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A Matroid Approach to Stable Matchings with Lower Quotas

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Abstract

In SODA'10, Huang introduced the laminar classified stable matching problem (**LCSM** for short) that is motivated by academic hiring. This problem is an extension of the well-known hospitals/residents problem in which a hospital has laminar classes of residents and it sets lower and upper bounds on the number of residents that it would hire in that class. Against the intuition that stable matching problems with lower quotas are difficult in general, Huang proved that this problem can be solved in polynomial time. In this paper, we propose a matroid-based approach to this problem and we obtain the following results. (i) We solve a generalization of the **LCSM** problem. (ii) We exhibit a polyhedral description for stable assignments of the **LCSM** problem, which gives a positive answer to Huang's question. (iii) We prove that the set of stable assignments of the **LCSM** problem has a lattice structure similarly to the ordinary stable matching model.

Keywords: stable marriages; college admission problem, matroids

1 Introduction

The *hospitals/residents problem* (**HR** for short) introduced by Gale and Shapley [6] is a many-to-one extension of the stable matching problem [6, 7, 11]. In this problem, the two sets that in the stable marriage problem correspond to men and women, here are the residents and hospitals, respectively. Each hospital has an *upper quota* on the number of residents that this hospital can accept. Many properties of stable

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matchings hold for the solutions of the **HR** problem. For example, any instance of this problem admits at least one stable matching, and we can efficiently find it.

Recently, several extensions of **HR**-type problems were studied with *lower quotas* for the hospitals. These problems can be motivated by e.g., academic hiring or project-type classes in universities [1]. In some cases, if the number of students allocated to some project p is less than the lower quota of p, then project p must to be cancelled [1]. Hamada, Iwama and Miyazaki [8] considered the following variant of the **HR** problem with lower quotas. We are given an instance of the **HR** problem in which preference lists are complete and each hospital has a lower quota. An assignment must satisfy all the lower and upper quotas and a solution is a matching with the minimum number of blocking pairs. The authors gave an inapproximability result and a polynomial-time solvable case. Biró, Fleiner, Irving and Manlove [1] considered a variant of the **HR** problem with lower quotas in which it is possible to close a hospital. More precisely, their stability definition allows a hospital not to satisfy its lower quota if no resident is assigned to it. The authors proved the $\mathcal{NP}$-completeness of deciding whether there exists a stable assignment. Huang introduced the *classified stable matching problem* in [9] that is an extension of the **HR** problem in which each hospital has lower and upper quotas for subsets of acceptable residents. The author proved the $\mathcal{NP}$-completeness of the problem of deciding whether there is a stable assignment. Furthermore, the author proved that if the quota sets form a laminar family for each hospital, then we can check the existence of a stable assignment in polynomial time. We shall call this latter problem the *laminar classified stable matching problem* (**LCSM** for short).

Huang's positive result is somewhat surprising since his model is quite natural and from other results it seems that stable matching problems with lower quotas are difficult in general. In this paper, we propose a matroid-based approach [3, 4, 5] to the **LCSM** problem and we obtain the following results.

- We solve a generalization of the **LCSM** problem.
- By exhibiting a polyhedral description for stable assignments of the **LCSM** problem, we give a positive answer to Huang's question in [9].
- We prove that similarly to the ordinary stable matchings, the set of stable assignments of the **LCSM** problem has a natural lattice structure.

The rest of this paper is organized as follows. In Section 2, we describe our model. We introduce known results about matroids in Section 3. In Section 4, we describe our matroid-based algorithm. In Section 5, we point out some interesting properties of stable assignments of our model. Section 6 concludes this paper.

Before describing our model, we introduce some definitions and notations. Let $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the set of non-negative reals and non-negative integers, respectively. Given a set U and $f \in \mathbb{R}_+^U$, we write $f(X)$ instead of $\sum_{x\in X} f(x)$ for a subset X of U and χ_X denotes the characteristic function of a subset X of U:

$$\chi_X(x) := \begin{cases} 0 & \text{if } x \notin X \\ 1 & \text{if } x \in X \ . \end{cases}$$

For $k \in \mathbb{Z}_+$, let $[k] = \{1, \ldots, k\}$. If X is a subset and x is an element of some ground set then we denote $X \cup \{x\}$ by $X + x$ and $X \setminus \{x\}$ by $X - x$. A family $\mathcal{F}$ of subsets of some ground set is called *laminar* if $X \cap Y = \emptyset$, or $X \subseteq Y$, or $Y \subseteq X$ for any $X, Y \in \mathcal{F}$. For graph $G = (V, E)$, vertex v of V and subset F of edge set E, notation $F(v)$ stands for the set of edges of F incident to v.

2 Problem Formulation

In this section, we introduce the *two-sided laminar classified stable matching problem* (**2LCSM** for short) that is a generalization of the **LCSM** problem. Roughly speaking, this is the same problem except that both sides can have quota sets (for the definition of the **LCSM** problem, see Section 5).

In the **2LCSM** problem, we are given a finite bipartite graph $G = (V, E)$ with colour classes P and Q. For each vertex v of V, there is a laminar family $\mathcal{C}_v$ of subsets of $E(v)$. Define

$$\mathcal{C}_P := \bigcup_{v \in P} \mathcal{C}_v, \quad \mathcal{C}_Q := \bigcup_{v \in Q} \mathcal{C}_v \quad \text{and} \quad \mathcal{C} := \mathcal{C}_P \cup \mathcal{C}_Q.$$

We are given lower and upper quota functions $l \colon \mathcal{C} \to \mathbb{Z}_+$ and $u \colon \mathcal{C} \to \mathbb{Z}_+$. In the sequel, we call a member C of $\mathcal{C}$ a *class*.

