership $(S, S_1, S_2, \ldots, S_l)$, and e_i is the replacement of e_{i+k} for all i. Thus $\{e_1, e_2, \ldots, e_m\}$ is indeed one of the S_j's.

To finish the proof, we settle the remaining case when sets $\{e_1, e_2, \ldots, e_m\}$ and $\{(e_1)^r, (e_2)^r, \ldots, (e_m)^r\}$ are disjoint. If $e_i \in S$ then $\{e_1, e_2, \ldots, e_m\} \subseteq S$ and 1-arc $e_i = uv$ is in $S^+(u)$ by Lemma 3.5. If, indirectly $e_i \in S_j$, so $e_i \notin S^-(u)$ then $(e_i)^r = uz \notin D_u(S^-(u))$ as $(e_i)^r$ is the $E(u)$-replacement of e_i. So either $(e_i)^r \in D_z(S^-(z))$ or $(e_i)^r \in S^-(u)$. In the former case, $e_{i+1} = (e_i)^r_r \in D_z(A^-(z) \cup \{(e_i)^r\})$. By property (1), $A^-(z) \subseteq S^-(z) \cup D_z(S^-(z))$, so by Lemma 2.3 we have $e_{i+1} \in D_z(S^-(z) \cup D_z(S^-(z)) \cup \{(e_i)^r\}) = D_z(S^-(z))$, that contradicts to $e_{i+1} \in S$. So $(e_i)^r \in S^-(u)$ holds. But $e_{i+1} \in S$ is the $E(v)$-replacement of $(e_i)^r$, and this can only happen if $e_i, (e_i)^r$ and e_{i+1} all belong to the same odd cycle S_j. The argument shows that S_j is exactly the cycle $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$, which is a contradiction, as $|S_j|$ is not odd. So $\{e_1, \ldots, e_m\} \subseteq S \setminus (S_1 \cup \ldots \cup S_l)$, as we claimed.

Consider $(S', S_1, \ldots, S_l)$. (Recall that $S' = S \setminus \{e_1, e_2, \ldots, e_m\} \cup \{e_1^r, e_2^r, \ldots, e_m^r\}$.) To see that it is a stable half-partnership we check the three properties of the definition. Fix a vertex v and let R^+ and R^- be the set of 1-arcs of rotation $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$ that leave and enter v, respectively. Let moreover $T^+ := \{(e_i)^r : e_i \in R^+\}$ and $T^- := \{(e_{i-1})^r : (e_{i-1})^r_r = e_i \in R^-\}$. Let S^* be one of the sets $S(v)$, $S^-(v)$ or $S(v) \setminus \{e\}$ for some edge e of some $S_j^+(v)$. To prove properties 1. and 2., we show that $C_v(S^* \setminus (R^+ \cup R^-) \cup T^+ \cup T^-) = C_v(S^*) \setminus (R^+ \cup R^-) \cup T^+ \cup T^-$.

The definition of replacements, property (1) and the monotonicity of D_v (Lemma 2.3) implies that $R^- \subseteq D_v(A^-(v) \cup T^-) \subseteq D_v(S^-(v) \cup D_v(S^-(v)) \cup T^-) \subseteq D_v(S^-(v) \cup T^-) \subseteq D_v(S^* \cup T^-)$, hence R^- is disjoint from $C_v(S^* \cup T^-)$ and hence $C_v(S^* \cup T^-) = C_v(S^* \cup T^- \setminus R^-)$. Substitutability of C_v gives $C_v(S^* \cup T^-) \cap S^* \subseteq C_v(S^*)$, thus $C_v(S^* \cup T^- \setminus R^-) = C_v(S^* \cup T^-) \subseteq C_v(S^*) \setminus R^- \cup T^-$. The increasing property of C_v implies that $|(C_v(S^*)| \le |C_v(S^* \cup T^-)| \le |C_v(S^*) \setminus R^- \cup T^-| = |C_v(S^*)| - |R^-| + |T^-|$, so from $|R^-| = |T^-|$ we get that $C_v(S^* \cup T^-) = C_v(S^* \cup T^- \setminus R^-) = C_v(S^*) \cup T^- \setminus R^-$.

