Supermodularity in Unweighted Graph Optimization II: Matroidal Term Rank Augmentation

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Abstract

Ryser's max term rank formula with graph theoretic terminology is equivalent to a characterization of degree sequences of simple bipartite graphs with matching number at least \( \ell \). In a previous paper [1] by the authors, a generalization was developed for the case when the degrees are constrained by upper and lower bounds. Here two other extensions of Ryser's theorem are discussed. The first one is a matroidal model, while the second one settles the augmentation version. In fact, the two directions shall be integrated into one single framework.

1 Introduction

Ryser [16] derived a formula for the maximum term rank of a \((0,1)\)-matrix with specified row- and column-sums. In graph theoretic terms, his theorem is equivalent to a characterization for the existence of a degree-specified simple bipartite graph (bigraph for short) with matching number at least \( \ell \). Several natural extensions, like the min-cost and the subgraph version, turned out to be \( \text{NP}\)-complete, but in a previous paper [1], we could extend Ryser's theorem to the degree-constrained case when, instead of exact degree-specifications, lower and upper bounds are imposed on the degrees of the bigraph. An even more general problem was also solved when, in addition, lower and upper bounds are imposed on the number of edges. The main tool in [1] for proving these extensions was a general framework for covering an intersecting supermodular function by degree-constrained simple bipartite graphs.

In the present paper we consider two other extensions of Ryser's theorem: the augmentation and the matroidal version. In the first one, a given initial bigraph is to be augmented to get a simple degree-specified bigraph with matching number at least \( \ell \). In original matrix terms, this means that some of the entries of the \((0,1)\)-matrix are specified to be 1. The solvability of this version is in sharp contrast with the

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1.1 Notions and notations

We use the notation of [1]. Here we briefly repeat the most important notions. For a family $T$ of sets, let $\cup T$ denote the union of the members of $T$. For a subpartition $T = \{T_1, \ldots, T_q\}$, we always assume that its members $T_i$ are non-empty but $T$ is allowed to be empty (that is, $q = 0$).

An arc $st$ enters or covers a set $X$ if $s \notin X$, $t \in X$. A digraph covers $X$ if it contains an arc covering $X$. Let $S$ and $T$ be two non-empty subsets of a ground-set $V$. By an $ST$-arc we mean an arc $st$ with $s \in S$ and $t \in T$. A set-function $p$ is called positively $T$-intersecting ($ST$-crossing) supermodular if the supermodular inequality

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

holds whenever $p(X) > 0$, $p(Y) > 0$, and $X \cap Y \cap T \neq \emptyset$ (respectively, $X \cap Y \cap T \neq \emptyset$ and $S - (X \cup Y) \neq \emptyset$). Two sets $X$ and $Y$ are $ST$-independent if $X \cap Y \cap T = \emptyset$ and $S - (X \cup Y) = \emptyset$, that is, no $ST$-arc enters both sets.

For a function $m : V \to \mathbb{R}$, the set-function $\tilde{m}$ is defined by $\tilde{m}(X) = \sum [m(v) : v \in X]$. A set-function $p$ can analogously extended to families $F$ of sets by $\tilde{p}(F) = \sum [p(X) : X \in F]$.

The following min-max theorem of Frank and Jordán [3] will be a basic tool in the proof of the main theorem.

**Theorem 1.1** (Supermodular arc-covering, set-function version). A positively $ST$-crossing supermodular function $p$ for which $p(V') \leq 0$ holds when no $ST$-arc enters $V'$ can be covered by $\gamma$ (possibly parallel) $ST$-arcs if and only if $\tilde{p}(I) \leq \gamma$ holds for every $ST$-independent family $I$ of subsets of $V$.

Henceforth we assume that $S$ and $T$ are two disjoint non-empty sets and $V := S \cup T$. Let $G^* = (S, T; E^*)$ denote the complete bipartite graph on bipartition $(S, T)$. Let $D^* = (S, T; A^*)$ be the digraph arising from $G^*$ by orienting each of its edges from $S$ to $T$, that is, $A^*$ consists of all $ST$-arcs. More generally, for a bigraph $H = (S, T; F)$,
Section 2: Matroidal covering and augmentation

Let \( \overrightarrow{H} = (S, T; \overrightarrow{F}) \) denote the digraph arising from \( H \) by orienting each of its edges from \( S \) toward \( T \).

Throughout we are given a simple bigraph \( H_0 = (S, T; F_0) \) serving as an initial bigraph to be augmented. For \( E_0 := E^* - F_0 \), the bigraph \( G_0 = (S, T; E_0) \) is called the \textbf{bipartite complement} of \( H_0 \), that is, \( F_0 \) and \( E_0 \) partition \( E^* \). Note that a bigraph \( G = (S, T; E) \) is a subgraph of \( G_0 \) precisely if the augmented bigraph \( G^+ = (S, T; F_0 + E) \) is simple. For \( X \subseteq S \) and \( Y \subseteq T \), let \( d_{G_0}(X, Y) \) denote the number of edges of \( G_0 \) connecting \( X \) and \( Y \).

