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On weighted kernels of two posets

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Abstract

A recent result of Aharoni Berger and Gorelik [1] is a weighted generalization of the well-known theorem of Sands Sauer and Woodrow [7] on monochromatic paths. The authors prove the existence of a so called weighted kernel for any pair of weighted posets on the same ground set. In this work, we point out that this result is closely related to the stable marriage theorem of Gale and Shapley [6], and we generalize Blair's theorem by showing that weighted kernels form a lattice under a certain natural order. To illustrate the applicability of our approach, we prove further weighted generalizations of the Sands Sauer Woodrow result.

Keywords posets, lattices, kernels, stable marriages; choice functions; deferred acceptance algorithm

1 Introduction

Sands Sauer and Woodrow proved an interesting generalization of the stable marriage theorem [6] by Gale and Shapley in [7]. Namely, if digraph D is the union of two acyclic digraphs, say D_1 and D_2 then there is a subset K of the vertices of D such that neither D_1 nor D_2 contains a directed path between two vertices of K but from any vertex of D outside K there is a directed path of D_1 or of D_2 to some vertex of K. The same result can also be formulated in terms of partially ordered sets as follows. If $\preceq_1$ and $\preceq_2$ are two partial orders on the same ground set V then there is a common antichain K of these posets such that for any element $v \in V \setminus K$ of the ground set there exists a vertex $k \in K$ such that $v \preceq_1 k$ or $v \preceq_2 k$ holds. This latter formulation comes from [3] by Fleiner (see also [4]) and the proof is based on a choice function framework and Tarski's well-known fixed point theorem [9]. Fleiner also described a generalization of the deferred acceptance algorithm by Gale and Shapley that finds

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such an antichain. Moreover, as an application of Blair's theorem in [2], it turned out that these antichains form a lattice under a natural partial order.

Recently, Aharoni Berger and Gorelik generalized the Sands Sauer Woodrow result to a weighted setting in [1]. They follow the terminology by Sands Sauer and Woodrow and call the above common antichain a kernel. They describe a generalized model, define weighted kernels for weighted posets and prove that for any integral weighted pair of posets, there exists an integral weighted kernel that has a so-called tame property. In this present work, after defining some new notions and recalling some well-known ones in Section 2, we build up a choice function based framework on lattices and generalize some notions well-known for set based choice functions to our setting in Section 3. In particular, we generalize the stable marriage theorem of Gale and Shapley by proving that a stable element always exists and can be found by an appropriate generalization of the deferred acceptance algorithm of Gale and Shapley. Furthermore, we also generalize Blair's result [2] by showing that these stable elements form a lattice under a certain natural order. In Section 4, we apply our framework and prove the result in [1] and as a corollary of our generalized version of Blair's theorem, we show that tame weighted kernels also form a lattice under a natural partial order. We illustrate the applicability of our approach in Section 5, where we describe other weighted generalizations of the Sands Sauer Woodrow theorem on monochromatic paths with the help of choice functions different than the ones we had in the proof of the the Aharoni Berger Gorelik result. We conclude in Section 6.

2 Preliminaries

A *partially ordered set* or *poset* is a pair $P = (V, \preceq)$ of a ground set V and a partial order (i.e. a reflexive, antisymmetric and transitive binary relation) $\preceq$ on V. Elements u and v of poset P are *comparable* if $u \preceq v$ or $v \preceq u$ holds, otherwise u and v are *incomparable*. A *chain* of the above poset P is a subset C of V such that its elements are pairwise comparable. Subset A of V is an *antichain* if now two different elements of A are comparable. The following result of Sands Sauer and Woodrow is a generalization of the stable marriage theorem of Gale and Shapley [6].

Theorem 2.1 (Sands, Sauer and Woodrow [7])**.** *If $P_1 = (V, \preceq)$ and $P_2 = (V, \preceq)$ are finite posets on the same ground set V then there exists a subset A of V that is a common antichain of P_1 and P_2 and for any element v of V there is some element a of A such that $v \preceq_1 a$ or $v \preceq_2 a$.*

Note that the original result of Sands Sauer and Woodrow in [7] has been formulated in terms of 2-arc colored digraphs and oriented paths and antichain A in Theorem 2.1 is called a kernel in their terminology. In fact the main result in [7] is somewhat more general than the above Theorem 2.1 as Theorem 2.1 corresponds to a certain acyclic case in the digraph terminology. Still, it is not difficult to deduce the Sands Sauer Woodrow result from Theorem 2.1. Note also that the marriage model of Gale and Shapley in [6] can also be translated to the language of Theorem 2.1. Namely, the common ground set of the two posets consists of all possible marriages and $\preceq_1$ is given by the men's and $\preceq_2$ by the women's preferences. This construction provides a bijection between stable marriage schemes and kernels A in Theorem 2.1.

Let us return to our model. Fix demand function $w : V \to \mathbb{R}_+$. Weight function $f : V \to \mathbb{R}_+$ is *$\preceq$-independent (with respect to w)* if $\widetilde{f}(C) := \sum\{f(c) : c \in C\} \leq max\{w(c) : c \in C\}$ holds for any chain C of P, that is, if the total weight of no chain exceeds the maximum demand of its elements. Clearly, if demand function $w \equiv \mathbb{1}$ then A is an antichain if and only if its characteristic function χ_A is independent (here $\mathbb{1}$ is the constant 1 function on V). Weight function f is *$\preceq$-tame (with respect to w)* if for every chain $C = \{c_1 \preceq c_2 \preceq \ldots \preceq c_k\}$ with $f(c_1) > 0$ we have $\widetilde{f}(C) \leq w(c_1)$, that is, if the total weight of no chain exceeds the demand of its minimal element unless this minimal element has weight zero. It is easy to see that if weight function f is $\preceq$-tame then f is $\preceq$-independent. We say that element v of V is *$\preceq$-dominated* by f if there is a chain $C = \{v = c_1 \preceq c_2 \preceq \ldots \preceq c_k\}$ such that $\widetilde{f}(C) \geq w(v)$, or in other words, if there is a chain starting at v of total weight not less than the demand of v.

Let $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ be posets on the same ground set V. A common antichain K of P_1 and P_2 is a *kernel* if each element v of $V \setminus K$ is *dominated* by K, that is there is an element k of K such that $v \preceq_1 k$ or $v \preceq_2 k$ holds. If $w : V \to \mathbb{R}_+$ is a demand function then weight function $f : V \to \mathbb{R}_+$ is a *weighted kernel* if f is both $\preceq_1$-independent and $\preceq_2$-independent and moreover each element v of V is $\preceq_1$-dominated or $\preceq_2$-dominated (or both). The above weight function f is called *integral* if $f : V \to \mathbb{Z}_+$. It is easy to see that for $w \equiv \mathbb{1}$ an integral weighted kernel is exactly the characteristic function of a kernel. The main result of Aharoni Berger and Gorelik states the following.

Theorem 2.2 (Aharoni, Berger, Gorelik [1]). *For any pair $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ of posets and for any demand function $w : V \to \mathbb{Z}_+$, there exists an integral weighted kernel $f : V \to \mathbb{Z}_+$ that is both $\preceq_1$-tame and $\preceq_2$-tame.*

Note that if common ground set V of the two posets is infinite then Theorem 2.2 might not hold, for example when $P_1 = P_2 = ([0, 1], \leq)$, $w(1) = 0$ and $w(x) = 1$ for each $0 \leq x < 1$. Although this condition is not stated in [1], the authors clearly require this assumption on finiteness throughout their paper.

A poset $L = (X, \preceq)$ is a *lattice* if any two elements x and y of X have a least common upper bound $x \vee y$ (the *join* of x and y) and a greatest common lower bound $x \wedge y$ (the *meet* of x and y). Lattice L is *complete* if any subset Y of X has a least common upper bound $\bigvee Y$ and a greatest common lower bound $\bigwedge Y$. Clearly, every complete lattice L has a unique maximal element $1 := \bigvee X$ and a unique minimal element $0 := \bigwedge X$. Lattice $L = (X, \preceq)$ is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1}$$

holds for any elements x, y, z of X. Note that condition (1) is equivalent to its dual, that is, for any $x, y, z \in X$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

holds. If lattice $L = (X, \preceq)$ is complete then L is called *infinitely distributive* if $x \wedge \bigvee Y = \bigvee\{x \wedge y : y \in Y\}$ holds for any element x and any subset Y of X. Note that unlike distributivity, infinite distributivity does not imply its dual.