Let M be a subset of E. We say that M *obeys* l (resp., u) *for a class* C *of* $\mathcal{C}$ if

$$l(C) \leq |M \cap C| \qquad (\text{resp., } |M \cap C| \leq u(C)).$$

We call M *feasible for a vertex* v *of* V if M obeys l and u for any class of $\mathcal{C}_v$, i.e.,

$$l(C) \leq |M \cap C| \leq u(C)$$

for any class C of $\mathcal{C}_v$. If M is feasible for any vertex of V, then M is an *assignment*.

Let M be an assignment. In our model, each vertex v has a strict linear order $<_v$ on $E(v)$. We think on this linear order as the preference order of v on its edges, the most preferred one is the $<_v$-smallest edge. An edge e of $E \setminus M$ is called *free for an endpoint* v *of* e if

$$M + e \text{ is feasible for } v, \text{ or}$$
$$\text{there is an edge } f \text{ of } M(v) \text{ such that } e <_v f \text{ and } M + e - f \text{ is feasible for } v.$$

An edge e of $E \setminus M$ *blocks* M if e is free for both endpoints of e. An assignment M of E is *stable* if no edge of $E \setminus M$ blocks M. Then, the **2LCSM** problem is to find a stable assignment if exists.

Remark. In the **LCSM** problem originally introduced in [9], the notion of blocking is defined for a *group* that consists of several vertices of P and one vertex of Q. Thus, it seems that the definitions of stability in the **2LCSM** problem and the **LCSM** problem are different. However, in Section 5 we prove that there is a blocking edge if and only if there is a blocking group, i.e., both definitions of stability are equivalent in the **LCSM** problem.

3 Matroid-Kernels

First we recall some basics on matroids. The expert readers may want to skip this part.

A pair $(U, \mathcal{I})$ is called a *matroid* if U is a finite set and $\mathcal{I}$ is a nonempty family of subsets of U satisfying the following conditions.

$$\text{If } I \in \mathcal{I} \text{ and } J \subseteq I, \text{ then } J \in \mathcal{I}. \tag{1}$$

$$\text{If } I, J \in \mathcal{I} \text{ and } |I| < |J|, \text{ then } I + e \in \mathcal{I} \text{ for some element } e \text{ of } J \setminus I. \tag{2}$$

Let $\mathcal{M} = (U, \mathcal{I})$ be a matroid. A subset I of U is called *independent* if $I \in \mathcal{I}$. A subset D of U is a *circuit* if $D \notin \mathcal{I}$, but any proper subset D' of D is independent. It is known [10] that if $I \in \mathcal{I}$ and e is an element of $U \setminus I$ such that $I + e \notin \mathcal{I}$, then $I + e$ contains a unique circuit D such that $e \in D$. Such a circuit D is called the *basic circuit of* e (with respect to I in $\mathcal{M}$). Obviously, the basic circuit D of e is the set of elements f of $I + e$ such that $I + e - f \in \mathcal{I}$. For a subset F of U, a subset B of F is called a *base of* F if B is an inclusionwise maximal independent subset of F. By (2), any two bases of a subset F of U have the same size, which is called the *rank of* F and denoted by $r_{\mathcal{M}}(F)$. We define the *span function* $\mathrm{span}_{\mathcal{M}} \colon 2^U \to 2^U$ by

$$\mathrm{span}_{\mathcal{M}}(F) := \{e \in U \mid r_{\mathcal{M}}(F + e) = r_{\mathcal{M}}(F)\}$$

for a subset F of U. Obviously, $F \subseteq \mathrm{span}_{\mathcal{M}}(F)$.

Lemma 3.1. *If $\mathcal{M} = (U, \mathcal{I})$ is a matroid, $I, J \in \mathcal{I}$ and $I \subseteq \mathrm{span}_{\mathcal{M}}(J)$, then $|I| \leq |J|$.*

Proof. Suppose $|I| > |J|$. By (2), $J + e \in \mathcal{I}$ for some element e of $I \setminus J$. This contradicts the fact that $e \in I \subseteq \mathrm{span}_{\mathcal{M}}(J)$. □

Let $\mathcal{M}_1 = (U_1, \mathcal{I}_1), \ldots, \mathcal{M}_k = (U_k, \mathcal{I}_k)$ be matroids such that $U_1, \ldots, U_k$ are pairwise disjoint. Let $U = U_1 \cup \cdots \cup U_k$, and define

$$\mathcal{I} := \{I \subseteq U \mid I \cap U_i \in \mathcal{I}_i \text{ for any } i \in [k]\}.$$

We call $\mathcal{M} = (U, \mathcal{I})$ the *direct sum of matroids* $\mathcal{M}_1, \ldots, \mathcal{M}_k$, and it can be easily checked that $\mathcal{M}$ is indeed a matroid.

3.1 Matroid-kernels

A triple $\mathcal{M} = (U, \mathcal{I}, <)$ is an *ordered matroid* if $(U, \mathcal{I})$ is a matroid and $<$ is a strict linear order on U. Let $\mathcal{M} = (U, \mathcal{I}, <)$ be an ordered matroid. We may not distinguish between $\mathcal{M}$ and a matroid $(U, \mathcal{I})$. An independent set I of $\mathcal{I}$ *dominates* an element e of $U \setminus I$ if $I + e \notin \mathcal{I}$ and $f < e$ for any element f of $D - e$, where D is the basic circuit of e with respect to I. The set of elements of U dominated by an independent set I of $\mathcal{I}$ is denoted by $\mathcal{D}_{\mathcal{M}}(I)$.

