Packing countably many branchings with prescribed root-sets in digraphs without backward-infinite paths

Attila Joó

July 2015
Packing countably many branchings with prescribed root-sets in digraphs without backward-infinite paths

Attila Joó

Abstract

We generalize an unpublished result of C. Thomassen. Let $D = (V, A)$ be a digraph without backward-infinite paths and let $\{V_i\}_{i \in \mathbb{N}}$ be a multiset of subsets of $V$. We show that if all $v \in V$ is simultaneously reachable from the sets $V_i$ by edge-disjoint paths, then there exists a system of edge-disjoint spanning branchings $\{B_i\}_{i \in \mathbb{N}}$ in $D$ where the root-set of $B_i$ is $V_i$.

1 Notations and background

The digraphs considered here may have multiple edges and arbitrary size. For $X \subseteq V$ let $\text{in}_D(X)$ and $\text{out}_D(X)$ be the set of ingoing and outgoing edges respectively of $X$ in $D$, and let $\varrho_D(X), \delta_D(X)$ be their respective cardinalities. The paths in this paper are directed, finite, simple paths. We say that the path $P$ goes from $X$ to $Y$ if $V(P) \cap X = \{\text{start}(P)\}$ and $V(P) \cap Y = \{\text{end}(P)\}$ ($\text{start}(P) = \text{end}(P)$ is allowed).

Let $\mathcal{V} = \{V_i\}_{i < \lambda}$ be a multiset of subsets of $V$, where $\lambda$ is a cardinal, and let $D = (V, A)$ be a digraph. We say that $v \in V$ is simultaneously reachable from $\mathcal{V}$ in $D$ if there is a system of edge-disjoint paths $\{P_i\}_{i < \lambda}$ in $D$ such that $P_i$ goes from $V_i$ to $v$. The system $\mathcal{V}$ satisfies the path condition in $D$ if all $v \in V$ is simultaneously reachable from $\mathcal{V}$. A digraph $B_0 = (U, E)$ is a branching if for all $u \in U$ there is a unique path from the root-set $\{w \in U : \varrho_{B_0}(w) = 0\}$ to $u$ in $B_0$. We call $B$ a multibranching in $D$ with respect to $\mathcal{V}$ if $B = \{B_i\}_{i < \lambda}$, where $B_i$'s are edge-disjoint branchings in $D$ and the root-set of $B_i$ is $V_i$. If in addition all the $B_i$'s are spanning branchings of $D$ (i.e. their vertex-set is $V$), then we call $B$ a spanning multibranching with respect to $\mathcal{V}$ in $D$. If there is a spanning multibranching $B$ in $D$ with respect to $\mathcal{V}$, then the system $\mathcal{V}$ obviously satisfies the path condition in $D$, since if $v \in V$, then $B_i$ contains a path $P_i$ from $V_i$ to $v$ ($i < \lambda$), and these paths are pairwise edge-disjoint because the branchings are pairwise edge-disjoint. If $\lambda$ is finite, then by Menger’s theorem one can formulate the path condition in the following equivalent form:

$$\forall X \subset V \left( X \neq \emptyset \implies \varrho_D(X) \geq |\{ i < \lambda : V_i \cap X = \emptyset \}| \right).$$  \hspace{1cm} (1)

July 2015
The (strong form of) Edmonds’ branching theorem (see [2] p. 349 Theorem 10.2.1) states that in the finite case ($\lambda$ and $D$ are finite) condition (1) is enough to assure the existence of a spanning multibranching. R. Aharoni and C. Thomassen proved by a construction (see [1]) that this theorem fails for infinite digraphs. Even so, one can weaken the finiteness condition for $D$ in Edmonds’ branching theorem, namely it is enough to assume that $D$ does not contain backward-infinite paths as showed by C. Thomassen (unpublished). The exclusion of backward-infinite paths can be replaced by exclusion of forward-infinite paths as we proved in [4]. In this paper we prove the strengthening of Thomassen’s result to countably many branchings, namely:

**Theorem 1.1.** Let $\{V_i\}_{i<\lambda}$ be a multiset of subsets of $V$ that satisfies the path condition in the digraph $D = (V, A)$. Assume that $D$ does not contain backward-infinite paths. Then there is a spanning multibranching in $D$ with respect to $\{V_i\}_{i<\lambda}$, i.e. there is a system of edge-disjoint spanning branchings $\{B_i\}_{i<\lambda}$ in $D$ such that the root-set of $B_i$ is $V_i$.

**Remark 1.2.** Instead of excluding backward-infinite paths it is enough to assume that any backward-infinite path of $D$ intersects all the $V_i$’s. See Remark [4.3].

As we have already mentioned Thomassen proved the theorem above for finitely many $V_i$’s (which motivated us to investigate this topic). The main idea of Thomassen’s proof is the following: construct first a spanning subgraph $D' = (V, A')$ of $D$ such that $D'$ also satisfies condition (1) and all vertices of $D'$ have finite in-degrees. After that, one can build the desired spanning multibranching in $D'$ using Edmonds’ branching theorem and compactness arguments. In the countably infinite case we have to use a very different approach.

## 2 Proof of the main Theorem

The key of the proof is the following lemma which actually works with arbitrary cardinal $\lambda$.

