On minimally $k$-rigid graphs

Viktória E. Kaszanitzky and Csaba Király

2012 November
On minimally $k$-rigid graphs

Viktória E. Kaszanitzky * and Csaba Király**

Abstract
A graph $G = (V, E)$ is called $k$-rigid in $\mathbb{R}^d$ if $|V| \geq k + 1$ and after deleting at most $k − 1$ arbitrary vertices the resulting graph is generically rigid in $\mathbb{R}^d$. A $k$-rigid graph $G$ is called minimally $k$-rigid if the omission of an arbitrary edge results in a graph that is not $k$-rigid. It was shown in [7] that the smallest possible number of edges is $2|V| − 1$ in a 2-rigid graph in $\mathbb{R}^2$. We generalize this result, provide an upper bound for the number of edges of minimally 2-rigid graphs (for any $d$) and give examples for minimally $k$-rigid graphs in higher dimensions.

1 Introduction
A graph $G = (V, E)$ is called $k$-rigid in $\mathbb{R}^d$ or shortly $[k, d]$-rigid if $|V| \geq k + 1$ and for any $U \subseteq V$ with $|U| \leq k − 1$ graph $G − U$ is generically rigid in $\mathbb{R}^d$. In this context we will call graphs that are rigid in $\mathbb{R}^d$ $[1, d]$-rigid. Every $[k, d]$-rigid graph is $[l, d]$-rigid by definition for $1 \leq l \leq k$. We remark that $G$ is $[k, d]$-rigid if and only if the deletion of $k − 1$ arbitrary vertices results in a graph that is generically rigid in $\mathbb{R}^d$.

$G$ is called minimally $[k, d]$-rigid if it is $[k, d]$-rigid but $G − e$ fails to be $[k, d]$-rigid for every $e \in E$. $G$ is said to be strongly minimally $[k, d]$-rigid if it is minimally $[k, d]$-rigid and there is no minimally $[k, d]$-rigid graph with $|V|$ vertices and less than $|E|$ edges. If $G$ is minimally $[k, d]$-rigid but not strongly minimally $[k, d]$-rigid then it is called weakly minimally $[k, d]$-rigid. Investigating the properties of $[k, d]$-rigid graphs is motivated by industrial applications, see [6, 8].

The following theorem gives a formula for the edge number of minimally rigid graphs.

Theorem 1.1 ([13]). Let $G = (V, E)$ be minimally rigid in $\mathbb{R}^d$. If $|V| \geq d + 1$ then $|E| = d|V| − \binom{d+1}{2}$.

A natural question to ask is whether there is a similar formula for the edge number of minimally $[k, d]$-rigid graphs for $k \geq 2$. The answer is no (see Section 6.1), there

---

*Eötvös University, Pázmány Péter sétány 1/C Budapest, Hungary H-1117. E-mail: viktoria@cs.elte.hu

**Eötvös University, Pázmány Péter sétány 1/C Budapest, Hungary H-1117. E-mail: csabi@cs.elte.hu
are minimally \([k,d]\)-rigid graphs for \(k \geq 2\) with different edge numbers, that is, the set of weakly minimally \([k,d]\)-rigid graphs is not empty if \(k \geq 2\).

To see a simple example consider the case \(d = 1\). It is well known that \(G\) is rigid in \(\mathbb{R}^1\) if and only if \(G\) is connected. Hence \(G\) is minimally \([k,1]\)-rigid if and only if it is minimally \(k\)-connected. Since there are \(k\)-connected graphs with the same number of vertices and different number of edges for \(k \geq 2\), weakly minimally \([k,1]\)-rigid graphs exist for every \(k \geq 2\).

It was shown in [7] that the smallest possible number of edges in a \([2,2]\)-rigid graph is \(2|V| - 1\). Later lower bounds were provided for the edge number of minimally \([k,d]\)-rigid graphs in [6, 8, 9] for some other values of \([k,d]\).

The main result of the present paper is a lower bound for the number of edges of \([k,d]\)-rigid graphs for every pair \([k,d]\) which is sharp for some values of \(k\) and \(d\).

We show that weakly minimally \([k,d]\)-rigid graphs exist for every pair \([k,d]\). We also provide an upper bound for the number of edges of minimally \([k,d]\)-rigid graphs for \(k = 2\).

### 1.1 Notation

In this paper we use the basic definitions and theorems of rigidity theory. All of the non-introduced definitions and non-proved statements can be found in the book of Graver et al. [3]. \(\mathcal{R}_d(G)\) denotes the \(d\)-dimensional generic rigidity matroid of \(G\).

We shall also use some standard notation from graph theory. \(\Delta(G)\) denotes the maximum degree in \(G\). \(K_n\) is the complete graph with \(n\) vertices. \(C_n\) denotes the cycle on \(n\) vertices. We will use the notation \(V(C_n) = \{v_1, \ldots, v_n\}\) and \(E(C_n) = \{v_iv_{i+1} : 1 \leq i \leq n\}\) where \(v_{n+1} := v_1\). \(C_n^d\) is the \(d\)th power of \(C_n\), or equivalently \(E(C_n^d) = \{v_iv_j : i - d \leq j \leq i + d\}\) where \(v_{n+i} := v_i\). \(P_n\) denotes the path on \(n\) vertices. We will use the notation \(V(P_n) = \{v_1, \ldots, v_n\}\) and \(E(P_n) = \{v_iv_{i+1} : 1 \leq i \leq n - 1\}\). \(P_n^d\) is the \(d\)th power of \(P_n\), or equivalently \(E(P_n^d) = \{v_iv_j : \min\{1, i - d\} \leq j \leq \max\{n, i + d\}\}\).

