# Nemlokális függést tartalmazó nemlineáris rendszerek

Doktori értekezés

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# On nonlinear systems containing nonlocal terms

PhD Dissertation

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Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

Bertrand Russell

# Contents

# Preface

> If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy.
>
> Alfréd Rényi

In this work we study systems of nonlinear parabolic differential equations. In particular, we consider equations containing nonlocal terms, in other words, functional differential equations. By "nonlocal term" we mean terms which may depend not only on the value of the unknown at a certain point but also on values at other points, for example, it may contain a delay or an integral of the unknown on a domain etc. Such problems may occur in some physical models. For instance, in some diffusion processes the diffusion coefficient may depend on the unknown in a nonlocal way, e.g., as in the following equation:

$$D_t u(t,x) - \operatorname{div}\left(g\Big(\int_\Omega u(t,x)\,dx\Big)\operatorname{grad} u(t,x)\right) = f(t,x) \tag{1}$$

for $t > 0$, $x \in \mathbb{R}^n$ where functions $f\colon (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$, $g\colon \mathbb{R}\to\mathbb{R}$ are given and $u\colon (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ is the unknown with initial condition $u(0,x) = \varphi(x)$ for $x \in \mathbb{R}^n$. One simple example for such diffusion process is, e.g., in population dynamics where the growing rate may depend on the size of the population, mathematically, on the integral of the density. Such nonlocal diffusion problems were considered in [21, 22], further, a related problem, the so-called cross-diffusion was demonstrated in [33].

We mention two other important applications. First, climatology. In [4, 25, 27, 28] functional differential equations arising in climatology were studied.

The other area where functional differential equations may occur is the modelling of fluid flow, especially in porous media. A porous medium is a solid medium with lots of tiny holes (e.g., limestone). The flow of a fluid through the medium is determined by the large surface of the solid matrix and the closeness of the holes. For a detailed introduction to this topic, see [7]. If the fluid carries dissolved chemical species, chemical reactions can occur, see [38]. Among these include reactions that can change the porosity. This process is modelled by a system of equations that contains three different types of equations: an ordinary, a parabolic and an elliptic one, see [23, 46].

For some other problems involving nonlocal differential equations, such as transmission problems, see [39, 40, 41], or nonlocal boundary conditions, see [56, 57, 65].

It is worth mentioning some monographs concerning functional differential equations (by means of mostly semigroups), see [6, 29, 36, 32, 52, 70]. We also note that instead of equations one may consider nonlocal variational inequalities. That type of problems occur in elasticity theory, see [12, 31].

In the following, we study two systems of differential equations containing nonlocal terms. The first one, that will be studied in Chapter 2, consists of equations of parabolic type that are generalizations of equation (1). We extend the results of [63] made on a single nonlocal parabolic equation to a system of equations.

The other system is the above mentioned one describing fluid flow in porous media and consists of three different types of differential equations that will be investigated in Chapter 3. Some numerical experiments concerning this type of problem were done in [23, 46], however, correct proof on existence of solutions were not made (and one can hardly find papers dealing with such kind of systems in rigorous mathematical way).

The main tool of our further investigations will be the theory of operators of monotone type. For a detailed explanation of this theory and its applications, see the classical monographs [15, 20, 24, 35, 44, 55, 71]. However, for the convenience of the Reader we shall recall some important assertions in Chapter 1. In particular, we shall apply some results of [8, 17, 43, 48, 49] related to pseudomonotone operators.

By using the above framework we shall show existence of weak solutions in time interval $(0, T)$ $(0 < T \leq \infty)$ for both systems, further, asymptotic properties will be studied such as the boundedness and stabilization (i.e., convergence to equilibrium) of solutions.

Besides the theoretical investigations of the above systems, we illustrate our results with a variety of examples.

The results of Chapter 2 and 3 were published by the author in papers [9, 10, 11, 14]. Further, some parts of Section 1.6 are also the author's results, see [13].

# Acknowledgment

If I have seen further it is by standing on the shoulders of Giants.

Isaac Newton

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I am indebted to my supervisor, László Simon for his support, trust and encouragement, for being an excellent teacher and a great man. The experience I gained under his supervision is truly unique and invaluable.

I am very grateful to my family for their continuous support and patience. Without them I could not have achieved my dreams...

The author

# Chapter 1

# Preliminaries

> Do not worry about your difficulties in mathematics. I can assure you mine are still greater.
>
> Albert Einstein

## 1.1 Inequalities

> Calculus has its limits.
>
> Folklore

In the following, the set of real numbers will be denoted by $\mathbb{R}$, further, $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. The space of all $n$-tuples ($n \geq 1$ integer) of real numbers will be denoted by $\mathbb{R}^n$. As ususal, $|\cdot|$ deontes the Euclidean norm.

Inequalities will play an important role in estimates. We briefly mention some of them that will appear later.

**Proposition 1.1** (triangle inequality)**.** *Let $a_1, a_2, \ldots, a_n \in \mathbb{R}^k$. Then*

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

**Proposition 1.2.** *Let $a_1, a_2, \ldots, a_n \in \mathbb{R}^k$ and $s > 1$ be a real number. Then*

$$|a_1 + a_2 + \cdots + a_n|^s \leq n^{s-1} \left( |a_1|^s + |a_2|^s + \cdots + |a_n|^s \right). \tag{1.1}$$

**Proposition 1.3.** *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers and $s > 1$ be a real number. Then*

$$a_1^s + a_2^s + \cdots + a_n^s \leq n^{-1}(a_1 + a_2 + \cdots + a_n)^s. \tag{1.2}$$

**Proposition 1.4** (Young's inequality)**.** *Let $p$, $q$ be finite conjugate exponents, i.e.,* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for* $a \geq 0,\ b \geq 0,$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

**Corollary 1.5** ($\varepsilon$-inequality)**.** *Let $p$, $q$ be finite conjugate exponents, i.e.,* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for* $a \geq 0,\ b \geq 0,\ \varepsilon > 0$ *it holds*

$$ab \leq \frac{\varepsilon^p a^p}{p} + \frac{b^q}{\varepsilon^q q}.$$

**Lemma 1.6.** *Let $b, c$ be arbitrary and $s \geq 0$ be a real number. Then*

$$\int_0^1 |c + \tau b|^s d\tau \geq \frac{|b|^s}{2^s(s+1)}. \tag{1.3}$$

*Proof.* The case $b = 0$ is obvious. Now let $b \neq 0$, further, without loss of generality we may assume $c < 0$. We have two cases. If $c + b > 0$, then by dividing interval $[0,1]$ with respect to the sign of $c + \tau b$ we obtain

$$\begin{aligned}\int_0^1 |c + \tau b|^s \, d\tau &= \int_0^{-\frac{c}{b}} (-c - \tau b)^s \, d\tau + \int_{-\frac{c}{b}}^1 (c + \tau b)^s \, d\tau \\ &= \frac{(-c)^{s+1} + (c+b)^{s+1}}{b(s+1)} \\ &\geq \frac{|b|^s}{2^s(s+1)}.\end{aligned}$$

In the last estimate we used inequality (1.1). In the other case $c + \tau b$ is negative for all $\tau \in [0,1]$ thus

$$\int_0^1 |c + \tau b|^s \, d\tau \geq \int_0^1 |\tau b|^s \, d\tau = \frac{|b|^s}{s+1} \geq \frac{|b|^s}{2^s(s+1)}.$$

Note that (1.3) is sharp, we have equality if $c = -\frac{b}{2}$. $\square$

**Proposition 1.7** (Gronwall's lemma)**.** *Let $t_0 \leq t_1$ be real numbers and $\phi, \psi$ $[t_0, t_1] \to \mathbb{R}^+$ be continuous functions such that*

$$\phi(t) \leq K + L \cdot \int_{t_0}^{t_1} \psi(s)\phi(s) \, ds$$

*holds for $t_0 \leq t \leq t_1$ with some positive contants $K, L$. Then*

$$\phi(t) \leq K \cdot e^{L \cdot \int_{t_0}^{t_1} \psi(s) \, ds}$$

*for $t_0 \leq t \leq t_1$.*

## 1.2 $L^p$ spaces, Sobolev spaces, product spaces

Nature laughs at the difficulties of integration.

Pierre-Simon Laplace

We introduce some abstract spaces that will serve for our investigations. For the details, see, e.g., [1]

Let $n \geq 1$ be a fixed natural number and denote by $\lambda$ the $n$ dimensional Lebesgue measure. We shall always work with this measure so we shall omit the notation $d\lambda$ in integrals. We use the abbreviations a.e and a.a. for the expressions *almost everywhere and* almost all that means except of a set with measure zero.

**Definition 1.8.** Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a $\lambda$-measurable set. Then $L^p(\Omega)$ denotes the space of measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p\right)^{\frac{1}{p}} < \infty.$$

**Definition 1.9.** If $p = \infty$, one defines the space $L^\infty(\Omega)$ to be the set of $\lambda$-measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\begin{aligned}\|f\|_{L^\infty(\Omega)} &:= \operatorname*{ess\,sup}_\Omega f \\ &:= \inf\left\{\sup_N |f| : N \subset \Omega, \lambda(N) = 0\right\} \\ &= \sup\left\{K \in \mathbb{R} : \exists A \subset \Omega : \lambda(A) > 0, f(x) > K \text{ for a.a. } x \in A\right\} < \infty.\end{aligned}$$

**Definition 1.10.** Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. In our further investigations we assume that the boundary is continuously differentiable (which will be sufficient, see [1].) Denote by $D_i$ the distributional differentiation with respect to the variable $x_i$ and let $D = (D_1, \ldots, D_n)$ (i.e. $Du = (D_1u, \ldots, D_nu)$ is the gradient). Then

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega)\colon D_iu \in L^p(\Omega),\ 1 \leq i \leq n\},$$

with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega \Big[|u|^p + |Du|^p\Big]\right)^{\frac{1}{p}}.$$

Let $C_0^\infty(\Omega)$ be the set of infinitely differentiable functions $\Omega\colon \to \mathbb{R}$ with compact support (i.e, identically zero outside of some compact subset of $\Omega$) then we define $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Then $W_0^{1,p}(\Omega)$ is a closed linear subspace of $W^{1,p}(\Omega)$.

*Remark* 1.11. In the sequel we use the above norm on $W^{1,p}(\Omega)$ which is equivalent with the other commonly used norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega \left[|u|^p + \sum_{i=1}^n |D_i u|^p\right]\right)^{\frac{1}{p}}.$$

The equivalence of the two norms follows from inequalities (1.1) and (1.2).

**Theorem 1.12.** *In case $1 < p < \infty$, $W^{1,p}(\Omega)$ is a reflexive Banach space.*

**Theorem 1.13.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $1 \le p < \infty$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

**Definition 1.14.** Let $1 \le p < \infty$ be a real, $N \ge 1$ a natural number and $\Omega \subset \mathbb{R}^n$ a $\lambda$-measurable domain. Then $(L^p(\Omega))^N$ denotes the set of measurable functions $u = (u^{(1)}, \dots, u^{(N)}) \colon \Omega \to \mathbb{R}^N$ such that $u^{(l)} \in L^p(\Omega)$ for every $1 \le l \le N$. We introduce on this space the following norm

$$\|u\|_{(L^p(\Omega))^N} := \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}.$$

*Remark* 1.15. One can readily verify that the above norm is equivalent with an other commonly used norm $\left(\sum_{l=1}^N \|u^{(l)}\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$. Note that for $p = 2$, $(L^2(\Omega))^N$ is a Hilbert space with the scalar product

$$(u, v)_{(L^2(\Omega))^N} := \sum_{l=1}^N \int_\Omega u^{(l)} v^{(l)}.$$

**Definition 1.16.** Let $1 \le p < \infty$ be a real, $N \ge 1$ a natural number and $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary. Then $(W^{1,p}(\Omega))^N$ denotes the space of measurable functions $u = (u^{(1)}, \dots, u^{(N)}) \colon \Omega \to \mathbb{R}^N$ such that $u^{(l)} \in W^{1,p}(\Omega)$ for $1 \le l \le N$. We introduce the following norm on this space

$$\|u\|_{(W^{1,p}(\Omega))^N} := \left(\int_\Omega |u|^p + |Du|^p\right)^{\frac{1}{p}},$$

where $Du = (D_1 u^{(1)}, \dots, D_1 u^{(N)}, \dots, D_n u^{(1)}, \dots, D_n u^{(N)})$.

**Theorem 1.17.** *In case $1 < p < \infty$, $(W^{1,p}(\Omega))^N$ is a reflexive Banach space.*

**Theorem 1.18.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $1 \le p < \infty$. Then the embedding $(W^{1,p}(\Omega))^N \hookrightarrow (L^p(\Omega))^N$ is compact.*

## 1.3 Equi-integrability

> There are 10 kinds of mathematicians. Those who can think binarily and those who can't.
>
> Folklore

Now we introduce a less known theorem about convergences in $L^p$ spaces. First we define the notion of equi-integrability. As before, let $\Omega \subset \mathbb{R}^n$ be a $\lambda$-measurable domain and $1 \leq p < \infty$ a real number.

**Definition 1.19.** Let $(f_k)_{k\in\mathbb{N}}$ a sequence of functions in $L^p(\Omega)$. Suppose that for every $\varepsilon > 0$ there exists a set $A_\varepsilon \subset \Omega$ of finite measure and $\delta(\varepsilon) > 0$ such that for every $k \in \mathbb{N}$ it holds

$$\int_{\Omega\setminus A_\varepsilon} |f_k|^p < \varepsilon, \tag{1.4}$$

furthermore, for every measurable set with measure less than $\delta(\varepsilon)$ it follows

$$\int_S |f_k|^p < \varepsilon.$$

Then we say that the sequence $(f_k)$ is *equi-integrable* in $L^p(\Omega)$.

*Remark* 1.20. In case of bounded $\Omega$, (1.4) is obviously satisfied with $A_\varepsilon = \Omega$.

*Remark* 1.21. It is worth noting the following. If $(f_k)$ and $(g_k)$ are sequences in $L^p(\Omega)$ such that $|f_k| \leq |g_k|$ for every $k$ and the sequence $(g_k)$ is equi-integrable in $L^p(\Omega)$ then the sequence $(f_k)$ is also equi-integrable in $L^p(\Omega)$.

**Proposition 1.22.** *If the sequence $(f_k)$ is convergent in $L^p(\Omega)$ then it is equi-integrable.*

**Theorem 1.23** (Vitali)**.** *Suppose that the sequence $(f_k)$ is equi-integrable in $L^p(\Omega)$ and $f_k \to f$ a.e. in $\Omega$. Then $f_k \to f$ in $L^p(\Omega)$ (strongly).*

The following theorem of choice is due to Frigyes Riesz.[1]

**Lemma 1.24** (Riesz)**.** *Let $(f_k)$ be a Cauchy sequence in $L^p(\Omega)$. Then there exists a subsequence $(\tilde{f}_k) \subset (f_k)$ and $f \in L^p(\Omega)$ such that $\tilde{f}_k \to f$ a.e. in $\Omega$.*

**Corollary 1.25.** *Assume that $f_k \to f$ in $L^p(\Omega)$. Then there exists a subsequence $(\tilde{f}_k) \subset (f_k)$ such that $\tilde{f}_k \to f$ a.e. in $\Omega$.*

*Remark* 1.26. Obviously the above statements holds not only for the Lebesgue measure but for every complete measure space.

[1] This statement is a part of the well-known proof of Riesz-Fischer theorem on the completeness of $L^p$ spaces.

## 1.4 Weak convergence

Mathematics consists of proving the most obvious thing in the least obvious way.

George Pólya

We shall use some properties of the weak convergence listed below.

**Theorem 1.27.** *In a normed space every weakly convergent sequence is bounded. In a dual Banach space every weak-star convergent sequence is bounded.*

**Theorem 1.28.** *In a reflexive Banach space (especially in a Hilbert space) every bounded sequence has a weakly convergent subsequence.*

**Theorem 1.29.** *Assume that $X$ is a normed space and $x_k \to x$ weakly in $X$. Then*

$$\|x\|_X \leq \liminf_{k\to\infty} \|x_k\|_X.$$

## 1.5 $L^p(0,T;V)$ spaces

I was $x$ years old in the year $x^2$. [In reply to a question about his age.]

Augustus de Morgan

We briefly introduce an abstract framework in order to treat evolution problems. For the details and proofs, see, e.g., [71].

**Definition 1.30.** Let $V$ be a Banach space, further, let $1 \leq p < \infty$ and $0 < T < \infty$. Then $L^p(0,T;V)$ denotes the set of measurable functions $u\colon (0,T) \to V$ such that

$$\int_0^T \|u(t)\|_V^p \, dt < \infty.$$

*Remark* 1.31. One can define analogously the spaces $L^p(a,b;V)$ for arbitrary $a < b$. The following theorems remain true in this case.

**Theorem 1.32.** *The space $L^p(0,T;V)$ is a Banach space with the norm*

$$\|u\|_{L^p(0,T;V)} = \left(\int_0^T \|u(t)\|_V^p \, dt\right)^{\frac{1}{p}}.$$

**Theorem 1.33.** *Let $V$ be a reflexive Banach space and let $p$, $q$ be finite conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^p(0,T;V)$ is a reflexive Banach space with its dual $L^q(0,T;V^*)$, in fact, a functional $v \in L^q(0,T;V^*)$ acts on $u \in L^p(0,T;V)$ in the way*

$$[v,u] := \int_0^T \langle v(t), u(t) \rangle \, dt.$$

*Remark* 1.34. In the sequel, the pairing between the spaces $V^*$and $V$, further, between $L^q(0,T;V^*)$ and $L^p(0,T;V)$, will be denoted by $\langle \cdot, \cdot \rangle$, $[\cdot, \cdot]$, respectively.

Let $V$ be a Banach space and $H$ a (real) Hibert space. Assume that $V \subseteq H$, $V$ is dense in $H$ and the embedding $V \hookrightarrow H$ is continuous. Denote by $\| \cdot \|_V$ the norm of $V$ and by $(\cdot, \cdot)_H$ the scalar product of $H$. Then to every $f \in H$ corresponds an $F \in V^*$ in the way $\langle F, \cdot \rangle := (f, \cdot)_H$. Since $V$ is dense in $H$, the converse is also true, every $f \in H$ is determined by an element $F \in V^*$. Hence we have a bijection between $H$ and a subspace of $V^*$ thus $H \subset V^*$. Moreover, it is also clear that the embedding $H \hookrightarrow V^*$ is continuous.

**Definition 1.35.** If the above conditions are satisfied, the triple $V \subseteq H \subseteq V^*$ is called an *evolution triple* (or *Gelfand triple*).

**Definition 1.36.** Let $u \in L^p(0,T;V)$ and assume that there exists $v \in L^q(0,T;V^*)$ such that for every $w \in V$ and $\varphi \in C_0^\infty(0,T)$,

$$\int_0^T \langle u(t), w \rangle \varphi'(t) \, dt = -\int_0^T \langle v(t), w \rangle \varphi(t) \, dt.$$

Then $v$ (which is unique if exists) is called the *distributional derivative* of $u$ and we write $v = u'$.

**Definition 1.37.** Let $V \subseteq H \subseteq V^*$ be an evolution triple and let $W^{1,p}(0,T;V,H)$ be the space of functions $u \in L^p(0,T;V)$ such that $u' \in L^q(0,T;V^*)$. We introduce the norm

$$\|u\|_{W^{1,p}(0,T;V,H)} := \|u\|_{L^p(0,T;V)} + \|u'\|_{L^q(0,T;V^*)}. \tag{1.5}$$

**Theorem 1.38.** *With the above norm* (1.5)*, $W^{1,p}(0,T;V,H)$ is a Banach space.*

**Theorem 1.39.** *Let $u \in W^{1,p}(0,T;V,H)$. Then the map $[0,T] \ni t \mapsto \|u(t)\|_H^2$ is continuous, moreover, it is absolutely continuous.*

**Theorem 1.40.** *The set $W^{1,p}(0,T;V,H)$ is a subset of $C([0,T],H)$, moreover, the embedding $W^{1,p}(0,T;V,H) \hookrightarrow C([0,T],H)$ is continuous. Precisely, for every $u \in W^{1,p}(0,T;V,H)$ there exists a unique continuous function $\bar{u} \colon [0,T] \to H$ such that $\bar{u} = u$ a.e. in $[0,T]$ and*

$$\max_{s \in [0,T]} \|\bar{u}(s)\|_H \le \text{const} \cdot \|u\|_{W^{1,p}(0,T;V,H)}.$$

**Corollary 1.41.** *If $V \subseteq H \subseteq V^*$ is an evolution triple and $u \in W^{1,p}(0,T;V,H)$ then the value of $u(t)$ makes sense for every $t \in [0,T]$ (it is some element of $H$), especially $u(0)$ makes sense.*

**Theorem 1.42** (integration by parts)**.** *Let $u$, $v \in W^{1,p}(0,T;V,H)$. Then for $0 \leq s \leq t \leq T$,*

$$\int_s^t \big(\langle u'(\tau), v(\tau)\rangle + \langle v'(\tau), u(\tau)\rangle\big) d\tau = (u(t), v(t))_H - (u(s), v(s))_H.$$

**Corollary 1.43.** *Let $u \in W^{1,p}(0,T;V,H)$. Then for $0 \leq s \leq t \leq T$,*

$$2\int_s^t \langle u'(\tau), u(\tau)\rangle d\tau = \|u(t)\|_H^2 - \|u(s)\|_H^2.$$

**Corollary 1.44.** *Let $u \in W^{1,p}(0,T;V,H)$ such that $u(0) = 0$. Then $[u', u] \geq 0$.*

*Remark* 1.45. Now we are able to give an abstract formulation of an evolution problem. Let $V \subseteq H \subseteq V^*$ be an evolution triple, further, let $A\colon L^p(0,T;V) \to L^q(0,T;V^*)$ be a (possibly nonlinear) operator and $b \in L^q(0,T;V^*)$. For given $u_0 \in H$ find $u \in W^{1,p}(0,T;V,H)$ such that

$$\begin{aligned} u' + A(u) &= b \\ u(0) &= u_0. \end{aligned}$$

By ensuring some special properties of operator $A$ (these properties will be discussed in Section 1.6) one obtains existence of solutions.

Finally, we mention some embedding theorems related to this topic.

**Theorem 1.46** (Minty)**.** *Let $V \subseteq H \subseteq V^*$ be an evolution triple where $V$ is a reflexive Banach space. Suppose that $B$ is a reflexive Banach space such that $V \subseteq B \subseteq V^*$ where the embedding $V \hookrightarrow B$ is compact and the embedding $B \hookrightarrow V^*$ is continuous. Then the embedding $W^{1,p}(0,T;V,H) \hookrightarrow L^p(0,T;B)$ is compact for $1 < p < \infty$.*

**Corollary 1.47.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, further, let $2 \leq p < \infty$ be a real and $N \geq 1$ a natural number. Then the embedding $W^{1,p}\big(0,T;(W^{1,p}(\Omega))^N,(L^2(\Omega))^N\big) \hookrightarrow L^p\big(0,T;(L^p(\Omega))^N\big)$ is compact.*

*Proof.* In case $2 \leq p < \infty$, $(W^{1,p}(\Omega))^N \subset (L^2(\Omega))^N \subset (W^{1,p}(\Omega)^*)^N$ is an evolution triple and $(W^{1,p}(\Omega))^N$ is reflexive thus Theorems 1.18 and 1.46 imply the desired statement. □

**Corollary 1.48.** *Assume $(u_k) \subset W^{1,p}\big(0,T;(W^{1,p}(\Omega))^N,(L^2(\Omega))^N\big)$ such that $u_k \to u$ weakly in $L^p\big(0,T;(W^{1,p}(\Omega))^N\big)$ and $u_k' \to u'$ weakly in $L^q\big(0,T;(W^{1,p}(\Omega)^*)^N\big)$. Then there exists a subsequence $(\tilde{u}_k) \subset (u_k)$ such that $\tilde{u}_k \to u$ in $L^p\big(0,T;(L^p(\Omega))^N\big)$.*

*Proof.* Due to the weak convegences, $(u_k)$ is bounded in $L^p\big(0,T;(W^{1,p}(\Omega))^N\big)$ and $(u_k')$ is bounded in $L^p\big(0,T;(W^{1,p}(\Omega)^*)^N\big)$ thus the sequence $(u_k)$ is bounded in $W^{1,p}\big(0,T;(W^{1,p}(\Omega))^N,(L^2(\Omega))^N\big)$. By using Corollary 1.47 one has a subsequence $(\tilde{u}_k)\subset(u_k)$ which is convergent in $L^p\big(0,T;(L^p(\Omega))^N\big)$ thus it is also weakly convergent there. Since $(u_k)$ is weakly convergent in $L^p\big(0,T;(W^{1,p}(\Omega))^N\big)$ so it is weakly convergent also in $L^p\big(0,T;(L^p(\Omega))^N\big)$. Now the uniqueness of the weak limit implies $\tilde{u}_k\to u$ in $L^p\big(0,T;(L^p(\Omega))^N\big)$. □

## 1.6 Special types of operators

> Mathematics is made of 50 percent formulas, 50 percent proofs, and 50 percent imagination.
>
> Folklore

In the above introduced framework of evolution problems the operators of monotone type play an important role. We define some properties of operators.

Let $X$ be a reflexive Banach space with its dual $X^*$. We use the notation $\langle\cdot,\cdot\rangle$ for the pairing between $X^*$ and $X$.