**Lemma 2.1.** Let $\{V_i\}_{i<\lambda}$ be a multiset of subsets of $V$ that satisfies the path condition in the digraph $D = (V, A)$. Assume that $D$ does not contain backward-infinite paths. Then for any $j < \lambda$ and $v \in V \setminus V_j$ there is a path $P$ from $V_0$ to $v$ in $D$ such that the path condition holds for $\{V_i'\}_{i<\lambda}$ in $D - A(P)$ where $V_i' = \begin{cases} V_i \cup V(P) & \text{if } i = j \\ V_i & \text{otherwise.} \end{cases}$

We show first how Theorem 1.1 follows from Lemma 2.1. If $B$ is a multibranching in $D$ with respect to $\{V_i\}_{i<\lambda}$, then let $D \setminus B = (V, A \setminus \bigcup_{i<\lambda} A(B_i))$. We say that the multibranching $B$ satisfies the path condition if path condition holds in $D \setminus B$ with respect to $\{V(B_i)\}_{i<\lambda}$. If $B$ satisfies the path condition, then Lemma 2.1 makes possible to extend a prescribed $B_i$ with a path in such a way that $B_i$ reaches a prescribed vertex and the new multibranching still satisfies the path condition. The plan is to start with the multibranching $\{(V_i, \emptyset)\}_{i<\lambda}$ and extend it to spanning
multibranching by a transfinite recursion using Lemma 2.1 in every successor step and
taking union in limit steps.

The only arising problem is that we may violate the path condition at limit steps.
We can handle easily this problem. Since we have just countably many branchings, we
can organize the recursion in such a way that if we extend a branching with a vertex
\( v \) in some step, then before the next limit step we extend with \( v \) all the branchings.
We claim that this ensures the path condition after limit steps.

Indeed, let \( \mathcal{B} \) be the multibranching that we have after some limit step and let \( u \in V \)
be arbitrary. We may fix a system of edge-disjoint paths \( \{ P_i \}_{i \in \mathbb{N}} \) in \( D \) such that \( P_i \)
goes from \( V_i \) to \( u \). Let \( v_i \) be the first vertex on \( P_i \) such that the terminal segment
\( P_i' \) of \( P_i \) that starts at \( v_i \) is still a path in \( D \setminus \mathcal{B} \). It is enough to show that \( v_i \in V(\mathcal{B}_i) \).
If \( v_i = \text{start}(P_i) \), then it is clear since \( \text{start}(P_i) \in V_i \subset V(\mathcal{B}_i) \). If \( v_i \neq \text{start}(P_i) \), then
by choice of \( v_i \) there is a successor step in which we extend some branching with \( v_i \),
but then before the next limit step we put \( v_i \) all the branchings thus \( v_i \in V(\mathcal{B}_i) \).

\( \blacksquare \)

**Question 2.2.** For uncountable \( \lambda \) the cheap trick that we use for the limit steps
works no more and we do not know yet if Theorem 1.1 remains true with arbitrary
cardinal instead of \( \mathbb{N} \).

Without loss of generality it is enough to prove Lemma 2.1 for \( j = 0 \). Before
the proof we have to generalize some phenomena that are well-known from finite
branching-packing theorems.

### 3 Generalization of tight and dangerous sets

In context of Edmonds’ branching theorem a set \( \emptyset \neq X \subset V \) is usually called tight
if \( g_0(X) = \{ i < \lambda : V_i \cap X \neq \emptyset \} \). For infinite \( \lambda \) the literal, cardinality-based general-
alization of tightness is unusable. In this section we give another generalization that
turns out to be useful and keeps all the nice properties of tight sets that are known
from the finite case.

Suppose, that \( \mathcal{V} = \{ V_i \}_{i<\lambda} \) satisfies the path condition in a digraph \( D = (V, A) \).
We call the set \( \emptyset \neq X \subset V \) tight (with respect to \( \mathcal{V} \) in \( D \)) if whenever \( \{ P_i \}_{i<\lambda} \)
is a system of edge-disjoint paths in \( D \) such that \( P_i \) goes from \( V_i \) to \( X \), then the paths \( P_i \)
necessarily use all the ingoing edges of \( X \) i.e. \( \text{in}_D(X) \subset \bigcup_{i<\lambda} A(P_i) \). (If \( V_i \cap X \neq \emptyset \)
then one may choose \( P_i \) as a path consisting of just one vertex, thus the definition is
really about those \( i \)'s for which \( X \cap V_i = \emptyset \).) A \( B \subset V \) is dangerous if it is tight
and \( V_0 \cap B \neq \emptyset \).

In this section we prove three simple consequences of dangerousness.

**Corollary 3.1.** Let \( B \) be dangerous and let \( v \in B \) arbitrary. If \( \{ P_i \}_{i<\lambda} \)
is a system of edge-disjoint paths in \( D \) such that \( P_i \) goes from \( V_i \) to \( v \), then \( V(P_0) \subset B \).
Furthermore if \( V_i \cap B = \emptyset \), then \( P_i \) uses exactly one ingoing edge of \( B \) i.e. \( |A(P_i) \cap \text{in}_D(B)| = 1 \).

**Proof:** Assume, to the contrary, that \( V(P_0) \nsubseteq B \). Then \( P_0 \) uses some edge \( e \in \text{in}_D(B) \).
Replace \( P_0 \) with a path \( P'_0 \) that consists of only one vertex \( u \) where \( u \in V_0 \cap B \). The
Section 4. Proof of the key-Lemma

modified path-system no more uses e but all its members have a vertex in $B$ hence by taking their appropriate initial segments we get a contradiction with the tightness of $B$.