2 Matroidal covering and augmentation

Let \( p_T \) be a positively intersecting supermodular function on \( T \). In [1], we studied the problem of finding a simple degree-specified bigraph \( G = (S, T; E) \) covering \( p_T \) in the sense that
\[
|\Gamma_G(Y)| \geq p_T(Y) \text{ for every subset } Y \subseteq T,
\]
where \( \Gamma_G(Y) \) denotes the set of neighbours of \( Y \). Here we consider a framework which is more general in two directions. First, for a given initial simple bigraph \( H_0 = (S, T; F_0) \) we want to find a degree-specified bigraph \( G \) in such a way that \( G^+ := G + H_0 \) is simple and covers \( p_T \). This kind of problems is often referred to as augmentation problem to be distinguished from the synthesis problem where \( F_0 \) is empty. If \( p_T \equiv 0 \), the augmentation problem is equivalent to finding a degree-specified subgraph of the bipartite complement of \( H_0 \).

Second, we extend the notion of covering to matroidal covering in the following sense. Let \( M_S = (S, r_S) \) be a matroid on \( S \) with rank function \( r_S \). A bigraph \( G \) is said to \textbf{\( M_S \)-cover} \( p_T \) if
\[
r_S(\Gamma_G(Y)) \geq p_T(Y) \text{ for every subset } Y \subseteq T. \tag{1}
\]
Clearly, when \( M_S \) is the free matroid, we are back at the original notion of covering by a bigraph.

2.1 Degree-specified matroidal augmentation

Let \( m_V = (m_S, m_T) \) be a degree-specification. Our main goal is to describe a characterization for the existence of a simple bigraph \( G \) fitting \( m_V \) so that \( G + H_0 \) is simple and \( M_S \)-covers \( p_T \). The more general problem, when there are upper and lower bounds on \( V \), is the subject of an ongoing research. The general degree-constrained case was solved in [1] in the special case when \( H_0 \) has no edges and \( M_S \) is the \( \ell \)-uniform matroid on \( S \).

**THEOREM 2.1.** We are given a simple bigraph \( H_0 = (S, T; F_0) \), a matroid \( M_S = (S, r_S) \), a positively intersecting supermodular function \( p_T \) on \( T \), and a degree-specification \( m_V = (m_S, m_T) \) for which \( \overline{m}_S(S) = \overline{m}_T(T) = \gamma \). There is a bigraph
2.1 Degree-specified matroidal augmentation

\[ G = (S, T; E) \] fitting \( m_V \) for which \( G^+ = G + H_0 \) is simple and \( M_S \)-covers \( p_T \) if and only if

\[
\begin{aligned}
\{ & \tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) + \tilde{p}_T(T) - |T| r_S(X) \leq \gamma \\
& \text{for } Y \subseteq T, X \subseteq S, \ T \text{ a subpartition of } T - Y \text{ with } d_{H_0}(S - X, \cup T) = 0,
\end{aligned}
\]

where \( G_0 \) is the bipartite complement of \( H_0 \) and \( d_{G_0}(X, Y) \) is the number of edges of \( G_0 \) connecting \( X \) and \( Y \).

**Proof.** Necessity. Suppose that there is a requested graph \( G \) and let \( G^+ = (S, T; E \cup F_0) \). Let \( T = \{T_1, \ldots, T_q\} \) be a subpartition occurring in (2). Since \( d_{H_0}(S - X, T_i) = 0 \), we have

\[
p_T(T_i) \leq r_S(\Gamma_{G^+}(T_i)) \leq r_S(\Gamma_{G^+}(T_i) \cap X) + r_S(\Gamma_{G^+}(T_i) - X) =
\]

\[
r_S(\Gamma_{G^+}(T_i) \cap X) + r_S(\Gamma_{G^+}(T_i) - X) \leq r_S(X) + |\Gamma_G(T_i) - X| \leq r_S(X) + d_G(T_i, S - X),
\]

from which \( d_G(T_i, S - X) \geq p_T(T_i) - r_S(X) \). Therefore \( G \) has at least \( \sum_{i=1}^q [p_T(T_i) - r_S(X)] \) edges connecting \( T - Y \) and \( S - X \), and has at least \( \tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) \) edges with end-nodes in \( X \) or in \( Y \), from which the inequality in (3) follows.

Let \( H_0 = \{V' \subseteq V : \text{no arc of } \overrightarrow{H_0} \text{ enters } V'\} \). Then \( H_0 \) is closed under taking union and intersection. Define a set-function \( p_0 \) on \( V \) as follows.

\[
p_0(V') := \begin{cases} 
\max\{p_T(y) - r_S(X), m_T(y) - |X| + d_{H_0}(y)\} & \text{if } V' = X + y \in H_0: \\
p_T(Y) - r_S(X) & \text{if } V' = X \cup Y \in H_0: \\
0 & \text{otherwise}.
\end{cases}
\]

Note that \( p_0(V') \) can be positive only if \( V' \in H_0 \) and \( \overrightarrow{G_0} \) covers \( V' \).