A most common example of a distributive complete lattice is lattice $L = (2^X, \subseteq)$ of subsets of a ground set X. For readers that are unfamiliar with lattices, the assumption that any lattice we work with is a subset lattice (and hence lattice operations $\vee$ and $\wedge$ are simply $\cap$ and $\cup$) may help to follow the arguments below.

If $L = (X, \preceq)$ is a lattice then function $\mathcal{F} : X \to X$ is a *choice function* if $\mathcal{F}(x) \preceq x$ holds for any element x of X. For choice function $\mathcal{F}$, element $x \in X$ is $\mathcal{F}$*-independent* if $\mathcal{F}(x) = x$. Mapping $\mathcal{D} : X \to X$ is a *determinant* of choice function $\mathcal{F}$ if $\mathcal{F}(x) = x \wedge \mathcal{D}(x)$ holds for any element x of X. Clearly, any choice function is a determinant of itself. Mapping $\mathcal{F} : X \to X$ is *monotone* if $x \preceq y$ implies $\mathcal{F}(x) \preceq \mathcal{F}(y)$ and $\mathcal{F}$ is *antitone* if $x \preceq y$ implies $\mathcal{F}(y) \preceq \mathcal{F}(x)$. Choice function $\mathcal{F} : X \to X$ is called *substitutable* if there is an antitone determinant $\mathcal{A} : X \to X$ of $\mathcal{F}$, that is, $\mathcal{F}(x) = x \wedge \mathcal{A}(x)$ holds for any element x of X. Choice function $\mathcal{F} : X \to X$ is *path-independent* if $\mathcal{F}(x \vee y) = \mathcal{F}(x \vee \mathcal{F}(y))$ holds for any elements x, y of X.

Observe that in case of subsetlattices, the above defined notions of substitutability and path-independence of a choice function correspond to the well-known notions of set-mappings under the same name.

An important result on complete lattices is the following theorem of Tarski.

Theorem 2.3 (Tarski [9]). *If $L = (X, \preceq)$ is a complete lattice and mapping $\mathcal{F} : X \to X$ is monotone then set $X_\mathcal{F} := \{x \in X : \mathcal{F}(x) = x\}$ of fixed points of $\mathcal{F}$ is nonempty and $(X_\mathcal{F}, \preceq|_{X_\mathcal{F}})$ is a lattice.* □

Note that if ground-set X of L is finite then it is easy to construct a fixed point by iterating $\mathcal{F}$, as $0 \preceq \mathcal{F}(0) \preceq \mathcal{F}(\mathcal{F}(0)) \preceq \ldots$, and $1 \succeq \mathcal{F}(1) \succeq \mathcal{F}(\mathcal{F}(1)) \succeq \ldots$, so any of these chains must eventually end in a fixed point of $\mathcal{F}$. (These fixed points actually are the least and the greatest elements of lattice $(X_\mathcal{F}, \preceq|_{X_\mathcal{F}})$, respectively.)

3 Path-independence, substitutability, stability

We observe some useful properties of substitutable and path-independent substitutable choice functions.

Lemma 3.1. *Assume that $L = (X, \preceq)$ is a complete lattice. If choice function $\mathcal{F} : X \to X$ is path-independent then*

$$\mathcal{F}(x) \preceq y \preceq x \textit{ implies } \mathcal{F}(x) = \mathcal{F}(y) \ . \tag{2}$$

On the other hand, if L is distributive and choice function $\mathcal{F} : X \to X$ is substitutable and has property (2) then $\mathcal{F}$ is path-independent.

Proof. If $\mathcal{F}(x) \preceq y \preceq x$ then $\mathcal{F}(y) = \mathcal{F}(y \vee \mathcal{F}(x)) = \mathcal{F}(y \vee x) = \mathcal{F}(x)$ and the first part follows. To see the second part, let $\mathcal{A}$ be an antitone determinant of $\mathcal{F}$. Then

$$\mathcal{F}(x \vee y) = (x \vee y) \wedge \mathcal{A}(x \vee y) \preceq (x \vee y) \wedge \mathcal{A}(y) = (x \wedge \mathcal{A}(y)) \vee (y \wedge \mathcal{A}(y)) \preceq x \vee \mathcal{F}(y) \preceq x \vee y \ , \tag{3}$$

Consequently, $\mathcal{F}(x \vee y) \preceq x \vee \mathcal{F}(y) \preceq x \vee y$ and $\mathcal{F}(x \vee \mathcal{F}(y)) = \mathcal{F}(x \vee y)$ by (2). So $\mathcal{F}$ is path-independent, indeed. □

The next lemma provides a sufficient condition on the determinant for path-independence of the corresponding choice function.

Lemma 3.2. *Assume that $L = (X, \preceq)$ is a distributive complete lattice and choice function $\mathcal{F} : X \to X$ has an antitone determinant $\mathcal{A}$ with the property that*

$$\mathcal{A}(x) = \mathcal{A}(\mathcal{F}(x)) \text{ holds for each element } x \text{ of } X. \tag{4}$$

Then $\mathcal{F}$ is path-independent.

Proof. As calculation in (3) is valid this case, we have $\mathcal{F}(x \vee y) \preceq x \vee \mathcal{F}(y) \preceq x \vee y$, hence

$$\mathcal{A}(x \vee y) \preceq \mathcal{A}(x \vee \mathcal{F}(y)) \preceq \mathcal{A}(\mathcal{F}(x \vee y)) = \mathcal{A}(x \vee y) \tag{5}$$

by property (4) and the antitonicity of $\mathcal{A}$. Thus we have equality throughout (5), in particular $\mathcal{A}(x \vee y) = \mathcal{A}(x \vee \mathcal{F}(y))$ holds. Now

$$\mathcal{F}(x \vee \mathcal{F}(y)) = (x \vee \mathcal{F}(y)) \wedge \mathcal{A}(x \vee \mathcal{F}(y)) \preceq (x \vee y) \wedge \mathcal{A}(x \vee y) = \mathcal{F}(x \vee y)$$

and

$$\begin{aligned}\mathcal{F}(x \vee y) = (x \vee y) \wedge \mathcal{A}(x \vee y) = (x \vee y) \wedge \mathcal{A}(x \vee y) \wedge \mathcal{A}(x \vee y) = \\ \mathcal{F}(x \vee y) \wedge \mathcal{A}(x \vee \mathcal{F}(y)) \preceq x \vee \mathcal{F}(y) \wedge \mathcal{A}(x \vee \mathcal{F}(y)) = \mathcal{F}(x \vee \mathcal{F}(y)) ,\end{aligned}$$

so $\mathcal{F}(x \vee y) = \mathcal{F}(x \vee \mathcal{F}(y))$, that is, $\mathcal{F}$ is path-independent, indeed. □

If choice function $\mathcal{F} : X \to X$ is substitutable then there might be several antitone determinants of $\mathcal{F}$. The following lemma states that there is a canonical one among them which is eventually the minimal one.

Lemma 3.3. *Assume that $L = (X, \preceq)$ is a complete lattice and $\mathcal{F} : X \to X$ is a substitutable choice function of it. Then for any $x, y \in L$*

$$\mathcal{F}(x \vee y) \wedge x \preceq \mathcal{F}(x) . \tag{6}$$

Furthermore, if L is infinitely distributive then

$$\mathcal{A}_{\mathcal{F}}(x) := \bigvee \{y \in X : y \preceq \mathcal{F}(x \vee y)\} \text{ is an antitone determinant of } \mathcal{F}. \tag{7}$$

If $\mathcal{A}$ is an antitone determinant of $\mathcal{F}$ then $\mathcal{A}_{\mathcal{F}}(x) \preceq \mathcal{A}(x)$ holds for any element x of X. At last, if $\mathcal{F}$ is path-independent then $\mathcal{A}_{\mathcal{F}}$ has property (4).