For the second part of the proposition, if $|A(P_i) \cap \text{in}_D(B)| > 1$ holds, then by replacing $P_i$ by its appropriate initial segment we get a contradiction in similar way. \hfill \bullet

**Corollary 3.2.** If $B_0, B_1 \subset V$ are dangerous sets with nonempty intersection, then $B_0 \cap B_1$ is also dangerous.

**Proof:** Let $\{P_i\}_{i<\lambda}$ be a system of edge-disjoint paths in $D$ such that $P_i$ goes from $V_i$ to $B_0 \cap B_1$. Suppose, to the contrary, that the paths do not use an edge $e \in \text{in}_D(B_0 \cap B_1)$. By symmetry we may assume that $e \in \text{in}_D(B_0)$. By taking appropriate initial segments of the paths we get a contradiction with tightness of $B_0$. Therefore $B_0 \cap B_1$ is tight.

Let $v \in B_0 \cap B_1$ and let $\{Q_i\}_{i<\lambda}$ be a system of edge-disjoint paths in $D$ such that $Q_i$ goes from $V_i$ to $v$. By using Corollary 3.1 with $B_0$ and with $B_1$ separately we get $V(Q_0) \subset B_0 \cap B_1$ hence $\text{start}(Q_0) \in V_0 \cap B_0 \cap B_1$ thus $B_0 \cap B_1$ is dangerous. \hfill \bullet

**Remark 3.3.** It is not too hard to prove that under the conditions of the corollary above $B_0 \cup B_1$ is also dangerous and there are no edges between $B_0 \setminus B_1$ and $B_1 \setminus B_0$ in any direction as it was the expectation from the finite case. In the current paper we do not need these facts.

For multisets $\mathcal{V}$ and $\mathcal{T}$ we denote by $\mathcal{V} \cup \mathcal{T}$ the multiset where the multiplicity of an element is the sum of its multiplicities in $\mathcal{V}$ and $\mathcal{T}$.

**Corollary 3.4.** Let $B$ be a dangerous set in $D$ with respect to $\mathcal{V} = \{V_i\}_{i<\lambda}$. Then the multiset $\mathcal{V}[B] \overset{\text{def}}{=} \{\text{end}(e) : e \in \text{in}_D(B)\} \cup \{V_i \cap B : i < \lambda, \ V_i \cap B \neq \emptyset\}$ satisfies the path condition in $D[B]$. Furthermore a set $X \subset B$ is dangerous with respect to $\mathcal{V}$ in $D$ iff $X$ is dangerous with respect to $\mathcal{V}[B]$ in $D[B]$.

**Proof:** Let $v \in B$ be arbitrary. The system $\mathcal{V}$ satisfies the path condition in $D$ thus we may fix a system of edge-disjoint paths $\{P_i\}_{i<\lambda}$ in $D$ such that $P_i$ goes from $V_i$ to $v$. The definition of tightness and Proposition 3.1 shows that the appropriate terminal segments of paths $\{P_i\}_{i<\lambda}$ certify that $v$ is simultaneously reachable from $\mathcal{V}[B]$ in $D[B]$.

Assume that $X \subset B$ is not dangerous with respect to $\mathcal{V}$ in $D$ and the path-system $\{Q_i\}_{i<\lambda}$ shows it. Cut the initial segments of the $Q_i$’s that are not in $B$. The resulting system shows that $X$ is not dangerous with respect to $\mathcal{V}[B]$ in $D[B]$. The other direction is similar. \hfill \bullet

4 Proof of the key-Lemma

Assume, seeking contradiction, that Lemma 2.1 is false and $D_0, V_0, u_0$ witness it (where $\mathcal{V}_0 = \{V_i\}_{i<\lambda}$ i.e. there is no path $P$ from $V_0$ to $u_0$ in $D_0$ such that the path condition holds for $\{V_0 \cup V(P)\} \cup \{V_i\}_{i \neq \lambda}$ in $D - A(P)$).
Let \( \{P_i\}_{i<\lambda} \) be a system of edge-disjoint paths in \( D_0 \) such that \( P_i \) goes from \( V_i \) to \( u_0 \). Consider the first edge \( e \) of \( P_0 \).

In the first case, assume that \( \{V_0 \cup \{\mathsf{end}(e)\} \cup \{V_i\}_{0<i<\lambda}\} \) satisfies the path condition in \( D_0 - e \). Then let \( V_1 = \{V_0 \cup \{\mathsf{end}(e)\} \cup \{V_i\}_{0<i<\lambda}\} \) and \( D_1 = D_0 - e \) and continue the process with the following edge of \( P_0 \). Otherwise extending with \( e \) violates the path condition. We need here a claim that we will prove in the next section.

**Claim 4.1.** Let \( \mathcal{V} = \{V_i\}_{i<\lambda} \) be a multiset of subsets of \( V \) that satisfies the path condition in the digraph \( D = (V,A) \). Let \( e \in \mathsf{out}_D(V_0) \) and let \( \mathcal{V}' = \{V'_i\}_{i<\lambda} \) where

\[
V'_i = \begin{cases} V_i \cup \{\mathsf{end}(e)\} & \text{if } i = 0 \\ V_i & \text{otherwise.} \end{cases}
\]

Then \( \mathcal{V}' \) violates the path condition in \( D' = D - e \) iff there exists a \( B \subset V \) such that \( B \) is dangerous with respect to \( \mathcal{V} \) in \( D \) and \( e \in \mathsf{in}_D(B) \).