Let $U = \{e_1, \ldots, e_n\}$ such that $e_1 < \cdots < e_n$. For a subset F of U, let $\mathcal{F}_{\mathcal{M}}(F)$ be a subset of F obtained by the following greedy algorithm. Define $\mathcal{F}^0_{\mathcal{M}}(F) = \emptyset$, and define

$$\mathcal{F}^i_{\mathcal{M}}(F) := \begin{cases} \mathcal{F}^{i-1}_{\mathcal{M}}(F) & \text{if } e_i \notin F \text{ or} \\ & \text{if } \mathcal{F}^{i-1}_{\mathcal{M}}(F) + e_i \notin \mathcal{I} \\ \mathcal{F}^{i-1}_{\mathcal{M}}(F) + e_i & \text{otherwise} \end{cases}$$

for $i \in [n]$. Define $\mathcal{F}_{\mathcal{M}}(F) := \mathcal{F}^n_{\mathcal{M}}(F)$.

Let $\mathcal{M}_1 = (U, \mathcal{I}_1, <_1)$ and $\mathcal{M}_2 = (U, \mathcal{I}_2, <_2)$ be ordered matroids. A common independent set K of $\mathcal{I}_1 \cap \mathcal{I}_2$ is an $\mathcal{M}_1\mathcal{M}_2$-*kernel* if

$$\mathcal{D}_{\mathcal{M}_1}(K) \cup \mathcal{D}_{\mathcal{M}_2}(K) = U \setminus K.$$

We denote by $\mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}$ the set of $\mathcal{M}_1\mathcal{M}_2$-kernels. Let $\mathrm{EO}_{\mathcal{M}_1\mathcal{M}_2}$ be the time required to compute $\mathcal{F}_{\mathcal{M}_1}(F), \mathcal{F}_{\mathcal{M}_2}(F)$ for any subset F of U.

Theorem 3.2 (Fleiner [3, 4, 5])**.** *If $\mathcal{M}_1, \mathcal{M}_2$ are ordered matroids on the same ground set U, then $\mathcal{K}_{\mathcal{M}_1\mathcal{M}_2} \neq \emptyset$ and we can find an $\mathcal{M}_1\mathcal{M}_2$-kernel in $O(|U|\mathrm{EO}_{\mathcal{M}_1\mathcal{M}_2})$ time.*

Theorem 3.3 (Fleiner [3, 4, 5])**.** *If $\mathcal{M}_1, \mathcal{M}_2$ are ordered matroids on the same ground set and $K, L \in \mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}$, then $\mathrm{span}_{\mathcal{M}_i}(K) = \mathrm{span}_{\mathcal{M}_i}(L)$ for any $i \in \{1, 2\}$.*

Define

$$\mathcal{B}_{\mathcal{M}_1\mathcal{M}_2} := \{B \subseteq U \mid B \cap K \neq \emptyset \text{ for any } K \in \mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}\},$$
$$\mathcal{A}_{\mathcal{M}_1\mathcal{M}_2} := \{A \subseteq U \mid |A \cap K| \leq 1 \text{ for any } K \in \mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}\}$$

the blocker and anti-blocker of matroid-kernels, respectively. Let

$$\mathcal{P}_{\mathcal{M}_1\mathcal{M}_2} := \mathrm{conv}\{\chi_K \mid K \in \mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}\}$$

be the convex hull of characteristic vectors of all $K \in \mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}$.

Theorem 3.4 (Fleiner [3, 4, 5])**.** *If $\mathcal{M}_1, \mathcal{M}_2$ are ordered matroids on the same ground set U, then*

$$\mathcal{P}_{\mathcal{M}_1\mathcal{M}_2} = \{x \in \mathbb{R}^U_+ \mid x(B) \geq 1 \text{ for any } B \in \mathcal{B}_{\mathcal{M}_1\mathcal{M}_2}, \quad x(A) \leq 1 \text{ for any } A \in \mathcal{A}_{\mathcal{M}_1\mathcal{M}_2}\}.$$

Furthermore, we can solve the separation problem over $\mathcal{P}_{\mathcal{M}_1\mathcal{M}_2}$ in time bounded by a polynomial in the input size and $\mathrm{EO}_{\mathcal{M}_1\mathcal{M}_2}$.

For subsets F_1, F_2 of U, define

$$F_1 \vee F_2 := \mathcal{F}_{\mathcal{M}_1}(F_1 \cup F_2) \quad \text{and} \quad F_1 \wedge F_2 := \mathcal{F}_{\mathcal{M}_2}(F_1 \cup F_2). \tag{3}$$

An $\mathcal{M}_1\mathcal{M}_2$-kernel K^* is $\mathcal{M}_1$-*optimal* if $K \vee K^* = K^*$ for any $K \in \mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}$.

Theorem 3.5 (Fleiner [3, 4, 5])**.** *If $\mathcal{M}_1, \mathcal{M}_2$ are ordered matroids on the same ground set U and $\vee, \wedge$ are defined by (3), then a triple $(\mathcal{K}_{\mathcal{M}_1\mathcal{M}_2}, \vee, \wedge)$ is a lattice and we can find the $\mathcal{M}_1$-optimal $\mathcal{M}_1\mathcal{M}_2$-kernel in $O(|U|\mathrm{EO}_{\mathcal{M}_1\mathcal{M}_2})$ time.*

4 Algorithm

In this section, we propose a matroid-based algorithm for the **2LCSM** problem. In our algorithm, we first construct ordered matroids $\mathcal{M}_P, \mathcal{M}_Q$ on E so that a subset M of E is a stable assignment if and only if M is an $\mathcal{M}_P\mathcal{M}_Q$-kernel and M obeys l for any class of $\mathcal{C}$. After that, we find an $\mathcal{M}_P\mathcal{M}_Q$-kernel K by using a generalization of the Gale-Shapley algorithm. If K obeys l for any class of $\mathcal{C}$, then the algorithm concludes that K is a stable assignment. Otherwise, i.e., if K does not obey l for some class C of $\mathcal{C}$, then the algorithm concludes that there is no stable assignment whatsoever. More precisely, we prove that in this case no $\mathcal{M}_P\mathcal{M}_Q$-kernel obeys l for C.