### 2 Operations preserving rigidity

Constructive characterizations are useful tools in combinatorial rigidity. Even though we do not have a constructive characterization theorem for the class of rigid graphs for \(d \geq 3\) it can be very useful to find operations that preserve rigidity. In this section we mention some of these operations.

The \(d\)-dimensional Henneberg-0 extension on \(G\) adds a new vertex and connects it to \(d\) distinct vertices of \(G\). The \(d\)-dimensional Henneberg-1 extension deletes an edge \(uv \in E\), adds a new vertex \(v\) and connects it to \(u, v\) and \(d - 1\) other vertices of \(G\). The \(d\)-dimensional Henneberg-0 extension is also called \(d\)-valent vertex addition.

**Theorem 2.1 ([10]).** If \(G\) is rigid in \(\mathbb{R}^d\) and \(G'\) is the graph that we get from \(G\) by a \(d\)-dimensional Henneberg-0 or Henneberg-1 extension then \(G'\) is rigid in \(\mathbb{R}^d\). □
Section 2. Operations preserving rigidity

As $d$-dimensional Henneberg extensions are used when we are in $\mathbb{R}^d$, we will simply call them Henneberg extensions if $d$ is clear from context. For $d = 2$ the following stronger statement holds:

**Theorem 2.2** ([10]). $G$ is minimally rigid in $\mathbb{R}^2$ if and only if it can be built up from the graph $K_2$ by a sequence of Henneberg-0 and Henneberg-1 extension.

If $G = (V, E)$ is minimally rigid in $\mathbb{R}^3$ then $|E| = 3|V| - 6$ by Theorem 1.1. Hence a minimally rigid graph in $\mathbb{R}^3$ does not necessarily have a vertex with degree 3 or 4. Thus for proving a 3-dimensional version of Theorem 2.1 one would need an operation that results in adding a vertex with degree 5. One such operation is the 3-dimensional X-replacement which deletes two non-adjacent edges $e = ab$ and $f = cd$ of $G$, chooses $w \in V$ different from $a, b, c, d$, adds a new vertex $v$ and connects it to $a, b, c, d, w$. It is not known whether the X-replacement preserves rigidity in $\mathbb{R}^3$.

**Conjecture 2.3** ([4]). Let $G$ be rigid in $\mathbb{R}^3$ and let $G'$ be the result of a 3-dimensional X-replacement applied to $G$. Then $G'$ is rigid in $\mathbb{R}^3$.

Conjecture 2.3 has been proved for some special cases of the 3-dimensional X-replacement (see [10] [12] for examples). We will use the special case when $a, b, w$ served in [9] that the 2-dimensional X-replacement preserves minimally $2$-rigidity in $\mathbb{R}^2$. (Note that points $v, a, b, w$ be the line of $\ell$.) Thus with notation $G_1 = G_0 + vw$.

Now we have to construct framework $(G', p)$ from $(G_0, p)$ by replacing edges $ab$ and $cd$ with $vw$ and $vd$, respectively. We shall also prove that $(G', p)$ is rigid. First add $vw$, let $G_1 = G_0 + vw$. There is a circuit in $(G_1, p)$ which is the $K_4$ induced by $v, a, b, w$. (Note that points $p(v), p(a), p(b), p(w)$ lie on a plane.) Thus with notation $G_1 − ab = G_2$ framework $(G_2, p)$ is independent. Using a similar argument it is not difficult to show that replacing $cd$ with $vd$ preserves independence.

It was shown in [7] that every strongly minimally $2$-rigid graph can be built up from a suitable base graph using Henneberg-1 extensions. The author also showed that 3-valent vertex addition preserves minimal $2$-$2$-rigidity under certain conditions.

There is a two-dimensional version of the X-replacement which is known to preserve rigidity in $\mathbb{R}^2$ [1]. The 2-dimensional X-replacement deletes two non-adjacent edges $e = ab$ and $f = cd$ of $G$, adds a new vertex $v$ and connects it to $a, b, c, d$. It was observed in [9] that the 2-dimensional X-replacement preserves minimally $2$-$2$-rigidity in specific cases. Summers, Yu and Anderson conjectured that the 3-valent vertex addition and the 2-dimensional X-replacement operations are sufficient to build up every weakly minimally $2$-$2$-rigid graph with at least nine vertices.
Conjecture 2.5 (§ 9). Let $G(V, E)$ be a minimally $[2,2]$-rigid graph with at least nine vertices. Then there exists either (a) a degree 4 vertex on which a reverse X-replacement operation can be performed to obtain a weakly minimally $[2,2]$-rigid graph or (b) there exists a degree three vertex on which a reverse 3-valent vertex addition can be performed to obtain a weakly minimally $[2,2]$-rigid graph.