Hence by Claim 4.1 there is a set \( B \) such that \( e \in \mathsf{in}_{D_0}(B) \), and \( B \) is dangerous with respect to \( V_0 \) in \( D_0 \). Note that \( V(P_0) \not\subset B \), thus by Corollary 3.4 \( u_0 \not\in B \). Let \( u_1 \) be the last vertex of \( P_0 \) which is in \( B \).

In the second case, we assume that the triple \( D_0[B], V_0[B], u_1 \) (see Corollary 3.4) is not a counterexample to Lemma 2.1. Let \( V_1 = \{V_0 \cup \mathcal{V}(P)\} \cup \{V_i\}_{0<i<\lambda} \) and \( D_1 = D_0 - P \), where \( P \) is a path in \( D_0[B] \) that goes from \( V_0 \cap B \) to \( u_1 \), and the extension does not violate the path condition with respect to \( V_0[B] \) in \( D_0[B] \). We claim that extending with path \( P \) does not violate the path condition with respect to \( V_0 \) in \( D_0 \) either (i.e. \( V_1 \) satisfies the path condition in \( D_1 \)). Suppose, to the contrary, that it does. Assume first that \( P \) consists of only one edge \( f \). Then by Claim 4.1 there is a set \( B' \) such that \( B' \) is dangerous with respect to \( V_0 \) and \( f \in \mathsf{in}_{D_0}(B') \). Then \( f \in \mathsf{in}_{D_0}(B \cap B') \), where \( B \cap B' \) is dangerous with respect to \( V_0 \) in \( D_0 \) by Corollary 3.2, thus by Corollary 3.4 \( B \cap B' \) is dangerous with respect to \( V_0[B] \) in \( D_0[B] \) too, therefore by Claim 4.1 extending with \( P \) does not keep the path condition with respect to \( V_0[B] \) in \( D_0[B] \) which contradicts the choice of \( P \). If \( 1 < |A(P)| \) then we may decompose the extension with \( P \) to a finite sequence of one-edge extensions and arguing in a similar way. Let \( P_0' \) be the terminal segment of \( P_0 \) that starts at \( u_1 \). Unfortunately path \( P \) may use edges from the \( P_i \)'s, that is why we need the following Proposition.

**Proposition 4.2.** The set \( \{P_i\}_{i<\lambda} \) can be extended to a system of edge-disjoint paths \( \{P_i\}_{i<\lambda} \) in \( D_1 \) such that \( P_i \) goes from \( V_i \) to \( u_0 \) for \( 0 \neq i < \lambda \).

Proposition 4.2 (that we will prove in the end of this section) makes possible to continue the process with the following edge i.e. the outgoing edge of \( u_1 \) in \( P_0' \).

In the remaining third case, the triple \( D_0[B], V_0[B], u_1 \) is a counterexample to Lemma 2.1. Note that this third case necessarily occurs after at most \( |A(P_0)| \) steps otherwise we would reach \( u_0 \) (since in the first two cases we reach a new vertex along \( P_0 \) and then the deleted edges would contain a unique path \( P \) from \( V_0 \) to \( u_0 \) that satisfies what Lemma 2.1 demands contradicting with the counterexample nature of the triple \( D_0, V_0, u_0 \).

To simplify the notations we may assume that the third case arise at the first step. Let us summarize that we get. We have a nontrivial path (namely \( P_0' \)) in \( D_0 \) from
the unreachable vertex of the new counterexample \((u_1)\) to the unreachable vertex of the original counterexample \((u_0)\) such that the vertex-set of the new counterexample \((B)\) contains just the first vertex of this path. Now we may forget the process above and just concentrate on the fact that we are able to find such a new counterexample from an arbitrary counterexample. By iterating it recursively we can find a sequence of counterexample triples \(D_i, V_i, u_i\) \((i \in \mathbb{N})\) where \(D_{i+1}\) is a subgraph of \(D_i\) and \(u_i \in V(D_i) \setminus V(D_{i+1})\) (see figure 1). We also find a sequence of paths \(Q_i\) \((i \in \mathbb{N})\) such that \(Q_i\) is a path in \(D_i\) and it goes from \(u_{i+1}\) to \(u_i\) in such a way that \(V(Q_i) \cap V(D_{i+1}) = \{u_{i+1}\}\). By uniting the paths \(Q_i\) we get a backward-infinite path contradicting with the assumption about \(D_0\).

![Figure 1: The sequence of the counterexample triples and the construction of the backward-infinite path in \(D_0\).](image)

**Remark 4.3.** If \(D_0\) may contain backward-infinite paths but all of them intersects all the \(V_i\)'s, then the proof above still works because we also find a backward-infinite path in \(D_1 = D_0[B]\) and it is still a contradiction since \(B\) is disjoint from some \(V_i\) (\(B\) is dangerous and \(e \in \text{in}_{D_0}(B)\)).

To prove Proposition 4.2 we need the following version of a well-known technique developed by L. R. Ford and D. R. Fulkerson (see [3]).