**Lemma 2.2.** The set-function \( p_0 \) is positively \( T \)-intersecting supermodular.

**Proof.** Let \( X_1, X_2 \) be subsets of \( S \) and let \( Y_1, Y_2 \) be subsets of \( T \) for which \( Y_1 \cap Y_2 \neq \emptyset \). Suppose that \( p_0(V_i) > 0 \) for \( V_i = X_i \cup Y_i \) \( (i = 1, 2) \). Then each of the sets \( V_1, V_2, V_1 \cap V_2, \) and \( V_1 \cup V_2 \) belongs to \( H_0 \). We distinguish three cases.

**Case 1** \( p_0(V_i) = p_T(Y_i) - r_S(X_i) \) for \( i = 1, 2 \). If \( |Y_1 \cap Y_2| \geq 2 \), then

\[
p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [p_T(Y_2) - r_S(X_2)] \leq \]

\[
p_T(Y_1) - r_S(X_1) + p_T(Y_1 \cup Y_2) - r_S(X_1 \cap X_2) = p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2).
\]

If \( Y_1 \cap Y_2 = \{y\} \) for some element \( y \in T \), then

\[
p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [p_T(Y_2) - r_S(X_2)] \leq \]

\[
p_T(Y_1 \cup Y_2) - r_S(X_1 \cap X_2) = p_0(V_1 \cup V_2).
\]

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\[ p_T(y) - r_S(X_1 \cap X_2) + p_0(V_1 \cup V_2) \leq p_0((X_1 \cap X_2) + y) + p_0(V_1 \cup V_2) = p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2). \]

**Case 2** \( p_0(V_i) > p_T(Y_i) - r_S(X_i) \) for \( i = 1, 2 \). Then \( Y_1 = Y_2 = \{ y \} \) for some \( y \in T \), and \( p_0(V_i) = m_T(y) - |X_i| + d_{H_0}(y) \). We have

\[ p_0(V_1) + p_T(V_2) = m_T(y) - |X_1| + d_{H_0}(y) + m_T(y) - |X_2| + d_{H_0}(y) = m_T(y) - |X_1 \cap X_2| + d_{H_0}(y) + m_T(y) - |X_1 \cup X_2| + d_{H_0}(y) \leq p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2). \]

**Case 3** \( p_0(V_1) = p_T(Y_1) - r_S(X_1) \) and \( p_0(V_2) > p_T(Y_2) - r_S(X_2) \). (The situation is analogous when the indices \( i = 1, 2 \) are interchanged.) Then \( Y_2 = \{ y \} \) for some \( y \in T \) and \( y \in Y_1 \). Since

\[ r_S(X_1 \cup X_2) - r_S(X_1) \leq |(X_1 \cup X_2) - X_1| = |X_2| - |X_1 \cap X_2|, \]

we have \(-r_S(X_1) - |X_2| \leq -r_S(X_1 \cup X_2) - |X_1 \cap X_2| \) and hence

\[ p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [m_T(y) - |X_2| + d_{H_0}(y)] = [p_T(Y_1 \cup Y_2) - r_S(X_1)] + [m_T(y) - |X_2| + d_{H_0}(y)] \leq p_T(Y_1 \cup Y_2) - r_S(X_1 \cup X_2) + m_T(y) - |X_1 \cap X_2| + d_{H_0}(y) \leq p_0(V_1 \cup V_2) + p_0(V_1 \cap V_2), \]

as required. ●

**Claim 2.3.** \( m_S(s) \leq d_{G_0}(s) \) for each \( s \in S \).

**Proof.** By applying (2) to \( Y = T, X = \{ s \} \), and \( T = \emptyset \), the claim follows. ●

For \( s \in S \), let \( V_s = \{ v \in V - s : sv \not\in F_0 \} \), and let a set-function \( p_1 \) on \( V \) be defined as follows.

\[ p_1(V') := \begin{cases} m_S(s) & \text{if } V' = V_s \text{ for some } s \in S \\ p_0(V') & \text{otherwise}. \end{cases} \]

(4)

Note that \( p_1(V') \) can be positive only if \( V' \in H_0 \) and \( \overrightarrow{G_0} \) covers \( V' \).

**Claim 2.4.** \( p_1(V_s) \geq p_0(V_s) \) holds for every \( s \in S \).

**Proof.** Consider first the case when \( V_s \cap T = \{ y \} \) for some \( y \in T \). By applying (2) to \( X = S - s \), to \( Y = \{ y \} \), and to \( T = \emptyset \), we get

\[ m_T(y) - |S - s| + d_{H_0}(y) = m_T(y) - d_{G_0}(S - s, y) \leq m_S(s). \]

By applying (2) to \( X = S - s \), to \( Y = \emptyset \), and to \( T = \{ y \} \), we get \( p_T(y) - r_S(S - s) \leq m_S(s) \) from which

\[ m_S(s) \geq \max\{p_T(y) - r_S(S - s), m_T(y) - |S - s| + d_{H_0}(y)\} = p_0(V_s). \]

Second, assume that \( |V_s \cap T| \geq 2 \). By applying (2) to \( X = S - s \), to \( Y = \emptyset \), and to \( T = \{ V_s \cap T \} \) we get

\[ p_0(V_s) = p_T(V_s \cap T) - r_S(S - s) \leq m_S(s). \]
Claim 2.5. The set-function $p_1$ is positively $ST$-crossing supermodular.