Assume that edge $(e_i)^r = uv \in T^+$ is in S. Property (1) shows that $A^-(u) \subseteq S^-(u) \cup D_u(S^-(u))$, so $e_{i+1} = (e_i)^r_r \in D_u(A^-(u) \cup \{(e_i)^r\}) \subseteq D_u(S^-(u) \cup D_u(S^-(u)) \cup \{(e_i)^r\}) = D_u(S^-(u) \cup \{(e_i)^r\})$. This contradicts $e_{i+1} \in S \setminus (S_1 \cup \ldots \cup S_l)$. This argument proves that $(e_i)^r \notin S$ and that $(e_i)^r \notin D_u(S(u))$. So $(e_i)^r$ has to be dominated at v: $(e_i)^r \in D_v(S^-(v))$. As $(e_i)^r$ was an arbitrary edge of T^+, we proved that T^+ is disjoint from S, moreover $T^+ \subseteq D_v(S^-(v)) \subseteq D_v(S^* \cup S^-) = D_v(S^*) \subseteq D_v(S^* \cup T^-) = D_v(S^* \cup T^- \setminus R^-)$ (we used the monotonicity of D_v), thus $C_v(S^* \cup T^- \setminus R^-) = C_v(S^* \cup T^- \setminus R^- \cup T^+)$. We use the substitutability of C_v for $S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+) \subseteq S^* \cup T^- \cup T^+ \setminus R^-$:

$$\begin{aligned} C_v(S^*) \cup T^- \setminus (R^- \cup R^+) &= \big(C_v(S^*) \cup T^- \setminus R^-\big) \cap \big(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)\big) \\ &= C_v(S^* \cup T^- \setminus R^-) \cap \big(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)\big) \\ &= C_v(S^* \cup T^- \cup T^+ \setminus R^-) \cap \big(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)\big) \\ &\subseteq C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+)) \ . \end{aligned} \tag{2}$$

As edges of T^+ are $E(v)$-replacements, it follows that $T^+ \subseteq C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))$, so with (2) we have $C_v(S^*) \cup T^- \cup T^+ \setminus (R^- \cup R^+) \subseteq C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))$. The increasing property of C_v gives that

$$\begin{aligned} |C_v(S^*)| + |T^-| + |T^+| - (|R^-| + |R^+|) &= |C_v(S^*) \cup T^- \cup T^+ \setminus (R^- \cup R^+)| \\ &\le |C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))| \\ &\le |C_v(S^* \cup T^- \cup T^+ \setminus R^-)| \\ &= |C_v(S^*) \cup T^- \setminus R^-| = |C_v(S^*)| + |T^-| - |R^-| \ , \end{aligned}$$

hence $C_v(S^*) \cup T^- \cup T^+ \setminus (R^- \cup R^+) = C_v(S^* \cup T^- \cup T^+ \setminus (R^- \cup R^+))$, as we claimed. So we justified properties 1. and 2. for $(S', S_1, \ldots, S_l)$.

For property 3., we have already seen that $C_v(S'(v))$ is disjoint from S', so it remains to check that any edge $e \in E \setminus S'$ is dominated at some vertex. There are two cases for e: either $e = e_i \in R^-$ is a 1-arc of our rotation. The above argument for $S^* = S^-(v)$ shows that $R^- \subseteq D_v((S')^-(v))$, so we may assume that $e \notin S$, hence $e \in D_v(S^-(v))$ for some vertex v. Again the above proof shows that everything that $S^-(v)$ is dominating according to C_v is also dominated by $(S')^-(v)$, except for R^+. This proves property 3. □

4th priority (rotation elimination) step:
Find a rotation $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$ with disjoint sets $\{e_1, e_2, \ldots, e_m\}$ and $\{e_1^r, e_2^r, \ldots, e_m^r\}$ and delete $\{e_1, e_2, \ldots, e_m\}$.

To justify the rotation elimination step, we only have to check that it does not create a new stable half-partnership.