4.1 Definitions

Recall that $\mathcal{C}_v$ is laminar for any vertex v of V. A class C of $\mathcal{C}$ is a *child of a class C' of $\mathcal{C}$* if C is a proper subset of C' and there is no class C° of $\mathcal{C}$ such that $C \subsetneq C^\circ \subsetneq C'$. Without loss of generality, we can make the following assumptions.

Assumption 1. For any vertex v of V and any edge e of $E(v)$, $\{e\} \in \mathcal{C}_v$.

(We can define $l(\{e\}) = 0$ and $u(\{e\}) = 1$.) By Assumption 1, for any class C of $\mathcal{C}$, either C has no child or children $C_1, \ldots, C_k$ of C form a partition of C, i.e. $C_1 \cup \cdots \cup C_k = C$.

Assumption 2. For any vertex v of V, $E(v) \in \mathcal{C}_v$.

(If $E(v) \notin \mathcal{C}_v$ then we can add $E(v)$ to $\mathcal{C}$ with $l(E(v)) = l(C_1) + l(C_2) + \ldots + l(C_k)$ and $u(E(v)) = |E(v)|$, where $C_1, \ldots, C_k$ are the inclusionwise maximal members of $\mathcal{C}_v$.)

Assumption 3. If a class C of $\mathcal{C}$ has children $C_1, \ldots, C_k$ then

$$l(C_1) + \cdots + l(C_k) \leq l(C) \leq u(C). \tag{4}$$

(We can do so because if the second relation does not hold then clearly there exists no assignment. If the first relations fails then we do not change the problem if we change $l(C)$ to $l(C_1) + \cdots + l(C_k)$.)

For a class C of $\mathcal{C}$, we denote by $\mathcal{C}_C$ the set of classes C' of $\mathcal{C}$ such that $C' \subseteq C$. The *level of a class C of $\mathcal{C}$* is the maximum integer k for which there are classes $C_1, \ldots, C_k$ of $\mathcal{C}$ such that $C_1 = C$, C_{i+1} is a child of C_i for any $i \in [k-1]$, and C_k has no child. For a class C of $\mathcal{C}$, we define a function $d_C \colon 2^C \to \mathbb{Z}_+$ as follows. If C has no child, then

$$d_C(F) := \max(|F|,\ l(C))$$

for a subset F of C. If C has children $C_1, \ldots, C_k$, then

$$d_C(F) := \max(d_{C_1}(F \cap C_1) + \cdots + d_{C_k}(F \cap C_k),\ l(C))$$

for a subset F of C. A subset F of a class C of $\mathcal{C}$ is *deficient on* C if the following conditions hold. If C has no child, then F does not obey l for C. If C has children $C_1, \dots, C_k$, then

$$d_{C_1}(F \cap C_1) + \cdots + d_{C_k}(F \cap C_k) < l(C).$$

Lemma 4.1. *Let C be a class of $\mathcal{C}$.*

(a) $d_C(F + e) \leq d_C(F) + 1$ for any subset F of C and any edge e of C.

(b) $d_C(F_1) \leq d_C(F_2)$ for any subsets F_1, F_2 of C such that $F_1 \subseteq F_2$.

(c) If a subset F of C obeys l for any class of $\mathcal{C}_C$, then $d_C(F) = |F|$.

(d) If a subset F of C is deficient on C, then $d_C(F + e) = d_C(F)$ for any edge e of C.

Proof. Statements (a) to (c) can be easily proved by induction on the level of C. Statement (d) follows from Statement (a). □

4.2 Matroids on edges

For a class C of $\mathcal{C}$, we define a family $\mathcal{I}_C$ of subsets I of C by

$$\mathcal{I}_C := \{I \subseteq C \mid d_{C'}(I \cap C') \leq u(C') \text{ for any } C' \in \mathcal{C}_C\}.$$

The goal of this subsection is to prove that $\mathcal{M}_C = (C, \mathcal{I}_C)$ is a matroid for any class C of $\mathcal{C}$.

Lemma 4.2. *If C is a class of $\mathcal{C}$, $I, J \in \mathcal{I}_C$ and $|I \cap C'| \geq |J \cap C'|$ for any class C' of $\mathcal{C}_C$ on which $I \cap C'$ is deficient, then $d_C(J) - d_C(I) \geq |J| - |I|$.*

Proof. We prove the lemma by induction on the level of C. If the level of C is one, that is, if C is a singleton then the lemma is straightforward. Assume that the lemma holds for any class with level at most r for some $r \geq 1$ and take a class C of level $r + 1$. If I is deficient on C, then $|I| \geq |J|$ by the condition in the lemma. So,

$$d_C(J) - d_C(I) = d_C(J) - l(C) \geq l(C) - l(C) = 0 \geq |J| - |I|,$$

where the first equality is due to that I is deficient on C.

