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On maximal independent arborescence-packing

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Abstract

In this paper, we generalize the results of Kamiyama, Katoh and Takizawa [7] to solve the following problem. Given a digraph $D = (V, A)$ and a matroid on an abstract set $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_k\}$ along with a map $\pi : \mathsf{S} \to V$; give k edge-disjoint arborescences $T_1, \ldots, T_k$ with roots $\pi(\mathsf{s}_1), \ldots, \pi(\mathsf{s}_k)$ such that for any $v \in V$ the set $\{\mathsf{s}_i : v \in T_i\}$ is independent and its rank reaches the theoretical maximum. We also give a simplified proof for the result of Fujishige [5] from the result of Kamiyama et al.

1 Introduction

The recent researches in rigidity theory showed that some extensions of the well known results of Tutte [13] and Nash-Williams [11, 12] on packing trees and covering with trees can be applied for some rigidity classes. The latest result is from Katoh and Tanigawa [8] who proved that minimally rigidity of 'bar-slider frameworks' are equivalent to some colored rooted-forest packing properties. This result inspired an extensive research on the possible extensions of Tutte's and Nash-Williams' results. Katoh and Tanigawa [9] proved a theorem on the existence of colored rooted-forest packings. In [10], they generalized this result with some constrains coming from matroids along with showing a wide overview of possible applications in rigidity theory.

Frank [4] showed how to derive Nash-Williams' [12] result from the weak form of Edmonds' theorem [2] on arborescence packings. Following this idea Gevigney, Nguyen and Szigeti [6] generalized Edmonds' weak theorem to give an alternative proof of the packing part of [10]. One can find that [6] also generalizes the strong form of the result of Edmonds. This implies the question whether the earlier extensions of [2] such as the one of Kamiyama, Katoh and Takizawa [7] and of Fujishige [5] can be generalized to such a form. We answer the question positively by extending [7] and showing that [5] is an easy consequence of [7]. (For a survey on tree and arborescence packing see [1] and [3, Chapter 10].)

We conclude the introduction by introducing some definitions used throughout the paper. In a digraph $D = (V, A)$, $\varrho_D(X)$ and $\delta_D(X)$ denotes the in-degree and the out-degree of a set $X \subseteq V$, respectively. For a non-empty set $R \subseteq V$, $B = (V, A')$ is said

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to be an R-**branching** if it consists of $|R|$ node-disjoint arborescences whose roots are in R. Let $D = (V, A)$ be a digraph. Then an R-branching is said to be **spanning** if it spans the node set V and it is said to be **maximal** if it spans all the nodes that are reachable from R in D. For non-empty sets $X, Z \subseteq V$, let $Z \mapsto X$ denote that X and Z are *disjoint* and X is reachable from Z, that is, there is a directed path from Z to X. For simplicity, we will denote the set $\{v\}$ by v.

Throughout the paper, $D = (V, A)$ is a digraph, $\mathcal{M}$ is a matroid on S with rank function $r_{\mathcal{M}}$ and $\pi : \mathsf{S} \to V$ is a (not necessarily injective) map. The independent sets of $\mathcal{M}$ are the sets $\mathsf{P} \subseteq \mathsf{S}$ with $|\mathsf{P}| = r_{\mathcal{M}}(\mathsf{P})$. For $\mathsf{P} \subseteq \mathsf{S}$, we will denote by $\mathrm{Span}_{\mathcal{M}}(\mathsf{P})$ the subset of S spanned by P, that is, the maximal set $\mathsf{X} \subseteq \mathsf{S}$ for which $\mathsf{P} \subseteq \mathsf{X}$ and $r_{\mathcal{M}}(\mathsf{X}) = \mathsf{r}_{\mathcal{M}}(\mathsf{P})$. For related definitions and properties of matroids we refer to [3]. We say that (D, S, π) is a **digraph with roots**. As in [6], π is called $\mathcal{M}$**-independent** if $\pi^{-1}(v)$ is independent in $\mathcal{M}$ for each $v \in V$. For $X \subseteq V$, we will denote by S_X the set $\pi^{-1}(X)$.

2 Preliminaries

The weak form of the result of Edmonds [2] asserts the following:

Theorem 2.1 (Edmonds' weak arborescence theorem)**.** *In a digraph $D = (V, A)$, there are k edge-disjoint spanning arborescences with root r_0 if and only if*

$$\varrho_D(X) \geq k \tag{1}$$

holds for every $\emptyset \neq X \subset V - r_0$.