We will disprove this conjecture by constructing weakly minimally $[2,2]$-rigid graphs on $n$ vertices that does not have such a vertex, where $n$ can be arbitrarily large.

3 On the number of edges in $[k,d]$-rigid graphs

First we present some results that apply to every dimension.

3.1 Lower bound for the number of edges

It was known that every $[2,2]$-rigid graph has at least $2|V| - 1$ edges, see [7]. In [6] Motevallian et al. gave a lower bound for the edge number of $[k,2]$-rigid graphs. We improve their results and extend it to every $d$. In Sections 4 and 5 we show that this lower bound is sharp for some values of $[k,d]$.

Theorem 3.1. If a graph $G = (V, E)$ is $[k,d]$-rigid with $|V| \geq d^2 + d + k$ then

$$|E| \geq d|V| - \left(\frac{d+1}{2}\right) + (k-1)d. \quad (1)$$

Proof. Observe that if a graph $H = (V', E')$ is $[1,d]$-rigid with $|V'| \geq d^2 + d$ then $\Delta(H) \geq 2d$. (To see this suppose that $\Delta(H) \leq 2d - 1$. Then $|E'| \leq |V'|d - \frac{|V'|}{2} < |V'|d - \left(\frac{d+1}{2}\right)$ which contradicts Theorem 1.1.) Let $v_1, v_2, \ldots, v_{k-1} \in V$ be such that $d_{G - \{v_1, \ldots, v_{\ell-1}\}}(v_{\ell}) = \Delta(G_{\ell})$ for every $1 \leq \ell \leq k - 1$ where $G_1 = G$ and $G_{\ell} = G - \{v_1, \ldots, v_{\ell-1}\}$. As $G_k$ is $[1,d]$-rigid,

$$|E(G_k)| \geq d(|V| - (k-1)) - \left(\frac{d+1}{2}\right) = d|V| - \left(\frac{d+1}{2}\right) - (k-1)d$$

by Theorem 1.1. Using this inequality, we have

$$|E| \geq d|V| - \left(\frac{d+1}{2}\right) - (k-1)d + (|E| - |E(G_k)|).$$

$G_{\ell}$ is $[1,d]$-rigid with $|V(G_{\ell})| = |V| - \ell + 1 \geq d^2 + d$ hence $\Delta(G_{\ell}) \geq 2d$ for every $1 \leq \ell \leq k$. This implies that $|E| - |E(G_k)| \geq (k-1)2d$. Thus $|E| \geq d|V| - \left(\frac{d+1}{2}\right) + (k-1)d$ as we claimed. \qed

3.2 Upper bound for $k = 2$

In this section we give an upper bound for the number of edges of minimally $[2,d]$-rigid graphs.
Theorem 3.2. Let $G = (V, E)$ be a minimally $[2, d]$-rigid graph. Then

$$|E| \leq 2d|V| - 3\left(\frac{d+1}{2}\right).$$

Proof. As $G$ is $[2, d]$-rigid, it is also $[1, d]$-rigid, thus it has a minimally $[1, d]$-rigid subgraph $H$ that has exactly $d|V| - \left(\frac{d+1}{2}\right)$ edges. Now, we count the edges in $E - E(H)$. For a vertex $v \in V$, let $E_v$ denote the set of edges in $E - E(H)$ for which $G - v - e$ is not $[2, d]$-rigid. By the minimality of $G$, $\bigcup_{v \in V} E_v = E - E(H)$. As $H$ is minimally rigid, the graph $H - v$ is independent in $\mathcal{R}_d(H - v)$ for any $v \in V$. By our assumption, $G - v$ is rigid for every $v \in V$ hence there is a set of edges $F_v \subseteq E(G - v)$ for which $(H - v) + F_v$ is minimally rigid. Since $|E(H - v)| = d|V| - \left(\frac{d+1}{2}\right) - d_H(v)$, we have $|F_v| = d_H(V) - d$. (Note that $d[H(v)] \geq d$ as $H$ is $[1, d]$-rigid.) The existence of $F_v$ ensures that $G - e - v$ is rigid for every $e \in (E - E(H)) - F_v$. Hence $E_v \subseteq F_v$ thus $|E_v| \leq d(H(v) - d)$. Therefore,

$$|E| = |E(H)| + \bigcup_{v \in V} E_v \leq |E(H)| + \sum_{v \in V} (d_H(v) - d) = 3|E(H)| - d|V| = 2d|V| - 3\left(\frac{d+1}{2}\right)$$

which completes the proof. \qed

The upper bound given in Theorem 3.2 is $4|V| - 9$ for $d = 2$. The number of edges of graph $W_{4,2}^2$ (to be defined in Section 6.1) is $3|V| - 7$ and this is the minimally $[2, 2]$-rigid graph with the highest number of edges that we know of, see [8, 9]. Hence it remains open if there are examples for minimally $[2, 2]$-rigid graphs with more edges or the bound given in Theorem 3.2 can be improved.

4 Strongly minimally $[2, d]$-rigid graphs

In this section we consider the case $k = 2$. We show that the lower bound given in Theorem 3.1 is sharp for $k = 2$ in any dimension and we disprove Conjecture 2.5.