**Definition 1.49.** Consider an operator $T\colon X\supseteq D(T)\to X^*$. Then (by using the terminology of [71])

- $T$ is *bounded* if it maps bounded sets (of $X$) into bounded sets (of $X^*$).
- $T$ is *hemicontinuous* if for arbitrary elements $u,\ v,\ w\in X$ the map $\mathbb{R}\ni\lambda\mapsto\langle T(u-\lambda v),w\rangle\in X$ is continuous.
- $T$ is *demicontinuous* if for every sequence $(u_k)\subset D(T)$ with the property $u_k\to u\in D(T)$ in $X$ it follows that $T(u_k)\to T(u)$ weakly in $X^*$.
- $T$ is *monotone* if $\langle T(u)-T(v),u-v\rangle\geq 0$ for every $u,v\in D(T)$. If equality holds only in case of $u=v$ then $T$ is said to be *strictly monotone.*
- $T$ is *uniformly monotone* if there exist constants $p>1$, $c>0$ such that $\langle T(u)-T(v),u-v\rangle\geq c\cdot\|u-v\|^p$ for every $u,v\in X$.
- $T$ is *maximal monotone* if $T$ is monotone, furthermore, if $u\in X$ and $b\in X^*$ are such that $\langle b-T(v),u-v\rangle\geq 0$ for every $v\in X$ then $T(u)=b$.

- $T$ is *pseudomonotone* if for every sequence $(u_k) \subset D(T)$ such that $u_k \to u$ in $X$ and $\limsup_{k\to\infty} \langle T(u_k), u_k - u\rangle \le 0$, it follows that $\lim_{k\to\infty} \langle T(u_k), u_k - u\rangle = 0$ and $T(u_k) \to T(u)$ weakly in $X^*$.
- $T$ is *coercive* if $\lim_{\|u\|_X\to\infty} \dfrac{\langle T(u), u\rangle}{\|u\|_X} = +\infty$.

*Remark* 1.50. If $T$ is a linear operator then its monotonicity is equivalent with $\langle T(u), u\rangle \ge 0$ for every $u \in D(T)$.

An important operator is the operator of the differentiation. Let $p, q$ be conjugate exponents and define

$$L\colon L^p(0,T;V) \supset D(L) \to L^q(0,T;V^*), \quad Lu = u' \tag{1.6}$$

where

$$D(L) := \{u \in L^p(0,T;V)\colon u' \in L^q(0,T;V^*),\ u(0) = 0\}, \tag{1.7}$$

or

$$D(L) := \{u \in L^p(0,T;V)\colon u' \in L^q(0,T;V^*),\ u(0) = u(T)\}. \tag{1.8}$$

**Theorem 1.51.** *Let $D(L)$ given by* (1.7) *or* (1.8)*. Then $L$ (defined by* (1.6)*) is a densely defined, closed, maximal monotone linear operator.*

The following convergence theorem will be useful in our investigations.

**Theorem 1.52.** *Suppose that $(u_k) \subset D(L)$ (where $D(L)$ is defined by* (1.7) *or* (1.8) *and $L$ is defined by* (1.6)*) such that $u_k \to u$ weakly in $L^p(0,T;V)$ and $Lu_k \to v$ weakly in $L^q(0,T;V^*)$ for some $v \in L^q(0,T;V^*)$. Then $u \in D(L)$ and $v = Lu$. Further, if $D(L)$ is defined by formula* (1.7) *then $u_k(0) \to u(0)$ and $u_k(T) \to u(T)$ weakly in $H$.*

*Proof.* From the definition of the distributional derivative it follows

$$\int_0^T \langle \varphi'(t)w, u_k(t)\rangle\, dt = -\int_0^T \langle Lu_k(t), w\rangle \varphi(t)\, dt,$$

for every $w \in V$ and $\varphi \in C_0^\infty(0,T)$. Passing to the limit as $k \to \infty$ yields

$$\int_0^T \langle \varphi'(t)w, u(t)\rangle\, dt = -\int_0^T \langle v(t), \varphi(t)w\rangle\, dt,$$

which means exactly that $v = u'$. Now note that for every $w(t) \equiv w \in V$ the integration by parts formula implies

$$\begin{aligned}\int_0^T \langle u_k', w\rangle &= \int_0^T \big(\langle u_k'(t), w(t)\rangle + \langle w'(t), u_k(t)\rangle\big)\, dt \\ &= (u_k(T) - u_k(0), w).\end{aligned}$$

By the weak convergence of $(u'_k)$,

$$\int_0^T \langle u'_k, w\rangle \to \int_0^T \langle u', w\rangle = (u(T) - u(0), w).$$

In case $D(L)$ is defined by (1.8), $u_k(T) - u_k(0) = 0$ hence from the above limit relation $(u(T) - u(0), w) = 0$ follows. This holds for all $w \in V$ thus the density of $V$ in $H$ implies $u(0) = u(T)$. If $D(L)$ is defined by (1.7) then it follows that $u_k(T) \to u(T) - u(0)$ weakly in $H$. By applying integration by parts formula we obtain

$$(u_k(T), \varphi(T)w) = \int_0^T \big(\langle \varphi'(t)w, u_k\rangle + \langle u'_k, \varphi(t)w\rangle\big)\, dt$$

where $\varphi \in C^\infty(0, T)$ and $w \in V$. Then by passing to the limit it follows

$$(u(T) - u(0), \varphi(T)w) = \int_0^T \big(\langle \varphi'(t)w, u\rangle + \langle u'\varphi(t)w\rangle\big)\, dt.$$

Now on the right hand side of the above equation we apply integration by parts formula for both terms. Then we may deduce

$$(u(T) - u(0), \varphi(T)w) = (u(T), \varphi(T)w) - (u(0), \varphi(0)w).$$

Choose $\varphi \in C^\infty(0, \infty) \cap C([0, T])$ such that $\varphi(0) = 1$ and $\varphi(T) = 0$ then by the density of $V$ in $H$ we conclude $u(0) = 0$. Finally, $u_k(T) \to u(T) - u(0) = u(T)$ weakly in $H$ and obviously $u_k(0) = 0 \to 0 = u(0)$ strongly in $H$. $\square$

Now we verify a sufficient condition for some of these properties in case of operators arising in weak formulation of partial differential equations.

**Definition 1.53.** Suppose that $X$ is a closed subspace of $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and define operator $A\colon X \to X^*$ by

$$\langle A(u), v\rangle = \int_\Omega \left( \sum_{i=1}^n a_i(x, u(x), Du(x)) D_i v(x) + a_0(x, u(x), Du(x)) v(x) \right) dx \quad (1.9)$$

where $v \in X$ and the following assumptions are fulfilled (a vector $\xi \in \mathbb{R}^{n+1}$ will have the coordinates $(\xi_0, \ldots, \xi_n)$):

(i) Functions $a_i\colon \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 0, \ldots, n)$ have the Carathéodory property, i.e., $a_i(x, \xi)$ is measurable in $x \in \Omega$ for all fixed $\xi \in \mathbb{R}^{n+1}$, and continuous in $\xi \in \mathbb{R}^{n+1}$ for a.a. fixed $x \in \Omega$.

(ii) There exist constants $p > 1$, $c > 0$ and a function $k_1 \in L^q(\Omega)$ such that for a.a. $x \in \Omega$ and every $\xi \in \mathbb{R}^{n+1}$,

$$|a_i(x, \xi)| \le c \cdot |\xi|^{p-1} + k_1(x), \qquad i = 0, \ldots, n.$$

(iii) For a.a. $x \in \Omega$ and every $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ such that $(\xi_1, \ldots, \xi_n) \neq (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$,

$$\sum_{i=1}^{n} \Big( a_i(x, \xi_0, \xi_1, \ldots, \xi_n) - a_i(x, \xi_0, \tilde{\xi}_1, \ldots, \tilde{\xi}_n) \Big) (\xi_i - \tilde{\xi}_i) \geq 0.$$

(iv) There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(\Omega)$ such that for a.a. $x \in \Omega$ and every $\xi \in \mathbb{R}^{n+1}$ ,

$$\sum_{i=0}^{n} a_i(x, \xi)\xi_i \geq c_2 |\xi|^p - k_2(x).$$

**Theorem 1.54.** *Suppose that conditions (i)–(iv) hold. Then $A$ is a bounded, demicontinuous, coercive and pseudomonotone operator.*

*Proof.* For the proof of the boundedness, demicontinuity and coverciveness see, e.g., the classical monographs [44, 53, 71]. For the pseudomonotonicity, see [17]. □

**Proposition 1.55.** *Assume $p \geq 2$ and conditions (i)–(ii). Further, suppose that functions $a_i$ $(i = 0, \ldots, n)$ are continuously differentiable in variable $\xi$ and there exists a constant $\delta > 0$ such that for a.a. $x \in \Omega$, every $\xi \in \mathbb{R}^{n+1}$, $(z_0, \ldots, z_n) \in \mathbb{R}^{n+1}$,*

$$\sum_{j=0}^{n} \sum_{i=0}^{n} D_j a_i(x, \xi) z_i z_j \geq \delta \cdot \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2. \tag{1.10}$$

*Then operator $A$ is uniformly monotone.*

*Proof.* Fix $x \in \Omega$, $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ and define functions $f_i \colon [0,1] \to \mathbb{R}$ by $f_i(\tau) = a_i(x, \tilde{\xi} + \tau(\xi - \tilde{\xi}))$ $(i = 0, \ldots, n)$. Then by applying assumption (1.10) and inequality (1.3) we may deduce

$$\begin{aligned}
\sum_{i=0}^{n} (a_i(x, \xi) - a_i(x, \tilde{\xi}))(\xi_i - \tilde{\xi}_i) &= \sum_{i=0}^{n} (f_i(1) - f_i(0))(\xi_i - \tilde{\xi}_i) \\
&= \sum_{i=0}^{n} \int_0^1 \sum_{j=0}^{n} D_j a_i(\tilde{\xi} + \tau(\xi - \tilde{\xi}))(\xi_j - \tilde{\xi}_j)(\xi_i - \tilde{\xi}_i)\, d\tau \\
&\geq \delta \cdot \sum_{i=0}^{n} \int_0^1 |\tilde{\xi} + \tau(\xi - \tilde{\xi})|^{p-2} (\xi_i - \tilde{\xi}_i)^2 d\tau \\
&\geq \frac{\delta}{2^{p-2}(p-1)} |\xi - \tilde{\xi}|^p.
\end{aligned}$$

Whence after integration we conclude

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \frac{\delta}{2^{p-2}(p-1)} \|u_1 - u_2\|_X^p.$$

□

Now we are able to give some example for the above functions $a_i$ such that the operator $A$ will be uniformly monotone. For more details, see [13].

**Proposition 1.56.** *Let $a_i(\xi) = \xi_i|\xi_i|^{p-2}$ with some $p \geq 2$ ($i = 0, \dots, n$). Then operator $A$ defined by* (1.9) *is uniformly monotone.*

*Proof.* Note that $A$ obviously satisfies conditions (i)–(ii). In this case of $a_i$,

$$\langle A(u), v\rangle = \int_\Omega \left( \sum_{i=1}^n D_i u D_i v |D_i u|^{p-2} + uv|u|^{p-2} \right) dx.$$

Simple calculations yield $D_i a_i(\xi) = (p-1)|\xi_i|^{p-2}$ and $D_j a_i(\xi) = 0$ ($j \neq i$). Hence

$$\sum_{j=0}^n \sum_{i=0}^n D_j a_i(\xi) z_i z_j = (p-1) \sum_{i=0}^n |\xi_i|^{p-2} z_i^2$$

thus Proposition 1.55 implies the uniform monotonicity of $A$. $\square$

**Proposition 1.57.** *Let $a_i(\xi) = \xi_i|(\xi_1, \dots, \xi_n)|^{p-2}$ for $i = 1, \dots, n$ and $a_0(\xi) = \xi_0|\xi_0|^{p-2}$ where $p \geq 2$. Then operator $A$ defined by* (1.9) *is uniformly monotone.*

*Proof.* Obviously, $A$ fulfils also conditions (i)–(ii). Now

$$\langle A(u), v\rangle = \int_\Omega \left( \sum_{i=1}^n D_i u D_i v |Du|^{p-2} + uv|u|^{p-2} \right),$$

i.e., $A$ is the weak form of operator $u \mapsto \Delta_p u + u|u|^{p-2}$, where $\Delta_p$ is called the $p$-Laplacian and has the form

$$\Delta_p u = \operatorname{div}(Du|Du|^{p-2}). \tag{1.11}$$

It is easy to see that

$$\begin{cases} D_j a_i(\xi) = (p-2)\xi_j \xi_i |(\xi_1, \dots, \xi_n)|^{p-4}, & \text{for } i, j > 0,\ i \neq j; \\ D_i a_i(\xi) = |(\xi_1, \dots, \xi_n)|^{p-2} + (p-2)\xi_i^2 |(\xi_1, \dots, \xi_n)|^{p-4}, & \text{for } i > 0; \\ D_j a_0(\xi) = D_0 a_i(\xi) = 0, & \text{for } j > 0,\ i > 0; \\ D_0 a_0(\xi) = (p-1)|\xi_0|^{p-2}. \end{cases}$$

Hence

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i z_i z_j &= \sum_{i=1}^{n} |(\xi_1,\ldots,\xi_n)|^{p-2} z_i^2 + (p-1)|\xi_0|^{p-2} z_0^2 \\
&\quad + (p-2)|(\xi_1,\ldots,\xi_n)|^{p-4} \cdot \sum_{j=1}^{n}\sum_{i=1}^{n} \xi_i\xi_j z_i z_j \\
&= \sum_{i=1}^{n} |(\xi_1,\ldots,\xi_n)|^{p-2} z_i^2 + (p-1)|\xi_0|^{p-2} z_0^2 \\
&\quad + (p-2)|(\xi_1,\ldots,\xi_n)|^{p-4} \cdot \left(\sum_{i=1}^{n} \xi_i z_i\right)^2 \\
&\geq \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2
\end{aligned}$$

thus from Proposition 1.55 it follows that $A$ is uniformly monotone. □

**Proposition 1.58.** *Suppose $2 \leq p \leq 4$ and let $a_i(\xi) = \xi_i|\xi|^{p-2}$ for $0 \leq i \leq k \leq n$ and $a_i(\xi) = \xi_i|(\xi_{k+1},\ldots,\xi_n)|^{p-2}$ for $k < i \leq n$. Then operator $A$ defined by* (1.9) *is uniformly monotone.*

*Proof.* It is easily seen that $A$ also satisfies conditions (i)–(ii). Now for brevity let $\zeta = (\xi_{k+1},\ldots,\xi_n)$. Clearly,

$$\begin{cases}
D_j a_i(\xi) = (p-2)\xi_i\xi_j|\xi|^{p-4}, & \text{for } 0 \leq i \leq k,\ 0 \leq j \leq n,\ j \neq i; \\
D_j a_i(\xi) = (p-2)\xi_i\xi_j|\zeta|^{p-4}, & \text{for } k < i \leq n,\ k < j < n,\ j \neq i; \\
D_j a_i(\xi) = 0, & \text{for } k < i \leq n,\ 0 \leq j \leq k \\
D_i a_i(\xi) = |\xi|^{p-2} + (p-2)\xi_i^2|\xi|^{p-4}, & \text{for } 0 \leq i \leq k; \\
D_i a_i(\xi) = |\zeta|^{p-2} + (p-2)\xi_i^2|\zeta|^{p-4}, & \text{for } k < i \leq n.
\end{cases}$$

Then

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j &= \sum_{i=0}^{k} |\xi|^{p-2} z_i^2 + (p-2)|\xi|^{p-4} \sum_{j=0}^{n}\sum_{i=0}^{k} \xi_i\xi_j z_i z_j \\
&\quad + \sum_{i=k+1}^{n} |\zeta|^{p-2} z_i^2 + (p-2)|\zeta|^{p-4} \sum_{j=k+1}^{n}\sum_{i=k+1}^{n} \xi_i\xi_j z_i z_j \\
&= \sum_{i=0}^{k} |\xi|^{p-2} z_i^2 + \sum_{i=k+1}^{n} |\zeta|^{p-2} z_i^2 + (p-2)|\xi|^{p-4} \left(\sum_{i=0}^{k} \xi_i z_i\right)^2 \\
&\quad + (p-2)|\zeta|^{p-4} \left(\sum_{i=k+1}^{n} \xi_i z_i\right)^2 + (p-2)|\xi|^{p-4} \sum_{j=k+1}^{n}\sum_{i=0}^{k} \xi_i\xi_j z_i z_j.
\end{aligned}$$

By using the estimate

$$\sum_{j=k+1}^{n}\sum_{i=0}^{k}\xi_i\xi_j z_i z_j = \left(\sum_{j=k+1}^{n}\xi_j z_j\right)\left(\sum_{i=0}^{k}\xi_i z_i\right)$$
$$\geq -\frac{1}{2}\left(\sum_{i=k+1}^{n}\xi_i z_i\right)^2 - \frac{1}{2}\left(\sum_{i=0}^{k}\xi_i z_i\right)^2.$$

and the fact that $|\zeta|^{p-4} \geq |\xi|^{p-4}$ since $p \leq 4$ we conclude

$$\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j = \sum_{i=0}^{k}|\xi|^{p-2}z_i^2 + \sum_{i=k+1}^{n}|\zeta|^{p-2}z_i^2 + \frac{1}{2}(p-2)|\xi|^{p-4}\left(\sum_{i=k+1}^{n}\xi_i z_i\right)^2$$
$$+\frac{1}{2}(p-2)|\xi|^{p-4}\left(\sum_{i=0}^{k}\xi_i z_i\right)^2 \geq \sum_{i=0}^{n}|\xi_i|^{p-2}z_i^2.$$

Now Proposition 1.55 yields the uniform monotonicity of $A$. □

*Remark* 1.59. In case $p > 4$ one may consider, e.g., the following functions

$$a_i(\xi) = \xi_i|(\xi_0,\ldots,\xi_k)|^{p-2} + \xi_i|\xi|^{r-2} \quad \text{if} \quad 0 \leq i \leq k \leq n,$$
$$a_i(\xi) = \xi_i|(\xi_{k+1},\ldots,\xi_n)|^{p-2} + \xi_i|(\xi_{k+1},\ldots,\xi_n)|^{r-2} \quad \text{if} \quad k < i < n,$$

where $2 \leq r \leq 4$. Then operator $A$ defined by (1.9) is uniformly monotone.

In non-time-dependent problems the following classical theorem states existence of solutions.

**Theorem 1.60.** *Let $X$ be a reflexive Banach space. Assume that operator $T\colon X \to X^*$ is bounded, hemicontinuous, pseudomonotone and coercive. Then for every $v \in X^*$ there exists $u \in X$ such that $T(u) = v$.*

We can ensure uniqueness by some stronger assumptions.

**Theorem 1.61.** *Let $X$ be a reflexive Banach space. Suppose that operator $T\colon X \to X^*$ is bounded, hemicontinuous, strictly monotone and coercive. Then for every $v \in X^*$ there exists a unique $u \in X$ such that $T(u) = v$.*

By supposing the uniform monotonicity of $A$, the continuous dependence of solutions follows.

**Proposition 1.62.** *If $A\colon X \to X^*$ is uniformly monotone then the solution $u$ of problem $A(u) = F$ is unique and depends continuously on $F \in X^*$.*

*Proof.* Uniqueness follows from the fact that if $A(u_1) = F = A(u_2)$ for $u_1, u_2 \in X$ then $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle = 0$ and the uniform monotonicity implies $u_1 = u_2$. Now let $F_1, F_2 \in X^*$ and $u_1, u_2 \in X$ be such that $A(u_i) = F_i$ $(i = 1, 2)$. Then

$$\begin{aligned}\|u_1 - u_2\|_X^p &\le \text{const} \cdot \langle A(u_1) - A(u_2), u_1 - u_2 \rangle \\ &\le \text{const} \cdot \|A(u_1) - A(u_2)\|_{X^*} \cdot \|u_1 - u_2\|_X \\ &= \text{const} \cdot \|F_1 - F_2\|_{X^*} \cdot \|u_1 - u_2\|_X\end{aligned}$$

thus $\|u_1 - u_2\|_X \le \text{const} \cdot \|F_1 - F_2\|_{X^*}^{\frac{1}{p-1}}$ which yields the continuous dependence. □

In time-dependent (evolution) problems we have operators of type $S = L + T$ where $L\colon X \supseteq D(L) \to X^*$ is (the operator of differentiation that is) a densely defined, closed, maximal monotone, linear operator, further, $T\colon X \to X^*$ is of monotone type.

**Definition 1.63.** Operator $T$ is *pseudomonotone with respect to* $D(L)$ if for every sequence $(u_k) \subset D(L)$ such that $u_k \to u$ weakly in $X$, $L(u_k) \to L(u)$ weakly in $X^*$ and $\limsup_{k\to\infty} \langle T(u_k), u_k - u \rangle \le 0$, it follows that $\lim_{k\to\infty} \langle T(u_k), u_k - u \rangle = 0$ and $T(u_k) \to T(u)$ weakly in $X^*$.

It is useful to rephrase this definition together with the definition of demicontinuity.

**Lemma 1.64** ("subsequence trick")**.** [2]

a) *Operator $T$ is demicontinuous if for every sequence $(u_k) \subset D(T)$ such that $u_k \to u \in D(T)$ in $X$, there exists a subsequence $(\tilde{u}_k) \subset (u_k)$ with the property $T(\tilde{u}_)) \to T(u)$ weakly in $X^*$.*

b) *Operator $T$ is pseudomonotone with respect to $D(L)$ if for every sequence $(u_k) \subset D(L)$ such that $u_k \to u$ weakly in $X$, $L(u_k) \to L(u)$ weakly in $X^*$ and $\limsup_{k\to\infty} \langle T(u_k), u_k - u \rangle \le 0$, there exists a subsequence $(\tilde{u}_k) \subset (u_k)$ with the properties $\lim_{k\to\infty} \langle T(\tilde{u}_k), \tilde{u}_k - u \rangle = 0$ and $T(\tilde{u}_k) \to T(u)$ weakly in $X^*$.*

*Proof.* We show the case a), the other can be treated similarly. We proceed by contradiction, suppose $T$ is not demicontinuous. Then there exist $\varepsilon > 0$, $v \in X$ and $(u_k) \subset D(T)$ such that $u_k \to u \in D(T)$ in $X$ and $\langle T(u_k) - T(u), v \rangle \ge \varepsilon$. But this implies that there is no $(\tilde{u}_k)$ subsequence of $(u_k)$ such that $T(\tilde{u}_k) \to T(u)$ holds. □

[2] This idea appears already in the works of Georg Cantor. He used the fact that a real sequence is convergent if and only if every subsequence of the sequence has a convergent subsequence.

In time-dependent problems the following theorem will be the key of existence of solutions, for the proof, see [8].

**Theorem 1.65.** *Let $X$ be a reflexive Banach space and $L\colon X \supseteq D(L) \to X^*$ a densely defined, closed, maximal monotone linear operator, further, let $T\colon X \to X^*$ be bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$. Then $(L+T)(D(L)) = X^*$.*

By modifying the definition of operator $A$ (see (1.9)) and conditions (i)–(iv) according to the time variable, one has a theorem analogous to 1.54.

**Definition 1.66.** Suppose that $V$ is a closed subspace of $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and let $X = L^p(0,T;V)$ for some $0 < T < \infty$ and $2 \le p < \infty$ (with its conjugate exponent $q$). Define operator $A\colon X \to X^*$ by

$$\begin{aligned}\langle A(u), v\rangle = &\int_{Q_T} \sum_{i=1}^{n} a_i(t,x,u(t,x),Du(t,x))D_i v(t,x)\,dt\,dx \\ &+ \int_{Q_T} a_0(t,x,u(t,x),Du(t,x))v(t,x)\,dt\,dx\end{aligned} \tag{1.12}$$

where $Q_T = (0,T)\times\Omega$, $v \in X$ and the following assumptions are fulfilled (a vector $\xi \in \mathbb{R}^{n+1}$ will have the coordinates $(\xi_0,\dots,\xi_n)$):

(i') Functions $a_i\colon Q_T \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 0,\dots,n)$ have the Carathéodory property, i.e., $a_i(t,x,\xi)$ is measurable in $(t,x) \in Q_T$ for all fixed $\xi \in \mathbb{R}^{n+1}$, and continuous in $\xi \in \mathbb{R}^{n+1}$ for a.a. fixed $(t,x) \in Q_T$.

(ii') There exist constants $p > 1$, $c > 0$ and a function $k_1 \in L^q(Q_T)$ such that for a.a. $(t,x) \in Q_T$ and every $\xi \in \mathbb{R}^{n+1}$,

$$|a_i(t,x,\xi)| \le c \cdot |\xi|^{p-1} + k_1(t,x), \qquad i = 0,\dots,n.$$

(iii') For a.a. $(t,x) \in Q_T$ and every $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ such that $(\xi_1,\dots,\xi_n) \ne (\tilde{\xi}_1,\dots,\tilde{\xi}_n)$,

$$\sum_{i=1}^{n} \Big(a_i(t,x,\xi_0,\xi_1,\dots,\xi_n) - a_i(t,x,\xi_0,\tilde{\xi}_1,\dots,\tilde{\xi}_n)\Big)(\xi_i - \tilde{\xi}_i) > 0.$$

(iv') There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(Q_T)$ such that for a.a. $(t,x) \in \Omega$ and every $\xi \in \mathbb{R}^{n+1}$ ,

$$\sum_{i=0}^{n} a_i(t,x,\xi)\xi_i \ge c_2|\xi|^p - k_2(t,x).$$

**Theorem 1.67.** *Assume that* $D(L)$ *is defined by* (1.7) *or* (1.8) *and operator* $L$ *is given by* (1.6) *for some* $0 < T < \infty$ *and* $2 \leq p < \infty$*). Further, assume that conditions (i')–(iv') are satisfied. Then operator* $A$ *(defined by* (1.12)*) is bounded, demicontinuous, coercive and pseudomonotone with respect to* $D(L)$.