The strong form of Edmonds' theorem considers the case when we are given k edge-disjoint subarborescences of D with root r_0 and we want to extend these arborescences to edge-disjoint spanning arborescences. We formulate this theorem in another but equivalent form using the notion of branchings. For a family $\mathcal{R} := \{R_1, ..., R_k\}$ of non-empty subsets of V and $X \subseteq V$, let us denote by $p_{\mathcal{R}}(X)$ the number of the members of $\mathcal{R}$ disjoint from X and let us denote by $p'_{\mathcal{R}}(X)$ the number of R_i's for which $R_i \mapsto X$.

Theorem 2.2 (Edmonds [2])**.** *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, ..., R_k\}$ be a family of non-empty subsets of V. Then there are edge-disjoint spanning R_i-branchings for $i = 1, ...k$ if and only if*

$$\varrho_D(X) \geq p_{\mathcal{R}}(X) \tag{2}$$

holds for every $X \subseteq V$. □

Kamijama, Katoh and Takizawa [7] extended the result of Edmonds. We formulate their theorem in the following form (as [3]), that seems to be a bit stronger for the first sight. However, it is easy to see that it is equivalent to the original form where each R_i consists of a single node.

Theorem 2.3 (Kamijama, Katoh, Takizawa [7]). *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, ..., R_k\}$ be a family of non-empty subsets of V. Then there are k edge-disjoint maximal R_i-branchings in D if and only if*

$$\varrho_D(X) \geq p'_{\mathcal{R}}(X) \tag{3}$$

holds for every $X \subseteq V$. □

Fujishige [5] extended this theorem. We present here a proof for Fujishige's theorem which shows that in fact it follows easily from Theorem 2.3. A set of nodes U is called convex if there is no node $v \in V - U$ for which $v \mapsto U$ and $U \mapsto v$.

Theorem 2.4 (Fujishige [5]). *In a digraph $D = (V, A)$, let $R := \{r_1, ..., r_k\} \subseteq V$ be a list of (possibly not distinct) nodes and let $U_i \subseteq V$ be convex sets with $r_i \in U$. Then there are edge-disjoint arborescences with root r_i spanning U_i in D for $i = 1, ..., k$ if and only if*

$$\varrho_D(X) \geq p_R^{\{U_1,...,U_k\}}(X) \tag{4}$$

holds for every $X \subseteq V$, where $p_R^{\{U_1,...,U_k\}}(X)$ denotes the number of U_i's for which $U_i \cap X \neq \emptyset$ and $r_i \notin X$.

Proof. As the the proof of the necessity of (4) is straightforward we only prove its sufficiency.

Let Z_i be the set of nodes reachable from r_i and let $R_i := Z_i - (U_i - r_i)$ for $i = 1, ..., k$. Then we claim that a maximal R_i-branching consists of the single nodes as roots in $R_i - \{r_i\}$ and an arborescence with root r_i spanning U_i for $i = 1, ..., k$. This statement follows from the following two observations. First, for $i = 1, ..., k$, $\delta_D(Z_i) = 0$ by definition thus $\delta_D(R_i - r_i) = \delta_D(Z_i - U_i) = 0$ by convexity of U_i as $U_i \mapsto v$ for $v \in R_i - r_i$. Second, for $i = 1, ..., k$, it is easy to see that there is an arborescence with root r_i spanning U_i by (4). Thus the existence of edge-disjoint R_i-branchings is equivalent to the existence of edge-disjoint arborescences with root r_i spanning U_i.

It is easy to see that $p_R^{\{U_1,...,U_k\}}(X) \geq p'_{\{R_1,...,R_k\}}(X)$ for $X \subseteq V$. Thus for $X \subseteq V$, if (4) holds, then (3) also holds. Therefore, the theorem follows by Theorem 2.3. □

Next we present the recent result of Gevigney, Nguyen and Szigeti [6] that generalizes Edmonds' results [2] in another direction. Following [6] we use the following definitions. The digraph with roots (D, S, π) is called **$\mathcal{M}$-connected**, if

$$\varrho_D(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{5}$$

holds for each $X \subseteq V$. An **$\mathcal{M}$-basic packing of arborescences** in (D, S, π) is a set $\{T_1, ..., T_{|\mathsf{S}|}\}$ of pairwise edge-disjoint arborescences in D such that T_i has root $\pi(\mathsf{s}_i)$ for $i = 1, ..., |\mathsf{S}|$ and the set $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}$ forms a base of $\mathcal{M}$ for each $v \in V$. The result of [6] is the following:

Theorem 2.5 (Gevigney, Nguyen, Szigeti [6]). *Let (D, S, π) be a digraph with roots and $\mathcal{M}$ be a matroid on S. There exists an $\mathcal{M}$-basic packing of arborescences in (D, S, π) if and only if π is $\mathcal{M}$-independent and (D, S, π) is $\mathcal{M}$-connected.* □

Let $\mathcal{R} = \{R_1, ..., R_k\}$ be a multiset such as in Theorem 2.2. If $\mathsf{S} := \bigcup \mathcal{R}$ (as a multiset), π maps each occurrence of r in S to the node $r \in V$, and $\mathcal{M}$ is the partition matroid on S given by $\mathcal{R}$ where a set $\mathsf{P} \subseteq \mathsf{S}$ is independent if and only if $|\mathsf{P} \cap R_i| \leq 1$ for $i = 1, ..., k$, then the problem of $\mathcal{M}$-basic packing of arborescences and that of edge-disjoint spanning R_i-branchings for $i = 1, ...k$ coincide. Moreover, in this case π is always $\mathcal{M}$-independent and (5) is equivalent to (2). Therefore, Theorem 2.2 follows from Theorem 2.5. However, Theorem 2.3 cannot be deduced from this theorem. In the next section we will extend Theorem 2.5 to a theorem from which Theorem 2.3 follows.

In our proof, we will use the following technical claim proved in [6]:

Claim 2.6 ([6])**.** *Let $\mathcal{M}$ be a matroid on S with rank function $r_\mathcal{M}$ and $\mathsf{P}, \mathsf{Q} \subseteq \mathsf{S}$ such that $r_\mathcal{M}(\mathsf{P}) + r_\mathcal{M}(\mathsf{Q}) = r_\mathcal{M}(\mathsf{P} \cap \mathsf{Q}) + r_\mathcal{M}(\mathsf{P} \cup \mathsf{Q})$. Then $Span_\mathcal{M}(\mathsf{P}) \cap Span_\mathcal{M}(\mathsf{Q}) \subseteq Span_\mathcal{M}(\mathsf{P} \cap \mathsf{Q})$.* □

3 Main result

In this section we prove our main result. Let $P(X) := X \cup \{v \in V - X : v \mapsto X\}$. We call a **maximal $\mathcal{M}$-independent packing of arborescences** a set $\{T_1, ..., T_{|\mathsf{S}|}\}$ of pairwise edge-disjoint arborescences for which T_i has root $\pi(\mathsf{s}_i)$ for $i = 1, ..., |\mathsf{S}|$, the set $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}$ is independent in $\mathcal{M}$ and $|\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}| = r_\mathcal{M}(\mathsf{S}_{P(v)})$. (We will also say that s_i is the root of T_i.)

Theorem 3.1. *Let (D, S, π) be a digraph with roots and $\mathcal{M}$ be a matroid on S with rank function $r_\mathcal{M}$. There exists a maximal $\mathcal{M}$-independent packing of arborescences in (D, S, π) if and only if π is $\mathcal{M}$-independent and*

$$\varrho_D(X) \geq r_\mathcal{M}(\mathsf{S}_{P(X)}) - r_\mathcal{M}(\mathsf{S}_X) \tag{6}$$

holds for each $X \subseteq V$.

One can see that Theorem 2.3 follows from this theorem in the same way as Theorem 2.2 did from Theorem 2.5.

Proof. As the necessity of (6) and $\mathcal{M}$-independency is straightforward we only prove the sufficiency.

For $X \subseteq V$, let $p(X) := r_\mathcal{M}(\mathsf{S}_{P(X)}) - r_\mathcal{M}(\mathsf{S}_X)$. X is called **tight** if $p(X) = \varrho_D(X)$. We call two sets X and Y **intersecting** if $X - Y, Y - X$ and $X \cap Y$ are non-empty sets. First we prove two lemmas.