Lemma 3.7. *Any stable half-partnership after a 4th priority step is also a stable half-partnership before this step.*

Proof. Observe that after the elimination, each edge $(e_i)^r$ becomes a 1-arc. Assume that $(S, S_1, \ldots, S_l)$ is a stable half-partnership after the step. As in the proof of Lemma 3.6, let R^- denote the set of deleted 1-arcs of our rotation that enter a fixed vertex v, let $T^- := \{(e_{i-1})^r : (e_{i-1})^r_r = e_i \in R^-\}$ be the new 1-arcs entering v and let A^- be the set of those 1-arcs that enter v and have not been deleted during the step.

By the definition of the rotation, from property (1) and the monotonicity of D_v we get $R^- \subseteq D_v(A^- \cup R^- \cup T^-) = D_v(A^- \cup T^-) \subseteq D_v(S^-(v) \cup D_v(S^-(v))) = D_v(S^-(v))$, and this is exactly what we wanted to prove. □

Theorem 3.8. *If no 1st, 2nd, 3rd and 4th priority step can be made on graph G then $(E(G), S_1, S_2, \ldots, S_l)$ is a stable half-partnership, where cycles S_i are given by the rotations.*

Proof. We have already seen that if all 1-arcs are bidirected then we have a stable partnership, which is a special case of a stable half-partnership. So assume that no further step can be executed but we still have a 1-arc e that is not bidirected. We have also seen that if we follow the alternating sequence of non bidirected 1-arcs and their replacements $e, e^r, e^r_r, (e^r_r)^r, (e^r_r)^r_r, \ldots$ then we find a rotation $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$, that must be an odd cycle S_i that cannot be eliminated. This means that $m = 2k + 1$, and $e_i = (e_{i+k})^r = (e_{i-1})^r_r$ for $1 \leq i \leq m$, where addition is modulo m. We shall prove that our starting point, 1-arc e is an edge of this rotation. Hence, if our algorithm cannot make a step then all 1-arcs are either bidirected or belong to a unique odd rotation.

To this end, we may assume that e^r_r is an edge of the rotation, namely $e^r_r = uv = e_i = (e_{i+k})^r$. That is, e_i is the $E(v)$-replacement of first choice e_{i+k}, that is $C_v(E(v) \setminus \{e_{i+k}\}) \cup D_v(E(v) \setminus \{e_{i+k}\}) = E(v) \setminus \{e_{i+k}\}$, in other words, if $e_i \in D_v(X)$ for some subset X of $E(v)$ then $e_{i+k} \in X$ must hold. But the definition of e^r_r gives that $e_i \in D_v(A^-(v) \cup \{e_r\})$, so $e_{i+k} \in A^-(v) \cup \{e_r\}$. 1-arc $e_{i+k} \in A^+(v)$ is not bidirected, hence $e_{i+k} \notin A^-(v)$, thus $e_{i+k} = e^r$ is an edge of the rotation, as well.

Similarly as above, $e_{i+k} = vw$ is the $E(w)$-replacement of first choice e_{i-1} of w, hence $C_w(E(w) \setminus \{e_{i-1}\}) \cup D_w(E(w) \setminus \{e_{i-1}\}) = E(w) \setminus \{e_{i-1}\}$. This means that if e_{i+k} is the $E(w)$-replacement of edge e then $e = e_{i-1}$ must hold. So e is an edge of our rotation, and all non bidirected 1-arcs of G belong to odd rotations.

Next we prove that all edges of G are 1-arcs. If, indirectly, $e = uv$ is not a 1-arc, then $e \in D_u(A^+(u))$ by the 1st priority step and $e \notin D_u(A^-(u))$ by the 2nd priority step. So u is incident with some nonbidirected 1-arcs such that these 1-arcs all belong to odd rotations, and each 1-arc of $A^-(u)$ is the $E(u)$-replacement of different 1-arcs of $A^+(u)$. As $e \notin D_u(A^-(u))$, we have $e \in C_u(A^-(u) \cup \{e\})$, and $|C_u(A^-(u) \cup \{e\})| \leq |C_u(E(u))| = |C_u(A^+(u))| = |C_u(A^-(u))|$ implies that there is a unique 1-arc $f \in A^-(u)$ that is dominated: $f \in D_u(A^-(u) \cup \{e\})$. But f is the $E(u)$-replacement of some other arc $g \in A^+(u)$, and we have already seen twice in this proof that this means that g is a member of each edge set that C_v-dominates f: $g \in A^-(u) \cup \{e\}$. But this is a contradiction as 1-arc g is not bidirected and leaves u and e is not a 1-arc.