*Idea of the proof.* The boundedness and coerciveness are similar to the non-time-dependent case (see Theorem 1.54). By using Corollary 1.48 instead of Theorem 1.18, the demicontinuity follows similarly to the non-time-dependent case. The pseudomonotonicity with respect to $D(L)$ can be proved the same way as the pseudomonotonicity for the non-time-dependent case in Theorem 1.54 by replacing Theorem 1.18 with Corollary 1.48 (see [8]). $\square$

# Chapter 2

# A system of parabolic equations

Should I refuse a good dinner simply because I do not understand the process of digestion? [Criticized for using formal mathematical manipulations, without understanding how they worked.]

Oliver Heaviside

## 2.1 Introduction

Obvious is the most dangerous word in mathematics.

Eric Temple Bell

In this chapter we consider the following nonlinear system containing $N$ parabolic differential equations:

$$\begin{aligned} &D_t u^{(l)}(\cdot) \\ &-\sum_{i=1}^n D_i\left[a_i^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u^{(1)}, \dots, u^{(N)})\right] \\ &+ a_0^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u^{(1)}, \dots, u^{(N)}) \\ &= f^{(l)}(\cdot), \end{aligned} \tag{2.1}$$

where $(\cdot)$ stands for the variable $(t, x) \in (0, T) \times \Omega$ and the terms after the symbol ";" represent the nonloacal variables ($l = 1, \dots, N$). We pose homogeneous Dirichlet or Neumann boundary condition. For instance, the homogeneous Neumann boundary condition is

$$\begin{aligned} &\sum_{i=1}^n a_i^{(l)}(t, x, u^{(1)}(t,x), \dots, u^{(N)}(t,x), Du^{(1)}(t,x), \dots, Du^{(N)}(t,x); u^{(1)}, \dots, u^{(N)}))\nu_i \\ &= 0 \end{aligned}$$

for $x \in \partial\Omega, t > 0$ where $\nu$ is the unit normal along the boundary ($l = 1, \ldots, N$). Clearly, we may assume the boundary conditions to be homogeneous by substracting a suitable function from the unknown.

Moreover, if $\partial\Omega = S_1 \cup S_2$ where $S_1 \cap S_2 = \emptyset$, then we may pose different boundary conditions on the elements of the partition.

Some physical motivations to this Chapter were demonstrated in the Preface. Nonlocal parabolic problems may occur, e.g., in population dynamics, climatology, fluid flow models, etc. In [21, 22] a simple nonloacal parabolic equation was studied which is similar to equation (1) shown in the Preface. A generalization of this equation which is similar to the above was investigated by L. Simon in [63]. These results were extended to systems of equations in [9].

In what follows, under some assumptions we shall define the weak form of the above system and prove existence of weak solutions in $(0, T)$ where $0 < T \leq \infty$, further, we show some properties of these solutions. Our assumptions are the generalizations of the classical Leray-Lions conditions. This chapter is devoted to be familiarized with monotone type operators in nonlinear differential equations. The gained knowledge will help us to deal with more complicate systems, for instance a system which contains three types of equations. Such a problem will be studied in Chapter 3. Some parts of the following section were published in [9].

### 2.1.1 Notation

To make easier the abstract formulation we introduce some notation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $0 < T < \infty$, $2 \leq p < \infty$ (with its conjugate exponent $q$) be real numbers. For brevity, denote $Q_T = (0, T) \times \Omega$. We use the definition of the space $W^{1,p}(\Omega)$ as it was introduced in Section 1.2. Denote $V_l \subset W^{1,p}(\Omega)$ ($l = 1, \cdots, N$) and let $V = V_1 \times \ldots \times V_N$, $H = (L^2(\Omega))^N$. Then for fixed $T$ we use the notion of spaces $L^p(0, T; V)$, $L^q(0, T; V^*)$, $W^{1,p}(0, T; V, H)$ as they were defined in Section 1.5. Briefly, let $X = L^p(0, T; V)$ and $Y = L^p(0, T; (L^p(\Omega))^N)$. The distributional derivative of a function $u \in L^p(0, T; V)$ will be denoted by $D_t u$. Precisely, a function $u \in L^p(0, T; V)$ has one variable ($t \in (0, T)$), however, it is often convenient to write it as a function of $(t, x)$ where $x \in \Omega$ (which sounds logic since the value of $u$ at each point $t$ is some element of $V$, i.e., a function depending on $x \in \Omega$). For $u \in X$ we shall write $u = (u^{(1)}, \ldots, u^{(N)})$ where $u^{(l)} \in L^p(0, T; V_l)$. A vector $\xi \in \mathbb{R}^{(n+1)N}$ will be written in the form $\xi = (\zeta_0, \zeta)$ where $\zeta_0 = (\zeta_0^{(1)}, \ldots, \zeta_0^{(N)}) \in \mathbb{R}^N$, $\zeta = (\zeta^{(1)}, \ldots, \zeta^{(N)}) \in \mathbb{R}^{nN}$ with $\zeta^{(l)} = (\zeta_1^{(l)}, \ldots, \zeta_n^{(l)}) \in \mathbb{R}^n$.

### 2.1.2 Assumptions

Suppose the following

(A1) Functions $a_i^{(l)} \colon Q_T \times \mathbb{R}^{(n+1)N} \times L^p(0,T;V) \to \mathbb{R}$ $(i = 0, \dots, n;\ l = 1, \dots, N)$ have the Carathéodory property for every fixed $v \in L^p(0,T;V)$, i.e., they are measurable in $(t,x) \in Q_T$ for every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and continuous in $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ for a.a. $(t,x) \in Q_T$

(A2) There exist bounded operators $g_1 \colon L^p(0,T;V) \to \mathbb{R}^+$ and $k_1 \colon L^p(0,T;V) \to L^q(Q_T)$ such that

$$|a_i^{(l)}(t,x,\zeta_0,\zeta;v)| \leq g_1(v) \left(|\zeta_0|^{p-1} + |\zeta|^{p-1}\right) + [k_1(v)]\,(t,x)$$

for a.a. $(t,x) \in Q_T$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p(0,T;V)$ $(i = 0, \dots, n;\ l = 1, \dots, N)$.

(A3) For a.a. $(t,x) \in Q_T$, every $\zeta \neq \tilde{\zeta} \in \mathbb{R}^{nN}$, $\zeta_0 \in \mathbb{R}^N$ and $v \in L^p(0,T;V)$,

$$\sum_{l=1}^{N} \sum_{i=1}^{n} \left(a_i^{(l)}(t,x,\zeta_0,\zeta;v) - a_i^{(l)}(t,x,\zeta_0,\tilde{\zeta};v)\right)(\zeta_i^{(l)} - \tilde{\zeta}_i^{(l)}) > 0.$$

(A4) There exist operators $g_2 \colon L^p(0,T;V) \to \mathbb{R}^+$ and $k_2 \colon L^p(0,T;V) \to L^1(Q_T)$ such that

$$\sum_{l=1}^{N} \sum_{i=0}^{n} a_i^{(l)}(t,x,\zeta_0,\zeta;v)\zeta_i^{(l)} \geq g_2(v)\left(|\zeta_0|^p + |\zeta|^p\right) - [k_2(v)](t,x)$$

for a.a. $(t,x) \in Q_T$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p(0,T;V)$. Further, operators $g_2$, $k_2$ have the following property:

$$\lim_{\|v\|_{L^p(0,T;V)} \to \infty} \|v\|_{L^p(0,T;V)}^{p-1} \left( g_2(v) - \frac{\|k_2(v)\|_{L^1(Q_T)}}{\|v\|_{L^p(0,T;V)}^p} \right) = +\infty.$$

(A5) If $u_k \to u$ weakly in $L^p(0,T;V)$ and strongly in $L^p(0,T;(L^p(\Omega))^N)$ then for every $i = 0, \dots, n;\ l = 1, \dots, N$,

$$\lim_{k \to \infty} \|a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u_k) - a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u)\|_{L^q(Q_T)} = 0.$$

(F1) $F \in L^q(0,T;V^*)$

Note that assumptions (A1)–(A4) are similar to the classical case (i.e., when there is no nonlocal term), see [44, 71] or Section 1.6. Condition (A5) means a kind of “continuity” in the nonlocal variable.

### 2.1.3 Weak formulation

Now we may define the weak form of system (2.1). Assuming conditions (A1), (A2), we may introduce operator $A\colon L^p(0,T;V) \to L^q(0,T;V^*)$ as follows. For $u = (u^{(1)}, \dots, u^{(N)}) \in L^p(0,T;V)$ and $v = (v^{(1)}, \dots, v^{(N)}) \in L^p(0,T;V)$ let

$$\begin{aligned}[A(u), v] := &\sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_i^{(l)}(t, x, u(t,x), Du(t,x); u) D_i v^{(l)}(t,x)\, dt\, dx \\ &+ \sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_0^{(l)}(t, x, u(t,x), Du(t,x); u) v^{(l)}(t,x)\, dt\, dx,\end{aligned} \tag{2.2}$$

where $D_i$ denotes the distributional derivative with respect to the variable $x_i$ and $D = (D_1, \dots, D_N)$ (see Section 1.2). Further, let $D(L) \to L^q(0,T;V^*)$ be the operator of differentiation (see Section 1.5):

$$D(L) := \{u \in L^p(0,T;V)\colon D_t u \in L^q(0,T;V^*),\ u(0) = 0\}, \quad Lu = D_t u. \tag{2.3}$$

Finally, in condition (F1) we consider general $F \in L^q(0,T;V^*)$ functionals, but it may have special form

$$[F, v] := \sum_{l=1}^{N} \int_{Q_T} f^{(l)}(t,x) v^{(l)}(t,x)\, dt\, dx$$

for $v \in L^p(0,T;V)$ where $f^{(l)} \in L^q(Q_T)$ $(l = 1, \dots, N)$.

By the operators above the weak form of system (2.1) is

$$Lu + A(u) = F. \tag{2.4}$$

Note that in equation (2.4) there is a "hidden" initial condition $u(0) = 0$ which is given in the domain of $L$. It is well-known (see, e.g., [44]) that one obtains the above weak form by taking sufficiently smooth solutions, using Green's theorem and finally considering the whole system in the space $L^p(0,T;V)$. Clearly, if the boundary condition is homogeneous Neumann then $V = W^{1,p}(\Omega)$ (since the boundary term vanishes in Green's theorem) and if we have homogeneous Dirichlet boundary condition then $V = W_0^{1,p}(\Omega)$ (in order to eliminate the boundary term in Green's theorem). Further, if we have a partition, for example in one dimension with homogenous Dirichlet and Neumann boundary conditions then $V = \{v \in W^{1,p_1}(0,1) : v(0) = 0, D_x v(1) = 0\}$.

## 2.2 Weak solutions in $(0,T)$

> A mathematician is a blind man in a dark room looking for a black cat which isn't there.
>
> Charles Robert Darwin

### 2.2.1 Existence

The following theorem states important properties of $A$. These will imply existence of solutions to problem (2.4).

**Theorem 2.1.** *Suppose that conditions (A1)–(A5) are satisfied. Then operator $A\colon X \to X^*$ is bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$.*

*Proof.* The proof is based mostly on techniques of estimates.

**Boundedness.** By Hölder's inequality for $i = 1, \dots, n$,

$$\begin{aligned} &\left| \int_{Q_T} a_i^{(l)}(t,x,u(t,x),Du(t,x);u) D_i v^{(l)}(t,x)\, dt\, dx \right| \\ &\le \left( \int_{Q_T} |a_i^{(l)}(t,x,u(t,x),Du(t,x);u)|^q \, dt\, dx \right)^{\frac{1}{q}} \left( \int_{Q_T} |D_i v^{(l)}(t,x)|^p \, dt\, dx \right)^{\frac{1}{p}}. \end{aligned} \tag{2.5}$$

(In case $i = 0$ we replace $D_i v^{(l)}$ with $v^{(l)}$.) The right hand side of (2.5) may be estimated by (A2) and inequality (1.1) which yields

$$\begin{aligned} &\left( \int_{Q_T} |a_i^{(l)}(t,x,u(t,x),Du(t,x);u)|^q \, dt\, dx \right)^{\frac{1}{q}} \\ &\le \text{const} \cdot \left( \int_{Q_T} \left[ g_1(u)^q \left( |u(t,x)|^p + |Du(t,x)|^p \right) + |[k_1(u)](t,x)|^q \right] dt\, dx \right)^{\frac{1}{q}} \\ &= \text{const} \cdot \left( g_1(u) \|u\|_X^{\frac{p}{q}} + \|k_1(u)\|_{L^q(Q_T)} \right). \end{aligned} \tag{2.6}$$

By summing the above estimates with respect to $i$ and $l$ we obtain

$$|[A(u), v]| \le \text{const} \cdot \left( g_1(u) \|u\|_X^{\frac{p}{q}} + \|k_1(u)\|_{L^q(Q_T)} \right) \|v\|_X. \tag{2.7}$$

Thus

$$\|A(u)\|_{X^*} \le \text{const} \cdot \left( g_1(u) \|u\|_X^{\frac{p}{q}} + \|k_1(u)\|_{L^q(Q_T)} \right).$$

Now the boundedness of operators $g_1$ and $k_1$ implies the boundedness of $A$.

**Demicontinuity.** Assume that $u_k \to u$ strongly in $X$. Then there exists a subsequence (for simplicity, it will be denoted as the original sequence) such that $u_k \to u$ and $Du_k \to Du$ for a.e. in $Q_T$. We shall show that $[A(u_k) - A(u), v] \to 0$ for every $v \in X$ then by using the "subsequence trick" the demicontinuity follows.

Now for fixed $w \in L^p(0,T;V)$ define operator $A_w\colon X \to X^*$ by

$$\begin{aligned} [A_w(v), z] := &\sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_i^{(l)}(t,x,v(t,x),Dv(t,x);w) D_i z^{(l)}(t,x)\, dt\, dx \\ &+ \sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_0^{(l)}(t,x,v(t,x),Dv(t,x);w) z^{(l)}(t,x)\, dt\, dx \end{aligned}$$

where $z = (z^{(1)}, \dots, z^{(N)}) \in L^q(0,T;V^*)$. We show that $A(u_k) - A_u(u_k) \to 0$ and $A_u(u_k) - A(u) \to 0$ weakly in $X^*$. By triangle and Hölder's inequalities it is sufficient to verify

$$\|a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u_k) - a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u)\|_{L^q(Q_T)} \to 0 \tag{2.8}$$

and

$$\|a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u) - a_i^{(l)}(\cdot\,, u(\cdot), Du(\cdot); u)\|_{L^q(Q_T)} \to 0. \tag{2.9}$$

The continuous embedding $X \to Y$ and condition (A5) imply (2.8). On the other hand, from condition (A1) and the almost everywhere convergence of $(u_k)$ and $(Du_k)$ it follows

$$a_i^{(l)}(t,x,u_k(t,x), Du_k(t,x); u) \to a_i^{(l)}(t,x,u(t,x), Du(t,x); u) \ \text{ a.e. in } \ Q_T.$$

Further,

$$\begin{aligned} &|a_i^{(l)}(t,x,u_k(t,x), Du_k(t,x); u)|^q \\ &\leq g_1(u)^q \left(|u_k(t,x)|^p + |Du_k(t,x)|^p\right) + |[k_1(u)](t,x)|^q. \end{aligned}$$

Denote by $f_k$ the right hand side of the above equation. Since $(u_k)$ is convergent in $X$, $(f_k)$ is convergent in $L^1(Q_T)$, consequently, it is equi-integrable in $L^1(Q_T)$, too. Hence functions $\left(a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u)\right)_{k\in\mathbb{N}}$ are equi-integrable in $L^q(Q_T)$. Then by Vitali's theorem we conclude

$$\lim_{k\to\infty} \|a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u) - a_i^{(l)}(\cdot\,, u(\cdot), Du(\cdot); u)\|_{L^q(Q_T)} = 0.$$

*Remark* 2.2. Observe that we have shown also the fact that $A(u_k) - A_u(u_k) \to 0$ weakly in $X^*$ and $[A(u_k) - A_u(u_k), v_k] \to 0$ for a bounded sequence $(v_k)$ in $X$.

**Coerciveness.** The first part of condition (A4) implies

$$\begin{aligned} [A(u), u] &\geq \int_{Q_T} \big(g_2(u)|u(t,x)|^p + g_2(u)|Du(t,x)|^p - [k_2(u)](t,x)\big)\, dt\, dx \\ &= g_2(u)\|u\|_X^p - \|k_2(u)\|_{L^1(Q_T)}, \end{aligned} \tag{2.10}$$

thus by using the second part of (A4) we may deduce

$$\lim_{\|u\|_X\to\infty} \frac{[A(u), u]}{\|u\|_X} \geq \lim_{k\to\infty} \left[ g_2(u)\|u\|_X^{p-1} - \frac{\|k_2(u)\|_{L^1(Q_T)}}{\|u\|_X} \right] = +\infty.$$

**Pseudomonotonicity.** Let us suppose that

$$u_k \to u \ \text{ weakly in } \ X \ \text{ and } \ Lu_k \to Lu \ \text{ weakly in } \ X^*, \tag{2.11}$$

further,

$$\limsup_{k\to\infty} [A(u_k), u_k - u] \leq 0. \tag{2.12}$$

By using the "subsequence trick" it is sufficient to show that for a suitable subsequence (denoted same as the original)

$$\lim_{k\to\infty}[A(u_k), u_k - u] = 0 \ \text{ and } \ A(u_k) \to A(u) \ \text{ weakly in } \ X^*.$$

Since the embedding $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact and $(u_k)$ is bounded in $X$, further, $(Du_k)$ is bounded in $X^*$ by its weak convergence, then Corollary 1.48 implies the existence of a subsequence $(u_k) \subset (u_k)$ such that $u_k \to u$ in $Y$. Now Remark 2.2 yields

$$\lim_{k\to\infty}[A(u_k) - A_u(u_k), u_k - u] = 0. \tag{2.13}$$

Comparing this with (2.12) it follows

$$\limsup_{k\to\infty}[A_u(u_k), u_k - u] \leq 0. \tag{2.14}$$

Now Theorem 1.67 implies that $A_u$ is pseudomonotone with respect to $D(L)$ hence from conditions (2.11) and (2.14) we obtain

$$\lim_{k\to\infty}[A_u(u_k), u_k - u] = 0 \ \text{ and } \ A_u(u_k) \to A_u(u)(= A(u)) \ \text{ weakly in } \ X^*. \tag{2.15}$$

Whence (2.13) yields $\lim_{k\to\infty}[A(u_k), u_k - u] = 0$. On the other hand, we have shown in the proof of demicontinuity that $A_u(u_k) - A(u_k) \to 0$ weakly in $X^*$, so that by using the second part of (2.15) we conclude $A(u_k) \to A(u)$ weakly in $X^*$. This completes the proof. □

**Corollary 2.3.** *Problem $Lu + A(u) = F$ has got a solution $u \in W^{1,p}(0,T;V,H)$ for every $F \in X^*$.*

*Proof.* Since operator $L$ is densely defined, closed, linear and maximal monotone (see Theorem 1.51), the statement follows from Theorem 1.65. (If in the definition of the domain of $L$ we pose $u(0) = u(T)$ instead of $u(0) = 0$ this Corollary remains true since Theorem 1.51 applies also in this case.) □

### 2.2.2 Modification of the problem

In this section we modify system (2.1) in order to be able to define the notion of periodic solutions and prove existence of them. In the following we admit only delay type of nonlocal variable. We introduce the usual notation. If $u \in L^p(-a,T;V)$ $(T \geq a)$ then let $u_t(s) = u(t+s)$ for $s \in [-a,0]$ and $t \in [0,T]$. Now consider system

$$\begin{aligned}
&D_t u^{(l)}(\cdot) \\
&- \sum_{i=1}^{n} D_i \left[ a_i^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u_t^{(1)}, \dots, u_t^{(N)}) \right] \\
&+ a_0^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u_t^{(1)}, \dots, u_t^{(N)}) \\
&= f^{(l)}(\cdot),
\end{aligned} \tag{2.16}$$

with some boundary condition and initial condition $u_0^{(l)}(s) = \varphi_l(s)$ for $s \in [-a, 0]$ where $\varphi_l \in L^p(-a, 0; V)$ $(l = 1, \dots, N)$. (As before, the exact form of the boundary condition determines the space $V$, see Section 2.1.3.)

We are interested in solutions $u \in L^p(-a, T; V)$ such that $D_t u \in L^q(-a, T; V^*)$ and $u$ is a weak solution of (2.16) for $t \in (0, T)$, further, $u(t) = u(t + T)$ for $t \in [-a, 0]$. We shall show existence of this type of solutions and at the end of Section 2.3.1 we shall extend them to a periodic weak solution of (2.16) in $(0, \infty)$ (see Theorem 2.13).

Now we modify conditions (A1)–(A5) according to the above modification of equation (2.16).

(A1') Functions $a_i^{(l)} \colon Q_T \times \mathbb{R}^{(n+1)N} \times L^p(-a, 0; V) \to \mathbb{R}$ $(i = 0, \dots, n;\ l = 1, \dots, N)$ have the Carathéodory property for every fixed $v \in L^p(-a, 0; V)$, i.e., they are measurable in $(t, x) \in Q_T$ for every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and continuous in $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ for a.a. $(t, x) \in Q_T$

(A2') There exist bounded operators $g_1 \colon L^p(-a, 0; V) \to \mathbb{R}^+$ and $k_1 \colon L^p(-a, 0; V) \to L^q(\Omega)$ such that

$$|a_i^{(l)}(t, x, \zeta_0, \zeta; v)| \le g_1(v)\left(|\zeta_0|^{p-1} + |\zeta|^{p-1}\right) + [k_1(v)](x)$$

for a.a. $(t, x) \in Q_T$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p(-a, T; V)$ $(i = 0, \dots, n;\ l = 1, \dots, N)$.

(A3') For a.a. $(t, x) \in Q_T$, every $\zeta \neq \tilde{\zeta} \in \mathbb{R}^{nN}$, $\zeta_0 \in \mathbb{R}^N$ and $v \in L^p(-a, 0; V)$,

$$\sum_{l=1}^{N} \sum_{i=1}^{n} \left(a_i^{(l)}(t, x, \zeta_0, \zeta; v) - a_i^{(l)}(t, x, \zeta_0, \tilde{\zeta}; v)\right)(\zeta_i^{(l)} - \tilde{\zeta}_i^{(l)}) > 0.$$

(A4') There exist a constant $g_2 > 0$ and a function $k_2 \in L^1(Q_T)$ such that

$$\sum_{l=1}^{N} \sum_{i=0}^{n} a_i^{(l)}(t, x, \zeta_0, \zeta; v)\zeta_i^{(l)} \ge g_2\left(|\zeta_0|^p + |\zeta|^p\right) - k_2(t, x) \qquad (2.17)$$

for a.a. $(t, x) \in Q_T$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p(-a, 0; V)$.

(A5') If $u_k \to u$ weakly in $L^p(-a, T; V)$ and strongly in $L^p(-a, T; (L^p(\Omega))^N)$ then for every $i = 0, \dots, n;\ l = 1, \dots, N$,

$$\lim_{k \to \infty} \|a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); (u_{k-1})_t) - a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u_t)\|_{L^q(Q_T)} = 0.$$

By supposing (A1'), (A2') we introduce operator $\tilde{A}\colon L^p(-a,T;V) \to L^q(0,T;V^*)$ as follows. For $u = (u^{(1)},\dots,u^{(N)}) \in L^p(-a,T;V)$, $v = (v^{(1)},\dots,v^{(N)}) \in L^p(0,T;V)$,

$$[\tilde{A}(u),v] := \sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_i^{(l)}(t,x,u(t,x),Du(t,x);u_t) D_i v^{(l)}(t,x)\,dt\,dx$$
$$+ \sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_0^{(l)}(t,x,u(t,x),Du(t,x);u_t) v^{(l)}(t,x)\,dt\,dx.$$

Further, let $L$ be the operator of differentiation:

$$D(L) := \{u \in L^p(0,T;V)\colon D_t u \in L^q(0,T;V^*),\ u(0) = u(T)\}, \quad Lu = D_t u.$$

(Notice that contrary to the previous section now we demand periodicity condition in the domain of $L$.) Finally, let $F \in L^q(0,T;V^*)$.

We want to find $u \in L^p(-a,T;V)$ such that $D_t u \in L^q(-a,T;V^*)$ and

$$Lu|_{(0,T)} + \tilde{A}(u) = F \tag{2.18}$$

$$u(t) = u(t+T) \quad \text{for} \quad t \in [-a,0]. \tag{2.19}$$

In the following if (2.19) holds we say that $u$ is periodic.