Proof. It follows from Claim 2.4 that $p_1$ arises from $p_0$ by increasing its values on sets $V_s$ ($s \in S$). Let $V' \subseteq V$ be a set with $ST$-crossing $V_s$. Then $S \not\subseteq V_s \cup V'$ and hence $V' \cap S \subseteq V_s \cap S$. Therefore $V' \cap T \subseteq V_s \cap T$, that is, there is an element $t \in (V' - V_s) \cap T$. Since $st$ is an arc of $H_0$ entering $V'$, we conclude that $p_1(V') = 0$, implying that $p_1$ is indeed positively $ST$-crossing supermodular. ⊠

Let $\nu$ denote the maximum total $p_1$-value of $ST$-independent sets.

Lemma 2.6. $\nu = \gamma$.

Proof. Since the family $\mathcal{L} = \{V_s : s \in S\}$ is $ST$-independent, $\nu \geq \tilde{p}_1(\mathcal{L}) = \tilde{m}_S(S) = \gamma$. Suppose indirectly that $\nu > \gamma$ and let $\mathcal{I}$ be an $ST$-independent family for which $\tilde{p}_1(\mathcal{I}) = \nu$. We can assume that $|\mathcal{I}|$ is minimal in which case $p_1(V') > 0$ for each $V' \in \mathcal{I}$.

Claim 2.7. There are no two $T$-intersecting members $V_1 = X_1 \cup Y_1$ and $V_2 = X_2 \cup Y_2$ of $\mathcal{I}$ for which $p_1(V_i) = p_0(V_i)$ ($i = 1, 2$).

Proof. Suppose indirectly the existence of such $V_1$ and $V_2$. Since $\mathcal{I}$ is $ST$-independent, we must have $X_1 \cup X_2 = S$ and hence $p_0(V_1 \cup V_2) = 0$. Since $p_0$ is $T$-intersecting,

$$p_1(V_1) + p_1(V_2) = p_0(V_1) + p_0(V_2) \leq p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2) = p_0(V_1 \cap V_2) \leq p_1(V_1 \cap V_2).$$

Now $\mathcal{I}' = \mathcal{I} - \{V_1, V_2\} + \{V_1 \cap V_2\}$ is also $ST$-independent and $\tilde{p}_1(\mathcal{I}') \geq \tilde{p}_1(\mathcal{I})$, but we must have equality by the optimality of $\mathcal{I}$, contradicting the minimality of $|\mathcal{I}|$. ⊠

We say that a member $V' \in \mathcal{I}$ is of Type I if $V' = X_t + t$ for some $t \in T$ and $X_t \subseteq S$ and

$$p_1(X_t + t) = p_0(X_t + t) = m_T(t) - |X_t| + d_{H_0}(t) > p_T(t) - r_S(X_t).$$

Let $\mathcal{I}_1$ ($\subseteq \mathcal{I}$) denote the family of sets of Type I. Claim 2.7 implies that if $X_1 + t_1 \in \mathcal{I}_1$ and $X_2 + t_2 \in \mathcal{I}_1$ ($X_1 \subseteq S, t_1, t_2 \in T$) then $t_1 \neq t_2$. Let

$$Y := \{t \in T : \text{there is a member } X_t + t \in \mathcal{I}_1\}.$$ 

Note that $|Y| = |\mathcal{I}_1|.$

We say that a member $V' \in \mathcal{I}$ is of Type II if

$$p_1(V') = p_0(V') = p_T(V' \cap T) - r_S(V' \cap S).$$

Let $\mathcal{I}_2 = \{V_1, V_2, \ldots, V_q\}$ ($\subseteq \mathcal{I}$) denote the family of set of Type II. Let

$$\mathcal{T} := \{T_1, \ldots, T_q\} \text{ where } T_i := V_i \cap T \text{ for } i = 1, \ldots, q.$$ 

Since $p_1(V_i) > 0$, the members of $\mathcal{T}$ are non-empty. Furthermore, Claim 2.7 implies that $\mathcal{T}$ is a subpartition.
Let $I_3 := I - (I_1 \cup I_2)$. The members of $I_3$ are called of Type III. Then each member $V'$ of $I_3$ is of form $V' = V_s$ for some $s \in S$ such that $m_S(s) = p_1(V') > p_0(V')$. Let

$$X := \{ s \in S : V_s \in I_3 \}.$$ 

It follows from the definitions that $I_1, I_2$, and $I_3$ form a partition of $I$.

Claim 2.8. Let $t \in Y$ and $X_t + t \in I_1$. Then $X \subset X_t$.

Proof. Suppose indirectly that there is an element $s \in X - X_t$. By the ST-independence of the sets $X_t + t$ and $V_s$, the element $t$ cannot be in $V_s$. Therefore the arc $st$ belongs to $\overrightarrow{F}_0$. Since $st$ enters $X_t + t$, we have $p_1(X_t + t) \leq 0$, a contradiction. •

Claim 2.9. $\sum [|X_t| - d_{H_0}(t) : t \in Y] \geq d_{G_0}(X,Y)$.