So if the algorithm cannot make any further step then our graph consists of bidirected 1-arcs and odd rotations $S_1, S_2, \ldots, S_k$. It is trivial from the definition that $(E(G), S_1, \ldots, S_k)$ is a stable half-partnership. □

4 Complexity issues

Irving's original algorithm [10] is very efficient: it runs in linear time. However, this algorithm is different from the one that we get if we apply our algorithm to an ordinary stable roommates problem. The difference is that Irving's algorithm has two phases: in the first phase it makes only 1st and 2nd priority steps, and after the 1st phase is over, it keeps on eliminating rotations, and never gets back to the 1st phase. The explanation is that our rotation elimination is more restricted than Irving's that does not only delete just first choices, but also removes some other edges.

Actually, it is rather straightforward to modify our algorithm to work similarly, and this improves even its time-complexity. The reason that we did not do this in the previous section is that the proof is more transparent this way. So what do we have to do to speed up the algorithm?

Observe that after a rotation elimination (4th priority) step, if 1-arc $e_i = uv$ is deleted, then e_i ceases to be a first choice of u. The new first choice instead of e_i will be its replacement $(e_i)^r$. So we can include (with no extra cost) that we orient each replacement edges. Of course, refusal (2nd priority) steps may still be possible, but only at those vertices that the newly created 1-arcs enter. The definition of a rotation implies that if we apply a refusal step at such a vertex then no 1-arc gets deleted, but we might delete some unoriented arcs. So if we modify the rotation elimination step in such a way that we also include these extra 1st and 2nd priority steps within the rotation elimination step, then once we start to eliminate rotations, we never go back to the first phase. That is, we shall never have to make a proposal or a rejection step again.

To analyse the above (modified) algorithm, we have to say something about the calculation of the choice function and the dominance function. Assume that functions C_v and D_v are given by an oracle for all vertices v of G, such that for an arbitrary subset X of $E(v)$ these oracles output $C_v(X)$ and $D_v(X)$ in unit time. Note that if we have only the oracle for D_v then we can easily construct one for C_v from the identity $C_v(X) = X \setminus D_v(X)$. Similarly, if we have an oracle for C_v then $D_v(X)$ can be calculated by $O(n)$ calls of the C_v-oracle, according to the definition of D_v. (As usual, n and m denotes the number of vertices and edges of G, respectively.)

The algorithm starts with n C-calls, and continues with n D-calls. After this, each deletion in a 2nd priority step involves one C-call at vertex u where we deleted, and, if this C-call generates a new 1-arc $e = uv$, then we also have to make one D-call at the other end v of e. So the first phase (the 1st and 2nd priority steps) uses at most $O(n + m)$ C-calls and $O(n + m)$ D-calls.

In the second phase, the algorithm makes 3rd priority steps and modified rotation elimination steps. We do it in such a way that we start from a nonbidirected 1-arc e and follow the sequence $e, e^r, e^r_r, (e^r_r)^r, \ldots$, until we find a rotation. The rotation will be a suffix of this sequence, and after eliminating this rotation, we reuse the prefix of this sequence, and continue the rotation search from there. This means that for the 3rd type steps we need altogether $O(m)$ C-calls. The modified rotation elimination steps consist of deleting each 1-arc $e_i = uv$ of the rotation, orienting edges $(e_i)^r = uw$ and applying refusal steps at vertices w. As we delete at most m edges in all rotation eliminations, this will add at most $O(m)$ D-calls. All additional work of the algorithm can be allocated to the oracle calls, so we got the following.

Theorem 4.1. *If we modify the rotation elimination step as described above, then our algorithm uses $O(n + m)$ C-calls and D-calls to find a stable half-partnership and runs in linear time.*

In the introduction, we indicated that in case of bipartite graphs there always exists a stable partnership for so called path independent substitutable choice functions (see [6]). That is, we do not have to require the increasing property of functions C_v if we want to solve the stable partnership problem on a bipartite graph. A natural question is if it is possible to generalize our result on stable partnerships to substitutable, but not necessarily increasing choice functions. In our proof, we heavily used the fact that if no proposal and rejection steps can be made then each 1-arc has a replacement and these replacements improve some other 1-arc at their other vertices. This property is not valid in the more general setting.