**Proposition 4.4.** Let \(\{P_i\}_{i \in I}\) be a system of edge-disjoint \(s \to t\) paths in a digraph \(D\) (where \(s \neq t \in V(D)\)) and denote the first and the last edge of \(P_i\) by \(e_i\) and by \(f_i\) respectively. Let \(\bar{D}\) be the digraph that we get by change the direction of edges \(\cup_{i \in I} A(P_i)\) in \(D\). We call these edges **backward-edges** of \(\bar{D}\) and we call **forward-edges** the others. Denote by \(U\) the set of the vertices in \(V(D)\) that are reachable in \(\bar{D}\) from \(s\). If \(t \in U\) and path \(R\) certifies it, where the first edge of \(R\) is \(e\) and the last edge of \(R\) is \(f\), then there is a system of edge-disjoint paths \(\{Q_j\}_{j \in J}\) in \(D\) such that the set of the first and the set of the last edges of paths \(Q_j\) \((j \in J)\) are \(\{e_i\}_{i \in I} \cup \{e\}\) and \(\{f_i\}_{i \in I} \cup \{f\}\) respectively.

If \(t \notin U\), then there is a system of edge-disjoint paths \(\{Q_h : h \in \text{out}_D(U)\}\) from \(U\) to \(t\) in \(D\) such that the first edge of \(Q_h\) is \(h\).

**Proof:** Consider a finite subgraph \(D'\) of \(D\) that contains \(R\) and all the \(P_i\)'s, say \(P_1, \ldots, P_k\), that give a backward-edge to \(R\). The \(P_1, \ldots, P_k\) paths determinate a flow
Section 4. Proof of the key-Lemma

in \( D' \) with amount \( k \) with respect to the constant 1 upper bound on the edges. By the technique of Ford and Fulkerson one can get by using \( R \) a flow in \( D' \) that has amount \( k + 1 \). By decomposing this flow to \( k + 1 \) edge-disjoint \( s \to t \) paths and keeping the untouched \( P_i \)'s we get the desired system \( \{ Q_i \}_{i \in I} \). For the second part of the Proposition the terminal segments of the paths \( P_i \) that go from \( U \) to \( t \) will be suitable, it follows immediately from the construction of \( \tilde{D} \) and from the definition of \( U \).

Now we are able to prove Proposition 4.2.

Proof of Proposition 4.2. If \( A(P_i) \cap \text{span}_D(B) = \emptyset \) for some \( 0 < i < \lambda \), then let \( P'_i = P_i \). Consider now \( I = \{ 0 < i < \lambda : A(P_i) \cap \text{span}_D(B) = \emptyset \} \). All but at most \( |A(P)| \) from the paths \( \{ P_i \}_{i \in I} \) are still paths in \( D_1 \). To simplify the notations we assume that the problematic paths (i.e. paths that has edge in \( A(P) \)) are \( P_1, \ldots, P_k \).

For \( i \in I \) let \( u_i \) denote the first and \( v_i \) the last intersection of \( P_i \) with \( B \). We construct a digraph \( G \) starting with \( D_1[B] \) (see figure 2). If for some \( w, z \in B \) there is an \( i \in I \) such that \( P_i \) has a \( w \to z \) segment such that the interior vertices of the segment are not in \( B \), then draw a new \( e_{iwz} \) edge from \( w \) to \( z \). Pick a new vertex \( t \) and draw an edge \( f_i \) from \( v_i \) to \( t \) \((i \in I) \). For all \( i \in I \) pick a new vertex \( s_i \). If path \( P_i \) starts inside \( B \), then draw one edge from \( s_i \) to all the elements of \( V_i \cap B \), otherwise draw an edge from \( s_i \) to \( u_i \). Construction of \( G \) is complete.

\[ D_1[B] \]

\[ s_i, s_i \]

\[ v_i = v_i \]

\[ f_i, f_i \]

\[ w, z \]

\[ e_{iwz} \]

\[ w, z \]

\[ D_1[B] \]

\[ s_i, s_i \]

\[ v_i = v_i \]

\[ f_i, f_i \]

\[ w, z \]

\[ e_{iwz} \]

\[ w, z \]

\[ D_1[B] \]

Figure 2: Construction of \( G \) from \( D_1[B] \). Here \( B \cap V_{i_3} \neq \emptyset \) but \( B \cap V_{i_2} = B \cap V_{i_3} = \emptyset \).

It is enough to show that there is a system of edge-disjoint \( s \to t \) paths \( \{ Q_i \}_{i \in I} \) in \( G \) such that \( Q_i \)'s first edge is \( ss_i \) because then we can construct the desired paths \( \{ P'_i \}_{i \in I} \) in a natural way. Indeed, let \( i \in I \) arbitrary and assume first that \( P_i \) starts in \( B \cap V_i \). Let the last edge of \( Q_i \) is \( f_j \) for some \( j \in I \). Let \( Q'_i \) the path that we get from \( Q_i \) by deleting the first two \((s \) and \( s_i) \) and the last \((t) \) vertices and replace the edges in the form \( e_{iwz} \) with the \( w \to z \) segment of \( P_i \) which instead we put \( e_{iwz} \) before. We get \( P'_i \) by uniting \( Q'_i \) with the terminal segment of \( P_j \) that starts at \( v_j \). If \( P_i \) starts outside \( B \) we do the same except we have to use also the initial segment of \( P_i \) that ends in \( u_i \) to the construction of \( P'_i \).
Section 5. Characterization of infeasible single-edge extensions