**Theorem 2.4.** *Let $T \geq a$ and assume that conditions (A1')–(A5') are satisfied. Then for every $F \in L^q(0,T;V^*)$ there exists $u \in L^p(-a,T;V)$ such that $D_t u \in L^q(-a,T;V^*)$ and* (2.18)–(2.19) *hold.*

*Proof.* The main idea is to apply the method of successive approximation (known from the theory of ordinary differential equations). We define a weakly convergent sequence of approximating solutions and we show that the weak limit of this sequence will be a solution that fulfills the requirements of the theorem.

To this end, for fixed $v \in L^p(-a,T;V)$ such that $D_t v \in L^q(-a,0;V^*)$ and $v(t) = v(t+T)$ for $t \in [-a,0]$ we introduce operator $\tilde{A}_v\colon L^p(0,T;V) \to L^q(0,T;V^*)$ as follows:

$$[\tilde{A}_v(u),w] := \sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_i^{(l)}(s,x,u(s,x),Du(s,x);v_t) D_i w^{(l)}(s,x)\,ds\,dx$$
$$+ \sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_0^{(l)}(s,x,u(s,x),Du(s,x);v_t) w^{(l)}(s,x)\,dt\,dx$$

where $u = (u^{(1)},\dots,u^{(N)}) \in L^p(0,T;V)$ and $w = (w^{(1)},\dots,w^{(N)}) \in L^p(0,T;V)$. In the following we show that for fixed periodic $v \in L^p(-a,T;V)$ operator $\tilde{A}_v$ is bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$. We proceed the same way as in the proof of Theorem 2.1. Similarly to (2.6), (2.7),

$$|[A_v(u),w]| \leq \text{const} \cdot \left( g_1(v_t) \|u\|_X^{\frac{p}{q}} + \int_0^T \|k_1(v_t)\|_{L^1(\Omega)}\,dt \right) \|w\|_X.$$

Since $v$ is periodic, $\|v_t\|_{L^p(-a,0;V)}$ is constant (hence bounded) in $t \in (0,T)$ thus $|g_1(v_t)|$ and $\|k_1(v_t)\|_{L^1(\Omega)}$ are bounded so the above inequality implies the boundedness of operator $A_v$.

To verify the demicontinuity we pick a sequence $(u_k) \subset L^p(0,T;V)$ such that $u_k \to u$ in $X$. We may assume that $u_k \to u$ and $Du_k \to Du$ a.e. in $Q_T$ thus

$$a_i^{(l)}(t,x,u_k(t,x),Du_k(t,x);v_t) \to a_i^{(l)}(t,x,u(t,x),Du(t,x);v_t) \text{ a.e. in } Q_T.$$

Further,

$$\begin{aligned}&|a_i^{(l)}(t,x,u_k(t,x),Du_k(t,x);v_t)|^q\\ &\leq g_1(v_t)^q\left(|u_k(t,x)|^p + |Du_k(t,x)|^p\right) + |[k_1(v_t)](x)|^q.\end{aligned}$$

The right hand side of the above inequality is equi-integrable in $L^1(Q_T)$ by the convergence of $(u_k)$ in $X$ and by the periodicity of function $v$. Whence by Vitali's theorem we conclude that

$$\|a_i^{(l)}(\cdot\,,u_k(\cdot),Du_k(\cdot);v_t) - a_i^{(l)}(\cdot\,,u(\cdot),Du(\cdot);v_t)\|_{L^q(Q_T)} \to 0.$$

which means $[\tilde{A}_v(u_k) - \tilde{A}_v(u), w] \to 0$ for every $w \in X$ so the demicontinuity follows.

The coerciveness follows by (A4') similarly to (2.10).

$$\lim_{\|u\|_X\to\infty} \frac{[A_v(u),u]}{\|u\|_X} \geq \lim_{k\to\infty}\left[g_2\|u\|_X^{p-1} - \frac{\|k_2\|_{L^1(Q_T)}}{\|u\|_X}\right] = +\infty.$$

Finally, the pseudomonotonicity with respect to $D(L)$ follows by using the classical arguments (combining with Theorem 1.48 and the boundedness of $v_t$), see [17, 44, 71].

Now let us define the sequence of approximating solutions $(u_k) \subset L^p(-a,T;V)$ by using a sequence $(\hat{u}_k) \subset L^p(0,T;V)$. Let $\hat{u}_0(s) = 0$ for $s \in [0,T]$ and $u_0(s) = 0$ for $s \in [-a,T]$. Suppose that $u_{k-1} \in L^p(-a,T;V)$ such that $D_t u_{k-1} \in L^q(-a,T;V^*)$ and $u_{k-1}(t) = u_{k-1}(t+T)$ for $t \in [-a,0]$. Then let $\hat{u}_k \in L^p(0,T;V)$ be a solution of

$$L\hat{u}_k + A_{u_{k-1}}(\hat{u}_k) = F. \tag{2.20}$$

Such solutions exist due to Theorem 1.65 and the properties of operator $A_{u_{k-1}}$ for fixed periodic $u_{k-1}$. Let $u_k(t) = \hat{u}_k(t)$ for $t \in [0,T]$ and $u_k(t) = \hat{u}_k(t+T)$ for $t \in [-a,0]$. Then $u_k$ is continuous and $D_t u_k \in L^q(-a,T;V^*)$.

Now we show that the sequence $(\hat{u}_k)$ is bounded in $L^p(0,T;V)$ (thus $(u_k)$ is bounded in $L^p(-a,T;V)$). Indeed, by integrating (2.17) in $Q_T$ (analogously to (2.10)),

$$[F,\hat{u}_k] = [L\hat{u}_k,\hat{u}_k] + [A_{u_{k-1}}(\hat{u}_k),\hat{u}_k] \geq g_2\|\hat{u}_k\|_X^p - \|k_2\|_{L^1(Q_T)}. \tag{2.21}$$

Since $\hat{u}_k(0) = \hat{u}_k(T)$, $[L\hat{u}_k, \hat{u}_k] \geq 0$ whence

$$\|F\|_{X^*} \geq g_2 \|\hat{u}_k\|_X^p - \|k_2\|_{L^1(Q_T)}.$$

Consequently, $(\hat{u}_k)$ is bounded in $L^p(0, T; V)$. Then due to the periodicity of $\hat{u}_k$, $(u_k)$ is bounded in $L^p(-a, T; V)$. So by using similar estimates as (2.5), (2.6) one obtains the boundedness of the sequence $(A_{u_{k-1}}(\hat{u}_k))$ in $L^q(0, T; V^*)$. Now from (2.21) we may deduce

$$\|L\hat{u}_k\|_{X^*} = \|F\|_{X^*} - \|A_{u_{k-1}}(\hat{u}_k)\|_{X^*} \leq \text{const.}$$

By applying Theorems 1.48 and 1.52 one has a subsequence of $(\hat{u}_k)$ (for simplicity denoted as the original) and a function $\hat{u} \in L^p(0, T; V)$ such that $\hat{u}(0) = \hat{u}(T)$ and

$$\hat{u}_k \to \hat{u} \ \text{ weakly in } \ L^p(0, T; V); \text{ strongly in } \ L^p(0, T; (L^2(\Omega))^N)$$
$$L\hat{u}_k \to L\hat{u} \ \text{ weakly in } \ L^q(0, T; V^*).$$

This implies that for a subsequence $\hat{u}_k \to \hat{u}$ a.e. in $Q_T$. Thus due to the periodic extension, there exists $u \in L^p(-a, T; V)$ such that $(u_{k-1})_t \to u_t$ a.e. in $[-a, 0] \times \Omega$ for every $t \in [0, T]$. Similarly to (2.9), Vitali's theorem implies

$$\lim_{k\to\infty} \|a_i^{(l)}(\cdot\,, \hat{u}_k(\cdot), D\hat{u}_k(\cdot); u_t) - a_i^{(l)}(\cdot\,, u(\cdot), Du(\cdot); u_t)\|_{L^q(Q_T)} = 0.$$

which means $A_u(\hat{u}_k) \to A_u(u)$ weakly in $L^q(0, T; V^*)$. Finally, by condition (A5'),

$$\|a_i^{(l)}(\cdot\,, \hat{u}_k(\cdot), D\hat{u}_k(\cdot); (u_{k-1})_t) - a_i^{(l)}(\cdot\,, \hat{u}_k(\cdot), D\hat{u}_k(\cdot); u_t)\|_{L^q(Q_T)} \to 0$$

so $A_{u_{k-1}}(\hat{u}_k) \to A_u(u)$ weakly in $L^q(0, T; V^*)$ hence $A_{u_{k-1}}(\hat{u}_k) \to A(u)$ weakly in $L^q(0, T; V^*)$. Now by passing to the limit as $k \to \infty$ from (2.20) we conclude that $L\hat{u} + A_{\hat{u}}(\hat{u}) = F$, further, by the a.e. convergence $u_t(s) = \hat{u}(s + T) = u(t + T)$ for $s \in [-a, 0]$.

$\square$

### 2.2.3 Examples

In this section we give examples for functions $a_i^{(l)}$ which fulfil conditions (A1)-(A5). We begin with a general form and we finish with concrete examples.

#### General case

Suppose that functions $a_i^{(l)}(t, x, \zeta_0, \zeta; v)$ have the form:

$$\begin{aligned} &a_i^{(l)}(t, x, \zeta_0, \zeta; v) \\ &= [H^{(l)}(v)](t, x) b_i^{(l)}(t, x, \zeta_0, \zeta) + [G^{(l)}(v)](t, x) d_i^{(l)}(t, x, \zeta_0, \zeta) \ \ (i \neq 0), \end{aligned} \tag{2.22}$$

$$\begin{aligned} &a_0^{(l)}(t, x, \zeta_0, \zeta; v) \\ &= [H^{(l)}(v)](t, x) b_0^{(l)}(t, x, \zeta_0, \zeta) + [G_0^{(l)}(v)](t, x) d_0^{(l)}(t, x, \zeta_0, \zeta), \end{aligned} \tag{2.23}$$

where $b_i^{(l)}, d_i^{(l)}, H^{(l)}, G^{(l)}, G_0^{(l)}$ have the following properties.

(K1) Functions $b_i^{(l)} \colon Q_T \times \mathbb{R}^{(n+1)N} \to \mathbb{R}$ and $d_i^{(l)} \colon Q_T \times \mathbb{R}^{(n+1)N} \to \mathbb{R}$ $(i = 1, \dots, n;$ $l = 1, \dots, N)$ are of Carathéodory type, i.e., they are measurable in $(t, x) \in Q_T$ for every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and continuous in $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ for a.a. $(t, x) \in Q_T$

(K2) There exist constants $c_1 > 0$, $0 \le r < p - 1$ and a function $k_1 \in L^q(\Omega)$ such that

a) $|b_i^{(l)}(t, x, \zeta_0, \zeta)| \le c_1(|\zeta_0|^{p-1} + |\zeta|^{p-1}) + k_1(x)$,

b) $|d_i^{(l)}(t, x, \zeta_0, \zeta)| \le c_1(|\zeta_0|^r + |\zeta|^r)$

for a.a. $(t, x) \in Q_T$ and every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ $(i = 1, \dots, n;\ l = 1, \dots, N)$.

(K3) For a.a. $(t, x) \in Q_T$, every $\zeta \ne \eta \in \mathbb{R}^{nN}$, $\zeta_0 \in \mathbb{R}^N$ and $l = 1, \dots, N$,

a) $\displaystyle\sum_{i=1}^{n} \left( b_i^{(l)}(t, x, \zeta_0, \zeta) - b_i^{(l)}(t, x, \zeta_0, \eta) \right) (\zeta_i^{(l)} - \eta_i^{(l)}) > 0$,

b) $\displaystyle\sum_{i=1}^{n} \left( d_i^{(l)}(t, x, \zeta_0, \zeta) - d_i^{(l)}(t, x, \zeta_0, \eta) \right) (\zeta_i^{(l)} - \eta_i^{(l)}) \ge 0$.

(K4) There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(\Omega)$ such that

a) $\displaystyle\sum_{i=0}^{n} b_i^{(l)}(t, x, \zeta_0, \zeta)\zeta_i^{(l)} \ge c_2(|\zeta_0^{(l)}|^p + |\zeta^{(l)}|^p) - k_2(x)$,

b) $\displaystyle\sum_{i=1}^{n} d_i^{(l)}(t, x, \zeta_0, \zeta)\zeta_i^{(l)} \ge 0$

for a.a. $(t, x)$ and every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ $(l = 1, \dots, N)$.

(K5) a) Operator $H^{(l)} \colon L^p(0, T; (L^p(\Omega))^N) \to L^\infty(Q_T)$ is bounded and continuous such that $[H^{(l)}(v)](t, x) \ge c_3 > 0$ holds for a.a. $(t, x) \in Q_T$ and every $v \in L^p(0, T; (L^p(\Omega))^N)$.

b) Operators $G^{(l)}, G_0^{(l)} \colon L^p(0, T; (L^p(\Omega))^N) \to L^{\frac{p}{p-r-1}}(Q_T)$ are bounded and continuous where $r$ is given in (K2)/b. Further, $[G^{(l)}(v)](t, x) \ge 0$ for a.a. $(t, x) \in Q_T$, every $v \in L^p(0, T; (L^p(\Omega))^N)$. In addition,

$$\lim_{\|v\|_{L^p(0,T;V)} \to \infty} \frac{\displaystyle\int_{Q_T} |G_0^{(l)}(v)(t, x)|^{\frac{p}{p-r-1}}\, dt\, dx}{\|v\|^p_{L^p(0,T;V)}} = 0, \quad l = 1, \dots, N. \tag{2.24}$$

**Proposition 2.5.** *Assume conditions (K1)–(K5). Then functions defined in* (2.22)–(2.23) *satisfy conditions (A1)–(A5).*

We need a technical lemma.

**Lemma 2.6.** *Let us introduce the following operators:*

$$H(v) = \sum_{l=1}^{N} |H^{(l)}(v)|, \quad G(v) = \sum_{l=1}^{N} |G^{(l)}(v)|, \quad G_0(v) = \sum_{l=1}^{N} |G_0^{(l)}(v)|.$$

*Then operators $H$, $G$ and $G_0$ fulfil the conditions formulated in (K5) on $H^{(l)}$, $G^{(l)}$ and $G_0^{(l)}$, respectively.*

*Proof.* Property (2.24) follows easily by using inequality (1.1), the other conditions are completely trivial. □

*Proof of Proposition 2.5.*
**Condition (A1)** Condition (K1) immediately implies (A1).
**Condition (A2)** Let $i > 0$ and $r > 0$. Obviously

$$|[H^{(l)}(v)](t,x)b_i^{(l)}(t,x,\zeta_0,\zeta)| \le \|H(v)\|_{L^\infty(Q_T)} \left(c_1\left(|\zeta_0|^{p-1} + |\zeta|^{p-1}\right) + k_1(x)\right).$$

On the other hand, by using Young's inequality with conjugate exponents $1 < p_1 = \dfrac{p-1}{r} < \infty$ and $q_1 = \dfrac{p-1}{p-r-1}$ one obtains

$$\begin{aligned} |[G^{(l)}(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)| &\le |[G(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)| \\ &\le \frac{|d_i^{(l)}(t,x,\zeta_0,\zeta)|^{p_1}}{p_1} + \frac{|[G(v)](t,x)|^{q_1}}{q_1}. \end{aligned} \tag{2.25}$$

Thus by using (K2)/b and inequality (1.1) we obtain

$$\begin{aligned} |[G^{(l)}(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)| &\le \text{const} \cdot (|\zeta_0|^{rp_1} + |\zeta|^{rp_1} + |[G(v)](t,x)|^{q_1}) \\ &= \text{const} \cdot \left(|\zeta_0|^{p-1} + |\zeta|^{p-1} + |[G(v)](t,x)|^{q_1}\right). \end{aligned} \tag{2.26}$$

Now by combining the above estimates we may deduce

$$\begin{aligned} |a_i^{(l)}(t,x,\zeta_0,\zeta;v)| &\le \text{const} \cdot \left(\|H(v)\|_{L^\infty(Q_T)} + 1\right)(|\zeta_0|^{p-1} + |\zeta|^{p-1}) \\ &\quad + \text{const} \cdot \left(\|H(v)\|_{L^\infty(Q_T)} k_1(x) + |[G(v)](t,x)|^{q_1}\right). \end{aligned}$$

Due to the boundedness of operator $H$ and by the continuous embedding $X \to Y$ it follows that $\|H(\cdot)\|_{L^\infty(Q_T)}$ is a bounded $X \to \mathbb{R}^+$ functional. Further, $k_1 \in L^q(\Omega)$ implies that $\|H(\cdot)\|_{L^\infty(Q_T)} k_1$ is a bounded $X \to L^q(Q_T)$ operator. Observe that $q_1 q = \frac{p}{p-r-1}$ so that

$$\begin{aligned} \int_{Q_T} (|[G(v)](t,x)|^{q_1})^q \, dt\, dx &= \int_{Q_T} |[G(v)](t,x)|^{\frac{p}{p-r-1}} \, dt\, dx \\ &= \left(\|G(v)\|_{L^{\frac{p}{p-r-1}}(Q_T)}\right)^{\frac{p}{p-r-1}}. \end{aligned} \tag{2.27}$$

Thus $|G(\cdot)|^{q_1}$ is a bounded $X \to L^q(Q_T)$ operator.

Now let $r = 0$. Observe that $q_1 = 1$, moreover, from (K2)/b it follows

$$|d_i^{(l)}(t, x, \zeta_0, \zeta)| \leq 2c_1.$$

So in this case we also have an inequality similar to (2.26),

$$|[G^{(l)}(v)](t, x) d_i^{(l)}(t, x, \zeta_0, \zeta)| \leq \text{const} \cdot |[G(v)](t, x)|^{q_1}.$$

This completes the proof in case $i > 0$. Case $i = 0$ is the same, we only have to replace $G$ with $G_0$.

**Condition (A3)** By using condition (K3) and (K5)/a, for every $\zeta \neq \eta$ we obtain

$$\begin{aligned}
&\sum_{l=1}^{N} \sum_{i=1}^{n} \left( a_i^{(l)}(t, x, \zeta_0, \zeta; v) - a_i^{(l)}(t, x, \zeta_0, \eta; v) \right) (\zeta_i^{(l)} - \eta_i^{(l)}) \\
&= \sum_{l=1}^{N} [H^{(l)}(v)](t, x) \sum_{i=1}^{n} \left( b_i^{(l)}(t, x, \zeta_0, \zeta) - b_i^{(l)}(t, x, \zeta_0, \eta) \right) (\zeta_i^{(l)} - \eta_i^{(l)}) \\
&\quad + \sum_{l=1}^{N} [G^{(l)}(v)](t, x) \sum_{i=1}^{n} \left( d_i^{(l)}(t, x, \zeta_0, \zeta) - d_i^{(l)}(t, x, \zeta_0, \eta) \right) (\zeta_i^{(l)} - \eta_i^{(l)}) \\
&> 0.
\end{aligned}$$

**Condition (A4)** Due to (K4) and (K5) it follows

$$\begin{aligned}
&\sum_{l=1}^{N} \sum_{i=0}^{n} a_i^{(l)}(t, x, \zeta_0, \zeta; v) \zeta_i^{(l)} \\
&\geq \sum_{l=1}^{N} c_3 c_2 (|\zeta_0^{(l)}|^p + |\zeta^{(l)}|^p) - c_3 k_2(x) + \sum_{l=1}^{N} [G_0^{(l)}(v)](t, x) d_0^{(l)}(t, x, \zeta_0, \zeta) \zeta_0^{(l)} \\
&\geq c_4 c_3 c_2 (|\zeta_0|^p + |\zeta|^p) - c_3 N k_2(x) + \sum_{l=1}^{N} [G_0^{(l)}(v)](t, x) d_0^{(l)}(t, x, \zeta_0, \zeta) \zeta_0^{(l)}.
\end{aligned} \tag{2.28}$$

In the last estimate we applied inequality (1.1). Put $c' = c_4 c_3 c_2$ and let us investigate the terms in the last sum. By applying the $\varepsilon > 0$-inequality with exponents $p$, $q$ and $\varepsilon > 0$ such that $\dfrac{\varepsilon^p}{p} < \dfrac{c'}{3N}$ it follows

$$\begin{aligned}
&|[G_0^{(l)}(v)](t, x) d_0^{(l)}(t, x, \zeta_0, \zeta) \zeta_0^{(l)}| \\
&\leq |[G_0(v)](t, x) d_0^{(l)}(t, x, \zeta_0, \zeta) \zeta_0^{(l)}| \\
&\leq \frac{\varepsilon^p}{p} |\zeta_0^{(l)}|^p + \frac{\varepsilon^{-q}}{q} |[G_0(v)](t, x) d_0^{(l)}(t, x, \zeta_0, \zeta)|^q.
\end{aligned} \tag{2.29}$$

The first term on the right hand side of (2.29) may be estimated from above by $\dfrac{c'}{3N} (|\zeta_0|^p + |\zeta|^p)$. In the second term, the $\varepsilon$-inequality with $\mu > 0$ (defined later) and exponents $p_1$, $q_1$ (similarly to (2.25), (2.26)) yields for $r > 0$

$$\begin{aligned}
&|[G_0^{(l)}(v)](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)|^q \\
&\le \mathrm{const}\cdot\left(\mu^{p_1}(|\zeta_0|^{p-1}+|\zeta|^{p-1})+\mu^{-q_1}|[G_0(v)](t,x)|^{q_1}\right)^q \qquad (2.30)\\
&\le c^*\mu^{p_1q}\,(|\zeta_0|^p+|\zeta|^p)+c^*\mu^{-q_1q}|[G_0(v)](t,x)|^{q_1q}.
\end{aligned}$$

Now choose $\mu$ such that $\dfrac{c^*\mu^{p_1q}\varepsilon^{-q}}{q} < \dfrac{c'}{3N}$. Then by substituting (2.29) and (2.30) into (2.28) one obtains

$$\begin{aligned}
&\sum_{l=1}^{N}\sum_{i=0}^{n} a_i^{(l)}(t,x,\zeta_0,\zeta;v)\zeta_i^{(l)} \qquad (2.31)\\
&\ge \frac{c'}{3}\,(|\zeta_0|^p+|\zeta|^p)-(c_3Nk_2(x)+Nd^*|[G_0(v)](t,x)|^{q_1q}).
\end{aligned}$$

Put

$$h(v) := c_3Nk_2(x)+Nd^*|[G_0(v)](t,x)|^{q_1q}$$

then $h(v)\in L^1(Q_T)$ due to (2.27) (and $k_2\in L^1(\Omega)$). Moreover,

$$\|h(v)\|_{L^1(Q_T)} \le c_3N\|k_2\|_{L^1(\Omega)} + Nd^*\int_{Q_T}|[G_0(v)](t,x)|^{\frac{p}{p-r-1}}\,dt\,dx.$$

Note that the this inequality holds also in case $r = 0$. From Lemma 2.6 it follows that $G_0$ fulfils (2.24) hence

$$\lim_{\|v\|_X\to\infty}\|v\|_X^{p-1}\left(\frac{c'}{3}-\frac{\|h(v)\|_{L^1(Q_T)}}{\|v\|_X^p}\right) = \lim_{\|v\|_X\to\infty}\frac{c'}{3}\|v\|_X^{p-1} = +\infty.$$

**Condition (A5)** Let $r > 0$. Suppose that $u_k \to u$ weakly in $X$ and strongly in $Y$. Then $(u_k)$ is bounded in $X$ therefore $\left(b_i^{(l)}(\cdot\,,u_k(\cdot),Du_k(\cdot))\right)_{k\in\mathbb{N}}$ is bounded in $L^q(Q_T)$, since similarly to (2.6)) one has the estimate

$$\int_{Q_T}|b_i^{(l)}(t,x,u_k(t,x),Du_k(t,x)|^q\,dt\,dx \le \mathrm{const}\cdot(\|u_k\|_X^p+\|k_1\|_{L^q(\Omega)}^q)\le K.$$

Further, observe that $\left(d_i^{(l)}(\cdot\,,u_k(\cdot),Du_k(\cdot))\right)_{k\in\mathbb{N}}$ is bounded in $L^{\frac{p}{r}}(Q_T)$, since by (K2)/b

$$\begin{aligned}
\int_{Q_T}|d_i^{(l)}(t,x,u_k(t,x),Du_k(t,x))|^{\frac{p}{r}}\,dt\,dx &\le \int_{Q_T}\left[|u_k(t,x)|^{r\frac{p}{r}}+|Du_k(t,x)|^{r\frac{p}{r}}\right]dt\,dx\\
&= \|u_k\|_X^p.
\end{aligned}$$

Whence by using the continuity of $H^{(l)}$ we may deduce

$$\begin{aligned}
&\int_{Q_T}|([H^{(l)}(u_k)](t,x)-[H^{(l)}(u)](t,x))b_i^{(l)}(t,x,u_k(t,x),Du_k(t,x))|^q\,dt\,dx\\
&\le \|H^{(l)}(u_k)-H^{(l)}(u)\|_{L^\infty(Q_T)}^q\int_{Q_T}|b_i^{(l)}(t,x,u_k(t,x),Du_k(t,x))|^q\,dt\,dx\\
&\le K\cdot\|H^{(l)}(u_k)-H^{(l)}(u)\|_{L^\infty(Q_T)}\\
&\to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