Consider the graph $C_n^d$ and its subgraph $L_d$ induced by vertices $v_{n-d+1}, \ldots, v_n$. (Note that $L_d$ is isomorphic to $K_d$.) $H_{n,2}^d = C_n^d - E(L_d)$ denotes the graph we get from $C_n^d$ after deleting the edge set of $L_d$. First we prove that $H_{n,2}^d$ is $[2, d]$-rigid.

Lemma 4.1. $H_{n,2}^d$ is $[2, d]$-rigid if $n \geq 3d$.

Proof. Let $v_i \in V(H_{n,2}^d)$ be arbitrary. We will prove that $H_{n,2}^d - v_i$ is $[1, d]$-rigid by constructing it from a subgraph isomorphic to $K_d$ using $(d$-dimensional) Henneberg-0 and Henneberg-1-extensions.

First suppose that $v_i \notin V(L_d)$. For simplicity, we can assume that $\left\lfloor \frac{n-d+1}{2} \right\rfloor \leq i \leq n - d$. Since $n \geq 3d$ we have $i \geq d + 1$. Vertices $v_1, \ldots, v_d$ induce a subgraph isomorphic to $K_d$ hence we can add $v_{d+1}, \ldots, v_{i-1}$ in this order using Henneberg-0 extensions which connect $v_j$ to vertices $v_{j-d+1}, \ldots, v_{j-1}$ for every $d + 1 \leq j \leq i - 1$. Therefore $v_1, \ldots, v_{i-1}$ induce a $[1, d]$-rigid subgraph.
Now we will add vertices $v_{i+1}, \ldots, v_{i+d}$ in this order using Henneberg-0 extensions. If $j \leq n - d$ then the extension connects $v_j$ to vertices $v_{j-d}, \ldots, v_{i+1}, v_{i+d}$ and to $v_1$. Note that $v_j v_1$ is not an edge of $H^d_{n,2} - v_i$ if $j \leq n - d$. We will apply Henneberg-1 extensions on these extra edges. If $j > n - d$ then it will be connected to $v_{j-d}, \ldots, v_{i+1}, v_{i+d} - n$ and to $v_1, \ldots, v_{i+d+n-j}$ all of which are edges of $H^d_{n,2} - v_i$.

From now on we will use Henneberg-1 extensions only for adding vertices $v_{i+d+1}, \ldots, v_n$ in this order. When adding $v_j$ for $j \leq n - d$ we apply the Henneberg-1 extension on edge $v_{j-d}v_1$ that connects $v_j$ to $v_{j-d+1}, \ldots, v_{j-1}$. In this case we remove the extra edge $v_{j-d}v_1$ and add a new one $v_jv_1$. If $j > n - d$ then similarly we apply the Henneberg-1 extension on edge $v_{j-d}v_1$ but we connect $v_j$ to $v_{j-d}, \ldots, v_{n-d}$ and to $v_2, \ldots, v_{d-n+j}$ and all of these edges are present in $H^d_{n,2} - v_i$. In this case the number of extra edges decreased by one.

Figure 1: Building up $C^3_{13} - E(L_3) - v_5$ using Henneberg operations.
If \( v \in V(L_d) \), then it is easy to see that \( H^d_{n,2} \) has a subgraph that can be built up using Henneberg-0-extensions only (we first build up the subgraph induced by vertices of \( H^d_{n,2} \) and then we add the nodes in \( V(L_d) - v \)).

If \( G = (V, E) \) is \([2, d]-rigid\) then \(|E| \geq d|V| - \binom{d+1}{2} + d = d|V| - \binom{d}{2} \) if \(|V| \geq d^2 + d + 2\) by Theorem 3.1. \(|E(H^d_{n,2})| = dn - \binom{d}{2} \) since \( C^d_n \) has \( dn \) edges if \( n \geq 2d + 1 \) and the deleted edges form a complete subgraph with \( d \) vertices. Hence by Lemma 4.1 we get the mail result of this section:

**Theorem 4.2.** If \( G = (V, E) \) is a strongly minimally \([2, d]-rigid\) graph with \(|V| \geq d^2 + d + 2\) then \(|E| = d|V| - \binom{d}{2} \).

## 5 Strongly minimally \([3, 3]\)-rigid graphs

In this section we show that the lower bound given in Theorem 3.1 is sharp when \( k = d = 3 \).

**Lemma 5.1.** \( C^3_n \) is \([3, 3]\)-rigid if \( n \geq 9 \).

**Proof.** Let \( v_i, v_j \in V(C^3_n) \) be arbitrary. We will prove that \( C^3_n - \{v_i, v_j\} \) is \([1, 3]\)-rigid by constructing it from a subgraph isomorphic to \( K_4 \) using 3-dimensional Henneberg-0 and Henneberg-1-extensions and \( \Delta-X \)-replacements.

We can assume that \( j = n \) and \( i \geq \left\lceil \frac{n+1}{2} \right\rceil \). \( n \geq 9 \) hence \( i \geq 5 \) and as in proof of Lemma 4.1 it can be seen easily that the subgraph induced by \( v_1, \ldots, v_{i-1} \) is rigid.

Let \( \ell = n - i - 1 \). We have to perform \( \ell \) more extension to add the remaining vertices. We split the proof into two cases depending on \( \ell \).