Proof. What we prove is that $|X_t| - d_{H_0}(t) \geq d_{G_0}(X,t)$ for $t \in Y$ and $X_t + t \in I_1$. Since no arc of $\overrightarrow{H}_0$ enters $X_t + t$ and since $X \subset X_t$ by Claim 2.8, we have

$$|X_t| - d_{H_0}(t) = |X_t| - d_{H_0}(X_t,t) = d_{G_0}(X_t,t) \geq d_{G_0}(X,t),$$

as required. •

Claim 2.10. $X \subset V_i \cap S$ holds for each $i = 1, \ldots, q$.

Proof. If, indirectly, there is an $s \in X - V_i$, then the ST-independence of $V_s$ and $V_i$ implies that $V_s \cap V_i \cap T = \emptyset$. In this case, an element $t \in V_i \cap T$ cannot be in $V_s$ implying that $st \in \overrightarrow{F}_0$. But in this case $p_0(V_i) = p_1(V_i) = 0$, contradicting the property $p_0(V') > 0$ for each $V' \in I$. •

By the ST-independence of $I$, family $T$ forms a subpartition of $T$ whose members are disjoint from $Y$. This and the last two claims imply

$$\gamma < \nu = \tilde{p}_T(I) = \tilde{p}_I(I_1) + \tilde{p}_I(I_2) + \tilde{p}_I(I_3) =$$

$$\sum [m_T(t) - |X_t| + d_{H_0}(t) : X_t + t \in I_1] + \sum_{i=1}^q [p_T(T_i) - r_S(V_i \cap S)] + \sum [m_S(s) : V - s \in I_3] \leq$$

$$\sum [m_T(t) : X_t + t \in I_1] - d_{G_0}(X,Y) + \sum_{i=1}^q [p_T(T_i) - r_S(X)] + \tilde{m}_S(X) =$$

$$\tilde{m}_T(Y) - d_{G_0}(X,Y) + \sum_{i=1}^q [p_T(T_i) - r_S(X)] + \tilde{m}_S(X),$$

in a contradiction with $[2]$, completing the proof of the lemma. • •

By Theorem [1,1] there is a digraph $D = (V, A)$ on $V$ with $\nu = \gamma$ ST-arcs that covers $p_1$, that is, $p_D(V') \geq p_1(V')$ for every subset $V' \subseteq V$. Let $G = (S,T ; E)$ denote the undirected bipartite graph underlying $D$. 

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Claim 2.11. \( d_G(s) = m_S(s) \) for every \( s \in S \) and \( d_G(t) = m_T(t) \) for every \( t \in T \).

**Proof.** Since \( d_G(s) = \delta_D(s) \geq \delta_D(V_s) \geq p_1(V_s) = m_S(s) \) for every \( s \in S \), we have \( \tilde{m}_S(S) = |E| = \sum [d_G(s) : s \in S] \geq \tilde{m}_S(S) \), from which \( d_G(s) = m_S(s) \) follows for every \( s \in S \).

Since \( d_G(t) = g_D(t) = g_D(\Gamma_{H_0}(t) + t) \geq p_1(\Gamma_{H_0}(t) + t) \geq m_T(t) \) for every \( t \in T \), we have \( \tilde{m}_S(S) = |E| = \sum [d_G(t) : t \in T] \geq \tilde{m}_T(T) = \tilde{m}_S(S) \), from which \( d_G(t) = m_T(t) \) follows for every \( t \in T \).

Claim 2.12. The bigraph \( G^+ = (S, T; E + F_0) \) is simple.

**Proof.** The minimality of \( D \) implies that each arc of \( D \) enters a subset \( V' \) with \( p_1(V') > 0 \). Since \( p_1(V') \) can be positive only if no arc of \( H_0 \) enters \( V' \), we can conclude that no edge of \( G \) is parallel to an edge of \( H_0 \).

Suppose indirectly that there are two parallel edges \( e \) and \( e' \) of \( G \) connecting \( s \) and \( t \) for some \( s \in S \) and \( t \in T \). Let \( X := \{ u \in S : ut \in F_0 \} \). Then \( p_1(X + t) \geq m_T(t) = g_D(t) \). For \( V' = X + s + t \), we have \( g_D(t) - 2 \geq g_D(V') \geq p_1(V') \geq p_1(X + t) - 1 \geq m_T(t) - 1 = g_D(t) - 1 \), a contradiction. \( \bullet \)

Claim 2.13. \( r_S(\Gamma_{G^+}(Y)) \geq p_T(Y) \) for every subset \( Y \subseteq T \).

**Proof.** Let \( X := \Gamma_{G^+}(Y) \) and \( V' := Y \cup X \). Then \( 0 = g_D(V') \geq p_1(V') \geq p_T(Y) - r_S(X) \), from which the claim follows. \( \bullet \)

We conclude that \( G \) meets all the requirements of the theorem, and the proof is complete. \( \bullet \bullet \bullet \)

2.2 Variations

2.2.1 Degree-specification only on \( S \)

With the proof technique used above, one can derive the following variation where the degrees are specified only for the nodes in \( S \).