On the other hand, Hölder's inequality with exponents $p_1, q_1$ yields

$$\begin{aligned}
&\int_{Q_T} |([G^{(l)}(u_k)](t,x) - [G^{(l)}(u)](t,x)) d_i^{(l)}(t,x,u_k(t,x), Du_k(t,x))|^q \, dt\, dx \\
&\le \left( \int_{Q_T} |d_i^{(l)}(t,x,u_k(t,x), Du_k(t,x)|^{\frac{p}{p-1}\frac{p-1}{r}} \, dt\, dx \right)^{\frac{1}{p_1}} \\
&\quad \times \left( \int_{Q_T} |[G^{(l)}(u_k)](t,x) - [G^{(l)}(u)](t,x)|^{\frac{p}{p-1}\frac{p-1}{p-r-1}} \, dt\, dx \right)^{\frac{1}{q_1}} \\
&\le K^{\frac{1}{p_1}} \|G^{(l)}(u_k) - G^{(l)}(u)\|_{L^{\frac{p}{p-r-1}}(Q_T)}^{\frac{p-r-1}{p}} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

This means that

$$\begin{aligned}
&\|a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u_k) - a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u)\|_{L^q(Q_T)} \\
&\le \text{const} \cdot \|(H^{(l)}(u_k) - H^{(l)}(u)) b_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot))\|_{L^q(Q_T)} \\
&\quad + \text{const} \cdot \|(G^{(l)}(u_k) - G^{(l)}(u)) d_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot))\|_{L^q(Q_T)} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{2.32}$$

If $r = 0$ then the first term on the right hand side of (2.32) tends to 0. Since $\dfrac{p}{p-r-1} = q$ (hence $G$ maps to $L^q(Q_T)$ continuously) and $|d_i^{(l)}(t,x,\zeta_0,\zeta)| \le 2c_1$ thus

$$\begin{aligned}
&\|(G^{(l)}(u_k) - G^{(l)}(u)) d_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot))\|_{L^q(Q_T)} \\
&\le 2c_1 \|(G^{(l)}(u_k) - G^{(l)}(u))\|_{L^q(Q_T)} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

So the second term on the right hand side of (2.32) tends to 0, too. Case $i = 0$ can be treated similarly, by replacing $G^{(l)}$ with $G_0^{(l)}$. □

**Concrete examples**

**Operator $H^{(l)}$**

Let $\Phi \colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\Phi \ge c > 0$ and introduce the following operators on $L^p(0,T;(L^p(\Omega))^N)$:

$$[\tilde{H}_1(v)](t,x) := \Phi\left( \int_{Q_t} \sum_{j=1}^N b_j v^{(j)} \right), \quad \text{where } \; b_j \in L^q(Q_T) \;\; (1 \le j \le N),$$

$$[\tilde{H}_2(v)](t,x) := \Phi\left( \left[ \int_{Q_t} |v|^\alpha \right]^{\frac{1}{\alpha}} \right), \quad \text{where } \; 1 \le \alpha \le p \; \text{ and } \; Q_t = (0,t) \times \Omega.$$

**Proposition 2.7.** *The above operators $\tilde{H}_i$ $(i = 1,2)$ fulfil condition (K5)/a.*

*Proof.* We start with the case of $\tilde{H}_1$. From Hölder's inequality it follows that $b_j v^{(j)} \in L^1(Q_T)$ so that $\tilde{H}_1$ is well-defined and obviously $\tilde{H}_1(v) \geq c > 0$. On the other hand, if $\|v\|_Y \leq K$ then

$$\left| \int_{Q_t} \sum_{j=1}^N b_j v^{(j)} \right| \leq \sum_{j=1}^N \int_{Q_T} |b_j v^{(j)}| \leq K \sum_{j=1}^N \|b_j\|_{L^q(Q_T)}.$$

Now the continuity of $\Phi$ yields the continuity and boundedness of $\tilde{H}_1$. Since, if $v_k \to v$ in $L^p(0,T;(L^p(\Omega))^N)$ then

$$\left| \int_{Q_t} \sum_{j=1}^N b_j v_k^{(j)} - \int_{Q_t} \sum_{j=1}^N b_j v^{(j)} \right| \leq \sum_{j=1}^N \left( \int_{Q_T} |b_j|^q \right)^{\frac{1}{q}} \left( \int_{Q_T} |v_k^{(j)} - v^{(j)}|^p \right)^{\frac{1}{p}} \to 0$$

as $k \to \infty$ therefore by continuity of $\phi$ it follows $\tilde{H}_1(v_k) \to \tilde{H}_1(v)$ in $L^\infty(Q_T)$. This completes the proof.

Clearly, operator $\tilde{H}_2$ is well-defined and maps to $L^\infty(Q_T)$ (that can be proved the same way as above). Now let $v_k \to v$ in $L^p(0,T;(L^p(\Omega))^N)$ then $v_k \to v$ a.e. in $Q_T$, further, they are equi-integrable in $L^\alpha(Q_T)$ for every $1 \leq \alpha \leq p$. Then Vitali's theorem yields the convergence in $L^\alpha(Q_T)$ so that $\tilde{H}_2(v_k) \to \tilde{H}_2(v)$, i.e., operator $\tilde{H}_1$ is continuous. □

### Operators $G^{(l)}$, $G_0^{(l)}$

Let $\psi \colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|\psi(y)| \leq \tilde{c} \cdot |y|^{p-r_0-1}$ holds for some constants $\tilde{c}$ and $0 \leq r_0 < p-1$. Let us introduce the following operators on $L^p(0,T;(L^p(\Omega))^N)$:

$$[\tilde{G}_1(v)](t,x) := \psi\left( \int_0^t \sum_{j=1}^N a_j(\tau,x) v^{(j)}(\tau,x) d\tau \right),$$

$$[\tilde{G}_2(v)](t,x) := \psi\left( \int_\Omega \sum_{j=1}^N a_j(t,\xi) v^{(j)}(t,\xi) d\xi \right),$$

$$[\tilde{G}_3(v)](t,x) := \psi\left( \left[ \int_0^t |v(\tau,x)|^\alpha d\tau \right]^{\frac{1}{\alpha}} \right)$$

where $a_j \in L^\infty(Q_T)$ $(1 \leq j \leq N)$ and $1 \leq \alpha \leq p$.

**Proposition 2.8.** *The above operators $\tilde{G}_i$ $(i = 1,2,3)$ fulfil conditions made on $G_0^{(l)}$ in (K5)/b with $0 \leq r < r_0$.*

*Proof.* We show the case of operator $\tilde{G}_1$, one can prove the other cases similarly.

Let $0 \le r < r_0 < p-1$ then obviously

$$\begin{aligned}
\int_{Q_T} |[\tilde{G}_1(v)](t,x)|^{\frac{p}{p-r-1}}\,dt\,dx &\le \text{const} \cdot \int_{Q_T} \left( \sum_{j=1}^{N} \int_0^T \|a_j\|_{L^\infty(Q_T)} |v^{(j)}(\tau,x)| d\tau \right)^{p\lambda} dt\,dx \\
&\le \text{const} \cdot \int_{Q_T} \left( \cdot \sum_{j=1}^{N} \int_0^T |v(\tau,x)| d\tau \right)^{p\lambda} dt\,dx \\
&= \text{const} \cdot \int_{Q_T} \left( \int_0^T |v(\tau,x)| d\tau \right)^{p\lambda} dt\,dx
\end{aligned}$$

where $0 < \lambda = \dfrac{p - r_0 - 1}{p - r - 1} < 1$. By using Hölder's inequality with exponents $p_1 = \dfrac{1}{\lambda}$ and $q_1 = \dfrac{p_1}{p_1 - 1}$ we obtain

$$\begin{aligned}
&\int_{Q_T} \left( \int_0^T |v(\tau,x)| d\tau \right)^{p\lambda} dt\,dx \\
&\le \text{const} \cdot \left( \int_{Q_T} \left( \int_0^T |v(\tau,x)| d\tau \right)^{p\lambda\frac{1}{\lambda}} dt\,dx \right)^{\lambda} \cdot \left( \int_{Q_T} 1^{q_1} \right)^{\frac{1}{q_1}} \\
&= \text{const} \cdot \left( \int_{Q_T} \left( \int_0^T |v(\tau,x)| d\tau \right)^{p} dt\,dx \right)^{\lambda}
\end{aligned}$$

Now we may estimate again by Hölder's inequality and then application of Fubini's theorem yields

$$\begin{aligned}
\int_{Q_T} \left( \int_0^T |v(\tau,x)| d\tau \right)^{p} dt\,dx &\le \int_{Q_T} \left[ \left( \int_0^T |v(\tau,x)|^p d\tau \right)^{\frac{1}{p}} \left( \int_0^T 1^q d\tau \right)^{\frac{1}{q}} \right]^p dt\,dx \\
&= \text{const} \cdot \int_{Q_T} \int_0^T |v(\tau,x)|^p d\tau\,dx\,dt \\
&= \text{const} \cdot \int_{Q_T} |v(t,x)|^p\,dt\,dx \\
&\le \text{const} \cdot \|v\|_X^p.
\end{aligned}$$

Summarizing the above estimates one obtains

$$\int_{Q_T} |[\tilde{G}_1(v)](t,x)|^{\frac{p}{p-r-1}}\,dt\,dx \le \text{const} \cdot \|v\|_X^{p\lambda}.$$

Now it is easily seen that $\tilde{G}_1$ is a bounded operator which maps to $L^{\frac{p}{p-r-1}}(Q_T)$. Further, due to $\lambda - 1 < 0$,

$$\lim_{\|v\|_X \to \infty} \frac{\int_{Q_T} |[\tilde{G}_1(v)](t,x)|^{\frac{p}{p-r-1}}\,dt\,dx}{\|v\|_X^p} = \lim_{\|v\|_X \to \infty} \|v\|_X^{p(\lambda-1)} = 0.$$

The continuity of the operator can be proved similarly to the previous theorem. $\square$

*Remark* 2.9. From Lemma 2.6 it follows easily that linear combinations of the above operators fulfil condtitions (K5)/a and (K5)/b, too.

**Functions $b_i^{(l)}$, $d_i^{(l)}$**

We begin with a little bit general but well-known example. Let $b_i^{(l)}(t,x,\zeta_0,\zeta) := \tilde{b}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$ $(i=0,\dots,n; l=1,\dots,N)$ be such that

(i) function $\tilde{b}_i^{(l)}\colon Q_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ has the Carathéodory property, i.e., it is measurable in $(t,x) \in Q_T$ for every $(\zeta_0,\zeta_i^{(l)}) \in \mathbb{R}^{N+1}$ and continuous in $(\zeta_0,\zeta_i^{(l)}) \in \mathbb{R}^{N+1}$ for a.a. $(t,x) \in Q_T$;

(ii) there exist a constant $c_1 > 0$ and a function $k_1 \in L^q(\Omega)$ such that

$$|\tilde{b}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})| \le c_1(|\zeta_0|^{p-1} + |\zeta_i^{(l)}|^{p-1}) + k_1(x)$$

for a.a. $(t,x) \in Q_T$ and every $(\zeta_0,\zeta_i^{(l)}) \in \mathbb{R}^{N+1}$;

(iii) function $\zeta_i^{(l)} \mapsto \tilde{b}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$ is strictly increasing for a.a. $(t,x) \in Q_T$ and every $\zeta_0 \in \mathbb{R}^N$;

(iv) there exist a constant $c_2 > 0$ and a function $k_2 \in L^1(\Omega)$such that

$$\tilde{b}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})\zeta_i^{(l)} \ge c_2|\zeta_i^{(l)}|^p - k_2(x)$$

for a.a. $(t,x) \in Q_T$ and every $(\zeta_0,\zeta_i^{(l)}) \in \mathbb{R}^{N+1}$.

Then $b_i^{(l)}$ obviously fulfils (K1), (K2)/a. Condition (K4)/a follows from inequality (1.1), further, the monotonicity yields (K3)/a.

Similarly, let $d_i^{(l)}(t,x,\zeta_0,\zeta) := \tilde{d}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$ if $i \ne 0$ and $d_0^{(l)} := d_0^{(l)}(t,x,\zeta_0,\zeta)$ $(i=1,\dots,n;\ l=1,\dots,N)$ be such that

(i) functions $\tilde{d}_i^{(l)}\colon Q_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ and $\tilde{d}_0^{(l)}\colon Q_T \times \mathbb{R}^{2N} \to \mathbb{R}$ are of Carathéodory type;

(ii) there exist constants $c_1 > 0$, $0 \le r < p-1$ and a function $k_1 \in L^q(Q_T)$ such that

$$|\tilde{d}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})| \le c_1(|\zeta_0|^r + |\zeta_i^{(l)}|^r) + k_1(x),$$
$$|d_0^{(l)}(t,x,\zeta_0,\zeta)| \le c_1(|\zeta_0|^r + |\zeta|^r) + k_1(x)$$

for a.a. $(t,x) \in Q_T$ and every $(\zeta_0,\zeta_i) \in \mathbb{R}^{2N}$;

(iii) function $\zeta_i^{(l)} \mapsto \tilde{d}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$ is nondecreasing and $\tilde{d}_i^{(l)}(t,x,\zeta_0,0) = 0$ for a.a. $(t,x) \in Q_T$ and every $(\zeta_0,\zeta_i) \in \mathbb{R}^{N+1}$.

Now conditions (K1), (K2)/b, (K3)/b obviously hold. To prove (K4)/b we only have to observe that (if $i \neq 0$) $\tilde{d}_i^{(l)}(t, x, \zeta_0, \zeta_i^{(l)})\zeta_i^{(l)} \geq 0$.

The simplest functions which satisfy the above general conditions are

$$b_i^{(l)}(t, x, \zeta_0, \zeta_i^{(l)}) = \zeta_i^{(l)}|\zeta_i^{(l)}|^{p-2}, \qquad d_i^{(l)}(t, x, \zeta_0, \zeta_i^{(l)}) = \zeta_i^{(l)}|\zeta|^{r-1}$$

for $i = 0, \ldots, n; l = 1, \ldots, N$ and for $r > 0$. If $r = 0$ let $d_i^{(l)} \equiv 0$ and $d_0^{(l)} \equiv 1$.

Other functions which fulfil the desired conditions (K1)–(K4) (but they do not fit in the above general case) are the following:

$$\begin{aligned} b_i^{(l)}(t, x, \zeta_0, \zeta) &= \zeta_i^{(l)}|\zeta|^{p-2} \quad (i \neq 0), \\ b_0^{(l)}(t, x, \zeta_0, \zeta) &= \zeta_0^{(l)}|\zeta_0|^{p-2}, \end{aligned} \qquad \text{or} \qquad \begin{aligned} b_i^{(l)}(t, x, \zeta_0, \zeta) &= \zeta_i^{(l)}|\zeta^{(l)}|^{p-2} \quad (i \neq 0), \\ b_0^{(l)}(t, x, \zeta_0, \zeta) &= \zeta_0^{(l)}|\zeta_0^{(l)}|^{p-2} \end{aligned}$$

and similarly for functions $d_i^{(l)}$ by replacing the exponent $p - 2$ with $r - 1$. In case of the second example one has

$$\sum_{i=1}^{n} D_i b_i^{(l)}(t, x, u, Du) = \sum_{i=1}^{n} D_i(D_i u^{(l)}|Du^{(l)}|^{p-2}) = \operatorname{div}\bigl(Du^{(l)}|Du^{(l)}|^{p-2}\bigr).$$

So we obtain the $p$-Laplacian (see (1.11)) as the operator $A$ of our original problem.

The above functions obviously satisfy conditions (K1)–(K4).

### Case of Theorem 2.4

Let functions $a_i^{(l)}(t, x, \zeta_0, \zeta; v)$ have the form:

$$\begin{aligned} &a_i^{(l)}(t, x, \zeta_0, \zeta; v) \\ &\quad = [H^{(l)}(v)](x) b_i^{(l)}(t, x, \zeta_0, \zeta) + [G^{(l)}(v)](x) d_i^{(l)}(t, x, \zeta_0, \zeta) \quad (i \neq 0), \end{aligned} \tag{2.33}$$

$$a_0^{(l)}(t, x, \zeta_0, \zeta; v) = [H^{(l)}(v)](x) b_0^{(l)}(t, x, \zeta_0, \zeta) + d_0^{(l)}(t, x, \zeta_0, \zeta). \tag{2.34}$$

By applying the arguments of Section 2.2.3 we have

**Proposition 2.10.** *Let $T \geq a$. Suppose that functions $b_i^{(l)}, d_i^{(l)}$ satisfy (K1)–(K4) ($i = 0, \ldots, n; l = 1, \ldots, N$). Further, $H^{(l)} \colon L^p(-a, 0; (L^p(\Omega))^N) \to L^\infty(Q_T)$ is bounded and continuous such that $[H^{(l)}(v)](t, x) \geq c_3 > 0$ holds for a.a. $(t, x) \in Q_T$ and every $v \in L^p(-a, 0; (L^p(\Omega))^N)$. In addition, $G^{(l)} \colon L^p(-a, 0; (L^p(\Omega))^N) \to L^{\frac{p}{p-r-1}}(Q_T)$ is bounded and continuous where $r$ is given in (K2)/b. Then functions* (2.33)–(2.34) *fulfil conditions (A1')–(A5').*

For functions $b_i^{(l)}, d_i^{(l)}$ consider the examples found in Section 2.2.3. Further, operators $H^{(l)}, G^{(l)}$ may have the following form. Let $\Phi \colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\Phi \geq c > 0$ and introduce the following operators on $L^p(0,T;(L^p(\Omega))^N)$:

$$[\tilde{H}_1(v)](x) := \Phi\left(\int_{-a}^{0}\sum_{j=1}^{N} b_j(s,x)v^{(j)}(s,x)\,ds\right), \quad \text{where} \;\; b_j \in L^q(Q_T) \;\; (1 \leq j \leq N),$$

$$[\tilde{H}_2(v)](t,x) := \Phi\left(\left[\int_{-a}^{0}|v(s,x)|^{\alpha}\,ds\right]^{\frac{1}{\alpha}}\right), \quad \text{where} \;\; 1 \leq \alpha \leq p.$$

Now let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|\psi(y)| \leq \tilde{c} \cdot |y|^{p-r_0-1}$ holds for some constants $\tilde{c}$ and $0 \leq r_0 < p-1$. Let us introduce the following operators on $L^p(0,T;(L^p(\Omega))^N)$:

$$[\tilde{G}_1(v)](t,x) := \psi\left(\int_{-a}^{0}\sum_{j=1}^{N} a_j(s,x)v^{(j)}(s,x)ds\right),$$
$$[\tilde{G}_2(v)](t,x) := \psi\left(\left[\int_{-a}^{0}|v(s,x)|^{\alpha}ds\right]^{\frac{1}{\alpha}}\right)$$

where $a_j \in L^\infty(Q_T)$ $(1 \leq j \leq N)$ and $1 \leq \alpha \leq p$.

**Proposition 2.11.** *The above operators $\tilde{H}_i, \tilde{G}_i$ $(i = 1,2)$ fulfil the conditions posed on them in Proposition 2.10.*

## 2.3 Solutions in $(0,\infty)$

> There's no sense in being precise when you don't even know what you're talking about.
>
> John von Neumann

In the previous section we showed existence of solutions in the time interval $(0,T)$. In what follows, we consider solutions in $(0,\infty)$. First of all, we define precisely the notion of solutions in $(0,\infty)$ then we show existence of these solutions and investigate the long-time behaviour of them. We shall obtain results on boundedness and stabilization as $t \to \infty$, see also [61, 64, 65].

### 2.3.1 Existence

Briefly, denote $Q_\infty = (0,\infty) \times \Omega$. Further, let the space $L^p_{\mathrm{loc}}(0,\infty;V)$ be the set of measurable functions $u\colon (0,\infty) \to V$ such that $u|_{(0,T)} \in L^p(0,T;V)$ for every $0 < T < \infty$. It is easy to see that if $u \in L^p_{\mathrm{loc}}(0,\infty;V)$ and for every $0 < T < \infty$ there exists $D_t(u|_{(0,T)}) \in L^q(0,T;V^\star)$ then $D_t u \in L^q_{\mathrm{loc}}(0,\infty;V^\star)$. Further, we denote by

$L^p_{\text{loc}}(Q_\infty)$ the space of measurable functions $v\colon Q_\infty \to \mathbb{R}$ such that $v|_{(0,T)} \in L^p(Q_T)$ for every $0 < T < \infty$.

In order to prove existence of weak solutions in $(0, \infty)$, one poses:

(Vol) Functions $a_i^{(l)}\colon Q_\infty \times \mathbb{R}^{(n+1)N} \times L^p_{\text{loc}}(0,\infty;V) \to \mathbb{R}$ $(i = 0,\dots,n; l = 1,\dots,N)$ have the so-called Volterra property, i.e., $a_i^{(l)}(t, x, \zeta_0, \zeta; , v)|_{(0,T)}$ depends only on $v|_{(0,T)}$ for every $0 < T < \infty$.

In addition, we suppose that conditions (A1)–(A5) are satisfied for every $T \in (0,\infty)$ by functions $a_i^{(l)}$ $(i = 0,\dots,n; l = 1,\dots,N)$. Precisely, we mean that for every $T \in (0,\infty)$, the restriction $a_i^{(l)}|_{(0,T)}\colon Q_T \times \mathbb{R} \times \mathbb{R}^{(n+1)N} \times L^p(0,T;V) \to \mathbb{R}$, which may be defined uniquely by the Volterra-property, satisfy the conditions (A1)–(A5) (not necessarily with the same $g_1, g_2, k_1, k_2$).

Finally, let

(F1*) $\mathcal{F} \in L^q_{\text{loc}}(0,\infty;V^*)$.

Now for every $0 < T < \infty$ define $L_T\colon D(L_T) \to L^q(0,T;V^*)$ by (2.3). By supposing the above conditions, for fixed $0 < T < \infty$ we may introduce operator $\mathcal{A}_T\colon L^p(0,T;V) \to L^q(0,T;V^*)$ by (2.2) (which will be bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L_T)$). Due to the Volterry property, there is an operator $\mathcal{A}\colon L^p_{\text{loc}}(0,\infty;V) \to L^q_{\text{loc}}(0,\infty;V^*)$ such that $\mathcal{A}_T(u|_{(0,T)}) = \mathcal{A}(u)|_{(0,T)}$ for every $0 < T < \infty$ and $u \in L^p_{\text{loc}}(0,T;V)$. Similarly, we write $\mathcal{F}_T = \mathcal{F}|_{(0,T)}$ for every $0 < T < \infty$.

We say that $u \in L^p_{\text{loc}}(0,\infty;V)$ is a weak solution of (2.1) in $(0,\infty)$ if

$$D_t u + \mathcal{A}(u) = \mathcal{F}$$

or, in other words, if for all $0 < T < \infty$,

$$L_T u|_{(0,T)} + \mathcal{A}_T(u|_{(0,T)}) = \mathcal{F}_T. \tag{2.35}$$

(Notice that initial condition $u(0) = 0$ is included in the above equations.) Observe that the Volterra property ensures that $\mathcal{A}_T(u|_{(0,T)})|_{(0,\tilde{T})} = \mathcal{A}_{\tilde{T}}(u|_{(0,\tilde{T})})$ for every $0 < \tilde{T} \le T < \infty$ and $u \in L^p_{\text{loc}}(0,\infty;V)$ thus if $u$ is a solution in $(0,T)$ then this it is also a solution in $(0,\tilde{T})$. In the sequel we omit the notation $|_{(0,T)}$ of the restriction of a function to a certain interval if it is not confusing, since the operators and the norms contain the information about the space.

**Theorem 2.12.** *Suppose that (Vol), (F1*) hold, further, conditions (A1)–(A5) are satisfied for every* $0 < T < \infty$. *Then there exists a weak solution* $u \in L^p_{\text{loc}}(0,\infty;V)$ *of* (2.1) *in* $(0,\infty)$.

*Proof.* The main idea is the following. By Corollary 2.3, for every $0 < T < \infty$ there exists a solution in $(0, T)$. Then the weak limit of a suitable weakly convergent subsequence of these solutions, that were chosen by using a diagonal process, will be a solution in $(0, \infty)$.

We briefly write $X_T = L^p(0, T; V)$ for $0 < T < \infty$. Let $(T_k)$ be a monotone increasing sequence of positive numbers such that $T_k \to +\infty$. Then by Corollary 2.3, for every $T_k$ there exists $u_k \in X_{T_k} \cap D(L_{T_k})$ such that

$$L_{T_k} u_k + \mathcal{A}_{T_k}(u_k) = \mathcal{F}_{T_k}.$$

Now we show that for fixed $m$, the sequence $\big(u_k|_{(0,T_m)}\big)_{k \geqslant m}$ is bounded in $X_{T_m}$. By the Volterra property $u_k|_{(0,T_m)}$ is a solution in $(0, T_m)$ for $k \geq m$, i.e.,

$$L_{T_m} u_k + \mathcal{A}_{T_m}(u_k) = \mathcal{F}_{T_m}.$$

By applying both sides to $u_k$ it follows

$$[L_{T_m} u_k, u_k] + [\mathcal{A}_{T_m}(u_k), u_k] = [\mathcal{F}_{T_m}, u_k].$$

The first term on the left hand side of the above equation is nonnegative, on the other hand, $|[\mathcal{F}_{T_m}, u_k]| \leq \|\mathcal{F}\|_{X^*_{T_m}} \cdot \|u_k\|_{X_{T_m}}$ hence

$$\frac{[\mathcal{A}_{T_m}(u_k), u_k]}{\|u_k\|_{X_{T_m}}} \leq \|\mathcal{F}\|_{X^*_{T_m}}.$$

Now the coerciveness of $\mathcal{A}_{T_m}$ in $X_{T_m}$ yields the boundedness of $\big(\|u_k\|_{X_{T_m}}\big)_{k \geqslant m}$. Further, the boundedness of operator $\mathcal{A}_{T_m}$ implies the boundedness of the sequence $(\mathcal{A}_{T_m}(u_k))_{k \geqslant m}$ in $X^*_{T_m}$.