If \( 1 \leq \ell \leq 3 \), we add \( v_{i+1} \) and connect it to \( v_1, v_{i-2}, v_{i-1} \). If \( \ell \geq 2 \) then we add \( v_{i+2} \) and connect it to \( v_1, v_{i-1}, v_{i+1} \). If \( \ell = 3 \) then we can add \( v_{i+3} \) performing a Henneberg-1 extension on edge \( v_{i+1}v_1 \) and connecting \( v_{i+3} \) to \( v_{i+2} \) and \( v_2 \).

If \( \ell \geq 4 \) then we will need a \( \Delta-X \)-replacement on edges \( v_2v_{n-3}, v_1v_{n-4} \). In this case we will add vertices \( v_{i+1}, v_{i+2}, v_{i+3} \) by Henneberg-0 extensions, \( v_{i+4}, \ldots, v_{n-2} \) by Henneberg-1 extensions. We will perform these operations such that after adding \( v_{n-2} \) edges \( v_2v_{n-3}, v_1v_{n-2}, v_1v_{n-4}, v_{n-2}v_{n-4} \) will be present in the resulting graph.

Let \( \sigma : \mathbb{Z} \to \{1, 2\} \) be a function with \( \sigma(t) := 2 \) if \( t \equiv \ell - 2 \) (mod 3) and \( \sigma(t) := 1 \) otherwise. We add \( v_{i+1} \) with Henneberg-0-extension that connects it to \( v_{i-2}, v_{i-1}, v_{\sigma(1)} \). Then add \( v_{i+2} \) with a Henneberg-0-extension that connects it to \( v_{i-1}, v_{i+1}, v_{\sigma(2)} \). Next, we add \( v_{i+3} \) with a Henneberg-0-extension that connects it to \( v_{i-1}, v_{i-2}, v_{\sigma(3)} \). Then we add \( v_{i+m} \) for \( 4 \leq m \leq \ell - 1 \) in sequence with Henneberg-1 extension on \( v_{i+m-3}v_{\sigma(m-3)} \) that connects it to \( v_{i+m-2}, v_{i+m-1} \). Finally, we add \( v_{n-1} \) with a \( \Delta-X \)-replacement on edges \( v_2v_{n-3}, v_1v_{n-4} \) as \( v_{n-2}v_1v_{n-1} \) is a triangle.

We have proved that \( C^3_n \) is \([3, 3]\)-rigid and clearly \( C^3_n \) has \( 3n \) edges if \( n \geq 7 \). This together with Theorem 3.1 gives the following:

**Theorem 5.2.** If \( G = (V, E) \) is a strongly minimally \([3, 3]\)-rigid graph with \(|V| \geq 9 \) then \(|E| = 3|V| \).
6 Higher dimensions revisited

Recall that \( L_d \) denotes the complete subgraph of \( C_n^d \) spanned by vertices \( v_{n-d+1}, \ldots, v_n \). Let \( L'_d \) denote the graph that we get from \( L_d \) by deleting the Hamiltonian cycle that consist of edges \( v_i v_{i+1} \) for \( n - d + 1 \leq i \leq n - 1 \) and \( v_{n-d+1} v_n \). Note that \( L'_3 \) is the empty graph on three vertices. Lemma 5.1 states that \( C_n^d - L'_d \) is strongly minimally \([3, 3]\)-rigid.

\[
|E(C_n^d - L'_d)| = dn - \binom{d}{2} + d = dn - \left(\frac{d+1}{2}\right) + 2d
\]

which motivates the second part of the following conjecture:

**Conjecture 6.1.** The lower bound given in Theorem 3.1 is sharp for \( k = 3 \) for any \( d \geq 3 \). Moreover, \( C_n^d - L'_d \) is a strongly minimally \([3, d]\)-rigid if \( n \) is sufficiently large.
It remains open if the lower bound given in Theorem 3.1 is tight for some pairs \([k, d]\) different from \([2, d]\) and \([3, 3]\). This question seems to be more complicated for larger values of \(k\) and \(d\) as there are just a few operations known that preserve rigidity in higher dimensions. Furthermore it was shown in [3] that the lower bound given in Theorem 3.1 is not tight for \(k = 3\) and \(d = 2\), a strongly minimally \([3, 2]\)-rigid graph on at least 6 vertices has \(2|V| + 2\) edges. Following their idea, the lower bound given in Theorem 3.1 can also be improved if the right-hand side of (1) is larger than \(d|V|\) because in this case \(\Delta(G) \geq 2d + 1\) holds.

6.1 Examples for minimally \([k, d]\)-rigid graphs

The question whether weakly minimally \([k, d]\)-rigid graphs exist for every pair \((k, d)\) can still be solved without knowing the edge count of strongly minimally \([k, d]\)-rigid graphs. There are examples for weakly minimally \([2, 2]\)-rigid graphs in [7, 8, 9] but the existence of weakly minimally \([k, d]\)-rigid graphs for other values of \(k\) and \(d\) was open so far. In this section we will give examples for minimally \([k, d]\)-rigid graphs with the same number of vertices but with different number of edges. Such a pair of graphs shows that the graph with the larger number of edges has to be weakly minimally \([k, d]\)-rigid.