Suppose that I is not deficient on C. Let $C_1, \dots, C_k$ be the children of C, $I_i = I \cap C_i$ and $J_i = J \cap C_i$. For any class C' of $\mathcal{C}_{C_i}$, $I \cap C' = I_i \cap C'$ and $J \cap C' = J_i \cap C'$. So, $I_i, J_i \in \mathcal{I}_{C_i}$ by $I, J \in \mathcal{I}_C$. Moreover, by the lemma assumption, $|I_i \cap C'| \geq |J_i \cap C'|$ for any class C' of $\mathcal{C}_{C_i}$ on which $I_i \cap C'$ is deficient. So, by the induction hypothesis, $d_{C_i}(J_i) - d_{C_i}(I_i) \geq |J_i| - |I_i|$. Thus,

$$\begin{aligned} d_C(J) - d_C(I) &= d_C(J) - \sum_{i \in [k]} d_{C_i}(I_i) \geq \sum_{i \in [k]} d_{C_i}(J_i) - \sum_{i \in [k]} d_{C_i}(I_i) \\ &\geq \sum_{i \in [k]} |J_i| - \sum_{i \in [k]} |I_i| = |J| - |I|, \end{aligned}$$

where the first equality follows from the fact that I is not deficient on C. □

Lemma 4.3. *If C is a class of $\mathcal{C}$, $I, J \in \mathcal{I}_C$ and $|I| < |J|$, then $I + e \in \mathcal{I}_C$ for some edge e of $J \setminus I$.*

Proof. We prove the lemma by induction on the level of C. If the level of C is one, then the lemma is straightforward as C is a singleton. Assume that the lemma holds if the level of C is at most r for some $r \geq 1$, and take a class C of level $r + 1$.

Case 1. $|I \cap C^*| < |J \cap C^*|$ for some class C^* of $\mathcal{C}_C$ on which $I \cap C^*$ is deficient. Let $I^* = I \cap C^*$ and $J^* = J \cap C^*$. By $I, J \in \mathcal{I}_C$, we have $I^*, J^* \in \mathcal{I}_{C^*}$. So, by the induction hypothesis, $I^* + e^* \in \mathcal{I}_{C^*}$ for some edge e^* of $J^* \setminus I^*$. Since I^* is deficient on C^*, $d_{C^*}(I^* + e^*) = d_{C^*}(I^*)$ by Lemma 4.1(d). ¿From this, we shall prove that $I + e^* \in \mathcal{I}_C$.

Let $L = I + e^*$ and $L' = L \cap C'$ for a class C' of $\mathcal{C}_C$. It suffices to prove that $d_{C'}(L') \leq u(C')$ for any class C' of $\mathcal{C}_C$. If $C' \cap C^* = \emptyset$, then this holds by $e^* \notin C'$ and $I \in \mathcal{I}_C$. If $C' \in \mathcal{C}_{C^*}$, then this holds by $I^* + e^* \in \mathcal{I}_{C^*}$. If $C^* \subseteq C'$, then this follows from $d_{C^*}(I^* + e^*) = d_{C^*}(I^*)$ and the fact that $d_{C'}(L')$ does not change if $d_{C^\circ}(L' \cap C^\circ)$ does not change for any child C° of C'.

Case 2. Assume that $|I \cap C'| \geq |J \cap C'|$ for any class C' of $\mathcal{C}_C$ on which $I \cap C'$ is deficient. By Lemma 4.2, $d_C(J) - d_C(I) \geq |J| - |I| > 0$. This implies that

$$d_C(I + e) \leq d_C(I) + 1 \leq d_C(J) \leq u(C) \tag{5}$$

for any edge e of $J \setminus I$, where the first inequality follows from Lemma 4.1(a) and the third from $J \in \mathcal{I}_C$. Let $C_1, \ldots, C_k$ be the children of C, $I_i = I \cap C_i$ and $J_i = J \cap C_i$. By $I, J \in \mathcal{I}_C$, we have $I_i, J_i \in \mathcal{I}_{C_i}$. Let N be the set of $i \in [k]$ such that $|I_i| < |J_i|$. Notice that $N \neq \emptyset$ by $|I| < |J|$. By the induction hypothesis, for any $i \in N$ there is an edge e_i of $J_i \setminus I_i$ such that $I_i + e_i \in \mathcal{I}_{C_i}$. So, by (5), $I + e_i \in \mathcal{I}_C$ for any $i \in N$. This completes the proof. □

We are now ready to prove the main result of this subsection.

Lemma 4.4. *For any class C of $\mathcal{C}$, $\mathcal{M}_C = (C, \mathcal{I}_C)$ is a matroid.*

Proof. By the first inequality of (4), $d_{C'}(\emptyset) = l(C')$ for any class C' of $\mathcal{C}_C$. So, by the second inequality of (4), $\emptyset \in \mathcal{I}_C$, i.e., $\mathcal{I}_C \neq \emptyset$. Furthermore, (1) and (2) follows from Lemmas 4.1(b) and 4.3, respectively. □

4.3 Algorithm

In this subsection, we describe our algorithm for the **2LCSM** problem. By Lemma 4.4, $\mathcal{M}_{E(v)}$ is a matroid for any vertex v of V. Let $\mathcal{M}_P = (E, \mathcal{I}_P, <_P)$ be an ordered matroid such that $(E, \mathcal{I}_P)$ is the direct sum of matroids $\mathcal{M}_{E(v)}$ for all vertices v of P and $<_P$ is a strict linear order defined in such a way that $e <_P f$ whenever $e <_v f$ for some vertex v of P. For the vertex class Q, we similarly define an ordered matroid $\mathcal{M}_Q = (E, \mathcal{I}_Q, <_Q)$. Then, our algorithm, called Algorithm_2LCSM, can be described as follows. Note that Step 1 of the algorithm is a natural generalization of the proposal algorithm of Gale and Shapley (with the choice function represented by the greedy algorithm), described in [5].

Algorithm_2LCSM

Step 1: Find an $\mathcal{M}_P\mathcal{M}_Q$-kernel K.

Step 2: If K obeys l for any class of $\mathcal{C}$, then we output K, i.e., K is a stable assignment. Otherwise, there is no stable assignment.