Proof. Let $\mathcal{A}$ be an antitone determinant of substitutable choice function $\mathcal{F}$. Then

$$\mathcal{F}(x \vee y) \wedge x = (x \vee y) \wedge \mathcal{A}(x \vee y) \wedge x = x \wedge \mathcal{A}(x \vee y) \preceq x \wedge \mathcal{A}(x) = \mathcal{F}(x)$$

proving (6). From (6) and the infinite distributivity of $\mathcal{F}$, we get

$$x \wedge \mathcal{A}_{\mathcal{F}}(x) = x \wedge \bigvee\{y \in X : y \preceq \mathcal{F}(x \vee y)\} = \bigvee\{x \wedge y : y \preceq \mathcal{F}(x \vee y)\} \preceq \bigvee\{x \wedge \mathcal{F}(x \vee y) : y \in X\} \preceq \bigvee\{\mathcal{F}(x) : y \in X\} = \mathcal{F}(x) \ . \quad (8)$$

Moreover, $\mathcal{F}(x) = \mathcal{F}(x \vee \mathcal{F}(x))$ as $\mathcal{F}(x) \preceq x$, so $\mathcal{F}(x) \preceq \mathcal{A}_{\mathcal{F}}(x)$, hence $\mathcal{F}(x) \preceq x \wedge \mathcal{A}_{\mathcal{F}}(x)$. Together with (8), this proves that $\mathcal{A}_{\mathcal{F}}$ is indeed a determinant of $\mathcal{F}$.

To show the antitone property of $\mathcal{A}_{\mathcal{F}}$, let $x_1 \preceq x_2$. Our goal is to prove that $\mathcal{A}_{\mathcal{F}}(x_2) \preceq \mathcal{A}_F(x_1)$, so assume that $y \preceq \mathcal{F}(x_2 \vee y)$. Again, (6) shows that

$$y \preceq \mathcal{F}(x_2 \vee y) \wedge (x_1 \vee y) = \mathcal{F}(x_1 \vee y \vee x_2) \wedge (x_1 \vee y) \preceq \mathcal{F}(x_1 \vee y) \ ,$$

and due to (7), this is exactly what we need.

To prove the minimal property of determinant $\mathcal{A}_{\mathcal{F}}$, assume that $\mathcal{A}$ is an antitone determinant of $\mathcal{F}$. Now if $y \preceq \mathcal{F}(x \vee y)$ then

$$y \preceq \mathcal{F}(x \vee y) = (x \vee y) \wedge \mathcal{A}(x \vee y) \preceq \mathcal{A}(x \vee y) \preceq \mathcal{A}(x)$$

hence $\mathcal{A}_{\mathcal{F}}(x) = \bigvee\{y \in X : y \preceq \mathcal{F}(x \vee y)\} \preceq \bigvee\{\mathcal{A}(x) : y \preceq \mathcal{F}(x \vee y)\} = \mathcal{A}(x)$.

At last, path-independence of $\mathcal{F}$ directly implies (4):

$$\mathcal{A}_{\mathcal{F}}(x) = \bigvee\{y : y \preceq \mathcal{F}(x \vee y)\} = \bigvee\{y : y \preceq \mathcal{F}(\mathcal{F}(x) \vee y)\} = \mathcal{A}_{\mathcal{F}}(\mathcal{F}(x)) \ ,$$

and this finishes the proof. □

Assume that $L = (X, \preceq)$ is a lattice and $\mathcal{F}_1$ and $\mathcal{F}_2$ are path-independent substitutable choice functions. Element s of X is $\mathcal{F}_1\mathcal{F}_2$-*stable* if

$$\mathcal{F}_1(s) = \mathcal{F}_2(s) = s \qquad \text{and} \quad (9)$$

$$\mathcal{F}_1(s \vee x) \wedge \mathcal{F}_2(s \vee x) \preceq s \qquad \text{holds for each element } x \text{ of } X. \quad (10)$$

Intuitively, this means that s is both $\mathcal{F}_1$-independent and $\mathcal{F}_2$-independent, moreover, if some other option x is offered together with s, then it is impossible that both these choice functions select x. We denote the set of $\mathcal{F}_1\mathcal{F}_2$-stable elements by $S(\mathcal{F}_1\mathcal{F}_2)$. The following lemma shows a handy characterization of $\mathcal{F}_1\mathcal{F}_2$-stability in terms of canonical determinants.

Lemma 3.4. *Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are substitutable choice functions on infinitely distributive complete lattice $L = (X, \preceq)$. Element s of X is $\mathcal{F}_1\mathcal{F}_2$-stable if and only if*

$$\mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) = s \quad (11)$$

Moreover, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are antitone determinants of $\mathcal{F}_1$ and $\mathcal{F}_2$ and $\mathcal{A}_1(s) \wedge \mathcal{A}_2(s) = s$ holds for some element s of X then s is $\mathcal{F}_1\mathcal{F}_2$-stable.

Proof. Assume first that s is $\mathcal{F}_1\mathcal{F}_2$-stable. From property (7), infinite distributivity of L, (6) and (10), we get

$$s = \mathcal{F}_1(s) \wedge \mathcal{F}_2(s) = s \wedge \mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) \preceq \mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) =$$
$$\bigvee\{y : y \preceq \mathcal{F}_1(s\vee y)\}\wedge\bigvee\{z : z \preceq \mathcal{F}_2(s\vee z)\} = \bigvee\{y\wedge z : y \preceq \mathcal{F}_1(s\vee y), z \preceq \mathcal{F}_2(s\vee z)\} \preceq$$
$$\bigvee\{y \wedge z : y \wedge z \preceq \mathcal{F}_1(s \vee y), y \wedge z \preceq \mathcal{F}_2(s \vee z)\} \preceq$$
$$\bigvee\{y \wedge z : y \wedge z \preceq \mathcal{F}_1(s \vee (y \wedge z)), y \wedge z \preceq \mathcal{F}_2(s \vee (y \wedge z))\} =$$
$$\bigvee\{x : x \preceq \mathcal{F}_1(s \vee x), x \preceq \mathcal{F}_2(s \vee x)\} =$$
$$\bigvee\{x : x \preceq \mathcal{F}_1(s \vee x) \wedge \mathcal{F}_2(s \vee x)\} \preceq \bigvee\{x : x \preceq s\} = s\ .$$

So we have equality throughout, in particular (11) holds.

Now assume that (11) holds. Then

$$s = s\wedge s \succeq \mathcal{F}_1(s)\wedge\mathcal{F}_2(s) = s\wedge\mathcal{A}_{\mathcal{F}_1}(s)\wedge s\wedge\mathcal{A}_{\mathcal{F}_2}(s) = s\wedge(\mathcal{A}_{\mathcal{F}_1}(s)\wedge\mathcal{A}_{\mathcal{F}_2}(s)) = s\wedge s = s,$$

hence we have equality throughout, in particular $\mathcal{F}_1(s) \wedge \mathcal{F}_2(s) = s$. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are choice functions, (9) follows. To see (10), let $x \in X$. By (7) and the antitone property of the canonical determinants, we see that

$$\mathcal{F}_1(s \vee x) \wedge \mathcal{F}_2(s \vee x) = (s \vee x) \wedge \mathcal{A}_{\mathcal{F}_1}(s \vee x) \wedge \mathcal{A}_{\mathcal{F}_2}(s \vee x) \preceq \mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) = s$$

and this finishes the proof of the first part of the Lemma.

To prove the second part, observe that

$$s \succeq \mathcal{F}_1(s) = s \wedge \mathcal{A}_1(s) \succeq s \wedge \mathcal{A}_1(s) \wedge \mathcal{A}_2(s) = s \wedge s = s$$

and a similar argument shows that $\mathcal{F}_2(s) = s$ as well. By Lemma 3.3 we get that

$$s \preceq \mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) \preceq \mathcal{A}_1(s) \wedge \mathcal{A}_2(s) = s\ ,$$

hence we have equality throughout, in particular $\mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) = s$, proving the $\mathcal{F}_1\mathcal{F}_2$-stability of s. □

The following consequence of Lemma 3.4 is an important characterization of $\mathcal{F}_1\mathcal{F}_2$-stable sets in case of substitutable choice functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are also path-independent.