First we generalize an example from [8, 9]. In the following lemma, \(P^0_n\) denotes the empty graph on \(n\) vertices.

**Lemma 6.2.** Let \(t, k\) and \(d\) be three positive integers with \(t \geq kd + 1\). Then there exists a minimally \([k, d]\)-rigid graph with \(t + k\) vertices and \((d + k - 1)t - \left(\frac{d}{2}\right)\) edges.

**Proof.** Let the graph \(W_{t,k}^d\) consist of \(P_t^d\) (with vertex set \(\{v_1, \ldots, v_t\}\)) and \(k\) additional vertices \(s_1, \ldots, s_k\) each of which is connected to all vertices of \(P_t^d\) (see Figure 3). We first prove that \(W_{t,k}^d\) is \([k, d]\)-rigid.

For \(k = 1\), we need to show that \(W_{t,1}^d\) is minimally \([1, d]\)-rigid. As \(t \geq d + 1\), \(s_1\) and \(v_1, \ldots, v_d\) form a complete graph with \(d + 1\) vertices. Starting with this subgraph, \(W_{t,1}^d\) can be built up by adding vertices \(v_{d+1}, \ldots, v_t\) with Henneberg-0 extensions. This proves case \(k = 1\).

Assume that \(k \geq 2\). First, we show that \(W_{t,k}^d - \{u_1, \ldots, u_{k-1}\}\) is \([1, d]\)-rigid if \(\{u_1, \ldots, u_{k-1}\} \subseteq \{v_1, \ldots, v_t\}\). As \(t \geq kd + 1\) there should be some integer \(1 \leq j \leq n - d + 1\) such that \(\{v_j, \ldots, v_{j+d-1}\} \cap \{u_1, \ldots, u_{k-1}\} = \emptyset\). Starting with the complete subgraph spanned by \(\{s_1, v_j, \ldots, v_{j+d-1}\}\) we can build up a subgraph of \(W_{t,k}^d - \{u_1, \ldots, u_{k-1}\}\) up by Henneberg-0 extensions. First we add \(s_2, \ldots, s_k\) one after one and then the vertices that are not deleted from \(\{v_{j+d}, \ldots, v_t, v_{j-1}, \ldots, v_1\}\) in this order.

Next observe that \(W_{t,k}^d - s_i\) is isomorphic to \(W_{t,k-1}^d\) for any \(i \in \{1, \ldots, k\}\). Thus by induction, \(W_{t,k}^d - \{s_i, u_1, \ldots, u_{k-2}\}\) is \([1, d]\)-rigid for every \(i \in \{1, \ldots, k\}\) and \(u_1, \ldots, u_{k-2}\) \(\subseteq \{v_1, \ldots, v_t\}\). So far we proved that \(W_{t,k}^d\) is \([k, d]\)-rigid.

Moreover, as subgraphs, \(\{W_{t,k}^d - s_i : i \in \{1, \ldots, k\}\}\) cover all edges of \(W_{t,k}^d\) and by induction these subgraphs are all minimally \([k - 1, d]\)-rigid graphs, \(W_{t,k}^d\) is minimally \([k, d]\)-rigid.
6.1 Examples for minimally \([k, d]\)-rigid graphs

Figure 3: \(W^3_{6,2}\).

Clearly \(|V(W^d_{t,k})| = t + k\), \(|E(W^d_{t,k})| = |E(P^d_{t-1})| + kt = (d + k - 1)t - \binom{d}{2}\) if \(t \geq kd + 1\) since in this case \(|E(P^d_{t-1})| = (d - 1)t - \binom{d}{2}\). This completes the proof.

The cone graph of \(G\) is the graph that arises from \(G\) by adding a new vertex \(s\) and edges \(sv\) for every \(v \in V\). The operation that creates the cone graph of \(G\) is called coning. The following claim states that one can construct \([k, d]\)-rigid graphs by coning \([k - 1, d]\)-rigid graphs. However these examples will not necessarily be minimal but by omitting some of their edges one can achieve minimality.

**Claim 6.3.** Let \(k \geq 2\) and \(d \geq 1\) integers. Let \(G = (V, E)\) be a \([k - 1, d]\)-rigid graph and let \(H = (V + s, E')\) be the cone graph of \(G\). Then \(H\) is \([k, d]\)-rigid.

**Proof.** We need to show that after omitting \(k - 1\) vertices \(H\) remains \([1, d]\)-rigid. If \(s\) is omitted, then we are done by the \([k - 1, d]\)-rigidity of \(G\). Otherwise, let \(u_1, \ldots, u_{k-1}\) be the omitted vertices. \(G - \{u_1, \ldots, u_{k-1}\}\) is \([1, d]\)-rigid and \(s\) is connected to every neighbor of \(v_{k-1}\). Hence \(H - \{u_1, \ldots, u_{k-1}\}\) has a subgraph isomorphic to the \([1, d]\)-rigid graph \(G - \{u_1, \ldots, u_{k-2}\}\) showing that it is \([1, d]\)-rigid. \(\square\)

Let \(H^d_{n,i}\) denote the cone graph of \(H^d_{n,(i-1)}\) for \(i \geq 3\). (For the definition of \(H^d_{n,2}\) see Section 4) By Claim \(6.3\) and Lemma \(4.1\) we get the following:

**Corollary 6.4.** Let \(t, d, k\) be three positive integers such that \(t \geq 3d\) and \(k \geq 2\). Then there exists a minimally \([k, d]\)-rigid graph \(H^d_{t,k}\) with \(t + k - 2\) vertices and at most \((d + k - 2)t - \binom{d}{2} + \binom{k-2}{2}\) edges.