For $k < i \in I$ we can construct from $P_i$ a $s \to t$ path $R_i$ in $G$ in a natural way. Indeed, take the segment of $P_i$ between $u_i$ and $v_i$ and replace the segments of it that leave $B$ with the appropriate edges $e_{uw}$, finally give two new initial and a new last vertex to it namely $s, s_i$ and $t$. Then $\{R_i\}_{i \in I}$ is a system of edge-disjoint $s \to t$ paths in $G$ such that the first edge of $R_i$ is $ss_i$ and $f_1, \ldots, f_k$ are unused by the path-system. Try to extend this system by using Proposition 4.4. If it succeeds, then iterate this with the resulting path-system. Assume that it do not. Then start $(f_1) \notin U$ (we use here the notations of Proposition 4.4) since $f_1$ is a forward edge of $G$. The vertex $\text{start}(f_1)$ is simultaneously reachable from $V_1$ in $D_1[B]$ which implies by construction of $G$ that there is a system of edge-disjoint $s \to \text{start}(f_1)$ paths in $G$ that uses all the outgoing edges of $s$. Take the initial segments of these paths that goes from $s$ to $V(G) \setminus U$ and extend them by using the second part of Proposition 4.4 to a system of edge-disjoint $s \to t$ paths that uses all the outgoing edges of $s$. 

\[ \square \]

5 Characterization of infeasible single-edge extensions

Our only debt is the proof of Claim 4.1. For finite $\lambda$ one can use the equivalent formulation of path condition (condition 1). In this case tightness of a set $X$ means that $X$ satisfies with equation the inequality at 1. Consider first Claim 4.1 with finite $\lambda$. Assume that by extending with edge $e$ we violate condition 1 and let $B$ be a witness of it. Then necessarily $g_D(B) < g_D(B)$ (thus $e \in \text{in}_D(B)$) and $|\{i < \lambda : V_i \cap B = \emptyset\}| = |\{i < \lambda : V_i' \cap B = \emptyset\}|$ hence $B$ intersects $V_\lambda$. Finally $g_D(B) = |i < \lambda : V_i \cap B = \emptyset|$ (because the extension can worsen the inequality 1 at most by one) which implies the dangerousness of $B$.

In the general case where the cardinal $\lambda$ can be infinite the Claim is less trivial. We present a proof of it in the rest of the section.

Proof of Claim 4.1. For the easy direction assume, seeking contradiction, that $\mathcal{V}'$ does not violate the path condition in $D'$, but there exists a $\mathcal{V}$-dangerous set $B$ such that $e \in \text{in}_D(B)$. Then there is a $\{P_i\}_{i < \lambda}$ system of edge-disjoint paths in $D'$ such that $P_i$ goes from $V_i'$ to $B$. We may assume that $P_0$ consists of one vertex $u$ where $u \in B \cap V_0$ hence $P_i$ goes from $V_i$ to $B$ in $D$ for all $i < \lambda$. The edge $e \in \text{in}_D(B)$ is unused by the paths $P_i$ which contradicts with the tightness of $B$.

For the other direction assume that $t$ certifies that $\mathcal{V}'$ violates the path condition in $D'$. We have to show that there is a $B \subset V$ such that $e \in \text{in}_D(B)$ and $B$ is dangerous with respect to $\mathcal{V}$.

Definition 5.1. Let $\mathcal{T} = \{V_i\}_{i \in I}$ be a multiset of nonempty subsets of $V$ and let $D = (V, A)$ be a digraph. Pick a new vertex $s_i$ for all $i \in I$ and draw one edge from $s_i$ to all the elements of $V_i$. Finally pick a new vertex $s$ and draw one edge from $s$ to $s_i$ ($i \in I$). We denote the resulting digraph by $D_T$.

Note that we can get $D_{\mathcal{T}}'$ from $D_\mathcal{T}$ by deleting $e$ and adding the edge $s_0$end($e$).
Let $\mathcal{P} = \{P_i\}_{i<\lambda}$ be a system of edge-disjoint $s \rightarrow t$ paths in $D_\mathcal{V}$ such that the first edge of $P_i$ is $ss_i$ (see Figure 3). Note that, there must be a $0 \neq i_p < \lambda$ such that $e \in A(P_{i_p})$ otherwise $t$ would be simultaneously reachable from $\mathcal{V}'$ in $D'$.

**Observation 5.2.** There is no system of edge-disjoint $s \rightarrow t$ paths in $D'_\mathcal{V}$ that uses all the outgoing edges of $s$ (since $t$ is not simultaneously reachable from $\mathcal{V}'$ in $D'$).

For $i_p \neq i < \lambda$ path $P_i$ is a path in $D'_\mathcal{V}$ too and it does not use edge $s_0\text{end}(e)$ since $s_0\text{end}(e)$ is not an edge of $D_\mathcal{V}$. We denote by $U = U(\mathcal{P})$ the set of vertices in $V(D'_\mathcal{V})$ that are reachable from $s$ in the digraph $H = H(\mathcal{P})$ what we get by changing the direction of edges $\cup_{i_p \neq i < \lambda} A(P_i)$ in $D'_\mathcal{V}$. On one hand, $\text{start}(e) \in U$, it is certified by the initial segment of $P_{i_p}$ that ends at $\text{start}(e)$. On the other hand, $t \notin U$ because of Observation 5.2 and Proposition 4.4. Furthermore the vertices of the terminal segment of $P_{i_p}$ that starts at $\text{end}(e)$ are disjoint from $U$ otherwise $t \in U$ would also hold. It implies that $s_0 \notin U$ since $s_0\text{end}(e)$ is a forward edge of $H$. It follows $V_0 \setminus U \neq \emptyset$ since the third vertex, say $v$, of $P_0$ cannot be in $U$ because $v s_0$ is a backward-edge of $H$.