Lemma 3.5. *Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are a path-independent substitutable choice functions on infinitely distributive complete lattice $L = (X, \preceq)$. Then $s \in X$ is $\mathcal{F}_1\mathcal{F}_2$-stable if and only if there exist elements a and b of X such that*

$$s = a \wedge b \text{ and } \mathcal{A}_{\mathcal{F}_1}(a) = b \text{ and } \mathcal{A}_{\mathcal{F}_2}(b) = a \tag{12}$$

hold. Furthermore, (12) *implies $\mathcal{F}_1(a) = \mathcal{F}_2(b) = s$ and $a = \mathcal{A}_{\mathcal{F}_2}(s)$, $b = \mathcal{A}_{\mathcal{F}_1}(s)$.*

Proof. For sufficiency, assume that $s = a \wedge b$ and $\mathcal{A}_{\mathcal{F}_1}(a) = b$ and $\mathcal{A}_{\mathcal{F}_2}(b) = a$. By Lemma 3.3, $\mathcal{A}_{\mathcal{F}_1}$ and $\mathcal{A}_{\mathcal{F}_2}$ have property (4) thus

$$b = \mathcal{A}_{\mathcal{F}_1}(a) = \mathcal{A}_{\mathcal{F}_1}(\mathcal{F}_1(a)) = \mathcal{A}_{\mathcal{F}_1}(a \wedge \mathcal{A}_{\mathcal{F}_1}(a)) = \mathcal{A}_{\mathcal{F}_1}(a \wedge b) = \mathcal{A}_{\mathcal{F}_1}(s)$$

and similarly

$$a = \mathcal{A}_{\mathcal{F}_2}(b) = \mathcal{A}_{\mathcal{F}_2}(\mathcal{F}_2(b)) = \mathcal{A}_{\mathcal{F}_2}(b \wedge \mathcal{A}_{\mathcal{F}_2}(b)) = \mathcal{A}_{\mathcal{F}_2}(b \wedge a) = \mathcal{A}_{\mathcal{F}_2}(s) \ ,$$

hence $\mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) = a \wedge b = s$. So s is $\mathcal{F}_1\mathcal{F}_2$-stable by Lemma 3.4.

To see necessity, assume that s is $\mathcal{F}_1\mathcal{F}_2$-stable, that is, $\mathcal{A}_{\mathcal{F}_1}(s) \wedge \mathcal{A}_{\mathcal{F}_2}(s) = s$ by Lemma 3.4. Define $a := \mathcal{A}_{\mathcal{F}_2}(s)$ and $b := \mathcal{A}_{\mathcal{F}_1}(s)$ and observe that

$$\mathcal{F}_1(a) = a \wedge \mathcal{A}_{\mathcal{F}_1}(a) \preceq a \wedge \mathcal{A}_{\mathcal{F}_1}(s) = a \wedge b = s \preceq a$$

so $\mathcal{F}_1(a) \preceq s \preceq a$, hence $\mathcal{F}_1(a) = \mathcal{F}_1(s) = s \wedge \mathcal{A}_{\mathcal{F}_1}(s) = s \wedge b = s$ by (2). Now property (4) implies that $\mathcal{A}_{\mathcal{F}_1}(a) = \mathcal{A}_{\mathcal{F}_1}(\mathcal{F}_1(a)) = \mathcal{A}_{\mathcal{F}_1}(s) = b$ and a similar argument shows that $\mathcal{A}_{\mathcal{F}_2}(b) = a$.

To see the second part, observe first that

$$\mathcal{F}_2(b) = b \wedge \mathcal{A}_{\mathcal{F}_2}(b) = b \wedge a = s \text{ and } \mathcal{F}_1(a) = a \wedge \mathcal{A}_{\mathcal{F}_1}(a) = a \wedge b = s \ ,$$

hence by (4) we get

$$a = \mathcal{A}_{\mathcal{F}_2}(b) = \mathcal{A}_{\mathcal{F}_2}(\mathcal{F}_2(b)) = \mathcal{A}_{\mathcal{F}_2}(s) \text{ and } b = \mathcal{A}_{\mathcal{F}_1}(a) = \mathcal{A}_{\mathcal{F}_1}(\mathcal{F}_1(a)) = \mathcal{A}_{\mathcal{F}_1}(s)$$

and this finishes the proof. □

The following lemma is a generalization of the stable marriage theorem.

Lemma 3.6. *If $L = (X, \preceq)$ is an infinitely distributive complete lattice and $\mathcal{F}_1$ and $\mathcal{F}_2$ are substitutable path-independent choice functions then there exists an $\mathcal{F}_1\mathcal{F}_2$-stable element s of X.*

Proof. Define mapping

$$\mathcal{M}(x) := \mathcal{A}_{\mathcal{F}_2}(\mathcal{A}_{\mathcal{F}_1}(x)). \tag{13}$$

As both $\mathcal{A}_{\mathcal{F}_1}$ and $\mathcal{A}_{\mathcal{F}_2}$ are antitone, $\mathcal{M}$ is monotone, and by Theorem 2.3 of Tarski there exists a fixed point a of $\mathcal{M}$. Define $b := \mathcal{A}_{\mathcal{F}_1}(a)$. Now $a = \mathcal{M}(a) = \mathcal{A}_{\mathcal{F}_2}(\mathcal{A}_{\mathcal{F}_1}(a)) = \mathcal{A}_{\mathcal{F}_2}(b)$, hence $s = a \wedge b$ is an $\mathcal{F}_1\mathcal{F}_2$-stable set by Lemma 3.5. □

Note that there is a generalization of the deferred acceptance algorithm of Gale and Shapley that finds an $\mathcal{F}_1\mathcal{F}_2$-stable element in case of lattice L is finite. This generalized algorithm finds a fixed point a of monotone function $\mathcal{M}$ in the proof of Lemma 3.6. This is done according to the remark after Theorem 2.3. Namely, if 0 is the least element of lattice L then $0 \preceq \mathcal{M}(0)$ implies $\mathcal{M}(0) \preceq \mathcal{M}(\mathcal{M}(0))$, and this yields $\mathcal{M}(\mathcal{M}(0)) \preceq \mathcal{M}(\mathcal{M}(\mathcal{M}(0)))$. So if L is a finite lattice then chain $0 \preceq \mathcal{A}_{\mathcal{F}_2}(\mathcal{A}_{\mathcal{F}_1}(0)) \preceq \mathcal{A}_{\mathcal{F}_2}(\mathcal{A}_{\mathcal{F}_1}(\mathcal{A}_{\mathcal{F}_2}(\mathcal{A}_{\mathcal{F}_1}(0)))) \preceq \ldots$ must converge to a fixed point a. Then $s := a \wedge \mathcal{A}_{\mathcal{F}_1}(a)$ is an $\mathcal{F}_1\mathcal{F}_2$-stable element of L.

Assume that $L = (X, \preceq)$ is a lattice and $\mathcal{F}$ is a substitutable path-independent choice function. We say that x is $\mathcal{F}$*-superior* to y (denoted by $y \leq_{\mathcal{F}} x$) if $\mathcal{F}(x \vee y) = x$ holds. Relation $\leq_{\mathcal{F}}$ is a partial order on $\mathcal{F}$-independent elements according to the following lemma.