Our next goal is to prove the correctness of Algorithm_2LCSM. By Lemma 4.1(c),

$$\text{a subset } M \text{ of } E \text{ is feasible for a vertex } v \text{ of } V \text{ if and only if} \\ M(v) \in \mathcal{I}_{E(v)} \text{ and } M \text{ obeys } l \text{ for any class of } \mathcal{C}_v. \tag{6}$$

Lemma 4.5. *A subset M of E is a stable assignment if and only if M is an $\mathcal{M}_P\mathcal{M}_Q$-kernel and M obeys l for any class of $\mathcal{C}$.*

Proof. We first prove sufficiency. Let M be an $\mathcal{M}_P\mathcal{M}_Q$-kernel obeying l for any class of $\mathcal{C}$. By (6), M is an assignment. Let e be an edge of $E \setminus M$. Since M is an $\mathcal{M}_P\mathcal{M}_Q$-kernel, without loss of generality, we can assume that $e \in \mathcal{D}_{\mathcal{M}_P}(M)$. Let v be the endpoint of e in P. By the definition of $\mathcal{M}_P$, $M(v) + e \notin \mathcal{I}_{E(v)}$. So, by (6), $M + e$ is not feasible for v. Let F be the set of arcs f of $M(v)$ such that $M + e - f$ is feasible for v. Now we prove that $f <_v e$ for any edge f of F. By (6), $M(v) + e - f \in \mathcal{I}_{E(v)}$, i.e., f is an edge of the basic circuit of e with respect to $M(v)$ in $\mathcal{M}_{E(v)}$ (also, M in $\mathcal{M}_P$). Since M is an $\mathcal{M}_P\mathcal{M}_Q$-kernel, we have $f <_P e$. So, by the definition of $<_P$, we have $f <_v e$.

For the necessity, let M be a stable assignment. By (6), $M \in \mathcal{I}_P \cap \mathcal{I}_Q$ and M obeys l for any class of $\mathcal{C}$. Let e be an edge of $E \setminus M$. Since M is a stable assignment, e is not free for at least one endpoint v of e. Without loss of generality, we can assume that $v \in P$. Now we prove that $e \in \mathcal{D}_{\mathcal{M}_P}(M)$. Since $M + e$ is not feasible for v, $M(v) + e \notin \mathcal{I}_{E(v)}$. Let D be the basic circuit of e with respect to $M(v)$ in $\mathcal{M}_{E(v)}$ (also, M in $\mathcal{M}_P$). Now we prove that $f <_P e$ for any edge f of $D - e$. For this, we need the following claim.

Claim 4.6. *$M + e - f$ obeys l for any class of $\mathcal{C}_v$.*

Proof. Let $M_1 = M + e$ and $M_2 = M + e - f$. Since $M(v) \in \mathcal{I}_{E(v)}$ and $M_1(v) \notin \mathcal{I}_{E(v)}$, there is a class C of $\mathcal{C}_v$ such that $e \in C$ and $d_C(M_1 \cap C) > u(C)$. By $M_2 \in \mathcal{I}_{E(v)}$, we have $f \in C$. So, $|M_2 \cap C'| = |M \cap C'| \geq l(C')$ for any class C' of $\mathcal{C}_v$ such that $C \subseteq C'$. Thus, it suffices to prove that M_2 obeys l for any class of $\mathcal{C}_C - C$. Assume that M_2 does not obey l for some class C^* of $\mathcal{C}_C - C$. Let $M_1^* = M_1 \cap C^*$ and $M_2^* = M_2 \cap C^*$. Since M_1 obeys l for C^*, we have $f \in C^*$. So, if we can prove that $d_{C^*}(M_2^*) = d_{C^*}(M_1^*)$, then $d_C(M_2 \cap C) = d_C(M_1 \cap C) > u(C)$, which contradicts the fact that $M_2(v) \in \mathcal{I}_{E(v)}$. Since M_2 does not obey l for C^* but M obeys l for C^*, we have $|M_1^*| = l(C^*)$. Moreover, since M_1 obeys l for any class of $\mathcal{C}_{C^*}$, we have $d_{C^*}(M_1^*) = |M_1^*|$ by Lemma 4.1(c). So,

$$l(C^*) \leq d_{C^*}(M_2^*) \leq d_{C^*}(M_1^*) = |M_1^*| = l(C^*),$$

where the second inequality follows from Lemma 4.1(b) and the fact that $M_2 \subseteq M_1$. This implies that $d_{C^*}(M_2^*) = d_{C^*}(M_1^*)$, which completes the proof. □

By Claim 4.6 and (6), $M + e - f$ is feasible for v. Since M is a stable assignment, we have $f <_v e$. So, by the definition of $<_P$, we have $f <_P e$. □

Lemma 4.7. *If an $\mathcal{M}_P\mathcal{M}_Q$-kernel K does not obey l for a class C of $\mathcal{C}_P$ but K obeys l for any class of $\mathcal{C}_C - C$, then*

$$\mathrm{span}_{\mathcal{M}_P}(K) \cap C = \mathrm{span}_{\mathcal{M}(C)}(K \cap C).$$

Proof. Obviously,

$$\mathrm{span}_{\mathcal{M}(C)}(K \cap C) \subseteq \mathrm{span}_{\mathcal{M}_P}(K) \cap C.$$