**THEOREM 2.14.** We are given a simple bigraph \( H_0 = (S, T; F_0) \), a matroid \( M_S = (S, r_S) \), a positively intersecting supermodular function \( p_T \) on \( T \), and a degree-specification \( m_S \) on \( S \) for which \( \tilde{m}_S(S) = \gamma \). There is a bigraph \( G = (S, T; E) \) fitting \( m_S \) for which \( G^+ = G + H_0 \) is simple and \( M_S \)-covers \( p_T \) if and only if

\[
m_S(s) + d_{H_0}(s) \leq |T| \text{ for every } s \in S \tag{5}
\]

and

\[
\begin{cases}
\tilde{m}_S(X) + \tilde{p}_T(T) - |T|r_S(X) \leq \gamma \\
\text{for } X \subseteq S, \ T \text{ a subpartition of } T \text{ with } d_{H_0}(S - X, \cup T) = 0.
\end{cases} \tag{6}
\]
2.2 Variations

2.2.2 Fully supermodular $p_T$

In the special case when $p_T \equiv 0$, it suffices to require the inequality in (2) only for the empty $T$, in which case Theorem 2.1 reduces to the following classic result (which actually holds for non-simple bigraphs, too).

Theorem 2.15 (Ore [12]). A simple bigraph $G_0 = (S, T; E_0)$ has a subgraph fitting a degree-specification $(m_S, m_T)$ with $\tilde{m}_S(S) = \tilde{m}_T(T) = \gamma$ if and only if

$$\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) \leq \gamma \text{ for } X \subseteq S, \ Y \subseteq T. \quad (7)$$

The content of the next result is that the condition in Theorem 2.1 can also be simplified when $p_T$ is fully supermodular.

THEOREM 2.16. We are given a simple bigraph $H_0 = (S, T; E_0)$, a matroid $M_S = (S, r_S)$, a fully supermodular function $p_T$ on $T$, and a degree-specification $m_V = (m_S, m_T)$ for which $\tilde{m}_S(S) = \tilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting $m_V$ for which $G^+ = G + H_0$ is simple and $M_S$-covers $p_T$ if and only if (7) holds and

$$\begin{align*}
\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) - p_T(T_0) - r_S(X) & \leq \gamma \\
\text{for } Y \subseteq T, \ X \subseteq S, \ T_0 \subseteq T - Y, \ \text{with } d_{H_0}(S - X, T_0) = 0,
\end{align*} \quad (8)$$

where $G_0$ is the bipartite complement of $H_0$ and $d_{G_0}(X, Y)$ is the number of edges of $G_0$ connecting $X$ and $Y$. If, in addition, $p_T$ is monotone non-decreasing, then $T_0$ in (8) can be chosen to by $T_0 = T - (Y \cup \Gamma_{H_0}(S - X))$, that is,

$$\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) + p_T(T - (Y \cup \Gamma_{H_0}(S - X))) - r_S(X) \leq \gamma \text{ for } X \subseteq S, \ Y \subseteq T. \quad (9)$$

Proof. Conditions (7) and (8) correspond the special cases of Condition (2) when $|T| = 0$ and $|T| = 1$, respectively. Therefore their necessity was proved earlier. To see sufficiency, by Theorem 2.1 it suffices to show that (2) holds in general. Suppose, indirectly, that there are $X, Y$, and $T$ violating (2). Assume that $|T|$ is minimal. Then (7) and (8) imply that $|T| \geq 2$. Let $T_1, T_2$ be two members of $T$. Since

$$p_T(T_1 \cup T_2) - r_S(X) \geq p_T(T_1) + p_T(T_2) - 2r_S(X),$$

the unchanged sets $X, Y$ and the partition $T'$ obtained from $T$ by replacing $T_1$ and $T_2$ with the single set $T_1 \cup T_2$ also violate (2), contradicting the minimal choice of $T$.

When $p_T$, in addition, is monotone non-decreasing, we can choose $T_0$ in (8) as large as possible, that is, $T_0$ is a maximal subset of $T - Y$ for which $d_{H_0}(S - X, T_0) = 0$. But then $T_0 = T - (Y \cup \Gamma_{H_0}(S - X))$.

It is worth formulating Theorem 2.16 in the special cases when $H_0$ has no edges.

Corollary 2.17. We are given a matroid $M_S = (S, r_S)$, a fully supermodular function $p_T$ on $T$, and a degree-specification $m_V = (m_S, m_T)$ for which $\tilde{m}_S(S) = \tilde{m}_T(T) = \gamma$. 