Lemma 3.7. *If $L = (X, \preceq)$ is a lattice and $\mathcal{F}$ is a substitutable path-independent choice function then $\leq_{\mathcal{F}}$ is a partial order on $\mathcal{F}$-independent elements of X.*

Proof. We need to prove that $\leq_{\mathcal{F}}$ is reflexive, antisymmetric and transitive. If x is $\mathcal{F}$-independent then $x = \mathcal{F}(x) = \mathcal{F}(x \vee x)$, that is, $x \leq_{\mathcal{F}} x$, proving reflexivity. Now if $x \leq_{\mathcal{F}} y \leq_{\mathcal{F}} x$, then $y = \mathcal{F}(x \vee y) = x$, hence $\leq_{\mathcal{F}}$ is antisymmetric, indeed. At last, if $x \leq_{\mathcal{F}} y \leq_{\mathcal{F}} z$ holds then by path-independence of $\mathcal{F}$ we have

$$\mathcal{F}(x \vee z) = \mathcal{F}(x \vee \mathcal{F}(y \vee z)) = \mathcal{F}(x \vee y \vee z) = \mathcal{F}(\mathcal{F}(x \vee y) \vee z)) = \mathcal{F}(y \vee z) = z \ ,$$

hence $x \leq_{\mathcal{F}} z$, proving the transitivity of $\leq_{\mathcal{F}}$. □

One can generalize Blair's theorem [2] on the lattice structure of stable matchings to our setting as follows.

Theorem 3.8. *Assume that $L = (X, \preceq)$ is an infinitely distributive complete lattice and $\mathcal{F}_1$ and $\mathcal{F}_2$ are substitutable path-independent choice functions. Then partial order $\leq_{\mathcal{F}_1}$ defines a lattice on the set $S(\mathcal{F}_1\mathcal{F}_2)$ of $\mathcal{F}_1\mathcal{F}_2$-stable elements of X. Moreover, $\leq_{\mathcal{F}_1}|_{S(\mathcal{F}_1\mathcal{F}_2)} = \geq_{\mathcal{F}_2}|_{S(\mathcal{F}_1\mathcal{F}_2)}$, that is, if s and s' are $\mathcal{F}_1\mathcal{F}_2$-stable elements then $s \leq_{\mathcal{F}_1} s'$ and $s' \leq_{\mathcal{F}_2} s$ are equivalent.*

Proof. It follows from Lemma 3.5 that $s \mapsto \mathcal{A}_{\mathcal{F}_2}(s)$ is a bijection between $S(\mathcal{F}_1\mathcal{F}_2)$ and set $X_{\mathcal{M}}$ of fixed points of mapping $\mathcal{M}$ defined in (13). As $\mathcal{M}$ is monotone, $(X_{\mathcal{M}}, \preceq)$ is a lattice according to Theorem 2.3. So to prove the first part of Theorem 3.8, it is enough to show that the bijection between $\mathcal{F}_1\mathcal{F}_2$-stable elements and fixed points of $\mathcal{M}$ is order-preserving.

Let $\mathcal{F}_1\mathcal{F}_2$-stable sets s and s' correspond to fixed points $a = \mathcal{A}_{\mathcal{F}_2}(s)$ and $a' = \mathcal{A}_{\mathcal{F}_2}(s')$. Assume first that $a \preceq a'$. By Lemma 3.5 $\mathcal{F}_1(a) = s$, and $\mathcal{F}_1(a') = s'$, and path-independence of $\mathcal{F}_1$ shows

$$\mathcal{F}_1(s \vee s') = \mathcal{F}_1(\mathcal{F}_1(a) \vee \mathcal{F}_1(a')) = \mathcal{F}_1(a \vee a') = \mathcal{F}_1(a') = s' \ .$$

Hence $a \preceq a'$ implies $s \leq_{\mathcal{F}_1} s'$.

If $s \leq_{\mathcal{F}_1} s'$ then $s' = \mathcal{F}_1(s \vee s')$ and $a = \mathcal{A}_{\mathcal{F}_2}(s)$, $a' = \mathcal{A}_{\mathcal{F}_2}(s')$, $b = \mathcal{A}_{\mathcal{F}_1}(s)$, $b' = \mathcal{A}_{\mathcal{F}_1}(s')$ by Lemma 3.5. By (4) and path-independence of $\mathcal{F}_1$, we get

$$b = \mathcal{A}_{\mathcal{F}_1}(s) \succeq \mathcal{A}_{\mathcal{F}_1}(s \vee s') = \mathcal{A}_{\mathcal{F}_1}(\mathcal{F}_1(s \vee s')) = \mathcal{A}_{\mathcal{F}_1}(s') = b' \ ,$$

thus $a' = \mathcal{A}_{\mathcal{F}_2}(b') \succeq \mathcal{A}_{\mathcal{F}_2}(b) = a$ by the antitone property of $\mathcal{A}_{\mathcal{F}_2}$. This proves the first part of Theorem 3.8.

To show the second part, observe that if $s \leq_{\mathcal{F}_1} s'$ then $a \preceq a'$. Moreover, $b' := \mathcal{A}_{\mathcal{F}_1}(a') \preceq \mathcal{A}_{\mathcal{F}_1}(a) = b$ by the antitone property of $\mathcal{A}_{\mathcal{F}_1}$. So path-independence of $\mathcal{F}_2$ shows that

$$\mathcal{F}_2(s \vee s') = \mathcal{F}_2(\mathcal{F}_2(b) \vee \mathcal{F}_2(b')) = \mathcal{F}_2(b \vee b') = \mathcal{F}_2(b) = s \ ,$$

i.e. $s' \leq_{\mathcal{F}_2} s$. □

4 Existence and structure of weighted kernels

To prove the Aharoni Berger Gorelik result in [1], we show that weighted kernels are exactly the $\mathcal{F}_1\mathcal{F}_2$-stable elements of an appropriate lattice for certain substitutable path-independent choice functions $\mathcal{F}_1$ and $\mathcal{F}_2$. First we define the lattice we work with.

Definition 4.1. Let V be a finite set and $w : V \to \mathbb{Z}_+$ be a demand function. Define poset $L^w := (\{f : V \to \mathbb{Z}_+, f \leq w\}, \leq)$ of weight functions on V.

Observation 4.2. Lattice L^w is an infinitely distributive complete lattice with lattice operations min and max. □

Let $P = (V, \preceq)$ be a finite poset and w and L^w as in Definition 4.1. We define a choice function $\mathcal{F}^w_\preceq$ on L^w that always picks a $\preceq$-tame weight function which is maximal in some sense. Namely, let $V = \{v_1, v_2, \ldots, v_n\}$ be a linear extension of $\succeq$, that is, if $v_i \preceq v_j$ then $j \leq i$ holds. (So v_1 is $\preceq$-maximal element of V and v_{i+1} is a $\preceq$-maximal element of $V \setminus \{v_1, v_2, \ldots, v_i\}$ for $i = 1, 2, \ldots$.) For any weight function f in L^w, define $\mathcal{F}^w_\preceq(f)$ for each of the values of $v_1, v_2, \ldots, v_n$ in this order in a certain greedy manner. By this we mean that after we calculated the values of $\left[\mathcal{F}^W_\preceq(f)\right](v_1), \ldots, \left[\mathcal{F}^W_\preceq(f)\right](v_{i-1})$, we determine value $\left[\mathcal{F}^W_\preceq(f)\right](v_i) = \alpha$ such that $\alpha \leq f(v_i)$ and α is maximal with the property that $\mathcal{F}^w_\preceq(f)$ is $\preceq$-tame on any chain $v_i \prec l_1 \prec l_2 \prec \ldots$ starting at v_i. More precisely,

$$\left[\mathcal{F}^w_\preceq(f)\right](v_i) = \min\left\{f(v_i), \max\left\{0, w(v_i) - \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v_i)\right\}\right\} \tag{14}$$

where $\left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v) = 0$ if v is a $\preceq$-maximal element of V, otherwise

$$\left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v) = \max\left\{\left[\mathcal{F}^w_\preceq(f)\right](u_1) + \left[\mathcal{F}^w_\preceq(f)\right](u_2) + \ldots : v \prec u_1 \prec u_2 \prec \ldots\right\} \tag{15}$$

By definition, $\left[\mathcal{F}^w_\preceq(f)\right](v_i) \leq f(v_i)$ holds for each element v_i of V, hence mapping $\mathcal{F}^w_\preceq$ is a choice function on L^w. Moreover, $\mathcal{F}^w_\preceq(f)$ is $\preceq$-tame for any weight $f \in L^w$ as we have chosen each value $[\mathcal{F}^w_\preceq(f)](v)$ such that every chain $v \preceq u_1 \preceq u_2 \preceq \ldots$ satisfies the property that tameness requires. The following Lemma describes a determinant of $\mathcal{F}^w_\preceq(f)$.