Let $m = 1$. Since $(u_k)$ and $(\mathcal{A}_{T_1}(u_k))$ are bounded sequences in reflexive Banach spaces, by Theorem 1.28 and Proposition 1.52 there exists a weakly convergent subsequence $(u_{1,k}) \subset (u_k)$ and there exist functions $u_{1,*} \in X_{T_1} \cap D(L_{T_1})$, $v_{1,*} \in X^*_{T_1}$ such that

$$\begin{aligned}
u_{1,k} &\to u_{1,*} \ \text{ weakly in } \ X_{T_1}, \\
L_{T_1} u_{1,k} &\to L_{T_1} u_{1,*} \ \text{ weakly in } \ X^*_{T_1}, \\
u_{1,k}(T_1) &\to u_{1,*}(T_1) \ \text{ weakly in } \ H, \\
\mathcal{A}_{T_1}(u_{1,k}) &\to v_{1,*} \ \text{ weakly in } \ X^*_{T_1}.
\end{aligned}$$

If $(u_{m-1,k})_{k \geq m-1}$ is given then $(u_{m-1,k})_{k \geq m-1}, (\mathcal{A}_{T_{m-1}} u_{m-1,k})_{k \geq m-1}$ are bounded in reflexive Banach spaces $X_{T_{m-1}}, (X_{T_{m-1}})^*$ thus Theorem 1.28 and Proposition 1.52

yield a subsequence $(u_{m,k}) \subset (u_{m-1,k})$ and functions $u_{m,*} \in X_{T_m} \cap D(L_{T_m}), v_{m,*} \in X^*_{T_m}$ such that

$$u_{m,k} \to u_{m,*} \text{ weakly in } X_{T_m}, \tag{2.36}$$

$$L_{T_m} u_{m,k} \to L_{T_m} u_{m,*} \text{ weakly in } X^*_{T_m}, \tag{2.37}$$

$$u_{m,k}(T_m) \to u_{m,*}(T_m) \text{ weakly in } H, \tag{2.38}$$

$$\mathcal{A}_{T_m}(u_{m,k}) \to v_{m,*} \text{ weakly in } X^*_{T_m}. \tag{2.39}$$

It is easy to see that for each fixed $l < m$ the above weak convergences hold in $X_{T_l}$, $X^*_{T_l}$, respectively, which implies $u_{m,*}|_{(0,T_l)} = u_{l,*}$ and $v_{m,*}|_{(0,T_l)} = v_{l,*}$ for $l < m$. Consequently, there exist unique functions $u, v\colon (0,\infty) \to V$ such that $u|_{(0,T_m)} = u_{m,*}, v|_{(0,T_m)} = v_{m,*}$ for every $m \in \mathbb{N}$. This means that $u \in L^p_{\text{loc}}(0,\infty;V)$ and $v \in L^q_{\text{loc}}(0,\infty;V^*)$.

Now fix $m \in \mathbb{N}$. In the sequel we shall work on interval $(0,T_m)$. We show that $u$ is a solution in this interval then the proof of the theorem will be complete.

At this point we already know that $u_{m,*} \in D(L_{T_m})$ and $L_{T_m} u_{m,*} + v_{m,*} = \mathcal{F}_{T_m}$. It remains to prove $v_{m,*} = \mathcal{A}(u_{m,*})$ then $u_{m,*}$ is a solution in $(0,T_m)$. By (2.39) it suffices to show that $\mathcal{A}_{T_m}(u_{m,k}) \to \mathcal{A}_{T_m}(u_{m,*})$ weakly in $X^*_{T_m}$. Now we use the fact that $\mathcal{A}_{T_m}$ is a pseudomonotone operator with respect to $D(L_{T_m})$, i.e., (2.36), (2.37) and

$$\limsup_{k\to\infty} [\mathcal{A}_{T_m}(u_{m,k}), u_{m,k} - u_{m,*}] \le 0 \tag{2.40}$$

imply that $\mathcal{A}_{T_m}(u_{m,k}) \to \mathcal{A}_{T_m}(u_{m,*})$ weakly in $X^*_{T_m}$. In the following we show that (2.40) holds. By using (2.39) we may deduce

$$\limsup_{k\to\infty} [\mathcal{A}_{T_m}(u_{m,k}), u_{m,k} - u_{m,*}] = \limsup_{k\to\infty} [\mathcal{A}_{T_m}(u_{m,k}), u_{m,k}] - [v_{m,*}, u_{m,*}]. \tag{2.41}$$

Further,

$$\begin{aligned}
[\mathcal{A}_{T_m}(u_{m,k}), u_{m,k}] &= [\mathcal{F}_{T_m}, u_{m,k}] - [L_{T_m} u_{m,k}, u_{m,k}] \\
&= [\mathcal{F}_{T_m}, u_{m,k}] - \frac{1}{2}\|u_{m,k}(T_m)\|_H^2 + \frac{1}{2}\|u_{m,k}(0)\|_H^2.
\end{aligned} \tag{2.42}$$

Now Lemma 1.29 and property (2.38) imply

$$\|u_{m,*}(T_m)\|_H \le \liminf_{k\to\infty} \|u_{m,k}(T_m)\|_H \tag{2.43}$$

so that by using (2.36) and (2.42) we conclude

$$\begin{aligned}
\limsup_{k\to\infty} [\mathcal{A}_{T_m}(u_{m,k}), u_{m,k}] &\le [\mathcal{F}_{T_m}, u_{m,*}] - \frac{1}{2}\|u_{m,*}(T_m)\|_H^2 \\
&= [\mathcal{F}_{T_m}, u_{m,*}] - [L_{T_m} u_{m,*}, u_{m,*}].
\end{aligned}$$

The above inequality and (2.41) together yield the desired relation (2.40). The proof of Theorem 2.12 is complete. □

In the following we are interested in periodic type of solutions in $(0, \infty)$. If the equations describes a periodic process for instance in biology then existence of periodic solutions is an improtant question. If the nonlocal variable may contain arbitrary long delay then it is not so clear how to define the notion of periodic solutions. The Volterra property ensures that at time $t$ the function depends only on the values before $t$. The notion of periodicity can make sense, e.g., by assuming that this delay is less than a certain time interval as in Section 2.2.2 for system (2.16).

We recall (2.16):

$$
\begin{aligned}
&D_t u^{(l)}(\cdot) \\
&- \sum_{i=1}^{n} D_i \left[ a_i^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u_t^{(1)}, \dots, u_t^{(N)}) \right] \\
&+ a_0^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u_t^{(1)}, \dots, u_t^{(N)}) \\
&= f^{(l)}(\cdot),
\end{aligned}
$$

with some boundary condition and initial condition $u_0^{(l)}(s) = \varphi_l(s)$ for $s \in [-a, 0]$ where $\varphi_l \in L^p(-a, 0; V)$ $(l = 1, \dots, N)$. (As before, the exact form of the boundary condition determines the space $V$, see Section 2.1.3.)

By supposing conditions (A1'), (A2') (see Section 2.2.2) we introduce operator $\tilde{A}_T \colon L^p(-a, T; V) \to L^q(0, T; V^*)$ as follows. For $u = (u^{(1)}, \dots, u^{(N)}) \in L^p(-a, T; V)$, $v = (v^{(1)}, \dots, v^{(N)}) \in L^p(0, T; V)$,

$$
\begin{aligned}
[\tilde{\mathcal{A}}_T(u), v] := &\sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_i^{(l)}(t, x, u(t, x), Du(t, x); u_t) D_i v^{(l)}(t, x) \, dt \, dx \\
&+ \sum_{l=1}^{N} \int_{Q_T} \sum_{i=1}^{n} a_0^{(l)}(t, x, u(t, x), Du(t, x); u_t) v^{(l)}(t, x) \, dt \, dx.
\end{aligned}
$$

Let $\tilde{\mathcal{A}} \colon L^p_{\text{loc}}(-a, \infty; V) \to L^q_{\text{loc}}(0, \infty; V^*)$ such that $\tilde{\mathcal{A}}(u)|_{(0,T)} = \tilde{\mathcal{A}}(u|_{(0,T)})$ for every $u \in L^p_{\text{loc}}(-a, \infty; V)$. In addition, let $\mathcal{F} \in L^q_{\text{loc}}(0, \infty; V^*)$ (and $\mathcal{F}_T = \mathcal{F}|_{(0,T)}$).

We want to find $u \in L^p_{\text{loc}}(-a, \infty; V)$ such that $D_t u \in L^q_{\text{loc}}(-a, \infty; V^*)$ and

$$
D_t u|_{(0,\infty)} + \tilde{\mathcal{A}}(u) = \mathcal{F} \tag{2.44}
$$

$$
u(t) = u(t + T) \quad \text{for} \;\; t \in [-a, \infty). \tag{2.45}
$$

**Theorem 2.13.** *Suppose that functions* $a_i^{(l)} \colon Q_\infty \times \mathbb{R}^{n+1} \times L^p(-a, 0; V) \to \mathbb{R}$ *(*$i = 1, \dots, n; l = 1, \dots, N$*) satisfy conditions (A1')–(A5') in* $(0, T)$ *for some* $T \geq a$*, further, they are* $T$*-periodic, i.e.,*

$$
a_i^{(l)}(t + T, x, \zeta_0, \zeta; v) = a_i^{(l)}(t, x, \zeta_0, \zeta; v)
$$

*for a.a.* $(t, x) \in Q_\infty$, *every* $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$ *and* $v \in L^p(-a, 0; V)$. *Then for every* $T$*-periodic* $\mathcal{F} \in L^q_{\text{loc}}(0, \infty; V^*)$ *there exists* $u \in L^p_{\text{loc}}(-a, \infty; V)$ *such that* $D_t u \in L^q_{\text{loc}}(-a, \infty; V^*)$ *and* (2.44)–(2.45) *hold.*

*Proof.* By applying Theorem 2.4 in interval $(0, T)$, there exist $u \in L^p(-a, T; V)$ such that $D_t u \in L^q(-a, T; V^*)$, further,

$$D_t u|_{(0,T)} + \tilde{\mathcal{A}}_T(u) = \mathcal{F}_T$$
$$u(t) = u(t + T) \quad \text{for} \quad t \in [-a, \infty).$$

Now we can apply Theorem 2.4 in interval $(T, 2T)$ and by the periodicity of $u$ we obtain the translation of $u$ as solution. By continuing the method on intervals $(kT, (k+1)T)$ we obtain the translations of $u$ which yields a periodic solution such that (2.44) holds. □

### 2.3.2 Boundedness

In this section we show the boundedness of solutions in $(0, \infty)$ formulated in Theorem 2.12. We modify condition (A4) and assume the boundedness of $\mathcal{F}$.

(A4*) There exist a constant $g_2 \in \mathbb{R}^+$ and an operator $k_2 \colon L^p_{\text{loc}}(0, \infty; V) \to L^1_{\text{loc}}(Q_\infty)$ of Volterra type such that

$$\sum_{l=1}^{N} \sum_{i=0}^{n} a_i^{(l)}(t, x, \zeta_0, \zeta; v)\zeta_i^{(l)} \geq g_2 \left(|\zeta_0|^p + |\zeta|^p\right) - [k_2(v)](t, x)$$

for a.a. $(t, x) \in Q_\infty$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p_{\text{loc}}(0, \infty; V)$. Further, for every $T > 0$,

$$\lim_{\|v\|_{L^p(0,T;V)} \to \infty} \frac{\|k_2(v)\|_{L^1(Q_T)}}{\|v\|^p_{L^p(0,T;V)}} = 0.$$

Finally, there exist constants $c_4 > 0$, $0 \leq p_1 < p$ and a continuous function $\varphi \colon [0, \infty) \to \mathbb{R}$ such that $\lim_{\tau \to \infty} \varphi(\tau) = 0$, further, if $v \in L^p_{\text{loc}}(0, \infty; V)$ and $D_t v \in L^q_{\text{loc}}(0, \infty; V^*)$ then for a.a. $t \in (0, \infty)$,

$$\begin{aligned} &\int_\Omega |[k_2(v)](t, x)| \, dx \\ &\leq c_4 \left( \sup_{\tau \in [0,t]} \|v(\tau)\|^{p_1}_{(L^2(\Omega))^N} + \varphi(t) \cdot \sup_{\tau \in [0,t]} \|v(\tau)\|^p_{(L^2(\Omega))^N} + 1 \right). \end{aligned} \tag{2.46}$$

(F1**) There exists $t_* \in (0, \infty)$ such that $\mathcal{F}|_{(t_*, \infty)} \in L^\infty(t_*, \infty; V^*)$.

*Remark* 2.14. The suprema in inequality (2.46) exist since $v \in L^p_{\text{loc}}(0, \infty; V)$ and $D_t v \in L^q_{\text{loc}}(0, \infty; V^*)$ imply $v \in C([0, T], (L^2(\Omega))^N)$ for every finite $T$.

Notice that condition (A4$^*$) implies (A4) for every $0 < T < \infty$. Now we have

**Theorem 2.15.** *Assume (Vol), further, suppose that (A1)–(A3), (A5) hold for every $0 < T < \infty$, and the modified conditions (A4$^*$), (F1$^{**}$) are fulfilled. Then for the solutions $u$ of problem* (2.35) *we have $u \in L^\infty(0, \infty; (L^2(\Omega))^N)$.*

*Proof.* For brevity, let $y(t) = \|u(t)\|_H^2$ (recall that $H = (L^2(\Omega))^N$). Note that $y \in C(0, \infty)$. We shall prove an integral inequality for $y$. By applying both sides of equation $D_t u(t) + [\mathcal{A}(u)](t) = \mathcal{F}(t)$ to $u(t)$ and integrating on interval $(T_1, T_2)$ where $t_* \le T_1 \le T_2 < \infty$ we obtain

$$\int_{T_1}^{T_2} \langle D_t u(t), u(t)\rangle \, dt + \int_{T_1}^{T_2} \langle [\mathcal{A}(u)](t), u(t)\rangle \, dt = \int_{T_1}^{T_2} \langle \mathcal{F}(t), u(t)\rangle \, dt. \tag{2.47}$$

The first term on the left hand side by Corollary 1.43 has the form

$$\int_{T_1}^{T_2} \langle D_t u(t), u(t)\rangle \, dt = \frac{1}{2}\left(\|u(T_2)\|_H^2 - \|u(T_1)\|_H^2\right) = \frac{1}{2}\left(y(T_2) - y(T_1)\right). \tag{2.48}$$

Further, one may estimate the second term from below as

$$\begin{aligned}\langle [\mathcal{A}(u)](t), u(t)\rangle &\ge \int_\Omega \left[g_2 \left(|u(t,x)|^p + |Du(t,x)|^p\right) - [k_2(u)](t,x)\right] dx \\ &= g_2 \|u(t)\|_V^p - \int_\Omega [k_2(u)](t,x)\, dx\end{aligned}$$

thus

$$\int_{T_1}^{T_2} \langle [A(u)](t), u(t)\rangle \ge g_2 \int_{T_1}^{T_2} \|u(t)\|_V^p \, dt - \int_{T_1}^{T_2} \int_\Omega [k_2(u)](t,x)\, dt\, dx.$$

By substituting (2.46) into the above inequality we may deduce

$$\begin{aligned}&\int_{T_1}^{T_2} \langle [\mathcal{A}(u)](t), u(t)\rangle \, dt \\ &\ge g_2 \int_{T_1}^{T_2} \|u(t)\|_V^p - c_4 \int_{T_1}^{T_2} \left[\sup_{\tau\in[0,t]} y(\tau)^{\frac{p_1}{2}} + \varphi(t) \cdot \sup_{\tau\in[0,t]} y(\tau)^{\frac{p}{2}} + 1\right] dt.\end{aligned} \tag{2.49}$$

Now let us estimate the right hand side of (2.47). Choose $\varepsilon > 0$ such that $\frac{\varepsilon^p}{p} < \frac{1}{2} g_2$ then by using the $\varepsilon$-inequality and the fact that $\mathcal{F} \in L^\infty(0, \infty; V^*)$ we conclude

$$\begin{aligned}\int_{T_1}^{T_2} \langle \mathcal{F}(t), u(t)\rangle \, dt &\le \int_{T_1}^{T_2} \|u(t)\|_V \|\mathcal{F}(t)\|_{V^*} \, dt \\ &\le \int_{T_1}^{T_2} \left[\frac{\varepsilon^p}{p} \|u(t)\|_V^p + \frac{1}{q\varepsilon^q} \|\mathcal{F}(t)\|_{V^*}^q\right] dt \\ &\le \frac{1}{2} g_2 \int_{T_1}^{T_2} \|u(t)\|_V^p \, dt + \frac{1}{q\varepsilon^q} \int_{T_1}^{T_2} \operatorname*{ess\,sup}_{(t_*,\infty)} \|\mathcal{F}\|_{V^*} \, dt.\end{aligned} \tag{2.50}$$

Then by substituting (2.48), (2.49), (2.50) into (2.47) one obtains

$$\frac{1}{2}\left(y(T_2)-y(T_1)\right)+\frac{1}{2}g_2\int_{T_1}^{T_2}\|u(t)\|_V^p\,dt$$
$$\leq c_4'\int_{T_1}^{T_2}\left[\sup_{\tau\in[0,t]}y(\tau)^{\frac{p_1}{2}}+\varphi(t)\cdot\sup_{\tau\in[0,t]}y(\tau)^{\frac{p}{2}}+1\right]dt.$$

Finally, the continuous embedding $(W^{1,p}(\Omega))^N\hookrightarrow(L^2(\Omega))^N$ implies

$$y(T_2)-y(T_1)+d_1\int_{T_1}^{T_2}y(t)^{\frac{p}{2}}\,dt\leq d_2\int_{T_1}^{T_2}\left[\sup_{\tau\in[0,t]}y(\tau)^{\frac{p_1}{2}}+\varphi(t)\cdot\sup_{\tau\in[0,t]}y(\tau)^{\frac{p}{2}}+1\right]dt$$

where the constants $d_1,d_2>0$ do not depend on the choice of $(T_1,T_2)$. We show that the above inequality implies the boundedness of $y$. We prove by contradiction. Suppose that for every $M>0$ there exists $t_M>0$ such that

$$M+1=y(t_M)=\sup_{\tau\in[0,t_M]}y(\tau). \tag{2.51}$$

(So $t_M$ is the first point where $y$ attains the value $M+1$.) Then by the continuity of $y$ there exists $\delta>0$ such that $y(t)>M$ for $t_M-\delta\leq t\leq t_M$. Now by choosing $T_1=t_M-\delta$ and $T_2=t_M$ in (2.51) it follows

$$y(t_M)-y(t_M-\delta)+d_1\delta M^{\frac{p}{2}}\leq d_2\delta(M+1)^{\frac{p_1}{2}}+d_2(M+1)^{\frac{p}{2}}\int_{t_M-\delta}^{t_M}\varphi(t)\,dt+d_2\delta. \tag{2.52}$$

On the right hand side $y(t_M)-y(t_M-\delta)\geq 0$, further, by the intermediate value theorem

$$\int_{t_M-\delta}^{t_M}\varphi(t)\,dt=\delta\cdot\sup_{t\in[t_M-1,t_M]}\varphi(t)=\delta\cdot\varphi(\hat{t})$$

for some $\hat{t}\in[t_M-\delta,t_M]$. Hence by (2.52),

$$d_1\left(\frac{M}{M+1}\right)^{\frac{p}{2}}\leq d_2(M+1)^{\frac{p_1-p}{2}}+d_2\varphi(\hat{t})+d_2(M+1)^{-\frac{p}{2}}.$$

Note that the left hand side converges to 1 as $M\to\infty$. On the other hand, $p_1<p$ and $\hat{t}\to\infty$ imply that the right hand side tends to 0. That is a contradiction, the proof our theorem is complete. $\square$

*Remark* 2.16. One may study also non-uniformly parabolic systems, when in condition (A4*) instead of a constant $g_2$ one has an operator $g_2\colon L^p_{\mathrm{loc}}(0,\infty;V)\to\mathbb{R}^+$ not necessarily bounded from below for all $v\in L^p_{loc}(0,\infty;V)$, see [67, 68, 69].

### 2.3.3 Stabilization

In this part we investigate the asymptotic properties as $t \to \infty$. In particular, we are interested in the stabilization of solutions, i.e., the convergence to a stationary state. To this end, suppose the following.

(A2$^+$) For every $v \in L^\infty(0,\infty;(L^2(\Omega))^N)$ there exist a constant $c_v > 0$ and a function $k_v \in L^q(\Omega)$ such that

$$|a_i^{(l)}(t,x,\zeta_0,\zeta;v)| \le c_v\left(|\zeta_0|^{p-1} + |\zeta|^{p-1}\right) + k_v(x)$$

for a.a. $(t,x) \in Q_\infty$, every $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$ $(i = 0,\dots,n; l = 1,\dots,N)$.

(A6) There exist Carathéodory functions $a_{i,\infty}^{(l)} \colon \Omega \times \mathbb{R}^{(n+1)N} \to \mathbb{R}$ $(i = 0,\dots,n;\ l = 1,\dots,N)$ such that for every fixed $v \in L^p_{\mathrm{loc}}(0,\infty;V) \cap L^\infty(0,\infty;(L^2(\Omega))^N)$, a.a. $x \in \Omega$ and every $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$,

$$\lim_{t\to\infty} a_i^{(l)}(t,x,\zeta_0,\zeta;v) = a_{i,\infty}^{(l)}(x,\zeta_0,\zeta). \tag{2.53}$$

(A7) There exists a constant $c_5 > 0$ such that for a.a. $x \in \Omega$, every $(\zeta_0,\zeta), (\tilde\zeta_0,\tilde\zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p_{\mathrm{loc}}(0,\infty;V)$,

$$\begin{aligned}&\sum_{l=1}^N \sum_{i=0}^n \left(a_i^{(l)}(t,x,\zeta_0,\zeta;v) - a_i^{(l)}(t,x,\tilde\zeta_0,\tilde\zeta;v)\right)(\zeta_i^{(l)} - \tilde\zeta_i^{(l)})\\ &\ge c_5\left(|\zeta_0 - \tilde\zeta_0|^p + |\zeta - \tilde\zeta|^p\right) - k_3(t,x,\zeta_0,\tilde\zeta_0;v),\end{aligned} \tag{2.54}$$

where $k_3 \colon Q_\infty \times \mathbb{R} \times \mathbb{R} \times L^p_{\mathrm{loc}}(0,\infty;V)$ satisfies

$$\lim_{t\to\infty} \int_\Omega k_3(t,x,u(t,x),\tilde u(t,x);v)\,dx = 0 \tag{2.55}$$

if $u, \tilde u, v \in L^\infty(0,\infty;(L^2(\Omega))^N)$.

(F2) There exists $\mathcal{F}_\infty \in V^*$ such that $\lim\limits_{t\to\infty} \|\mathcal{F}(t) - \mathcal{F}_\infty\|_{V^*} = 0$.

*Remark* 2.17. Precisely, by the convergence $s(t) \to 0$ as $t \to \infty$ where $s\colon \mathbb{R}^+ \to M$ is a measurable function and $M$ is a normed space, we mean that for all $\varepsilon > 0$ there exists $t_0$ such that $\|s(t)\|_M \le \varepsilon$ for a.a. $t > t_0$.

Now we may define operator $\mathcal{A}_\infty \colon V \to V^*$ by

$$\begin{aligned}\langle \mathcal{A}_\infty(v), w\rangle := &\sum_{l=1}^N \int_\Omega \sum_{i=1}^n a_{i,\infty}^{(l)}(x,v(x),Dv(x)) D_i w^{(l)}(x)\,dx\\ &+ \sum_{l=1}^N \int_\Omega \sum_{i=1}^n a_{0,\infty}^{(l)}(x,v(x),Dv(x)) w^{(l)}(x)\,dx\end{aligned} \tag{2.56}$$

where $v = (v^{(1)},\dots,v^{(N)})$, $w = (w^{(1)},\dots,w^{(N)}) \in V$.

Our main result is

**Theorem 2.18.** *Assume (Vol). In addition, suppose that conditions (A1)–(A3), (A5) hold for every* $0 < T < \infty$*, further, (A2*$^+$*), (A4*$^*$*), (A6), (A7), (F2) are satisfied. Then there exists a unique solution* $u_\infty \in V$ *of problem*

$$\mathcal{A}_\infty(u_\infty) = \mathcal{F}_\infty.$$

*In addition,* $u_\infty$ *possesses the following stabilization relation:*

$$\lim_{t\to\infty} \|u(t) - u_\infty\|_{(L^2(\Omega))^N} = 0$$

*where* $u$ *is a solution of problem* (2.35)*.*

Before the proof it is worth emphasizing some properties of operator $\mathcal{A}_\infty$.