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There is a simple bigraph $G = (S,T;E)$ fitting $m_V$ and $M_S$-covering $p_T$ if and only if (7) holds and

$$\tilde{m}_S(X) + \tilde{m}_T(Y) - |X||Y| + p_T(T_0) - r_S(X) \leq \gamma \quad \text{for } Y \subseteq T, \ X \subseteq S, \ T_0 \subseteq T - Y,$$

(10)

If, in addition, $p_T$ is monotone non-decreasing, then $T_0$ in (10) can be chosen to by $T_0 = T - Y$, that is,

$$\tilde{m}_S(X) + \tilde{m}_T(Y) - |X||Y| + p_T(T - Y) - r_S(X) \leq \gamma \quad \text{for } X \subseteq S, \ Y \subseteq T. \ \bullet$$

(11)

## 3 Matroidal max term rank

Let $\mathcal{G}(m_S,m_T)$ denote the set of simple bigraphs $G = (S,T;E)$ fitting a degree-specification $(m_S,m_T)$ with $\tilde{m}_S(S) = \tilde{m}_T(T) = \gamma$. It follows from Theorem 2.15 that $\mathcal{G}(m_S,m_T)$ is non-empty if and only if

$$\tilde{m}_S(X) + \tilde{m}_T(Y) - |X||Y| \leq \gamma \quad \text{for } X \subseteq S, \ Y \subseteq T.$$ 

(12)

In [1] (Theorem 7.1), we reformulated Ryser’s classic max term rank formula in graph theoretic language.

**Theorem 3.1** (Ryser). Let $\ell \leq |T|$ be an integer. Suppose that $\mathcal{G}(m_S,m_T)$ is non-empty. Then $\mathcal{G}(m_S,m_T)$ has a member $G$ with matching number $\nu(G) \geq \ell$ if and only if

$$\tilde{m}_S(X) + \tilde{m}_T(Y) - |X||Y| + (\ell - |X| - |Y|) \leq \gamma \quad \text{whenever } X \subseteq S, \ Y \subseteq T.$$ 

(13)

Moreover, (13) holds if the inequality in it is required only when $X$ consists of the $i$ largest values of $m_S$ and $Y$ consists of the $j$ largest values of $m_T$ ($i = 0, 1, \ldots, |S|$, $j = 0, 1, \ldots, |T|$).

We keep using graph terminology, but the original expression max term rank of Ryser is retained. Our present goal is to extend Ryser’s theorem in two directions. In the augmentation version an initial bigraph is to be augmented while in the matroidal form the matching is expected to cover a basis of a matroid $M_S$ on $S$ and a basis of matroid $M_T$ on $T$. Actually, we shall integrate the two generalizations into one single framework.

In what follows, $M_S = (S,r_S)$ and $M_T = (T,r_T)$ will be matroids of rank $\ell$. In [1], the complementary set-function $p$ of a set-function $b$ was defined by $p(Y) := b(S) - b(S - Y)$. Clearly, $b$ is submodular if and only if $p$ is supermodular, and $p$ is monotone non-decreasing if and only if $b$ is monotone non-decreasing. The complementary function $p_T$ of the rank function $r_T$ of $M_T$ is called the co-rank function of $M_T$. It can easily be shown that $p_T(Y) = \min\{|Y \cap B| : B \text{ a basis of } M_T\}$.

The following extension of Edmonds’ matroid intersection theorem [3] will be used. For notational convenience, the bipartite graph in the theorem is denoted by $G^+$. 

---

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Theorem 3.2 (Brualdi, [3]). Let $G^+ = (S, T; E^+)$ be a bigraph with a matroid $M_S = (S, r_S)$ on $S$ and with a matroid $M_T = (T, r_T)$ on $T$ for which $r_S(S) = r_T(T) = \ell$. There is a matching of $G^+$ covering bases of $M_S$ and $M_T$ if and only if

$$\begin{align*}
\begin{cases}
r_S(X) + r_T(Z) \geq \ell \\
\text{holds whenever } X \cup Z \text{ hits every edge of } G^+ \quad (X \subseteq S, Z \subseteq T).
\end{cases}
\end{align*}$$

We need the following equivalent version of Theorem 3.2.

Lemma 3.3. We are given a bigraph $G^+ = (S, T; E^+)$, a matroid $M_S$ on $S$ with rank function $r_S$ and a matroid $M_T$ on $T$ with a co-rank function $p_T$ for which $r_S(S) = p_T(T) = \ell$.

There is a matching of $G^+$ covering bases of $M_S$ and $M_T$ if and only if

$$r_S(\Gamma_{G^+}(Y)) \geq p_T(Y) \quad \text{for every } Y \subseteq T. \quad (15)$$

Proof. The necessity is straightforward. The sufficiency follows from Theorem 3.2 once we show that $r_S(X) + r_T(Z) \geq \ell$ holds. Suppose, indirectly, that there are $X$ and $Z$ for which $r_S(X) + r_T(Z) < \ell$, that is, $r_S(X) < \ell - r_T(Z) = p_T(T - Z)$. Since $X \cup Z$ hits every edge of $G^+$, for $Y := T - Z$ we have $\Gamma_{G^+}(Y) \subseteq X$. Therefore $r_S(\Gamma_{G^+}(Y)) \leq r_S(X) < p_T(Y)$, contradicting (15).