We claim that $s_i \in U$ iff $V'_i \subset U$. For $i_p$ it is obviously holds since $\{s_{i_p}\} \cup V'_{i_p} \subset U$. Let $i_p \neq i < \lambda$. Assume that $s_i \in U$. All but one edges in $\{s_i v : v \in V'_{i_p}\}$ are forward edges of $H$, and we necessarily reach from $s$ in $H$ the exception too, because the only ingoing edge of $s_i$ in $H$ comes from there. For the other direction if $s_i \notin U$, then the third vertex, say $v$, of the path $P_i$ cannot be in $U$ because then $v s_i$ would be an outgoing edge of $U$ in $H$ which contradicts with the definition of $U$.

Let $Q = Q(\mathcal{P}) = \{Q_f : f \in \text{out}(U)\}$ where $Q_f$ is the terminal segment of the unique $P_i$ that contains $f$ such that $Q_f$’s first edge is $f$. We claim that it is well defined and $V(Q_f) \cap U = \{\text{start}(f)\}$. On the one hand for $e$ we have $e \in A(P_{i_p})$ and we have already observed that the terminal segment of $P_{i_p}$ that starts at $\text{end}(e)$ is disjoint from $U$. On the other hand for an $e \neq f \in \text{out}_{D_\mathcal{V}}(U)$: since $\text{out}_H(U) = \emptyset$ there is an $i < \lambda$ such that $f \in A(P_i)$; and $A(Q_f) \cap \text{in}_{D_\mathcal{V}}(U) = \emptyset$ also holds since $\text{in}_{D_\mathcal{V}}(U) = \text{in}_{D'_\mathcal{V}}(U)$ (because neither $e$ in $D_\mathcal{V}$ nor $s_0\text{end}(e)$ in $D'_\mathcal{V}$ is an ingoing edge of $U$) and $\text{out}_H(U) = \emptyset$. 

---

**EGRES Technical Report No. 2015-11**
Figure 3: Some parts of $D_P$ that are relevant with respect to the construction above. The dotted edge from $s_0$ to end($e$) is not in $D_P$ but in $D'_P$. The figure contains four paths from $P$. Their thickened terminal segments are elements of $Q$.

We define by transfinite recursion two sequences $U_\alpha$ and $Q_\alpha$. We demand the following properties for all $i < \lambda$ and for all $\beta < \alpha$:

1. (a) $s_0, t \notin U_\beta \subseteq V(D_P), e \in \text{out}_{D_P}(U_\beta)$, $V_0 \setminus U_\beta \neq \emptyset$,
   (b) $s_i \in U_\beta \iff V'_i \subseteq U_\beta$ (iff $V_i \subseteq U_\beta$ because of 1(a)),
   (c) $U_\beta \supseteq U_\alpha$,

2. (a) $Q_\alpha = \{Q_f^\alpha : f \in \text{out}_{D_P}(U_\alpha)\}$ is a system of edge-disjoint paths from $U_\alpha$ to $t$ in $D_P$ such that the first edge of $Q_f^\alpha$ is $f$.
   (b) $\text{in}_{D_P}(U_\beta) \cap A(Q_f^\alpha) = \emptyset$,
   (c) the terminal segment of $Q_f^\alpha$ that goes from $U_\beta$ to $t$ is an element of $Q_\beta$.

Let $P = \{P_i\}_{i<\lambda}$ be a system of edge-disjoint $s \rightarrow t$ paths in $D_P$ such that the first edge of $P_i$ is $ss_i$, and let $U_0 = U(P)$ and $Q_0 = Q(P)$. We have already showed that it satisfies properties 1(a,b), 2(a) (for $\alpha = 0$ that is all what we need). Suppose that $U_\beta$ and $Q_\beta$ have already defined if $\beta < \alpha$ for some $0 < \alpha$ in such a way that it satisfies the properties 1(a,b,c), 2(a,b,c).

Assume first that $\alpha$ is a limit ordinal. Let $U_\alpha = \bigcap_{\beta<\alpha} U_\beta$. The properties 1(a,b,c) obviously hold. If $f \in \text{out}_{D_P}(U_\alpha)$, then there is a $\delta < \alpha$ such that $f \in \text{out}_{D_P}(U_\delta)$, so we may let $Q_f^\alpha \overset{\text{def}}{=} Q_f^\delta$. Here $V(Q_f^\alpha) \cap U_\alpha = \{\text{start}(f)\}$ since $V(Q_f^\delta) \cap U_\delta = \{\text{start}(f)\}$ and $U_\delta \supseteq U_\alpha$ thus $Q_f^\alpha$ goes from $U_\alpha$ to $t$.