To prove the opposite direction, let e be an edge of $(\mathrm{span}_{\mathcal{M}_P}(K)\cap C)\setminus K$ and $L = K+e$. By the definition of e, $d_{C^*}(L \cap C^*) > u(C^*)$ for some class C^* of $\mathcal{C}_P$. Recall that K obeys l for any class of $\mathcal{C}_C - C$. So, if C has children $C_1, \dots, C_k$, then

$$d_{C_1}(K \cap C_1) + \cdots + d_{C_k}(K \cap C_k) = |K \cap C|$$

by Lemma 4.1(c). Thus, since K does not obey l for C, $K \cap C$ is deficient on C. So, $d_C(L\cap C) = d_C(K\cap C)$ by Lemma 4.1(d). This implies that $d_{C'}(L\cap C') = d_{C'}(K\cap C')$ for any class of C' of $\mathcal{C}_P$ such that $C \subseteq C'$. So, by $K \in \mathcal{I}_P$, we have $C^* \in \mathcal{C}_C$, i.e., $e \in \mathrm{span}_{\mathcal{M}(C)}(K \cap C)$. □

Lemma 4.8. *If an $\mathcal{M}_P\mathcal{M}_Q$-kernel K does not obey l for some class C of $\mathcal{C}$ but K obeys l for each class of $\mathcal{C}_C - C$, then no $\mathcal{M}_P\mathcal{M}_Q$-kernel obeys l for C.*

Proof. Without loss of generality, we can assume that $C \in \mathcal{C}_P$. For any $\mathcal{M}_P\mathcal{M}_Q$-kernel L,

$$L \cap C \subseteq \mathrm{span}_{\mathcal{M}_P}(L) \cap C = \mathrm{span}_{\mathcal{M}_P}(K) \cap C = \mathrm{span}_{\mathcal{M}(C)}(K \cap C),$$

where the first equality follows from Theorem 3.3 and the second from Lemma 4.7. Since $K \cap C$ and $L \cap C$ are independent sets of $\mathcal{I}_C$, $|L \cap C| \leq |K \cap C| < l(C)$ by Lemma 3.1. □

Theorem 4.9. Algorithm_2LCSM *can solve the* **2LCSM** *problem in $O(|E|^3)$ time.*

Proof. Since $\mathrm{EO}_{\mathcal{M}_P\mathcal{M}_Q}$ is clearly $O(|E|^2)$, Algorithm_2LCSM halts in $O(|E|^3)$ time by Theorem 3.2. If Algorithm_2LCSM outputs a subset M of E, then M is a stable assignment by Lemma 4.5. On the other hand, if Algorithm_2LCSM outputs no assignment, then there can not be a stable assignment by Lemmas 4.5 and 4.8. □

5 Properties of Stable Assignments

In this section, we point out some interesting properties of stable assignments of our model.

First we exhibit a polyhedral description for stable assignments of the **2LCSM** problem. We use implicit constraints differently to the stable matching polytope described by Vande Vate [13] and Rothblum [12]. In [9], Huang raised an open problem about a polyhedral description for stable assignments of the **LCSM** problem. Thus, our result gives a positive answer for this open problem. Let $\mathcal{S}$ be the set of stable assignments of the **2LCSM** problem. Let $\mathcal{P}_\mathcal{S}$ be the convex hull of the characteristic vectors for all stable assignments of $\mathcal{S}$.

Theorem 5.1.

$$\begin{aligned} \mathcal{P}_\mathcal{S} &= \mathcal{P}_{\mathcal{M}_P\mathcal{M}_Q} \cap \{x \in \mathbb{R}^E_+ \mid x(C) \geq l(C) \text{ for any } C \in \mathcal{C}\} && (7)\\ &= \{x \in \mathbb{R}^E_+ \mid x(B) \geq 1 \text{ for any } B \in \mathcal{B}_{\mathcal{M}_P\mathcal{M}_Q}, && (8)\\ &\qquad x(A) \leq 1 \text{ for any } A \in \mathcal{A}_{\mathcal{M}_P\mathcal{M}_Q},\\ &\qquad x(C) \geq l(C) \text{ for any } C \in \mathcal{C}\}. \end{aligned}$$

Furthermore, we can solve an optimization problem over $\mathcal{P}_\mathcal{S}$ with a linear cost function in polynomial time.

Proof. Recall that an extreme point of $\mathcal{P}_{\mathcal{M}_P\mathcal{M}_Q}$ corresponds to an $\mathcal{M}_P\mathcal{M}_Q$-kernel by Theorem 3.4. We first consider the case of $\mathcal{S} = \emptyset$. In this case, it suffices to prove that $\mathcal{P}_\mathcal{S} = \emptyset$. By Lemma 4.8, there is some class of $\mathcal{C}$ for which no $\mathcal{M}_P\mathcal{M}_Q$-kernel obeys. This implies that $\mathcal{P}_\mathcal{S} = \emptyset$.

Assume now that $\mathcal{S} \neq \emptyset$. By Lemmas 4.5 and 4.8, $\mathcal{S} = \mathcal{K}_{\mathcal{M}_P\mathcal{M}_Q}$, so it suffices to prove that $\mathcal{P}_\mathcal{S} = \mathcal{P}_{\mathcal{M}_P\mathcal{M}_Q}$. Lemmas 4.5 and 4.8 imply that any $\mathcal{M}_P\mathcal{M}_Q$-kernel obeys l for any class of $\mathcal{C}$. This follows that $\mathcal{P}_\mathcal{S} = \mathcal{P}_{\mathcal{M}_P\mathcal{M}_Q}$, proving (7).

Validity of description (8) follows from (7) and Theorem 3.4. □

Next we prove that the set $\mathcal{S}$ of stable assignments of the **2LCSM** problem has a lattice structure similarly to the ordinary stable matching problem [2]. For subsets F_1, F_2 of E, we define

$$F_1 \vee F_2 := \mathcal{F}_{\mathcal{M}_P}(F_1 \cup F_2) \quad \text{and} \quad F_1 \wedge F_2 := \mathcal{F}_{\mathcal{M}_Q}(F_1 \cup F_2). \qquad (9)$$

Recall that if $\mathcal{S} \neq \emptyset$, then $\mathcal{S} = \mathcal{K}_{\mathcal{M}_P\mathcal{M}_Q}$. We call the stable assignment corresponding to the $\mathcal{M}_P$-optimal $\mathcal{M}_P\mathcal{M}_Q$-kernel the *$P$-optimal stable assignment*. The P-optimal stable assignment is a generalization of the "man-optimal" stable matching in the ordinary stable matching problem. Theorem 3.5 in our settings gives the following result.