Below we show that the stable partnership problem for substitutable choice functions is NP-complete by reducing the 3-SAT problem to it.

Theorem 4.2. *For any* 3*-CNF expression* ϕ*, we can construct a graph* G_ϕ *and substitutable choice functions* C_v *on the stars of* G_ϕ *in polynomial time in such a way that* ϕ *is satisfyable if and only if there exists a stable partnership in* G_ϕ *for the choice functions* C_v*.*

Proof. Define directed graph D_ϕ such that D_ϕ has three vertices a_C, b_C and v_C for each clause C of ϕ and two vertices t_x and f_x for each variable x of ϕ. The arc set of D_ϕ consists of arcs t_xf_x and f_xt_x for each variable x of ϕ, arcs of type v_Ct_x (and v_Cf_x) if literal x (literal $\bar{x}$) is present in clause C of ϕ. Moreover, we have arcs a_Cb_C, b_Cv_C and v_Ca_C for each clause C of ϕ. If A is a set of arcs incident with some vertex v of D_ϕ then $C'_v(A) = A$ if no arc of A leaves v, otherwise $C'_v(A)$ is the set of arc of A that leave v. It is easy to check that choice function C'_v is substitutable. Let G_ϕ be the undirected graph that corresponds to D_ϕ and let C_v denote the choice function induced by $C'(v)$ on the undirected edges of G_ϕ. We shall show that ϕ is satisfiable if and only if there is a stable partnership of G_ϕ for choice functions C_v, that is, if and only if there is a subset S of arcs of D_ϕ such that S does not contain two consecutive arcs and for any arc uv outside S there is an arc vw of S.

Assume now that ϕ is satisfiable, and consider an assignment of logical values to the variables of ϕ that determine a truth evaluation of ϕ. If the value of variable x is true then add arc f_xt_x, if it is false, then add arc t_xf_x to S. Do this for all variables of ϕ. Furthermore, add all arcs a_Cb_C to S. If variable x is true then add all arcs v_ct_x to S for all clauses that contain variable x. If variable y is false then add all arcs v_cf_x to S for all clauses that contain negated variable $\bar{y}$. Clearly, the hence defined S does not contain two consecutive arcs. If some arc of type t_xf_x or f_xt_x is not in S then it is dominated by the other, which is in S. Each arc of type v_Ca_C is dominated by arc a_Cb_C of S and each arc of type b_Cv_C is dominated by some arc of type xt_x or yf_y as C has a variable that makes C true.

To finish the proof, assume that S is a stable partnership of D_ϕ. Observe that for each variable x either t_xf_x or f_xt_x belongs to S, as no other arc dominates these arcs. If $t_xf_x \in S$ then set varible x to be false, else assign logical value true to x. We have to show that for this asssigment the evaluation of each clause C is true, that is, there is an arc of S from v_C to some t_x or f_x. Indirectly, if there is no such arc then the corresponding edges of S form a stable partnership on directed circuit $v_Ca_Cb_C$, which is impossible.

So the decision problem of the existence of a stable partnership is NP-complete. □

5 A coloring property of stable partnerships

In this section we prove a generalization of a result by Cechlárová and Fleiner [3].

Theorem 5.1. *Let* S *be a stable partnership for graph* $G = (V, E)$ *and increasing substitutable choice functions* C_v*. For each vertex* v *it is possible to partition* $E(v)$ *into (possibly empty) parts* $E_0(v), E_1(v), E_2(v), \ldots, E_{|S(v)|}(v)$ *in such a way that for any stable partnership* S' *we have* $S' \cap E_0(v) = \emptyset$ *and* $|S' \cap E_i(v)| = 1$ *holds for* $i = 1, 2, \ldots, |S(v)|$*.*