We shall also use Claim \(6.3\) in the proof of the following lemma.
6.1 Examples for minimally \([k,d]\)-rigid graphs

**Lemma 6.5.** Let \( t \geq 2 \), \( k \geq 1 \) and \( d \geq 3 \) be three integers. There exists a minimally \([k,d]\)-rigid graph with \( t + k + d - 2 \) vertices and \( (d + k - 1)t + \left(\frac{k+d-2}{2}\right) - 1 \) edges.

**Proof.** Define graph \( M_{t}^{k+d-2} \) as follows. Take the disjoint union of a path \( P_{t} \) (on vertex set \( \{v_{1}, \ldots, v_{t}\} \)) and a complete graph \( K_{k+d-2} \) (on vertex set \( \{w_{1}, \ldots, w_{k+d-2}\} \)) and add edges \( v_{i}w_{j} \) for every pair \( 1 \leq i \leq t, 1 \leq j \leq k + d - 2 \) (see Figure 4).

![Figure 4: \( M_{6}^{3} \).](image)

First we show that \( M_{t}^{k+d-2} \) is minimally \([k,d]\)-rigid. If \( k = 1 \) then \( v_{1}, v_{2}, w_{1}, \ldots, w_{k+d-2} \) form a complete subgraph with \( d + 1 \) vertices. Starting with this subgraph, \( M_{t}^{d-1} \) can be built up by adding \( v_{3}, \ldots, v_{t} \) with Henneberg-0 extensions.

For \( k \geq 2 \) graph \( M_{t}^{k+d-2} \) is \([k,d]\)-rigid by induction and Claim 6.3. Moreover, \( M_{t}^{k+d-2} - w_{j} \) is isomorphic to \( M_{t}^{k+1+d-2} \) for any \( 1 \leq j \leq k + d - 2 \) that is minimally \([k-1,d]\)-rigid by induction. As \( d \geq 3 \) these subgraphs cover \( M_{t}^{k+d-2} \) showing the minimality.

Clearly, \(|V(M_{t}^{k+d-2})| = t + k + d - 2\) and \(|E(M_{t}^{k+d-2})| = (t - 1) + \left(\frac{k+d-2}{2}\right) + (k + d - 2)t = (d + k - 1)t + \left(\frac{k+d-2}{2}\right) - 1\). \( \square \)

Let \( k, d, n \) be integers such that \( k \geq 2 \), \( d \geq 3 \) and \( n \geq k(d + 1) + 1 \). Put \( t_{1} = n - k \) and \( t_{2} = n - k - d + 2 \). With this notation \( n = |V(W_{t_{1},k}^{d})| = |V(M_{t_{2}}^{k+d-2})| \). We will prove that \(|E(W_{t_{1},k}^{d})| < |E(M_{t_{2}}^{k+d-2})|\) which shows that \( M_{t_{2}}^{k+d-2} \) is weakly minimally \([k,d]\)-rigid. By Lemmas 6.2 and 6.5, we have to prove that

\[
(d + k - 1)(n - k) - \binom{d}{2} < (d + k - 1)(n - k - d + 2) + \binom{k + d - 2}{2} - 1.
\]

By subtracting \((d + k - 1)(n - k) - \binom{d}{2}\) from each side, we get

\[
0 < \frac{dd-1}{2} + (d + k - 1)(-d + 2) + \frac{(k + d - 2)(k + d - 3)}{2} - 1,
\]

that is,

\[
0 < \frac{k^2 - k}{2}
\]

that holds for \( k \geq 2 \).

---

EGRES Technical Report No. 2012-21
Now, let $k, d, n$ be positive integers such that $k \geq 2$ and $n \geq \max\{k(d+1)+1, 3d+k-2, 3k+2d-k-4\}$. Put $t_0 = n-k+2$. With this notation $n = |H_{t_0,k}^d| = |V(W_{t_0,k}^d)|$.

We will prove that $|E(H_{t_0,k}^d)| < |E(W_{t_0,k}^d)|$ which shows that $W_{t_0,k}^d$ is weakly minimally $[k,d]$-rigid. By Lemma 6.2 and Corollary 6.4 it is enough to prove that

$$(d+k-2)(n-k+2) - \binom{d}{2} + \binom{k-2}{2} < (d+k-1)(n-k) - \binom{d}{2}$$

By subtracting $(d+k-2)(n-k+2) - \binom{d}{2} + \binom{k-2}{2}$ from each side, we get

$$0 < n - 2d - 3k + 4 - \binom{k-2}{2}$$

that holds because of the choice of $n$.

We have proved the following theorem:

**Theorem 6.6.** Let $d$ and $k$ be positive integers with $k \geq 2$. Then there are weakly minimally $[k,d]$-rigid graphs, that is, there are minimally $[k,d]$-rigid graphs that are not strongly minimally $[k,d]$-rigid.