Lemma 3.2. *Let X and Y be two intersecting tight subsets of V. If $v \mapsto X \cap Y$ for every $v \in Y - X$, then $X \cap Y$ is tight and $Span_\mathcal{M}(\mathsf{S}_X) \cap Span_\mathcal{M}(\mathsf{S}_Y) \subseteq Span_\mathcal{M}(\mathsf{S}_{X \cap Y})$.*

Proof. Let X and Y be two intersecting tight subsets of V for which $Y - X \mapsto X \cap Y$. First we prove that

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y). \tag{7}$$

As $v \mapsto X \cap Y$ for every $v \in Y - X$ and the reachability is transitive, we get $P(Y) \subseteq P(X \cap Y)$. Furthermore, it is easy to see that $P(X) \subseteq P(X \cup Y)$. Thus by the monotonicity of the rank function,

$$r_{\mathcal{M}}(\mathsf{S}_{P(X)}) + r_{\mathcal{M}}(\mathsf{S}_{P(Y)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(X\cup Y)}) + r_{\mathcal{M}}(\mathsf{S}_{P(X\cap Y)}). \tag{8}$$

It is easy to see that $\mathsf{S}_X \cap \mathsf{S}_Y = \mathsf{S}_{X\cap Y}$ and $\mathsf{S}_X \cup \mathsf{S}_Y = \mathsf{S}_{X\cup Y}$. Thus by the submodularity of the rank function,

$$r_{\mathcal{M}}(\mathsf{S}_X) + r_{\mathcal{M}}(\mathsf{S}_Y) \geq r_{\mathcal{M}}(\mathsf{S}_{X\cup Y}) + r_{\mathcal{M}}(\mathsf{S}_{X\cap Y}). \tag{9}$$

Subtracting (9) from (8) we get (7).

From the tightness of X and Y, (7), (6) and submodularity of ϱ_D, we get

$$\varrho_D(X) + \varrho_D(Y) = p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \leq$$

$$\leq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \leq \varrho_D(X) + \varrho_D(Y). \tag{10}$$

Hence $p(X \cap Y) + p(X \cup Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y)$. Thus $X \cap Y$ and $X \cup Y$ are tight. Moreover, $p(X) + p(Y) = p(X \cap Y) + p(X \cup Y)$. Hence in (9) equality must hold. Thus by Claim 2.6 and $\mathsf{S}_X \cap \mathsf{S}_Y = \mathsf{S}_{X\cap Y}$, we get $\mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X) \cap \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y) \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$. □

Lemma 3.3. *Let X and Y be two intersecting subsets of V such that X is tight and $\varrho_D(Y) = 0$. If $v \mapsto X \cap Y$ for every $v \in Y - X$, then $X \cap Y$ is tight and $\mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X) \cap \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y) \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$.*

Proof. By (6) and the monotonicity of the rank function, we get

$$0 = \varrho_D(Y) \geq r_{\mathcal{M}}(\mathsf{S}_{P(Y)}) - r_{\mathcal{M}}(\mathsf{S}_Y) \geq 0.$$

Thus Y is tight and the claim follows by Lemma 3.2. □

Following [6], we introduce some definitions. For $X, Y \subseteq V$, Y **dominates** X if $\mathsf{S}_X \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y)$. It is easy to see that domination is a transitive relation. An edge $uv \in A$ is said to be **good** if v dominates u, otherwise it is **bad**. We call a bad edge **s-bad**, if $\pi(\mathsf{s}) = u$ and $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$. Note that a bad edge is s-bad for at least one $\mathsf{s} \in \mathsf{S}$. For $\mathsf{s} \in \mathsf{S}$, we call a *tight* set $X \subseteq V$ **s-critical** if $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X)$ and there exists a node v for which $\pi(\mathsf{s})v$ is an s-bad edge of D that enters X.

By transitivity of the domination, if there is no bad edge, then $\mathsf{S}_{P(v)} \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$. Thus the arborescences consisting of the roots, (that is, each node v is an arborescence consisting of this single node $|\mathsf{S}_v|$ times), form a maximal $\mathcal{M}$-independent packing of arborescences. (Note that the $\mathcal{M}$-independecy of π ensures the independency of the roots.) From now on, we will prove by induction on $|A|$.

Let $\pi(\mathsf{s})v$ be an s-bad edge such that it enters no s-critical set where $\mathsf{s} \in \mathsf{S}$. Let $D' := D - \pi(\mathsf{s})v$, $\mathsf{S}' := \mathsf{S} \cup \{\mathsf{s}'\}$ where $\mathsf{s}' \notin \mathsf{S}$, $\pi' : \mathsf{S}' \to V$ such that $\pi'|_{\mathsf{S}} \equiv \pi$ and $\pi'(\mathsf{s}') = v$ and let $\mathcal{M}'$ be the matroid on S' that is obtained from $\mathcal{M}$ by considering

s' as an element parallel to s. For $X \subseteq V$, let $P'(X) := X \cup \{v \in V : v \mapsto_{D'} X\}$ and $\mathsf{S}'_X := \pi^{-1}(X)$.