Proof. Let us find some stable partnership S by the algorithm in the previous section. Fix a vertex v and determine the partition of $E(v)$ in the following manner. Each element of $S(v)$ will belong to a different part. Follow the algorithm backwards, that is, we start from S and we build up the original G by adding edges according to the deletions of the algorithm. If we add an edge that is not incident with v, then we do not do anything. If we add an edge

e of $E(v)$ that was deleted by a 2nd priority step, then we put e into part $E_0(v)$. This is a good choice, since e is contained in no stable partnership. If e was deleted in a 4th priority step along a rotation then this rotation contains another (replacement) edge f incident with v. Lemma 3.6 shows that if we assign e to that part $E_i(v)$ that contains f, then still no stable partnership can contain two edges of the same part $E_i(v)$. Let us buid up the graph by backtracking the algorithm. This way, we find a part for each edge of $E(v)$, and this partition clearly has the property we need. $\square$

The following corollary describes the phenomenon that is called the "rural hospital theorem" in the stable matching literature. This states that if a hospital cannot fill up its quota with residents in some stable outcome, then no matter which stable outcome is selected, it always receives the same applicants.

Corollary 5.2. *If S and S' are stable partnerships then $|S(v)| = |S'(v)|$ for any vertex v of the underlying graph.*

There is an aesthetic problem with Theorem 5.1, namely, that part $E_0(v)$ of the star of v is redundant in the following sense. If we remove all edges from $E_0(v)$ and independently from one another we assign each of them to an arbitrary part $E_i(v)$ (for $1 \leq i \leq |S(v)|$) then the resulted partition also satisfies the requirements of Theorem 5.1 and $E_0(v) = \emptyset$ for all vertices v. In what follows, by proving a strengthening of Theorem 5.1, we exhibit an interesting connection between the stable partnership problem and the stable roommates problem.

If $G = (V, E)$ is a graph and v is a vertex of it then *detaching v into k parts* is the inverse operation of merging k vertices into one vertex. That is, we delete vertex v, introduce new vertices $v^1, v^2, \ldots, v^k$ and each edge that was originally incident with v will be incident with one of $v^1, v^2, \ldots, v^k$. If $k : V \to \{1, 2, 3, \ldots\}$ is a function then a *k-detachment of G* is a graph G^k that we get by detaching each vertex v of G into $k(v)$ parts. Clearly, there is a natural correspondance between the edges of G and that of G^k. With this notation, there is an equivalent formulation of Theorem 5.1: if G is a graph, and increasing substitutable choice function C_v is given for each vertex v of G and S is a stable partnership then there exists a k-detachment G^k of G in such a way that any stable partnership of G corresponds to a matching of G^k, where $k(v) := |S(v)|$ for each vertex v of G.

Theorem 5.3. *Let S be a stable partnership for graph $G = (V, E)$ and increasing substitutable choice functions C_v. Let $k(v) := \max\{|S(v)|, 1\}$. There is a k-detachment G^k of G and there are linear orders $<_{v_i}$ on the stars of G^k such that any stable partnership of G corresponds to a stable matching of G^k.*

Proof. Just like in the proof of Theorem 5.1, we start from a stable partnership S', produced by our algorithm and we build up G^k and construct orders $<_{v_i}$ by following the algorithm backwards.

Let $G_i = (V, E_i)$ denote the actual graph after the ith step of our algorithm, that is, $G_0 = G$ and $G_t = (V, S')$ for some t. Assume that we have a k-detachment G_i^k of G_i and suitable linear orders such that any stable partnership of G_i (for the restricted choice functions $C_v|_{E_i}$) corresponds to a stable matching of G_i^k. We show how to find a k-detachment G_{i-1}^k of G_{i-1} and extensions of the linear orders such that any stable partnership of G_{i-1} corresponds to a matching of G_{i-1}^k. If we do this, then G_0^k with the constructed linear orders is a k-detachment we look for.

First we construct G_t^k by detaching G_t into a matching. This means that each vertex v incident with S' is detached into $|S'(v)| = |S(v)| = k(v)$ parts (and we do not detach isolated vertices of G_t). As each degree of G_t^k is 0 or 1, the linear orders are trivial. Clearly the only stable partnership S' of G corresponds to the unique stable matching of G_t^k. So assume we have have already constructed G_i^k and the linear orders. If the ith step of the algorithm was 1st or 3rd type then $G_{i-1} = G_i$, hence we can choose $G_{i-1}^k = G_i^k$ and the same linear orders on the stars.