7 A counterexample for Conjecture 2.5

In this section we disprove Conjecture 2.5 by constructing minimally $[2,2]$-rigid graphs that do not have a vertex at which the reverse degree 3 vertex addition or the reverse X-replacement can be performed. To give such an example we will need the following simple observation.

**Claim 7.1.** Let $G = (V,E)$ be a graph. Suppose $v \in V$ with $d(v) = 4$ is contained in a $K_4$ subgraph of $G$. Then every possible reverse X-replacement at $v$ creates a parallel pair of edges.

We define an operation called $K_4$-extension that preserves $[2,2]$-rigidity although the resulting graph may not be minimally $[2,2]$-rigid. Let $G = (V,E)$ be a graph with $|V| \geq 4$, and let $v_1,v_2,v_3,v_4 \in V$ be four distinct vertices. The $K_4$-extension adds four new vertices $u_1,u_2,u_3,u_4$ to $G$, connects $v_i$ to $u_i$ for every $1 \leq i \leq 4$ and $u_k$ to $u_l$ for every pair $1 \leq k,l \leq 4$.

**Claim 7.2.** If $G = (V,E)$ is $[2,2]$-rigid then $G' = (V',E')$ obtained by a $K_4$-extension is also $[2,2]$-rigid. Furthermore $G' - e$ is not $[2,2]$-rigid for any $e \in E' - E$.

**Proof.** Clearly, $G' - v$ is rigid for any $v \in V'$.

Consider the graph $G' - e$ for some $e \in E' - E$. Let $u_i \in V' - V$ be such that $e$ is not incident to $u_i$. We claim that $G'' = G' - u_i - e$ is not rigid. $G''$ consist of $G$ and a set of three vertices that is incident to five edges only. Hence there are only $2|V| - 3 + 5 = 2|V'| - 4$ independent edges in $G''$ thus $G''$ is not rigid as we claimed. □
Now let $G_0 = (V_0, E_0)$ be a $[2, 2]$-rigid graph with $V_0 \geq 4$. Apply some $K_4$-extensions to vertices of $V_0$, let the resulting graph be $G_1 = (V_1, E_1)$ (see Figure 5). Suppose that every vertex in $V_0$ is incident to at least five edges from $E_1 - E_0$. After the extensions delete edges from $E_1$ (if necessary) to obtain a minimally $[2, 2]$-rigid graph $G_2 = (V_1, E_2)$. By Claim 7.1 deleting any edge from $E_1 - E_0$ results in a graph that is not $[2, 2]$-rigid hence the minimum degree in $G_2$ is four and all the degree four vertices are in $V_1 - V_0$. Clearly we cannot perform the reverse degree 3 vertex addition in $G_2$. But every vertex of $V_1 - V_0$ is contained in a $K_4$ subgraph of $G_2$ and by Claim 7.1 every reverse X-replacement on one of these vertices creates a parallel pair of edges. Thus no reverse X-replacement operation preserves minimal $[2, 2]$-rigidity of $G_2$ which disproves Conjecture 2.5.

Figure 5: A counterexample $G_c$ for Conjecture 2.5 that we get by performing five $K_4$-extensions on the subgraph induced by vertices $a, b, c, d$. Clearly, $K_4$ is minimally $[2, 2]$-rigid hence $G_c$ is $[2, 2]$-rigid by Claim 7.2. It can be easily seen that deleting any of the edges $bc, cd, db$ from graph $G_c - a$ results in a flexible graph. By symmetry the deletion of any edge of the starting graph results in a graph that is not $[2, 2]$-rigid. This implies that $G_c$ is minimally $[2, 2]$-rigid.

**Remark 7.3.** We also remark that for any positive integer $t$ graph $G_1$ can be constructed such that every vertex in $V_0$ is incident to at least $t$ edges from $E_1 - E_0$. Hence $G_2$ has vertices of degree four and the rest of its vertices has degree at least $t$. Since $t$ can be arbitrarily large this example shows that it may be difficult to find a constructive characterization that only uses operations that add low-degree vertices.
8 Concluding remarks

The results presented in this paper are about the edge numbers of minimally \([k,d]\)-rigid graphs. Similar questions were asked about minimally globally \([k,d]\)-rigid graphs in \cite{8} where \(G = (V,E)\) is globally \([k,d]\)-rigid if \(|V| \geq k + 1\) and after deleting at most \(k - 1\) arbitrary vertices the resulting graph is globally rigid in \(\mathbb{R}^d\).

Other version of the problem is \([k,d]\)-edge rigidity (and global \([k,d]\)-edge rigidity) where instead of at most \(k - 1\) vertices we delete at most \(k - 1\) edges of the graph. Proving similar results on these variants of the problem considered is a possible direction of future research.

A different direction is to characterize inductively the class of graphs mentioned above for some values of \([k,d]\) which seems to be an interesting and difficult open question.

Acknowledgments

The authors received a grant (no. CK 80124) from the National Development Agency of Hungary, based on a source from the Research and Technology Innovation Fund. Research was supported by the MTA-ELTE Egerváry Research Group.

The authors thank Zsuzsanna Jankó and János Geleji for the inspiring discussions and Tibor Jordán for posing the interesting questions solved in this paper.

References