Lemma 4.3. *Let $P = (V, \preceq)$ be a finite poset and w and L^w as in Definition 4.1. For any $f \in L^w$ and $v \in V$ define $\left[\mathcal{M}^w_\preceq(f)\right](v) = 0$ if v is $\preceq$-maximal otherwise let*

$$\begin{aligned}\left[\mathcal{M}^w_\preceq(f)\right](v) := \max\{&f'(c_1) + f'(c_2) + \ldots + f'(c_k) : \\ &f' \leq f \text{ and } f' \text{ is } \preceq\text{-tame and } v \prec c_1 \prec c_2 \prec \ldots \prec c_k\}\end{aligned} \tag{16}$$

as the maximum total f'-weight of a chain above v where f' is a $\preceq$-tame lower bound of f. Then $\mathcal{A}^w_\preceq := \max\{0, w - \mathcal{M}^w_\preceq\}$ is a determinant of $\mathcal{F}^w_\preceq$, that is

$$\left[\mathcal{F}^w_\preceq(f)\right](v) = \min\left\{f(v), \max\{0, w(v) - \left[\mathcal{M}^w_\preceq(f)\right](v)\}\right\} . \tag{17}$$

Moreover, $\mathcal{A}^w_\preceq = \mathcal{A}_{\mathcal{F}^w_\preceq}$, i.e. $\mathcal{A}^w_\preceq$ is the canonical determinant of $\mathcal{F}^w_\preceq$.

Proof. To show (17), according to (14), (15) and (16), it is enough to prove that for each element v_j of V

$$\left[\mathcal{M}^w_{\preceq}(f)\right](v_j) = \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](v_j) \tag{18}$$

holds. We apply induction on j. (Recall that $v_1, v_2, \dots$ is a linear extension of reverse order $\succeq$ of $\preceq$.) If v_j is $\preceq$-maximal in V (e.g. if $j = 1$) then both sides of (18) equals 0 by definition. Assume now that v_j is not $\preceq$-maximal in V and (18) holds for $1, 2, \dots, j-1$, in particular for all elements of V above v_j. We have seen that $\mathcal{F}^w_{\preceq}(f)$ is $\preceq$-tame, so the right hand side of (18) is a lower bound of the left hand side. To show the opposite inequality, pick chain $v_j \prec c_1 \prec c_2 \prec \dots \prec c_k$ and weight function $f' \in L^w$ that achieves the maximum in (16). We may assume that $f'(c_1) > 0$. We distinguish two cases. If $f(c_1) + \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](c_1) \geq w(c_1)$ then by (14) and the $\preceq$-tame property of f' we have

$$\left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](v_j) \geq \left[\mathcal{F}^w_{\preceq}(f)\right](c_1) + \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](c_1) \geq w(c_1) \geq \\ f'(c_1) + f'(c_2) + \dots + f'(c_k) = \left[\mathcal{M}^w_{\preceq}(f)\right](v_j)\ .$$

Otherwise, if $f(c_1) + \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](c_1) < w(c_1)$ then $\left[\mathcal{F}^w_{\preceq}(f)\right](c_1) = f(c_1)$ again by (14). As (18) holds for c_1 by induction, we get

$$\left[\mathcal{M}^w_{\preceq}(f)\right](v_j) = f'(c_1) + f'(c_2) + \dots + f'(c_k) \leq f(c_1) + f'(c_2) + \dots + f'(c_k) \leq \\ f(c_1) + \left[\mathcal{M}^w_{\preceq}(f)\right](c_1) = f(c_1) + \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](c_1) = \left[\mathcal{F}^w_{\preceq}(f)\right](c_1) + \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](c_1) \leq \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](v_j)$$

This proves the induction step and justifies (17), hence $\mathcal{A}^w_{\preceq}$ is indeed a determinant of $\mathcal{F}^w_{\preceq}$. The fact that $\mathcal{A}^w_{\preceq}$ is the canonical determinant of $\mathcal{F}^w_{\preceq}$ immediately follows from definition (7) and the observation that the value $\left[\mathcal{A}^w_{\preceq}(f)\right](v)$ does not depend on $f(v)$. This finishes the proof. □

Lemma 4.4. *Mapping* $\mathcal{M}^w_{\preceq}$ *in* (16) *is monotone and choice function* $\mathcal{F}^w_{\preceq}$ *is substitutable and path-independent.*

Proof. Monotonicity of $\mathcal{M}^w_{\preceq}$ directly follows from its definition (16). Namely, if $f, g \in L^w$, $f \leq g$, $v \in V$ and weight function $f' \in L^w$ defines the value of $\left[\mathcal{M}^w_{\preceq}(f)\right](v)$ then $f' \leq f$ implies $f' \leq g$, so $\left[\mathcal{M}^w_{\preceq}(f)\right](v) \leq \left[\mathcal{M}^w_{\preceq}(g)\right](v)$ holds. The monotone property of $\mathcal{M}^w_{\preceq}$ and (17) immediately implies that $\mathcal{A}^w_{\preceq}$ is antitone, hence it is a determinant of some substitutable choice function $\mathcal{F}$ and $\mathcal{F} = \mathcal{F}^w_{\preceq}$ by to Lemma 4.3.

As $\mathcal{F}^w_{\preceq}(f)$ is $\preceq$-tame for any weight function $f \in L^w$, (18) implies that $\mathcal{M}^w_{\preceq}(f) = \widehat{\mathcal{F}^w_{\preceq}}(f) = \mathcal{M}^w_{\preceq}\left(\mathcal{F}^w_{\preceq}(f)\right)$ and consequently, $\mathcal{A}^w_{\preceq}(f) = \mathcal{A}^w_{\preceq}(\mathcal{F}^w_{\preceq}(f))$ holds for $\mathcal{A}^w_{\preceq}$ which is an antitone determinant of $\mathcal{F}^w_{\preceq}$ by Lemma 4.3. Path independence of $\mathcal{F}^w_{\preceq}$ follows directly from Lemma 3.2. □

The following lemma together with Lemma 3.6 immediately imply Theorem 2.2 of Aharoni, Berger and Gorelik.

Lemma 4.5. *Assume that $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ are posets, $w : V \to \mathbb{Z}_+$ is a demand function and weight function f is $\preceq_1$-tame and $\preceq_2$-tame. Then f is a weighted kernel if and only if f is $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stable.*

Proof. Assume first that f is a weighted kernel. To show that f is $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stable, it is enough to prove by Lemma 3.4 that

$$f = \min\left\{\mathcal{A}_{\mathcal{F}^w_{\preceq_1}}(f), \mathcal{A}_{\mathcal{F}^w_{\preceq_2}}(f)\right\} \ . \tag{19}$$

As f is $\preceq_1$-tame and $\preceq_2$-tame, $f = \mathcal{F}^w_{\preceq_1}(f) = \mathcal{F}^w_{\preceq_2}(f)$, so $f \le \min\left\{\mathcal{A}_{\mathcal{F}^w_{\preceq_1}}(f), A_{\mathcal{F}^w_{\preceq_2}}(f)\right\}$ by the definition of the determinant. Now pick any $v \in V$. As f is a weighted kernel, v is either $\preceq_1$-dominated or $\preceq_2$-dominated by f (or both). In the first case $\left[\mathcal{A}^w_{\preceq_1}(f)\right](v) = f(v)$ and in the second one $\left[\mathcal{A}^w_{\preceq_2}(f)\right](v) = f(v)$ holds, that is

$$f \ge \min\left\{\mathcal{A}^w_{\preceq_1}(f), \mathcal{A}^w_{\preceq_2}(f)\right\} = \min\left\{\mathcal{A}_{\mathcal{F}^w_{\preceq_1}}(f), \mathcal{A}_{\mathcal{F}^w_{\preceq_2}}(f)\right\} \ ,$$

by Lemma 4.3. This proves (19) hence the $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stability of f.