THEOREM 3.4. We are given a simple bigraph $H_0 = (S, T; F_0)$, matroids $M_S = (S, r_S)$ and matroid $M_T = (T, r_T)$ with $r_S(S) = r_T(T) = \ell$, and a degree-specification $m_V = (m_S, m_T)$ for which $m_S(S) = m_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting $m_V$ for which $G^+ = G + H_0$ is simple and includes a matching covering a basis of $M_S$ and a basis of $M_T$ if and only if

$$\begin{align*}
\begin{cases}
\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_S(X') - r_T(Y') \leq \gamma \\
\text{for } X \subseteq X' \subseteq S, Y \subseteq Y' \subseteq T, \quad X' \cup Y' \text{ hits all the edges of } H_0,
\end{cases}
\end{align*}$$

where $G_0$ is the bipartite complement of $H_0$ and $d_{G_0}(X, Y)$ is the number of edges in $G_0$ connecting $X$ and $Y$.

Proof. Necessity. Suppose that the requested bigraph $G$ and its $\ell$-element matching $M$ exist. The number of edges of $G$ with at least one end-node in $X \cup Y$ is at least $\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y)$. The number of edges in $M$ with at least one end-node in $X' \cup Y'$ is at most $r_S(X') + r_T(Y')$. Therefore $M$ has at least $\ell - r_S(X') - r_T(Y')$ elements connecting $S - X'$ and $T - Y'$. But these elements must be in $E$ since $X' \cup Y'$ hits all edges of $H_0$. Therefore the total number of edges of $G$ is at least $\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_S(X') - r_T(Y')$, and (16) follows.

Sufficiency. Let $p_T$ denote the co-rank function of $M_T$, that is, $p_T(Z) = \ell - r_T(Z)$ for $Z \subseteq T$. This is fully supermodular and monotone non-decreasing and the inequality in Condition 9 transforms to

$$\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_T(Y \cup \Gamma_{H_0}(S - X)) - r_S(X) \leq \gamma. \quad (17)$$

No sets $X \subseteq S, Y \subseteq T$ can violate this inequality since then by letting $Y' := Y \cup \Gamma_{H_0}(S - X)$ and $X' := X$, the quadruple $(X, Y, X', Y')$ would violate (16).
By the second part of Theorem 2.16 there is bigraph \( G \) fitting \( m_V \) for which \( G^+ = G + H_0 \) is simple and \( M_S \)-covers \( p_T \). The last property, by definition, means that (15) holds, and therefore Lemma 3.3 implies that \( G^+ \) has a requested matching.

**Remarks** It follows that requiring inequality (17) for every pair of sets \( X \subseteq S \) and \( Y \subseteq T \) is also a necessary and sufficient condition for the existence of the bigraph \( G \) described in the theorem. This condition has the advantage that it is more compact than (16) in the sense that \( X' \) and \( Y' \) play no role in it. On the other hand, Condition (16) has the advantage that it is symmetric in \( S \) and \( T \).

When \( m_V \equiv 0 \) and \( \gamma = 0 \), it suffices to require (16) only for \( X = Y = \emptyset \) in which case it transforms to

\[
\begin{align*}
\ell - r_S(X') - r_T(Y') & \leq 0 \\
\text{for } X' \subseteq S, Y' \subseteq T, \ X' \cup Y' \text{ hits all the edges of } H_0,
\end{align*}
\]

which is the same as (14). In other words, Theorem 3.4 may be considered as a generalization of Brualdi’s theorem.

By specializing Theorem 3.4 to the case when \( F_0 = \emptyset \), one obtains the following.

**Corollary 3.5.** Let \( S \) and \( T \) be two disjoint sets and \( m_V = (m_S, m_T) \) a degree-specification on \( S \cup T \) for which \( m_S(S) = m_T(T) = \gamma \) and \( G(m_S, m_T) \) is non-empty, that is, (12) holds. Let \( M_S = (S, r_S) \) and matroid \( M_T = (T, r_T) \) be matroids for which \( r_S(S) = r_T(T) = \ell \). There is a simple graph fitting \( m \) that includes a matching covering bases of \( M_S \) and \( M_T \) if and only if

\[
\tilde{m}_S(X) + \tilde{m}_T(Y) - |X||Y| + \ell - r_S(X) - r_T(Y) \leq \gamma
\]

holds for every \( X \subseteq S \) and \( Y \subseteq T \). ●

By specializing Theorem 3.4 to the case when \( M_S \) and \( M_T \) are \( \ell \)-uniform matroids on \( S \) and \( T \), respectively, one obtains the following.

**Corollary 3.6.** We are given a simple bigraph \( H_0 = (S, T; F_0) \), an integer \( \ell \), and a degree-specification \( m_V = (m_S, m_T) \) for which \( \tilde{m}_S(S) = \tilde{m}_T(T) = \gamma \). There is a bigraph \( G = (S, T; E) \) fitting \( m_V \) for which \( G^+ = G + H_0 \) is simple and includes an \( \ell \)-element matching if and only if

\[
\begin{align*}
\tilde{m}_S(X) + \tilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - |X'| - |Y'| & \leq \gamma \\
\text{for } X \subseteq X' \subseteq S, Y \subseteq Y' \subseteq T, \ X' \cup Y' \text{ hits all the edges of } H_0,
\end{align*}
\]

where \( G_0 \) denotes the bipartite complement of \( H_0 \), and \( d_{G_0}(X, Y) \) is the number of edges in \( G_0 \) connecting \( X \) and \( Y \). ●

**References**