Observe that $Q_f^\alpha = Q_f^\delta$ for all $\delta < \gamma < \alpha$, it is implied by property 2(c) for $\delta$ and $\gamma$ and by the fact that the terminal segment of $Q_f^\gamma$ that goes from $U_\delta$ to $t$ is $Q_f^\gamma$ itself. We claim that the paths in $Q_\alpha$ are edge-disjoint. Indeed, let $Q_f^\beta, Q_g^\gamma \in Q_\alpha$ arbitrary where $\beta \leq \gamma < \alpha$ and assume that $h \in A(Q_f^\beta) \cap A(Q_g^\gamma)$. By our observation above $Q_g^\beta = Q_g^\gamma$ thus $h \in A(Q_f^\beta) \cap A(Q_g^\gamma)$ hence by property 2(a) necessarily $Q_g^\beta = Q_g^\gamma$.

EGRES Technical Report No. 2015-11
therefore \( Q^\gamma_\beta = Q^\beta_f \). To prove 2(b,c) pick a \( \gamma \) such that \( \beta < \gamma < \alpha \) and \( Q^\gamma_\beta = Q^\gamma_f \) and apply property 2(b,c) to \( \gamma \) and \( \beta \).

Let now \( \alpha = \gamma + 1 \). If \( V \setminus U_\gamma \) is tight in \( D \) with respect to \( \mathcal{V} \), then by property 1(a) we know that \( e \in \partial(D(V \setminus U_\gamma)) \) and \( \mathcal{V}_0 \setminus \mathcal{V}_\gamma \neq \emptyset \) therefore the set \( B \equiv V \setminus U_\alpha \) satisfies the conditions of Claim 4.1 and we are done. Otherwise \( V \setminus U_\gamma \) is not tight in \( D \) with respect to \( \mathcal{V} \), thus let \( I = \{ i < \lambda : V_i \subset U_\gamma \} \) and we may fix a system of edge-disjoint paths \( \{ P^i_\gamma \} \subset \mathcal{I} \) in \( D \) such that \( P^i_\gamma \) goes from \( V_i \) to \( V \setminus U_\gamma \) and there is an edge-set \( \emptyset \neq \mathcal{F} \subset \partial(D(U_\gamma)) \) such that the elements of \( \mathcal{F} \) are unused by the paths \( P^i_\gamma \). Let \( \mathcal{P}^i_\gamma = \mathcal{F} P^i_\gamma \) if \( i \in I \) and \( \mathcal{P}^i_\gamma \) otherwise. By property 1(b) \( s_i \in U_\gamma \) if \( V_i \subset U_\gamma \) thus \( \{ \mathcal{P}^i_\gamma \} \subset \mathcal{I} \) is a system of edge-disjoint paths in \( D_\gamma \) such that \( P^i_\gamma \) goes from \( s \) to \( V \setminus U_\gamma \) and the first edge of \( P^i_\gamma \) is \( ss_i \). For all \( i < \lambda \) unite \( P^i_\gamma \) with the unique element of \( \mathcal{Q} \), whose first edge is the last edge of \( P^i_\gamma \) and denote the result by \( P^\gamma_\gamma \). Let \( \mathcal{P}_\gamma = \{ P^\gamma_\gamma \} \subset \mathcal{I} \), \( U_{\gamma + 1} = U(\mathcal{P}_\gamma) \) and \( \mathcal{Q}_{\gamma + 1} = \mathcal{Q}(\mathcal{P}_\gamma) \) (see Figure 4).

Properties 1(a,b) and 2(a) hold as we have seen it for arbitrary \( \mathcal{P} \). From the construction it is clear that paths \( P^\gamma_\gamma \) have no ingoing edges to \( U_\gamma \) hence \( \partial_{D_\gamma}(U_\gamma) \) = \( F \). If for some \( f \in F \) the vertex \( \text{start}(f) \) would be in \( U_{\gamma + 1} \), then \( Q^\gamma_f \) shows that \( t \in U_{\gamma + 1} \) which is false for arbitrary \( \mathcal{P} \). This proves \( U_\gamma \supset U_{\gamma + 1} \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image}
\caption{There are four paths of \( \mathcal{P}_\gamma \) in the figure. The dashed paths are elements of \( \mathcal{Q}_\gamma \), two of them are not used in the construction of \( \mathcal{P}_\gamma \) and \( f_0, f_1 \in F \).}
\end{figure}

To show 2(b) for \( \gamma + 1 \) let \( \beta < \gamma + 1 \) and assume (reductio ad absurdum) that \( h \in \mathcal{A}(Q^\gamma_{\gamma + 1}) \cap \partial_{D_\gamma}(U_\beta) \). Since \( \mathcal{I}_{D_\gamma}(U_\beta) \subset \mathcal{I}_{D_\gamma}(U_\gamma) \) from the construction of \( \mathcal{Q}_{\gamma + 1} \), it follows that \( h \in Q^\gamma_\beta \) for some \( Q^\gamma_\beta \in \mathcal{Q}_\gamma \). But then \( \beta \) and \( \gamma \) violates property 2(b) if \( \beta < \gamma \) and \( \gamma \) violates 2(a) if \( \beta = \gamma \). The proof of 2(c) for \( \gamma + 1 \) is similar. This concludes the description of the recursion.

On the one hand, the recursion stops if we find the desired \( B \) in some successor step. On the other hand property 1(c) ensures that it will happen after less than \( |V(D_\gamma)|^+ \) steps since there is no sequence of strictly decreasing subsets of \( V(D_\gamma) \) with length \( |V(D_\gamma)|^+ \). \qed
References