Now suppose that f is $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stable. As $f = \mathcal{F}^w_{\preceq_1}(f) = \mathcal{F}^w_{\preceq_2}(f)$, f is both $\preceq_1$-tame and $\preceq_2$-tame. Moreover, (19) holds by (11). Pick any $v \in V$. Now Lemma 4.3 implies

$$f(v) = \min\left\{\left[\mathcal{A}_{\mathcal{F}^w_{\preceq_1}}(f)\right](v), \left[\mathcal{A}_{\mathcal{F}^w_{\preceq_2}}(f)\right](v)\right\} = \min\left\{\left[\mathcal{A}^w_{\preceq_1}(f)\right](v), \left[\mathcal{A}^w_{\preceq_2}(f)\right](v)\right\}$$

So either $f(v) = \left[\mathcal{A}^w_{\preceq_1}(f)\right](v)$ or $f(v) = \left[\mathcal{A}^w_{\preceq_2}(f)\right](v)$ (or both). In the first case v is $\preceq_1$-dominated by f and in the second case v is $\preceq_2$-dominated by f according to the definition (7) of the canonical determinant and Lemma 3.3. This proves that f is indeed a weighted kernel. □

Theorem 3.8 and Lemma 4.5 yields the following generalization of Theorem 2.2.

Corollary 4.6. *For any posets $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ and for any demand function $w : V \to \mathbb{Z}_+$, there exists a $\preceq_1$-tame and $\preceq_2$-tame weighted kernel. Moreover, $\preceq_1$-tame and $\preceq_2$-tame weighted kernels form a complete lattice under $\le_{\mathcal{F}^w_{\preceq_1}}$.* □

Although $\mathcal{F}$-superiority in Corollary 4.6 is defined before Lemma 3.7 in a general setting, the following lemma provides a choice function free characterization of $\le_{\mathcal{F}^w_{\preceq_1}}$.

Lemma 4.7. *Assume that $P = (V, \preceq)$ is a poset, $w : V \to \mathbb{Z}_+$ is a demand function and f and g are $\preceq$-tame weights in L^w. Then $f \le_{\mathcal{F}^w_{\preceq}} g$ if and only if*

$$f(v) \ge g(v) \text{ holds whenever } v \text{ is not dominated by } f. \tag{20}$$

Proof. Assume first that$f \le_{\mathcal{F}^w_{\preceq}} g$, that is, $\mathcal{F}^w_{\preceq}(\max\{f, g\}) = f$. As $\mathcal{F}^w_{\preceq}$ is path-independent and substitutable by Lemma 4.4, we get

$$\mathcal{A}^w_{\preceq}(f) = \mathcal{A}^w_{\preceq}(\mathcal{F}(\max(f, g)) = \mathcal{A}^w_{\preceq}(\max\{f, g\})$$

by Lemma 3.3. To justify (20), suppose that v is not dominated by f. Consequently, $\left[\mathcal{A}^w_{\preceq}(f)\right](v) > f(v)$ as $\mathcal{A}^w_{\preceq}$ is the canonical determinant of $\mathcal{F}^w_{\preceq}$ by Lemma 4.3. Now

$$f(v) = \left[\mathcal{F}^w_{\preceq}(\max\{f,g\}\right](v) = \min\left\{\left[\mathcal{A}^w_{\preceq}(\max\{f,g\})\right](v), \max\{f(v),g(v)\}\right\} = \\ \min\left\{\left[\mathcal{A}^w_{\preceq}(f)\right](v), \max\{f(v),g(v)\}\right\} = \max\{f(v),g(v)\}\ ,$$

where the last equality holds because $\left[\mathcal{A}^w_{\preceq}(f)\right](v)$ is greater than $f(v)$ which is the left hand side. Therefore, $f(v) \geq g(v)$ and (20) follows.

To show the opposite implication, assume that (20) holds for $\preceq$-tame weight functions f and g and suppose indirectly that $f \not\leq_{\mathcal{F}^w_{\preceq}} g$, that is, $\left[\mathcal{F}^w_{\preceq}(\max\{f,g\})\right](v) \neq f(v)$ for some element v of V. Pick a $\preceq$-maximal v with the above property, that is,

$$\left[\mathcal{F}^w_{\preceq}(\max\{f,g\})\right](u) = f(u) \text{ whenever } v \prec u. \tag{21}$$

Now $\max\{f(v),g(v)\} \geq \left[\mathcal{F}^w_{\preceq}(\max\{f,g\})\right](v) > f(v)$ by (21) and (17), so $g(v) > f(v)$. Hence v is dominated by f due to (20). Now (21) and (14) implies that $\left[\mathcal{F}^w_{\preceq}(\max\{f,g\})\right](v) = f(v)$, a contradiction. This concludes the proof. □

5 Further generalizations

Our approach can be applied to prove other generalizations of the Sands Sauer Woodrow result than the one by Aharoni Berger and Gorelik. To do so, we may define other lattices than lattice L^w we used in Section 4. A natural extension is if we define a "continuous" version of L^w on functions $f : V \to \mathbb{R}_+$ and we allow the demand function $w : V \to \mathbb{R}_+$ to be nonintegral. Aharoni Berger and Gorelik remark in [1] that a nonintegral analogue of Theorem 2.2 holds by the well-known Scarf lemma [8]. Note that we get the same result by applying our framework. To do so, we only need to copy the argument word by word in Section 4 and replacing lattice L^w by $(L^w)' := \{f : V \to \mathbb{R}_+, f \leq w\}$ that is also an infinitely distributive complete lattice. As a side product of this approach, one can deduce the lattice property of weighted kernels that does not seem to follow from the application of the Scarf lemma.

We may also use our framework to deduce the many-to-one and many-to-many generalizations of the stable marriage theorem of Gale and Shapley. There, we have given a bipartite graph G with color classes U and V and a quota function $b : U \cup V \to \mathbb{Z}_+$ and each vertex v of G has a linear preference order $\preceq_v$ on the set $E(v)$ of edges that are incident with v. A subset M of $E(G)$ is a *b-matching* if each vertex v of G is incident with at most $b(v)$ edges of M. A b-matching M is *stable* if for any edge $e = uv$ of $E(G) \setminus M$ there exist either $b(u)$ edges of M that are all preferred to e by u or there exist $b(v)$ edges of M that are all preferred to e by v (or both conditions hold). It is easy to see that if $b \equiv \mathbb{1}$ then a stable b-matching is exactly a stable matching. The generalization of the stable marriage theorem of Gale and Shapley states that for any quota function b, there exists a stable b-matching. We can deduce this result from our framework by defining two partial orders on ground set $E(G)$. The first order corresponds to preferences in U and the second to preferences in V. Define two demand functions w_1 and w_2 on E such that if edge e has end vertices u

and v in U and V, respectively then $w_1(e) = b(u)$ and $w_2(e) = b(v)$. We work on the lattice $L^{\mathbb{1}}$ (hence all weight functions are characteristic functions of sets of edges) and define choice functions $\mathcal{F}_1$ and $\mathcal{F}_2$ by determinants $\mathcal{A}^{w_1}_{\preceq_1}$ and $\mathcal{A}^{w_2}_{\preceq_2}$, respectively. (As we work in $L^{\mathbb{1}}$, these choice functions can be interpreted as ordinary set choice functions). In this model, it is easy to see that characteristic vectors of stable b-matchings are exactly the $\mathcal{F}_1\mathcal{F}_2$-stable weight functions of $L^{\mathbb{1}}$ that form a nonempty complete lattice by Theorem 2.3 by Tarski.