Now π' is $\mathcal{M}'$-independent since $\pi(\mathsf{s})v$ was an s-bad edge. We claim that

$$\varrho_{D'}(X) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) - r_{\mathcal{M}'}(\mathsf{S}'_X) \tag{11}$$

holds for every $X \subseteq V$, hence there exist a maximal $\mathcal{M}'$-independent packing of arborescences $\mathcal{P}'$ in (D', S', π') by induction. Take an arbitrary set $X \subseteq V$. To prove (11), first observe that $P'(X) \subseteq P(X)$ and they are not equal if and only if $v \in P'(X)$ and $\pi(\mathsf{s}) \notin P'(X)$ both hold. Thus $r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(X)})$ by the definition of $\mathcal{M}'$. Also by definition, $r_{\mathcal{M}}(\mathsf{S}'_X) \geq r_{\mathcal{M}}(\mathsf{S}_X)$. Thus the right side of (6) does not increase in (11). Therefore, if $X \subseteq V$ is not tight, then (11) holds trivially as $\varrho_{D'}(X) + 1 \geq \varrho_D(X)$ and (6) holds with '$>$'; if $X \subseteq V$ is tight but $\pi(\mathsf{s})v$ does not enter X, then (11) holds trivially as $\varrho_{D'}(X) = \varrho_D(X)$. If X is tight and $\pi(\mathsf{s})v$ enters X, then $r_{\mathcal{M}}(\mathsf{S}'_X) > r_{\mathcal{M}}(\mathsf{S}_X)$ because $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}'}(\mathsf{S}'_X)$ as $s' \in \mathsf{S}'_X$ but $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X)$ since $\pi(\mathsf{s})v$ enters no s-critical set. Thus in this case, $\varrho_D(X) = \varrho_{D'}(X) + 1$ and $r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) - r_{\mathcal{M}'}(\mathsf{S}'_X) + 1$ hence (11) is again a consequence of (6).

Since s and s' are parallel in $\mathcal{M}'$, the arborescences $T, T' \in \mathcal{P}'$ rooted at s and s' are node-disjoint. Therefore, $T \cup T' \cup \pi(\mathsf{s})v$ is an arborescence rooted at $\pi(\mathsf{s})$ and $\mathcal{P} = \mathcal{P} - \{T, T'\} \cup \{T \cup T' \cup \pi(\mathsf{s})v\}$ is a packing of arborescences rooted at S in D. To see that $\mathcal{P}$ is a maximal $\mathcal{M}$-independent packing of arborescences, first observe that the $\mathcal{M}$-rank of the root set of the arborescences in $\mathcal{P}$ covering an arbitrary node u is the same as the $\mathcal{M}'$-rank of the root set of the arborescences in $\mathcal{P}'$ covering u by the definitions of $\mathcal{M}'$ and $\mathcal{P}'$. As $\mathcal{P}'$ is a maximal $\mathcal{M}'$-independent packing of arborescences this latter value is equal to $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)})$.

To prove that $\mathcal{P}$ is a maximal $\mathcal{M}$-independent packing of arborescences, we show that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ for all $u \in V$. Observe that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ is obvious hence we only prove '$\geq$'. Suppose to the contrary, that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) < r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ for a given $u \in V$. Since $r_{\mathcal{M}'}(\mathsf{S}'_Q) \geq r_{\mathcal{M}}(\mathsf{S}_Q)$ holds for any $Q \subseteq V$, $P'(u) \neq P(u)$ in this case. Hence $v \in P'(u)$ but $\pi(\mathsf{s}) \notin P'(u)$ because D and D' differs only on the edge $\pi(\mathsf{s})v$. Therefore, $\pi(\mathsf{s})v$ is the single edge of D that enters $P'(u)$. Thus writing up (6) for $X = P'(u)$, we get

$$1 = \varrho_D(P'(u)) \geq r_{\mathcal{M}}(\mathsf{S}_{P(P'(u))}) - r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P(u)}) - r_{\mathcal{M}}(\mathsf{S}_{P'(u)})$$

hence by our assumption,

$$r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1 \geq r_{\mathcal{M}}(\mathsf{S}_{P(u)}) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) + 1 \geq r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1.$$

Therefore, equality holds. From $r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1 = r_{\mathcal{M}}(\mathsf{S}_{P(u)})$, we get that $P'(u)$ is tight, and by $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P'(u)})$, we get that $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{P'(u)})$. Thus $P'(u)$ is s-critical, a contradiction.