Assume that the ith step is a 1st type rejection step, that is, we delete some edges incident with some vertex v, say $vu_1, vu_2, \ldots, vu_p$. Define k-detachment G_{i-1}^k by adding p edges to the G_i^k in such a way that the edge corresponding to vu_i will be edge (say) $v^1u_j^1$. The extended linear orders on the stars of G_{i-1}^k will be the same as those of G_i^k, except for we append the new edges $v^1u_j^1$ to the end of these orders, that is, these new edges will be the least preferred ones of the vertices. Lemma 3.1 implies that the set of stable partnerships of G_i and of G_{i-1} is the same, so it is enough to check that no stale matching S^k of G_i^k that corresponds to a a stable parnership S of G_i is blocked by some edge $v^1u_j^1$.

Edge vu_j is not blocking S, hence at least one of v and u_j is covered by S. Corollary 5.2 implies that $|S(v)| = |S'(v)|$ and $S(u_j)| = |S'(u_j)|$, so this means that S^k has an edge e that covers v^1 or u_j^1. The definition of the linear orders on the stars of G_{i-1} implies $v^1u_j^1$ is dominated by e, so $v^1u_j^1$ cannot block stable matching S^k.

The remaining case is that the ith step of the algorithm is a 4th type rotation elimination. Assume the eliminated rotation is $(e_1, (e_1)^r, e_2, (e_2)^r, \ldots, e_m, (e_m)^r)$, so we delete edges $e_1, e_2, \ldots, e_m$ where 1-arc $e_j = u_jv_j$ is a first choice of u_j. After the elimination, each edge $(e_j)^r = u_jv_{j+1}$ becomes a first choice of u_j for $j = 1, 2, \ldots, m$.

To construct G_{i-1}^k, we add an edge e_j^k to G_i^k that correspond to e_j for $j = 1, 2, \ldots, m$. Assume that edges $(e_{j-1})^r = u_{j-1}v_j$ and $(e_j)^r = u_jv_{j+1}$ of G_i correspond to edges $u_{j-1}^t v_j^{t'}$ and $u_j^s v_{j+1}^{s'}$, of G_i^k, respectively. Then the edge of G_{i-1}^k that corresponds to e_j will be $e_j^k := u_j^s v_j^t$. In other words, we choose k-detachment G_{i-1}^k in such a way that edges of the rotation correspond to a cycle. We insert e_j^k into the linear order of u_j^s in such a way that e_j^k and $((e_j)^r)^r$ are consecutive and e_j^k is preceeding $((e_j)^r)^r$. We insert e_j^k into the linear order of v_j^t in such a way that and e_j^k and $((e_{j-1})^r)^k$ will also be consecutive according to the order of v_j^t, but e_j^k succeeds $((e_{j-1})^r)^k$. We do this for all $j = 1, 2, \ldots, m$, hence determining k-detachment G_{i-1}^k and linear orders on its stars.

First we prove that for any eliminated edge e_j of the rotation, edge $((e_j)^r)^k$ is the first edge in the linear order of u_j^s in G_i^k. By the definition of the replacement and rotation elimination, $(e_j)^r$ is a first choice of u_j in G_i. So after the $(i-1)$st step of the algorithm, $(e_j)^r$ never could be an u_j-replacement of another edge. This means, that from the ith step on, we never inserted an edge right before $(e_j)^r$ in the linear order of u_j^s. So if $(e_j)^r$ is an edge of stable partnership S' produced by our algorithm then $(e_j)^r$ is still the most preferred edge of u_j^s. If $(e_j)^r$ is deleted after the $(i-1)$st step then we had to delete it in the lth step, in a rotation elimination, as a first choice of u_j. This means on one hand that $((e_j)^r)^k$ is first in the linear order of u_j^s, in G_l^k. As we did not insert any edge before $((e_j)^r)^k$ during the construction of $G_{l-1}^k, G_{l-2}^k, \ldots, G_i^k$, we see that $((e_j)^r)^k$ is first in the linear order of u_j^s in G_i^k.