**Lemma 2.19.** *Operator* $\mathcal{A}_\infty \colon V \to V^*$*, defined by* (2.56)*, is bounded, hemicontinuous, uniformly monotone and coercive.*

*Proof.* Let $w(t) \equiv w \in V$ then $w \in L^\infty(0,\infty;(L^2(\Omega))^N)$. From condition (A2$^+$) it follows

$$|a_i^{(l)}(t,x,\zeta_0,\zeta;w)| \le c_w\left(|\zeta_0|^{p-1} + |\zeta|^{p-1}\right) + k_w(x). \tag{2.57}$$

Hence by passing to the limit as $t \to \infty$ we may deduce

$$|a_{i,\infty}^{(l)}(x,\zeta_0,\zeta)| \le c_w\left(|\zeta_0|^{p-1} + |\zeta|^{p-1}\right) + k_w(x). \tag{2.58}$$

From the above estimate, the boundedness of operator follows by the classical argument, see the proof of Theorem 2.1 or the monographs [44, 71].

The hemicontinuity follows from the above estimate, as well. Indeed, let $\lambda_k \to \lambda$ be a real sequence then for arbitrary $u$, $v$, $w \in V$,

$$\begin{aligned}
&\langle \mathcal{A}_\infty(u - \lambda_k v), w\rangle \\
&= \sum_{l=1}^N \sum_{i=1}^n \int_\Omega a_{i,\infty}^{(l)}\big(x, u(x) - \lambda_k v(x), Du(x) - \lambda_k Dv(x)\big) D_i w^{(l)}(x)\,dx \\
&\quad + \sum_{l=1}^N a_{0,\infty}^{(l)}\big(x, u(x) - \lambda_k v(x), Du(x) - \lambda_k Dv(x)\big) w^{(l)}(x)\,dx.
\end{aligned} \tag{2.59}$$

Clearly, the integrand on the right hand side of the above equation converges pointwise. Further, by using Young's inequality combined with inequalities (2.57), (1.1) we may deduce

$$\begin{aligned}
&|a_{i,\infty}^{(l)}(x, u - \lambda_k v, Du - \lambda_k Dv) D_i w^{(l)}| \\
&\le \frac{1}{q}|a_{i,\infty}^{(l)}(x, u - \lambda_k v, Du - \lambda_k Dv)|^q + \frac{1}{p}|D_i w^{(l)}|^p \\
&\le \text{const}\cdot\left(|u - \lambda_k|^{(p-1)q} + |Du - \lambda_k v|^{(p-1)q} + |k_1(w)|^q + |Dw|^p\right) \\
&\le \text{const}\cdot\left(|u|^p + |\lambda_k v|^p + |Du|^p + |\lambda_k Dv|^p + |k_1(w)|^q + |Dw|^p\right) \\
&\le \text{const}\cdot\left(|u|^p + |v|^p + |Du|^p + |Dv|^p + |k_1(w)|^q + |Dw|^p\right).
\end{aligned}$$

The right hand side of the above inequality consists of functions of $L^1(\Omega)$ thus the integrands of (2.59) have integrable majorants hence by Lebesgue's theorem we conclude

$$\lim_{k\to\infty}\langle \mathcal{A}_\infty(u-\lambda_k v), w\rangle = \langle \mathcal{A}_\infty(u-\lambda v), w\rangle$$

that is exactly the hemicontinuity of $\mathcal{A}_\infty$.

Now fix $w(t) \equiv w \in V$. Then for arbitrary $v, v_* \in V$ it holds

$$\begin{aligned}
&\sum_{l=1}^{N}\sum_{i=1}^{n}\int_\Omega \Big(a_i^{(l)}(t,x,v(x),Dv(x);w) - a_i^{(l)}(t,x,v_*(x),Dv_*(x);w)\Big)\\
&\qquad \times (D_i v^{(l)}(x) - D_i v_*^{(l)}(x))\,dx\\
&\quad + \sum_{l=1}^{N}\int_\Omega \Big(a_0^{(l)}(t,x,v(x),Dv(x);w) - a_0^{(l)}(t,x,v_*(x),Dv_*(x);w)\Big)\\
&\qquad \times (v^{(l)}(x) - v_*^{(l)}(x))\,dx\\
&\ge c_5\int_\Omega (|v(x)-v_*(x)|^p + |Dv(x)-Dv_*(x)|^p)\,dx - \int_\Omega k_3(t,x,v(x),v_*(x);w)\,dx.
\end{aligned}$$

Similarly to the previous paragraph we may use Lebesgue's theorem thus by applying (2.53), (2.57) and (2.55), as $t\to\infty$ it follows

$$\begin{aligned}
&\sum_{l=1}^{N}\sum_{i=1}^{n}\int_\Omega \Big(a_{i,\infty}^{(l)}(x,v(x),Dv(x)) - a_{i,\infty}^{(l)}(x,v_*(x),Dv_*(x))\Big)\\
&\qquad \times (D_i v^{(l)}(x) - D_i v_*^{(l)}(x))\,dx\\
&\quad + \sum_{l=1}^{N}\int_\Omega \Big(a_{0,\infty}^{(l)}(x,v(x),Dv(x)) - a_{0,\infty}^{(l)}(x,v_*(x),Dv_*(x))\Big)\\
&\qquad \times (v^{(l)}(x) - v_*^{(l)}(x))\,dx\\
&\ge c_5\int_\Omega (|v(x)-v_*(x)|^p + |Dv(x)-Dv_*(x)|^p)\,dx.
\end{aligned}$$

The above inequality reads in abstract formulation as

$$\langle \mathcal{A}_\infty(v) - \mathcal{A}_\infty(v_*), v-v_*\rangle \ge c_5\|v-v_*\|_V^p \tag{2.60}$$

for arbitrary $v$, $v_* \in V$, i.e., $\mathcal{A}_\infty$ is uniformly monotone.

The coerciveness follows from the uniform monotonicity. Indeed, by choosing $v_* = 0$ in the above (2.60) inequality, it follows

$$\langle \mathcal{A}_\infty(v) - \mathcal{A}_\infty(0), v\rangle \ge c_5\|v\|_V^p,$$

hence

$$\frac{\langle \mathcal{A}_\infty(v), v\rangle}{\|v\|_V} \ge c_5\|v\|_V^{p-1} - \frac{\langle \mathcal{A}_\infty(0), v\rangle}{\|v\|_V} \ge c_5\|v\|_V^{p-1} - \|\mathcal{A}_\infty(0)\|_{V^*}.$$

Observe that the right hand side of the above inequality tends to $+\infty$ as $t\to+\infty$ due to $p>1$. The proof of the lemma is complete. $\square$

*Proof of Theorem 2.18.* By Lemma 2.19 the conditions of Theorem 1.61 are satisfied thus there exists a unique $u_\infty \in V$ such that $\mathcal{A}_\infty(u_\infty) = \mathcal{F}_\infty$. Let $u$ be a solution of problem (2.35) (with arbitrary initial condition) and let $y(t) := \int_\Omega |u(t,x) - u_\infty|^2\,dx$. Observe that condition (F2) implies (F1*) so the conditions of Theorem 2.15 are fulfilled therefore $u \in L^\infty(0,\infty;(L^2(\Omega))^N)$ hence $y$ is bounded as well. Note that $y$ is also continuous that can be readily verified by using the fact $u \in C([0,T],(L^2(\Omega))^N)$. In the sequel we proceed similarly as in the proof of Theorem 2.15, we verify an integral inequality for $y$.

The facts that $u$ is a solution of (2.35) and $\mathcal{A}_\infty(u_\infty) = \mathcal{F}_\infty$ together yield

$$D_t(u(t) - u_\infty) + [\mathcal{A}(u)](t) - \mathcal{A}_\infty(u_\infty) = \mathcal{F}(t) - \mathcal{F}_\infty$$

for a.a. $t \in (0,\infty)$. One applies both sides of the above equation to $(u(t) - u_\infty)$ then it follows

$$\begin{aligned} &\langle D_t(u(t) - u_\infty), u(t) - u_\infty\rangle + \langle [\mathcal{A}(u)](t) - \mathcal{A}_\infty(u_\infty), u(t) - u_\infty\rangle \\ &= \langle \mathcal{F}(t) - \mathcal{F}_\infty, u(t) - u_\infty\rangle. \end{aligned} \tag{2.61}$$

The first term on the left hand side is $y'(t)$. Further, let us divide the second term into two terms by the following way

$$\begin{aligned} \langle [\mathcal{A}(u)](t) - \mathcal{A}_\infty(u_\infty), u(t) - u_\infty\rangle = \langle [\mathcal{A}(u)](t) - [\mathcal{A}_u(u_\infty)](t), u(t) - u_\infty\rangle \\ + \langle [\mathcal{A}_u(u_\infty)](t) - \mathcal{A}_\infty(u_\infty), u(t) - u_\infty\rangle. \end{aligned} \tag{2.62}$$

where for fixed $w \in L^p_{\text{loc}}(0,\infty;V)$ and $t > 0$, functional $[\mathcal{A}_w(\cdot)](t)\colon L^p_{\text{loc}}(0,\infty;V) \to L^q_{\text{loc}}(0,\infty;V^*)$ is given by

$$\begin{aligned} &\langle [\mathcal{A}_w(v)](t), z\rangle \\ &:= \sum_{l=1}^N \int_\Omega \sum_{i=1}^n a_i^{(l)}(t,x,v(t,x),Dv(t,x);w) D_i z^{(l)}(t,x)\,dx \\ &\quad + \sum_{l=1}^N \int_\Omega \sum_{i=1}^n a_0^{(l)}(t,x,v(t,x),Dv(t,x);w) z^{(l)}(t,x)\,dx, \end{aligned}$$

with $v \in L^p_{\text{loc}}(0,\infty;V)$, $z \in L^q_{\text{loc}}(0,\infty;V^*)$. The first term on the right hand side of the above equation may be estimated from below by using the uniform monotonicity of $\mathcal{A}_\infty$, then one obtains

$$\begin{aligned} &\langle [\mathcal{A}(u)](t) - [\mathcal{A}_u(u_\infty)](t), u(t) - u_\infty\rangle \\ &\geq c_5\|u(t) - u_\infty\|_V^p - \int_\Omega k_3(t,x,u(t,x),u_\infty(x);u)\,dx. \end{aligned} \tag{2.63}$$

Further, by estimating from above the second term on the right hand side of (2.62) by the $\varepsilon$-inequality it follows

$$\begin{aligned} &|\langle [\mathcal{A}_u(u_\infty)](t) - \mathcal{A}_\infty(u_\infty), u(t) - u_\infty\rangle| \\ &\leq \frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p + \frac{1}{q\varepsilon^q}\|[\mathcal{A}_u(u_\infty)](t) - \mathcal{A}_\infty(u_\infty)\|_{V^*}^q. \end{aligned} \tag{2.64}$$

Finally, for the right hand side of (2.61) the $\varepsilon$-inequality implies

$$|\langle \mathcal{F}(t) - \mathcal{F}_\infty, u(t) - u_\infty\rangle| \leq \frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p + \frac{1}{q\varepsilon^q}\|\mathcal{F}(t) - \mathcal{F}_\infty\|_{V^*}^q. \tag{2.65}$$

Now choose $\varepsilon > 0$ sufficiently small (in fact, $\frac{\varepsilon^p}{p} < \frac{c_5}{3}$) then by substituting estimates (2.63), (2.64), (2.65) into (2.61) we conclude

$$\begin{aligned} y'(t) + \frac{c_5}{3}\|u(t) - u_\infty\|_V^p \leq \text{const} \cdot \|[\mathcal{A}_u(u_\infty)](t) - \mathcal{A}_\infty(u_\infty)\|_{V^*}^q \\ + \text{const} \cdot \|\mathcal{F}(t) - \mathcal{F}_\infty\|_{V^*}^q \\ + \int_\Omega k_3(t, x, u(t,x), u_\infty(x); u)\,dx. \end{aligned} \tag{2.66}$$

We claim that the right hand side of the above inequality tends to 0 as $t \to \infty$. Indeed, the convergence of the second term is clear. Further, the third term tends to 0 by condition (2.55). In addition, Hölder's inequality implies the following upper estimate of the third term:

$$\begin{aligned} &\|[\mathcal{A}_u(u_\infty)](t) - \mathcal{A}_\infty(u_\infty)\|_{V^*} \\ &\leq \sum_{l=1}^N \sum_{i=0}^n \left(\int_\Omega \left|a_i^{(l)}(t, x, u_\infty(x), Du_\infty(x), u) - a_{i,\infty}^{(l)}(x, u_\infty(x), Du_\infty(x))\right|^q dx\right)^{\frac{1}{q}}. \end{aligned}$$

The integrands on the right hand side of the above estime converge pointwise to 0 by (2.53), moreover, due to (2.58) and (2.57), they have integrable majorants. Thus Lebesgue's theorem yields

$$\lim_{t\to\infty} \|[\mathcal{A}_u(u_\infty)](t) - \mathcal{A}_\infty(u_\infty)\|_{V^*}^q = 0.$$

So we have proved that

$$y'(t) + \frac{c_5}{3}\|u(t) - u_\infty\|_V^p \leq \phi(t)$$

where $c > 0$ and $\phi(t) \to 0$ as $t \to +\infty$. Note that the embedding $(W^{1,p}(\Omega))^N \hookrightarrow (L^2(\Omega))^N$ is continuous thus it follows with some constant $c > 0$ that

$$y'(t) + c \cdot y(t)^{\frac{p}{2}} \leq \phi(t). \tag{2.67}$$

We show that the above inequality implies $y(t) \to 0$ as $t \to \infty$. We proceed similarly as in the proof of Theorem 2.15, we prove by contradiction. Suppose that there exists a nonnegative sequence $t_k \to \infty$ and $\varepsilon > 0$ such that $y(t_k) > \varepsilon$. Then the set $M = \{t \in \mathbb{R}^+ : y(t) > \varepsilon\}$ is non-empty and it has arbitrary large elements. On the other hand, it is an open set by the continuity of $y$ so there exist countably many open intervals $(a_k, b_k)$ such that $\cup_{k=1}^{\infty}(a_k, b_k) = M$. By the continuity of $y$, $y(a_k) = y(b_k) = \varepsilon$ for every $k$. Thus by integrating (2.67) on $(a_k, b_k)$ it follows

$$c\varepsilon^{\frac{p}{2}}(a_k - b_k) \leq c\int_{a_k}^{b_k} y(s)^{\frac{p}{2}}\,ds \leq \int_{a_k}^{b_k} \phi(s)\,ds \leq \|\phi\|_{L^\infty(a_k,b_k)}(a_k - b_k)$$

that is a contradiction since $\phi(t) \to 0$ as $t \to \infty$. $\square$

*Remark* 2.20. Since operator $\mathcal{A}_\infty$ is uniformly monotone, Proposition 1.62 implies that $u_\infty$ depends continuously on $\mathcal{F}_\infty$.

One may study the "speed" of the above convergences. We pose concrete formulae on the convergences in conditions (A6), (A7), (F2), namely,

(Est) There exist constants $k^* \geq 0$, $\beta > 1$ such that

$$\|a_i^{(l)}(t,\cdot,u(\cdot),Du(\cdot);v) - a_{i,\infty}^{(l)}(\cdot,u(\cdot),Du(\cdot))\|_{L^q(\Omega)}^q \leq k^* t^{-\beta}, \tag{2.68}$$

for a.a. $t \in (0,\infty)$ and every $u \in V$, $v \in L^\infty(0,\infty;(L^2(\Omega))^N)$ $(i = 0,\dots,n; l = 1,\dots,N)$,

$$\int_\Omega |k_3(t,x,u(t,x),\tilde{u}(t,x);v)|\,dx \leq k^* t^{-\beta}, \tag{2.69}$$

for a.a. $t \in (0,\infty)$ and every $u, \tilde{u}, v \in L^\infty(0,\infty;(L^2(\Omega))^N)$,

$$\|\mathcal{F}(t) - \mathcal{F}_\infty\|_{V^*}^q \leq k^* t^{-\beta}. \tag{2.70}$$

**Proposition 2.21.** *Assume (Vol). In addition (A1)–(A3), (A5) hold for every $0 < T < \infty$, further, assumptions (A2$^+$), (A4$^*$), (A6), (A7), (F2) are satisfied with further assumption (Est). Then for the solutions $u, u_\infty$ formulated in Theorem 2.18, $y(t) = \int_\Omega |u(t,x) - u_\infty(x)|^2\,dx$ has the asymptotics*

$$\int_t^\infty y(s)^\alpha\,ds \leq \text{const} \cdot t^{\frac{1}{1-\alpha}}$$

*for $t > 0$ sufficiently large where*

$$\alpha = \max\left\{\frac{p}{2}, 1 + \frac{1}{\beta - 1}\right\}. \tag{2.71}$$

*Proof.* Our starting equation is (2.66). Clearly, assumption (Est) and the continuous embedding $(W^{1,p}(\Omega))^N \hookrightarrow (L^2(\Omega))^N$ imply (with some constant $c^* > 0$)

$$y'(t) + c^* \cdot y(t)^{\frac{p}{2}} \leq \text{const} \cdot t^{-\beta}.$$

By using the fact $y(t) \to 0$ as $t \to \infty$ that was proved in the previous theorem, integrating the above inequality in the interval $(t, \infty)$ (with $t$ sufficiently large), we obtain

$$c^* \int_t^\infty y(s)^\alpha \, ds \leq c^* \int_t^\infty y(s)^{\frac{p}{2}} \, ds \leq \text{const} \cdot \frac{t^{-\beta+1}}{\beta - 1} + y(t).$$

Now denote $g(t) = t^{-\beta+1}$. Observe that $\alpha \geq 1 + \dfrac{1}{\beta - 1}$ implies for $t \geq 1$ that

$$\int_t^\infty g(s)^\alpha \, ds = \frac{t^{\alpha(-\beta+1)+1}}{\alpha(\beta - 1) - 1} \leq \frac{g(t)}{\alpha(\beta - 1) - 1}.$$

Thus

$$\begin{aligned}\int_t^\infty (y(s) + g(s))^\alpha \, ds &\leq \text{const} \int_t^\infty y(s)^\alpha \, ds + \text{const} \int_t^\infty g(s)^\alpha \, ds \\ &\leq \text{const} \cdot (y(t) + g(t)).\end{aligned}$$

Put

$$h(t) = \int_t^\infty (y(s) + g(s))^\alpha \, ds$$

then $h'(t) = -(y(t) + g(t))^\alpha$ whence $h(t)^\alpha \leq -\tilde{c} \cdot h'(t)$ for some constant $\tilde{c} > 0$. Consequently,

$$h(t) \leq \text{const} \cdot t^{\frac{1}{1-\alpha}}$$

so by the nonnegativity of function $g$ we conclude

$$\int_t^\infty y(t)^\alpha \leq \text{const} \cdot t^{\frac{1}{1-\alpha}}.$$

□

### 2.3.4 Examples

In this part we show some examples which fulfil the conditions of the preceding theorems.

**Case of Theorem 2.12**

Suppose that functions $a_i^{(l)}\colon Q_\infty \times \mathbb{R}^{(n+1)N} \times L^p_{\text{loc}}(0,\infty;V) \to \mathbb{R}$ have the form (2.22)–(2.23), i.e.,

$$\begin{aligned} & a_i^{(l)}(t,x,\zeta_0,\zeta;v) \\ & = [H^{(l)}(v)](t,x)b_i^{(l)}(t,x,\zeta_0,\zeta) + [G^{(l)}(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta) \ \text{ if } \ (i \neq 0), \end{aligned} \tag{2.72}$$

$$\begin{aligned} & a_0^{(l)}(t,x,\zeta_0,\zeta;v) \\ & = [H^{(l)}(v)](t,x)b_0^{(l)}(t,x,\zeta_0,\zeta) + [G_0^{(l)}(v)](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta). \end{aligned} \tag{2.73}$$

Assume that functions $b_i^{(l)}, d_i^{(l)}$ $(i = 0,\dots,n; l = 1,\dots,N)$ satisfy conditions (K1)–(K4) for all $0 < T < \infty$ (in the same sense as for functions $a_i^{(l)}$ mentioned before Theorem 2.12). Further, operators

$$H^{(l)}\colon L^p_{\text{loc}}\big(0,\infty;(L^p(\Omega))^N\big) \to L^\infty_{\text{loc}}(Q_\infty),$$
$$G^{(l)}, G_0^{(l)}\colon L^p_{\text{loc}}\big(0,\infty;(L^p(\Omega))^N\big) \to L^{\frac{p}{p-1-r}}_{\text{loc}}(Q_\infty)$$

are of Volterra type. The restrictions $H^{(l)}(v)|_{L^p(0,T;(L^p(\Omega))^N)}\colon L^p(0,T;(L^p(\Omega))^N) \to L^\infty(Q_T)$, $G^{(l)}|_{L^p(0,T;(L^p(\Omega))^N)}, G_0^{(l)}|_{L^p(0,T;(L^p(\Omega))^N)}\colon L^p(0,T;(L^p(\Omega))^N) \to L^{\frac{p}{p-r-1}}(Q_T)$ are bounded and continuous for every $0 < T < \infty$ (where $r$ is given in (K2)). Finally, $[H^{(l)}(v)](t,x) \geq c_3$, $[G^{(l)}(v)](t,x) \geq 0$ for a.a. $(t,x) \in Q_\infty$ and (2.24) holds for every $0 < T < \infty$. Then one can easily see that the above functions (2.72)–(2.73) satisfy the conditions of Theorem 2.35. By extending the concrete examples for $H^{(l)}, G^{(l)}, G_0^{(l)}, b_i^{(l)}, d_i^{(l)}$ given in Section 2.2.3 to all $t \in (0,\infty)$ (from $t \in (0,T)$) they will satisfy the above conditions. E.g., define the following operators on $L^p(0,T;(L^p(\Omega))^N)$:

$$[\tilde{H}(v)](t,x) := \Phi\left(\int_{Q_t} \sum_{j=1}^N b_j v^{(j)}\right),$$
$$[\tilde{G}(v)](t,x) := \psi\left(\left[\int_0^t |v(\tau,x)|^\alpha d\tau\right]^{\frac{1}{\alpha}}\right)$$

where $1 \leq \alpha \leq p$, $b_j \in L^q(Q_T)$ $(1 \leq j \leq N)$, further, $\Phi, \psi\colon \mathbb{R} \to \mathbb{R}$ are continuous functions such that $\Phi \geq c > 0$, $|\psi(y)| \leq \tilde{c} \cdot |y|^{p-r_0-1}$ holds for some constants $\tilde{c}$ and $0 \leq r_0 < p-1$.

**Case of Theorem 2.13**

Let $T \geq a$. Assume that functions $a_i^{(l)} \colon Q_\infty \times \Omega \times \mathbb{R}^{(n+1)N} \times L^p_{\text{loc}}(-a, \infty; V) \to \mathbb{R}$ have the form (2.22)–(2.23), i.e.,

$$\begin{aligned} &a_i^{(l)}(t, x, \zeta_0, \zeta; v) \\ &= [H^{(l)}(v)](x) b_i^{(l)}(t, x, \zeta_0, \zeta) + [G^{(l)}(v)](x) d_i^{(l)}(t, x, \zeta_0, \zeta) \text{ if } (i \neq 0), \end{aligned} \tag{2.74}$$

$$a_0^{(l)}(t, x, \zeta_0, \zeta; v) = [H^{(l)}(v)](x) b_0^{(l)}(t, x, \zeta_0, \zeta) + d_0^{(l)}(t, x, \zeta_0, \zeta). \tag{2.75}$$

Suppose that (the restrictions of) functions $b_i^{(l)}, d_i^{(l)}$ $(i = 0, \dots, n; l = 1, \dots, N)$ satisfy conditions (K1)–(K4) in $(0, T)$ and they are $T$-periodic, (i.e., $b_i^{(l)}(t, x, \zeta_0, \zeta) = b_i^{(l)}(t + T, x, \zeta_0, \zeta)$ for $t \in (-a, \infty)$ and similarly for $d_i^{(l)}$). Further, operators

$$\begin{aligned} &H^{(l)} \colon L^p\big(-a, 0; (L^p(\Omega))^N\big) \to L^\infty(Q_T), \\ &G^{(l)} \colon L^p\big(-a, 0; (L^p(\Omega))^N\big) \to L^{\frac{p}{p-1-r}}(Q_T) \end{aligned}$$

are bounded and continuous (where $r$ is given in (K2)) and $[H^{(l)}(v)](t, x) \geq c_3$, $[G^{(l)}(v)](t, x) \geq 0$ for a.a. $(t, x) \in Q_\infty$. Then one can easily see that the above functions (2.74)–(2.75) satisfy the conditions of Theorem 2.13. For such operators see Section 2.2.3. For periodic $b_i^{(l)}$ consider, e.g., functions

$$\begin{aligned} &b_i^{(l)}(t, x, \zeta_0, \zeta) = k(t, x) \zeta_i^{(l)} |\zeta|^{p-2} \quad (i \neq 0), \\ &b_0^{(l)}(t, x, \zeta_0, \zeta) = k(t, x) \zeta_0^{(l)} |\zeta_0|^{p-2}, \end{aligned}$$

and similarly for functions $d_i^{(l)}$ by replacing the exponent $p - 2$ with $r - 1$ where $k$ is a $T$-periodic function in $L^\infty$).