But playing with the underlying lattice is not the only option to find a generalization of Theorem 2.1. We may work on our well known lattice L^w with path-independent substitutable choice functions other than $\mathcal{F}^w_{\preceq}$. One possibility is to replace the sum operation with maximization in the definition of $\mathcal{F}^w_{\preceq}$. More precisely, if $P = (V, \preceq)$ is a poset such that $v_1, v_2, \ldots, v_n$ is a linear extension of $\succeq$, $w : V \to \mathbb{Z}_+$ is a demand function and $f \in L^w$ is a weight function then observe that (14) can be rewritten as

$$\left[\mathcal{F}^w_{\preceq}(f)\right](v_i) = \begin{cases} 0 & \text{if } \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](v_j) \geq w(v_i) \\ \min\left\{f(v_i), w(v_i) - \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](v_j)\right\} & \text{otherwise.} \end{cases} \qquad (22)$$

Now consider the following modification of (22)

$$\left[\mathcal{G}^w_{\preceq}(f)\right](v_i) = \begin{cases} 0 & \text{if } \max\left\{\left[\mathcal{G}^w_{\preceq}(f)\right](v_j) : v_i \prec v_j\right\} \geq w(v_i) \\ \min\left\{f(v_i), w(v_i)\right\} & \text{otherwise} \end{cases} \qquad (23)$$

Similarly as we did in Section 4, one can prove that $\mathcal{G}^w_{\preceq}$ is substitutable and path-independent. To motivate choice function $\mathcal{G}^w_{\preceq}$, let us say that a weight function f of L^w is $\preceq$*-reasonable* if $f(v) > 0$ implies that $f(u) \leq w(v)$ holds whenever $v \preceq u$. We say that weight function f $\preceq$*-covers* v if there is an element u of V such that $v \preceq u$ and $f(u) \geq w(v)$. With these notions, for every v_i choice function $\mathcal{G}^w_{\preceq}$ picks the maximum value $\left[\mathcal{G}^w_{\preceq}(f)\right](v_i)$ such that choice $\mathcal{G}^w_{\preceq}(f)$ is $\preceq$-reasonable. Theorem 3.8 yields the following generalization of Theorem 2.2.

Theorem 5.1. *For any pair $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ of posets and for any demand function $w : V \to \mathbb{Z}_+$, there exists a weight function f such that f is both $\preceq_1$-reasonable and $\preceq_2$-reasonable and any element v of V is $\preceq_1$-covered or $\preceq_2$-covered by f.* □

We may also mix the two kinds of choice functions we have seen so far.

Theorem 5.2. *For any pair $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ of posets and for any demand function $w : V \to \mathbb{Z}_+$, there exists a weight function f such that f is $\preceq_1$-tame and $\preceq_2$-reasonable and any element v of V is $\preceq_1$-dominated or $\preceq_2$-covered by f.* □

Clearly, Theorem 2.1 is a special case of Theorems 5.1, 5.2 and 4.6 for $w \equiv \mathbb{1}$. Note that one can define further interesting path-independent substitutable choice functions that provide nontrivial result when plugged into Theorem 3.8. For example, if $0 < \alpha \leq 1$ then we can modify the definition (14) as

$$\left[\mathcal{F}^{w,\alpha}_{\preceq}(f)\right](v_i) = \min\left\{f(v_i), \max\left\{0, w(v_i) - \alpha \cdot \left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](v_i)\right\}\right\} .$$

Lemma 5.3. *For any poset $P = (V, \preceq)$, any demand function $w : V \to \mathbb{Z}_+$ and for any $0 < \alpha \leq 1$, choice function $\mathcal{F}_{\preceq}^{w,\alpha} : L^w \to L^w$ is substitutable and path-independent.*

Sketch of the proof. Observe that

$$\left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right](v) = \max_{v \prec u} \left\{ \left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right](u) + \left[\mathcal{F}_{\preceq}^{w,\alpha}(f)\right](u) \right\} =$$
$$\max_{v \prec u} \left\{ \left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right](u) + \min\left\{ f(u), \max\left\{ 0, w(u) - \alpha \cdot \left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right](u) \right\}\right\}\right\} =$$
$$\max_{v \prec u} \left\{ \min\left\{ \left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right](u) + f(u), \max\left\{ \left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right](u), w(u) + (1-\alpha) \cdot \left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right](u) \right\}\right\}\right\}$$

From this formula, it is easy to prove by induction on $|V|$ that $\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}$ is monotone in f. Consequently, determinant $\mathcal{A}_{\preceq}^{w,\alpha}$ of $\mathcal{F}_{\preceq}^{w,\alpha}$ is antitone in f where $\left[\mathcal{A}_{\preceq}^{w,\alpha}\right](f) = \max\left\{0, w - \alpha \cdot \left[\widehat{\mathcal{F}_{\preceq}^{w,\alpha}}(f)\right]\right\}$. So $\mathcal{F}_{\preceq}^{w,\alpha}$ is substitutable. Path-independence of $\mathcal{F}_{\preceq}^{w,\alpha}$ follows the same way as we proved it for $\mathcal{F}_{\preceq}^{w}$: the value of the $\left[\mathcal{A}_{\preceq}^{w,\alpha}(f)\right](v)$ depends only on the $\mathcal{F}_{\preceq}^{w}(f)$-values of elements u with $v \prec u$, hence $\mathcal{A}_{\preceq}^{w,\alpha}(f) = \mathcal{A}_{\preceq}^{w,\alpha}(\mathcal{F}_{\preceq}^{w,\alpha}(f))$ holds for any $f \in L^w$, and $\mathcal{F}_{\preceq}^{w,\alpha}$ is path-independent by Lemma 3.2. □

We encourage the motivated reader to construct further nontrivial examples of substitutable path-independent choice functions on L^w.

6 Conclusion

In this work, we described a fairly general framework that allowed us to generalize a result of Aharoni, Berger and Gorelik. Theorem 3.8, our main result is based on Tarski's fixed point theorem and it is a formal generalization of Blair's theorem on the lattice structure of stable matchings to lattice-based choice functions. These choice functions act on lattices unlike "traditional" ones that are plain set functions. It turned out that several notions and results can be generalized to our framework, like substitutability or path-independence. The key to our results is a novel approach that culminates in the definition of substitutability of choice functions by so-called determinants. It turns out that these determinants are particularly handy tools when we work with choice functions. An example is the proof of Theorem 3.8 that seems to be more direct than other known proofs. Substitutability of a choice function is the antitone property of its determinant and path-independence also have a practical characterization in terms of determinants. Here we note that this determinant-approach is not new. Fleiner Jankó and Tamura used a very similar one in [5].

A special case of Theorem 3.8 is an extension of the above mentioned Aharoni Berger Gorelik result: beyond proving the result itself, we also point out that weighted kernels do not only exist but form a lattice under a natural partial order. In this work, we also demonstrated that by plugging in other substitutable and path-independent choice functions in Theorem 3.8, one can manufacture further nontrivial results that generalize the stable marriage theorem of Gale and Shapley. We also demonstrated that the continuous version of the Aharoni Berger Gorelik result one does not require strong tools like Scarf's lemma, as one can prove a strengthening of it in our framework.

Although in this work, we did not emphasize algorithmic issues, we did point out that the well-known deferred acceptance algorithm of Gale and Shapley turn out to be a monotone function iteration in our framework, hence if the common ground set of the lattices we work with is finite then it provides an algorithm to find the extremal stable solutions. The number of evaluations of choice functions is at most twice the length of the longest chain in the lattice, so for L^w we calculate the choices at most $\widetilde{w}(V) = \sum\{w(v) : v \in V\}$ times.

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References

[1] Ron Aharoni, Eli Berger, and Irina Gorelik. Kernels in Weighted Digraphs. *Order*, 31(1):35–43, 2014.

[2] Charles Blair. The lattice structure of the set of stable matchings with multiple partners. *Math. Oper. Res.*, 13(4):619–628, 1988.

[3] Tamás Fleiner. A fixed point approach to stable matchings and some applications. *Mathematics of Operations Research*, 28(1):103–126, 2003.

[4] Tamás Fleiner. Stable and crossing structures, August, 2000. PhD dissertation, `http://www.renyi.hu/~fleiner`.

[5] Tamás Fleiner, Zsuzsanna Jankó, and Akihisa Tamura. Stability of generalized network flows. Unpublished manuscript, 2012.

[6] D. Gale and L.S. Shapley. College admissions and stability of marriage. *Amer. Math. Monthly*, 69(1):9–15, 1962.

[7] B. Sands, N. Sauer, and R. Woodrow. On monochromatic paths in edge-coloured digraphs. *J. Combin. Theory Ser. B*, 33(3):271–275, 1982.

[8] Herbert E. Scarf. The core of an N person game. *Econometrica*, 35:50–69, 1967.

[9] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific J. of Math*, 5:285–310, 1955.