We prove that any stable parnership of G_{i-1} corresponds to a stable matching of G_{i-1}^k. If S is a stable parnership of G_{i-1} then either $e_1, e_2, \ldots, e_m \in S$ or S is a stable partnership of G_i by Lemma 3.5. In the first case, Lemma 3.6 implies that

$$S \setminus \{e_1, e_2, \dots, e_m\} \cup \{(e_1)^r, (e_2)^r, \dots, (e_m)^r\}$$

is a stable partnership, hence it corresponds to a stable matching of G_i^k by the induction hypothesis. As we have chosen edges e_j^k and $((e_j)^r)^k$ and e_j^k and $((e_{j-1})^r)^k$ consective in the linear orders of u_j^s and of v_j^t, we see that no edge can block the matching that corresponds to S in G_{i-1}^k.

In the second case, when S is a stable partnership of G_i, we have to show that the stable matching of G_i^k that corresponds to S (by the induction hypothesis) is not blocked by edge e_j^k. If $((e_j)^r)^k$ is in the stable matching then it dominates e_j^k at v_j^t. If $((e_j)^r)^k$ does not belong to the stable matching then it cannot block it, hence, as $((e_j)^r)^k$ is the best edge of $u_{j+1}^{t'}$, the stable matching dominates it at v_j^t. So this matching that corresponds to S in G_{i-1}^k also dominates e_j^k. This completes the proof. $\square$

Note that Theorem 5.3 is not an equivalence: it is not true that for any stable partnership problem there exists a detachment with appropriate linear orders in such a way that stable partnerships correspond bijectively to stable matchings. A counterexample is a graph on two vertices, four parallel edges with opposite linear orders on the vertices. The choice function of both vertices is the best two edges of the offered set, that is the stable partnership problem is a stable 2-matching problem. It is easy to see that there are exatly 3 different stable partnerships, but any 2-detachment has 1, 2 or 4 stable matching.

6 Conclusion, open questions

In this work, we extended Tan's result [16] from stable matchings to a much more general framework. An interesting, and perhaps not well enough understood feature of Tan's characterization is that it follows from Scarf's lemma [13] just like its generalization to stable b-matchings [2]. Our present extension is a characterization of a very similar kind. A most natural question to ask is whether our result can also be deduced from an appropriate generalization of Scarf's lemma.

Note that Scarf's proof of his well known lemma is algorithmic. However, Scarf's algorithm is not known to be polynomial. For special applications, like Tan's characterization there are known polynomial algorithms, like the one of Tan and Hsueh [17]. Another natural question is whether the Tan-Hsueh algorithm can be extended to an efficient algorithm that finds a stable half-partnership. In fact, the Tan-Hsueh algorithm can be generalized to the stable b-matching problem (that is, to the many-to-many stable roommates problem) the following way. We assign quota $b(v) = 0$ to each agent v, such that $\emptyset$ is a stable b-matching. Then, in each phase of the algorithm, we rise one quota by one and find a new stable half-b-matching. The work we do during such a phase is essentially the same that the Tan-Hsueh algorithm does in one phase. We do this until we reach the real quota for each agent. Note that this generalized Tan-Hsueh algorithm can be extended to our stable partnership setting. We sketch it below.

For a linear order $\prec_v$ on $E(v)$, define truncated choice function C_v^i on subset X of $E(v)$ as the $\prec_v$-smallest i elements of $C_v(X)$. If C_v is increasing and substitutable then it is not difficult to choose $\prec_v$ in such a way that C_v^i is also an increasing substitutable choice function for any i. So $\emptyset$ is a stable half-partnership for choice functions C_v^0. In the beginning of a phase, we start with a stable half-partnership for truncated choice functions

$C_{v_1}^{i_1}, C_{v_2}^{i_2}, \ldots, C_{v_n}^{i_n}$. We pick a vertex v_j such that $i_j < n$, and find a stable half-partnership for the same truncated choice functions, except that we use $C_{v_j}^{i_j+1}$ instead of $C_{v_j}^{i_j}$. To do this, we use a straightforward generalization of the Tan-Hsueh algorithm. After $i_j = n$ for all vertices v_i, we have a stable half-partnership for the original problem.

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