To finish our proof, we must show that there exists an $\mathsf{s} \in \mathsf{S}$ and an s-bad edge of the form $\pi(\mathsf{s})v$ such that it enters no s-critical set. First we show the following lemma where $D[X]$ denotes the subgraph of D spanned by X:

Lemma 3.4. *For* $\mathsf{s} \in \mathsf{S}$*, let* $\pi(\mathsf{s})v$ *be an* s*-bad edge and* X *be a minimal* s*-critical set with* $v \in X$*. Then* $X \subseteq P(v)$*. Moreover,* v *is reachable from all points of* X *in* $D[X]$*.*

Proof. Assume for a contradiction that X is not a subset of $Y := P(v)$. Then X and Y are intersecting sets, $\varrho_D(Y) = 0$ and $v \in X \cap Y$ is reachable from all points of $Y - X$. As X is tight, $X \cap Y$ is also tight by Lemma 3.3. Moreover, X is s-critical hence $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_X$; furthermore, $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_Y$ as $\pi(\mathsf{s}) \in Y$. Thus $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$ also holds by Lemma 3.3. Therefore, $X \cap Y \subset X$ is an s-critical set such that $\pi(\mathsf{s})v$ enters it, a contradicting the minimality of X.

To prove the second part, assume for a contradiction that v is not reachable from all nodes of X in $D[X]$. Let Y' denote the subset of X from which v is reachable in $D[X]$. Then $\varrho_D(Y') \leq \varrho_D(X)$. Furthermore, $P(Y') = P(X) = Y$ by the first part. As Y' is not s-critical by the minimality of X, $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{Y'})$ and thus $r_{\mathcal{M}}(\mathsf{S}_{Y'}) < r_{\mathcal{M}}(\mathsf{S}_X)$. Therefore,

$$\varrho_D(Y') \leq \varrho_D(X) = r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) < r_{\mathcal{M}}(\mathsf{S}_{P(Y')}) - r_{\mathcal{M}}(\mathsf{S}_{Y'}),$$

contradicting (6). □

Suppose that for all $\mathsf{s} \in \mathsf{S}$ and for each s-bad edge, there exists an s-critical set that is entered by $\pi(\mathsf{s})v$. We call a set **critical** if there exists an $\mathsf{s} \in \mathsf{S}$ for which it is s-critical. Choose a minimal critical set X. We can assume that, say, X is s-critical for $\mathsf{s} \in \mathsf{S}$ and $\pi(\mathsf{s})v$ is an s-bad edge entering X.

If there is no bad edge spanned by X, then $\mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X) \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$ by the minimality of X and Lemma 3.4. Moreover, as X is s-critical $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_X$. Thus $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_v$, that is, v dominates $\pi(\mathsf{s})$ contradicting that $\pi(\mathsf{s})v$ is an s-bad edge.

Thus there is a bad edge $u'v'$ spanned by X. By our assumption, there is an s'-critical set X' for any $\mathsf{s}' \in \mathsf{S}_{u'} - \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{v'})$ such that $u'v'$ enters X'. Let us use the notation $Z := P(v')$. Then using Lemma 3.3, one can prove that $Z \cap X$ is tight as $\varrho_D(Z) = 0$ and $v' \in Z \cap X$ is reachable from all points of $Z - X$. Moreover, since $(Z \cap X) - X' \subseteq Z - v' = P(v') - v' \mapsto \{v'\} \subseteq Z \cap X \cap X'$, we get that $Z \cap X \cap X'$ is tight by Lemma 3.2. Thus $Z \cap X \cap X'$ is s'-critical and $Z \cap X \cap X' \subset X$ because $u' \in X - X'$ and $v' \in Z \cap X \cap X'$. Therefore, $Z \cap X \cap X'$ is a proper critical subset of X, a contradiction.

This finishes the proof of Theorem 3.1. □ □

One can see that the proof of Theorem 3.1 give rise to an algorithm if the matroid is given by an oracle for the rank function.

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