Theorem 5.2. *If $\vee, \wedge$ are defined by* (9)*, then a triple* $(\mathcal{S}, \vee, \wedge)$ *is a lattice. Moreover, if $\mathcal{S} \neq \emptyset$, then we can find the P-optimal stable assignment in $O(|E|^3)$ time.*

Finally, we prove that the definition of stability in the **LCSM** problem originally introduced by Huang [9] and our definition of stability are equivalent. The **LCSM** problem is a special case of the **2LCSM** problem with $\mathcal{C}_p = \{E(p)\}$, $l(E(p)) = 0$ and $u(E(p)) = 1$ for any vertex p of P. Namely, a vertex of P is assigned to at most one vertex of Q. Let M be an assignment and $q \in Q$. A subset B of $E(q)$ is a *blocking group for q with respect to M* if it satisfies the following conditions. Let $|M(q)| = m$, $|B| = n$, $M(q) = \{e_1, \dots, e_m\}$ such that $e_1 <_q \cdots <_q e_m$ and $B = \{b_1, \dots, b_n\}$ such that $b_1 <_q \cdots <_q b_n$. Let p_i be the endpoint of b_i in P. If $M(p_i)$ is not empty, then we do not distinguish between $M(p_i)$ and its element. For a vertex v of V, $e \leq_v f$ means that $e <_v f$ or $e = f$. For convenience, we define $e <_v \emptyset$ for a vertex v of V and an edge e of $E(v)$. Then, B is a blocking group for q if it satisfies the following four conditions.

$$n \geq m \text{ and } B \text{ is feasible for } q, \tag{10}$$

$$b_i \leq_{p_i} M(p_i) \text{ for all } i \in [n], \tag{11}$$

$$b_i \leq_q e_i \text{ for all } i \in [m], \text{ and} \tag{12}$$

$$n > m, \text{ or there is } i \in [m] \text{ such that } b_i <_{p_i} M(p_i) \text{ and } b_i <_q e_i. \tag{13}$$

In the **LCSM** problem, M is stable if there is no blocking group for any vertex of Q.

Theorem 5.3. *In the* **LCSM** *problem, there is an edge blocking an assignment M of E if and only if there is a blocking group for a vertex of Q with respect to M.*

Proof. Obviously, if there is an edge blocking M, then there is a blocking group for some vertex of Q. So, we prove the other direction. Assume that there is a blocking group B for a vertex q of Q. Since B is feasible for q, we have $B \in \mathcal{I}_{E(q)}$ by (6). If $|B| > |M(q)|$, then by (2), $M(q) + b \in \mathcal{I}_{E(q)}$ for some edge b of $B \setminus M(q)$. So, by (11), b blocks M.

Assume that the size of any blocking group B for q is equal to $|M(q)|$. Let $|M(q)| = |B| = m$, $M(q) = \{e_1, \ldots, e_m\}$ such that $e_1 <_q \cdots <_q e_m$ and $B = \{b_1, \ldots, b_m\}$ such that $b_1 <_q \cdots <_q b_m$. By the definition of a blocking group, $M(q) \setminus B$ is not empty. Let j be the maximum index i of $[m]$ such that $e_i \in M(q) \setminus B$. Since $e_{j+1}, \ldots, e_m \in B$ and (12), we have $b_i = e_i$ for any $i \in \{j+1, \ldots, m\}$. By (1) and (2), $M(q) + b_k - e_j \in \mathcal{I}_{E(q)}$ for some edge b_k of $B \setminus M(q)$. Notice that $k \leq j$ and $b_k \leq_q b_j <_q e_j$. So, if $M(q) + b_k \in \mathcal{I}_{E(q)}$, then b_k blocks M. Suppose $M(q) + b_k \notin \mathcal{I}_{E(q)}$. Since $M(q)$ is feasible for q and $M(q) + b_k \notin \mathcal{I}_{E(q)}$, we can prove that $M + b_k - e_j$ obeys l for any class of $\mathcal{C}_q$ by the same way as Claim 4.6. So, $M + b_k - e_j$ is an assignment and b_k blocks M by $b_k <_q e_j$. This completes the proof. □

6 Conclusion

In the above work, we generalized previous results of Huang in [9]. We applied known matroid-generalizations of stable matching related theorems for a particular generalization of Huang's model. We think that an advantage of our approach is that in the matroid framework, instead of proving each step by a lengthy proof, we only have to deduce the result from more general ones. This way, our work illustrates two phenomena. On one hand, it shows the applicability of some earlier findings in fairly general (choice function based) models that may seem far from practical problems at the first glance. On the other hand, it points out that quite unusual matroids may bear practical significance: we feel that the "right" approach to Huang's very natural model is based on our weird matroids defined in Section 4.2. Though one can prove all our results without these matroids, it is hard to imagine an appealing argument along the "traditional" lines. However, there is at least one shortage of our approach of reducing the general model to the special case: it does not help so much to find those practical models where it is applicable, or, in more general, those models, where we have a chance to prove stability-related theorems. Exploration of such models does need certain insight that sometimes is ingenious. This insight often represents the most nontrivial part of the work leading to a positive result. We were lucky that we could avoid this part of the work. This is the reason that we are indebted to Huang without whom we probably would never have completed the above work.

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