**Case of Theorem 2.15**

Consider functions (2.72)–(2.73). By using our earlier investigations on these functions, see estimate (2.31), we have

$$\begin{aligned} &\sum_{l=1}^{N} \sum_{i=0}^{n} a_i^{(l)}(t, x, \zeta_0, \zeta; v) \zeta_i^{(l)} \\ &\geq \frac{c'}{3} (|\zeta_0|^p + |\zeta|^p) - (c_3 N k_2(x) + N d^* |[G_0(v)](t, x)|^{q_1 q}). \end{aligned}$$

with some positive constants $c_3', c_3$ and $G_0 = \sum_{l=1}^{N} G_0^{(l)}$. Put

$$h(v) := c_3 N k_2(x) + N d^* |[G_0(v)](t, x)|^{q_1 q}$$

then

$$\|h(v)\|_{L^1(Q_T)} \leq c_3 N \|k_2\|_{L^1(\Omega)} + N d^* \int_{Q_T} |[G_0(v)](t, x)|^{\frac{p}{p-r-1}} \, dt dx.$$

We showed that this implies (3.8) by (2.24). Now assume that there exist a constant $c_4 > 0$ and a function $\varphi\colon (0,\infty) \to \mathbb{R}$ such that $\lim_{\tau\to\infty} \varphi(\tau) = 0$ and if $v \in L^p_{\text{loc}}(0,\infty;V)$ then for a.a. $(t \in (0,\infty)$

$$\int_\Omega |[G_0^{(l)}(v)](t,x)|^{\frac{p}{p-1-r}}\,dx \le c_4 \left( \sup_{\tau\in[0,t]} \|v(\tau)\|_H^{p_1} + \varphi(t) \sup_{\tau\in[0,t]} \|v(\tau)\|_H^p + 1 \right). \tag{2.76}$$

Then it is clear that condition (A4*) is satisfied. In the following we give some examples fulfilling the above condition (2.76).

Let the continuous functions $\psi, \chi, \varphi\colon [0,\infty) \to \mathbb{R}$ be such that $|\psi(\tau)| \le \text{const} \cdot |\tau|^{p-1-r_0}$, $|\chi(\tau)| \le \text{const} \cdot |\tau|^{p-1-r}$ and $\lim_{\tau\to\infty} \varphi(\tau) = 0$ where $0 \le r < r_0 < p-1$. Then consider operators defined on $L^p_{\text{loc}}\left(0,T;(L^p(\Omega))^N\right)$ by

$$[\tilde{G}_1(v)](t,x) := \psi\left( \left| \int_\Omega \sum_{j=1}^N a_j(t,\xi)|v^{(j)}(t,\xi)|^\alpha d\xi \right|^{\frac{1}{\alpha}} \right),$$

$$[\tilde{G}_2(v)](t,x) := \varphi(t)\chi\left( \left| \int_\Omega \sum_{j=1}^N a_j(t,\xi)|v^{(j)}(t,\xi)|^2 d\xi \right|^{\frac{1}{2}} \right)$$

where $a_j \in L^\infty(Q_\infty)$ $(1 \le j \le N)$, $0 < \alpha \le 2$.

**Proposition 2.22.** *The above $\tilde{G}_1$, $\tilde{G}_2$ have the property* (2.76).

*Proof.* First consider operator $\tilde{G}_1$. It is clear that

$$\begin{aligned} |[\tilde{G}_1(v)](t,x)|^{\frac{p}{p-1-r}} &\le \text{const} \cdot \left( \sum_{j=1}^N \int_\Omega \|a_j\|_{L^\infty(Q_\infty)} |v^{(j)}(t,\xi)|^\alpha d\xi \right)^{\frac{p\lambda}{\alpha}} \\ &\le \text{const} \cdot \left( \sum_{j=1}^N \int_\Omega |v^{(j)}(t,\xi)|^\alpha d\xi \right)^{\frac{p\lambda}{\alpha}} \\ &= \text{const} \cdot \left( \int_\Omega |v(t,\xi)|^\alpha d\xi \right)^{\frac{p\lambda}{\alpha}}, \end{aligned}$$

where $0 < \lambda = \dfrac{p-1-r_0}{p-1-r} < 1$. By applying Hölder's inequality it follows

$$\int_\Omega |v(t,x)|^\alpha\,dx \le \left( \int_\Omega |v(t,x)|^{\alpha\frac{2}{\alpha}}\,dx \right)^{\frac{\alpha}{2}} \cdot \left( \int_\Omega 1 \right)^{\frac{2-\alpha}{2}} = \text{const} \cdot \|v(t)\|_H^\alpha.$$

(In case $\alpha = 2$ the above inequality is obvious.) Thus

$$[\tilde{G}_1(v)](t,x)|^{\frac{p}{p-1-r}} \le \text{const} \cdot \|v(t)\|_H^{p\lambda}$$

hence

$$\int_\Omega [\tilde{G}_1(v)](t,x)|^{\frac{p}{p-1-r}}\,dx \le \text{const} \cdot \|v(t)\|_H^{p\lambda} \le \text{const} \cdot \sup_{\tau\in[0,t]} \|v(\tau)\|_H^{p\lambda}.$$

This means that operator $\tilde{G}_1$ have the property (2.76) with $p_1 = p\lambda$.

The case of $\tilde{G}_2$ can be treated similarly. □

**Case of Theorem 2.18**

Let the functions in (2.72)–(2.73) have the form

$$b_i^{(l)}(t,x,\zeta_0,\zeta) := \zeta_i^{(l)}|\zeta^{(l)}|^{p-2},\ (i \neq 0), \tag{2.77}$$
$$b_0^{(l)}(t,x,\zeta_0,\zeta) := \zeta_0^{(l)}|\zeta_0^{(l)}|^{p-2}, \tag{2.78}$$
$$d_i^{(l)}(t,x,\zeta_0,\zeta) :\equiv 0, \quad (i=1,\dots,n); l=1,\dots,N) \tag{2.79}$$
$$d_0^{(l)}(t,x,\zeta_0,\zeta) :\equiv 1 \quad (l=1,\dots,N) \tag{2.80}$$

In addition, define the following operators on $L^p_{\mathrm{loc}}(0,T;(L^p(\Omega))^N)$:

$$[H^{(l)}(v)](t,x) := k(x), \tag{2.81}$$
$$[G_0^{(l)}(v)](t,x) := \varphi(t)\cdot\chi\left(\left[\int_\Omega \sum_{j=1}^N a_j(t,\xi)|v^{(j)}(t,\xi)|^2 d\xi\right]^{\frac{1}{2}}\right) \tag{2.82}$$

where $k \in L^\infty(\Omega)$ such that $k(x) \geq c^* > 0$, further, $a_j \in L^\infty(Q_\infty)$ $(1 \leq j \leq N)$, $\varphi$, $\chi\colon [0,\infty) \to \mathbb{R}$ are nonnegative functions such that $\lim_{\tau\to\infty}\varphi(\tau) = 0$, $\chi(\tau) \leq \mathrm{const}\cdot|\tau|^{p-1}$. (Due to (2.80) we do not need operators $G^{(l)}$.)

Now we show that these functions satisfy the conditions of Theorem 2.18. Obviously, conditions (A1$^*$) holds with $c_v = 1$ and $k_v = k$, further, (A6) is fulfilled due to (2.82) since the second factor of the product on the right hand side is bounded. Moreover, (2.58) holds, too, since $H^{(l)}(v), G^{(l)}(v) \in L^\infty(Q_\infty)$ for every $L^p_{\mathrm{loc}}(0,T;(L^p(\Omega))^N)$. Thus $a_i^{(l)}$ $(i=0,\dots,n)$ can be estimated from above by $\mathrm{const}\cdot\left(|\zeta_0^{(l)}|^{p-1} + |\zeta^{(l)}|^{p-1}\right)$. Furthermore, it is obvious that $a_{i,\infty}^{(l)} = k \cdot b_i^{(l)}$ for $i = 0,\dots,n$.

Property (2.54) follows from Proposition 1.57. Indeed,

$$\begin{aligned}
&\sum_{l=1}^N \sum_{i=0}^n \left(a_i^{(l)}(t,x,\zeta_0,\zeta;v) - a_i^{(l)}(t,x,\eta_0,\eta;v)\right)(\zeta_i^{(l)} - \eta_i^{(l)}) \\
&= \sum_{l=1}^N [H^{(l)}(v)](t,x) \sum_{i=1}^n \left(\zeta_i^{(l)}|\zeta^{(l)}|^{p-2} - \eta^{(l)}|\eta_i^{(l)}|^{p-2}\right)(\zeta_i^{(l)} - \eta_i^{(l)}) \\
&\quad + \sum_{l=1}^N [H^{(l)}(v)](t,x)\left(\zeta_0^{(l)}|\zeta_0^{(l)}|^{p-2} - \eta_0^{(l)}|\eta_0^{(l)}|^{p-2}\right) \\
&\quad + \sum_{l=1}^N [G_0^{(l)}(v)](t,x)(\zeta_0^{(l)} - \eta_0^{(l)}) \\
&\geq c^*\cdot\sum_{l=1}^N \left(|\zeta^{(l)} - \eta^{(l)}|^p + |\zeta_0^{(l)} - \eta_0^{(l)}|^p\right) + \sum_{l=1}^N [G_0^{(l)}(v)](t,x)(\zeta_0^{(l)} - \eta_0^{(l)}).
\end{aligned} \tag{2.83}$$

The second term on right hand side of the above relation can be estimated from above by the $\varepsilon$-inequality, if $\varepsilon > 0$ is small enough (especially, $\frac{\varepsilon^p}{p} \leq \frac{c^*}{2N}$) then by using the estimate in the proof of Proposition 2.22 we may deduce

$$
\begin{aligned}
&|[G_0^{(l)}(v)](t,x)(\zeta_0^{(l)}-\eta_0^{(l)})| \\
&\le \frac{\varepsilon^p}{p}|\zeta_0^{(l)}-\eta_0^{(l)}|^p + \frac{1}{\varepsilon^q q}|[G_0^{(l)}(v)](t,x)|^q \\
&\le \frac{c^*}{2N}|\zeta_0^{(l)}-\eta_0^{(l)}|^p + c'\cdot|\varphi(t)|^q \sup_{\tau\in[0,t]}\left(\int_\Omega |v(\tau,x)|^2\,dx\right)^{\frac{p}{2}}.
\end{aligned}
\tag{2.84}
$$

Let

$$
\frac{c'}{N}\cdot|\varphi(t)|^q \sup_{\tau\in[0,t]}\left(\int_\Omega |v(\tau,x)|^2\,dx\right)^{\frac{p}{2}} := k_3(t,x,\zeta_0,\eta_0;v) = k_3(x;v). \tag{2.85}
$$

Then by applying (2.84) we may estimate from below the left hand side of (2.83) as follows

$$
\begin{aligned}
&c^*\cdot\sum_{l=1}^N\left(|\zeta^{(l)}-\eta^{(l)}|^p + |\zeta_0^{(l)}-\eta_0^{(l)}|^p\right) + \sum_{l=1}^N [G_0^{(l)}(v)](t,x)(\zeta_0^{(l)}-\eta_0^{(l)}) \\
&\ge \frac{c^*}{2}\cdot\sum_{l=1}^N\left(|\zeta^{(l)}-\eta^{(l)}|^p + |\zeta_0^{(l)}-\eta_0^{(l)}|^p\right) + k_3(t;v) \\
&\ge \tilde{c}\cdot(|\zeta-\eta|^p + |\zeta_0-\eta_0|^p) + k_3(t;v).
\end{aligned}
\tag{2.86}
$$

Since $\lim_{t\to\infty}\varphi(t)=0$, $\lim_{t\to\infty}\int_\Omega k_3(t;v)\,dx = 0$ if $v\in L^\infty(0,\infty;(L^2(\Omega))^N)$. So condition (A7) also holds.

### Case of Proposition 2.21

We repeat the example of the previous section and we add further assumptions on them. So let the functions in (2.72)–(2.73) have the form

$$
b_i^{(l)}(t,x,\zeta_0,\zeta) := \zeta_i^{(l)}|\zeta^{(l)}|^{p-2},\ (i\neq 0), \tag{2.87}
$$

$$
b_0^{(l)}(t,x,\zeta_0,\zeta) := \zeta_0^{(l)}|\zeta_0^{(l)}|^{p-2}, \tag{2.88}
$$

$$
d_i^{(l)}(t,x,\zeta_0,\zeta) :\equiv 0, \quad (i=1,\dots,n); l=1,\dots,N) \tag{2.89}
$$

$$
d_0^{(l)}(t,x,\zeta_0,\zeta) :\equiv 1 \quad (l=1,\dots,N) \tag{2.90}
$$

In addition, define the following operators on $L^p_{\mathrm{loc}}(0,T;(L^p(\Omega))^N)$:

$$
[H^{(l)}(v)](t,x) := k(x), \tag{2.91}
$$

$$
[G_0^{(l)}(v)](t,x) := \varphi(t)\cdot\chi\left(\left[\int_\Omega \sum_{j=1}^N a_j(t,\xi)|v^{(j)}(t,\xi)|^2 d\xi\right]^{\frac{1}{2}}\right) \tag{2.92}
$$

where $k\in L^\infty(\Omega)$ such that $k(x)\ge c^*>0$, further, $a_j\in L^\infty(Q_\infty)$ $(1\le j\le N)$, $\varphi,\ \chi\colon [0,\infty)\to\mathbb{R}$ are nonnegative functions such that $\varphi(\tau)=\mathrm{const}\cdot\tau^{-\beta}$, $\chi(\tau)\le \mathrm{const}\cdot|\tau|^{p-1}$. (Due to (2.80) we do not need operators $G^{(l)}$.)

We show that the above functions satisfy condition (2.68)–(2.69). Obviously, (2.68) holds for $i > 0$, further, if $u, v \in L^\infty(0,\infty;(L^2(\Omega)^N)$ then

$$|[G_0^{(l)}(v)](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)| \le t^{-\beta q} \cdot \|v(t,\cdot)\|_{(L^2(\Omega))^N}^{p-1} \le \text{const} \cdot t^{-\beta q}$$

thus (due to $q > 1$)

$$\|[G_0^{(l)}(v)](t,\cdot)d_0^{(l)}(t,\cdot;v)\|_{L^q(\Omega)}^q \le \text{const} \cdot t^{-\beta q} \le \text{const} \cdot t^{-\beta}$$

so that (2.68) holds also in case $i = 0$.

Now we may repeat the deduction of (2.84), (2.85), (2.86) and we obtain that function $k_3$ included in condition (2.69) may be chosen as follows:

$$k_3(x;v) := k_3(t,x,\zeta_0,\eta_0;v) := \frac{c'}{N} \cdot |\varphi(t)|^q \sup_{\tau\in[0,t]} \left( \int_\Omega |v(\tau,x)|^2 \, dx \right)^{\frac{p}{2}}.$$

Whence

$$\int_\Omega |k_3(x;v)| \, dx \le \text{const} \cdot t^{-\beta} \le \text{const} \cdot t^{-\beta}$$

so condition (2.69) is also satisfied.

*Remark* 2.23. Generally, condition (2.68) is satisfied, e.g., if

$$|a_i^{(l)}(t,x,\zeta_0,\zeta;v) - a_{i,\infty}^{(l)}(x,\zeta_0,\zeta)| \le \Phi(t)(|\zeta_0|^{p-1} + |\zeta|^{p-1})$$

for every $v \in L^\infty(0,\infty;(L^2(\Omega))^N)$ where $\Phi(t) \le \text{const} \cdot t^{-\frac{\beta}{q}}$.

Condition (2.69) is fulfilled, e.g., in the following general case. Suppose that we have $a_i^{(l)}$ such that $a_0^{(l)} = \hat{a}_0^{(l)} + \bar{a}_0^{(l)}$ $(l = 1,\dots,N)$ and there exists a constant $c_5 > 0$ such that for a.a. $x \in \Omega$, every $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p_{\text{loc}}(0,\infty;V)$,

$$\begin{aligned}
&\sum_{l=1}^N \sum_{i=1}^n \left( a_i^{(l)}(t,x,\zeta_0,\zeta;v) - a_i^{(l)}(t,x,\eta_0,\eta;v) \right) (\zeta_i^{(l)} - \eta_i^{(l)}) \\
&\quad + \sum_{l=1}^N \left( \hat{a}_0^{(l)}(t,x,\zeta_0,\zeta;v) - \hat{a}_0^{(l)}(t,x,\eta_0,\eta;v) \right) (\zeta_0^{(l)} - \eta_0^{(l)}) \\
&\ge c_5 \left( |\zeta_0 - \eta_0|^p + |\zeta - \eta|^p \right).
\end{aligned} \tag{2.93}$$

Further, there is a continuous function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $|\psi(t)| \le \text{const} \cdot t^{-\beta}$ and

$$|\bar{a}_i^{(l)}(t,x,\zeta_0,\zeta;v)| \le \psi(t)(|\zeta_0| + 1) \tag{2.94}$$

for a.a. $(t,x) \in Q_\infty$ and every $(\zeta_0,\zeta) \in \mathbb{R}^{N+1}$, $v \in L^\infty(0,\infty;(L^2(\Omega))^N))$. Then (2.69) holds. Indeed, by (2.93),

$$\begin{aligned}
&\sum_{l=1}^N \sum_{i=0}^n \left( a_i^{(l)}(t,x,\zeta_0,\zeta;v) - a_i^{(l)}(t,x,\eta_0,\eta;v) \right) (\zeta_i^{(l)} - \eta_i^{(l)}) \\
&\ge c_5 \left( |\zeta_0 - \eta_0|^p + |\zeta - \eta|^p \right) \\
&\quad - \sum_{l=1}^N \left| \left( \bar{a}_0^{(l)}(t,x,\zeta_0,\zeta;v) - \bar{a}_0^{(l)}(t,x,\eta_0,\eta;v) \right) (\zeta_0^{(l)} - \eta_0^{(l)}) \right|.
\end{aligned}$$

Further, by (2.94) for $l = 1, \dots, N$,

$$
\begin{aligned}
&\left| \int_\Omega \left( \bar{a}_0^{(l)}(t, x, \zeta_0, \zeta; v) - \bar{a}_0^{(l)}(t, x, \tilde{\zeta}_0, D\tilde{\zeta}_0; v) \right) (\zeta_0 - \tilde{\zeta}_0) \right| \\
&\le \psi(t) \int_\Omega (|\zeta_0| + |\tilde{\zeta}_0| + 2)(|\zeta_0| + |\tilde{\zeta}_0|) \\
&\le \text{const} \cdot \psi(t) \int_\Omega (|\zeta|^2 + |\tilde{\zeta}_0|^2 + 1).
\end{aligned}
$$

So that function $k_3$ can be chosen as follows:

$$
k_3(t, x, \zeta_0, \eta_0; v) = \tilde{c}\psi(t)(\zeta_0^2 + \eta_0^2 + 1)
$$

with some positive constant $\tilde{c}$. Now for $u, \tilde{u} \in L^\infty(0, \infty; (L^2(\Omega))^N)$ it follows

$$
\begin{aligned}
&\int_\Omega |k_3(t, x, u(t, x), \tilde{u}(t, x); v)|\, dx \\
&\le \tilde{c}\psi(t)(1 + \|u(t, \cdot)\|^2_{(L^2(\Omega))^N} + \|\tilde{u}(t, \cdot)\|^2_{(L^2(\Omega))^N}) \\
&\le \text{const} \cdot t^{-\beta}
\end{aligned}
$$

so that (2.69) holds.

# Chapter 3

# A system containing three types of equations

If only I had the theorems! Then I should find the proofs easily enough.

Georg Friedrich Bernhard Riemann

## 3.1 Introduction

That sometimes clear... and sometimes vague stuff... which is... mathematics.

Imre Lakatos

This chapter is devoted to the investigation of a nonlinear system which consists of there different types of differential equations: an ordinary, a parabolic and an elliptic one. This kind of problem is motivated by a model of fluid flow in porous medium. A porous medium, roughly speaking, is a solid medium with lots of tiny holes. For example think of limestone. Such medium consists of two parts, the solid matrix and the holes. The flow of a fluid through the medium is influenced by the relatively large surface of the solid matrix and the closeness of the holes. If the fluid carries dissolved chemical species, a variety of chemical reactions can occur. Among these include reactions that can change the porosity. This process was modelled by J. Logan, M. R. Petersen, T. S. Shores in [46] by the following system of equations in one dimension:

$$\omega(t,x)D_t u(t,x) \\ = D_x(\alpha|v(t,x)|u_x(t,x)) + K(\omega(t,x))D_x p(t,x)u_x(t,x) - ku(t,x)g(\omega(t,x)) \tag{3.1}$$

$$D_t\omega(t,x) = bu(t,x)g(\omega(t,x)) \tag{3.2}$$

$$D_x(K(\omega(t,x))D_x p(t,x)) = bu(t,x)g(\omega(t,x)), \tag{3.3}$$

$$v(t,x) = -K(\omega(t,x))D_x p(t,x) \tag{3.4}$$

for $t > 0,\ x \in (0,1)$ with initial and boundary conditions

$$u(0,x) = u_0(x), \quad \omega(0,x) = \omega_0(x) \quad x \in (0,1),$$
$$u(t,0) = u_1(t), \quad D_x u(t,1) = 0 \quad t > 0,$$
$$p(t,0) = 1, \quad p(t,1) = 0 \quad t > 0$$

where $\omega$ is the porosity, $u$ is the concentration of the dissolved chemical solute carried by the fluid, $p$ is the pressure, $v$ is the velocity, further, $\alpha$, $k$, $b$ are given constants, $K$ and $g$ are given real functions. For the details of making this model and on flow in such media, see the monograph [7] and papers [23, 46]. Observe that $v$ is explicitly given by $\omega$ and $p$ in equation (3.4) thus we may eliminate equation (3.4) by substituting it into (3.1). Further, for fixed $u$ equation (3.2) is an ordinary differential equation with respect to the function $\omega$; for fixed $\omega$ and $p$ equation (3.1) is a parabolic problem with respect to the function $u$; and for fixed $\omega$ and $u$ equation (3.3) is an elliptic problem with respect to the function $p$.

This argument shows that the above system is a hybrid evolutionary/elliptic problem thus theorems of "classical" systems of partial differential equations do not work. In [23] a similar model was considered by using the method of Rothe, further, some numerical experiments were done, however correct proof on existence of solutions was not made (and one can hardly find papers dealing with such kind of systems in rigorous mathematical way).

In what follows, we investigate a generalization of this model where also the main parts may contain functional dependence on the unknown functions. We show existence and some properties of weak solutions by using the theory of operators of monotone type.

The main idea consists of two parts. First the choice of the appropriate spaces for the weak solutions (for the elliptic equation it will be not the usual space because of the time dependence). The second is the idea of the proof which is to apply the so-called successive approximation (known, e.g., from the theory of ordinary differential equations) and combine this with some methods of the theory of monotone operators that were demonstrated in the previous chapters. Especially, we lean on the results of Chapter 2. Finally, some examples are given. Most of the following part was published by the author in papers [10, 11, 14].

### 3.1.1 Notation

We introduce some further notation and for the convenience of the reader we repeat some earlier one of Section 2.1.1 that we shall use in the sequel.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary (for example, continuously differentiable is sufficient), further, let $0 < T < \infty$, $2 \le p_1, p_2 < \infty$ be real numbers. As before, we use the notation $Q_T := (0,T) \times \Omega$, $Q_\infty := (0,\infty) \times \Omega$ and the notion of the Sobolev space $W^{1,p_i}(\Omega)$ $(i = 1,2)$. In addition, let $V_i$ be a closed linear subspace of the space $W^{1,p_i}(\Omega)$ which contains $W_0^{1,p_i}(\Omega)$ and let $X_i := L^{p_i}(0,T;V_i)$. The pairing between $V_i^*$ and $V_i$, further, between $X_i^*$ and $X_i$ will be denoted by $\langle \cdot,\cdot \rangle$ and $[\cdot,\cdot]$, respectively, as before. As in the previous chapter we use the convention that a function $v \in L^p(0,T;V)$ can be considered also as a function with variables $(t,x)$ (however $v$ has only a time variable $t$).

### 3.1.2 Formulation of the problem

Let us consider the following system of equations:

$$D_t\omega(t,x) = f(t,x,\omega(t,x),u(t,x);u), \qquad \omega(0,x) = \omega_0(x), \tag{3.5}$$

$$\begin{aligned} &D_t u(t,x) \\ &- \sum_{i=1}^n D_i\left[a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})\right] \\ &+ a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) \\ &= g(t,x), \qquad u(0,x) = 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\sum_{i=1}^n D_i[b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})] \\ &+ b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) \\ &= h(t,x) \end{aligned} \tag{3.7}$$

with homogeneuos Dirichlet or Neumann type boundary condition (we may assume them to be homogeneuos by subtracting a suitable function). (The variable $\mathbf{p}$ is written by boldface letter for the purpose of distinguishing it from exponents $p_1, p_2$.) Moreover, if $\partial\Omega = S_1 \cup S_2$ where $S_1 \cap S_2 = \emptyset$ then we may pose different boundary conditions on the elements of the partition. That is the case in the model (3.1)–(3.4) where the partitions are the endpoints of the interval $[0,1]$.

Functions $a_i, b_i, f$ may contain nonlocal dependence on the unknown functions $\omega, u, \mathbf{p}$ which are written after the symbol ";". The above system is a generalization of the model (3.1)–(3.4). Indeed, as we mentioned in the introduction, $v$ can be eliminated form (3.1)–(3.4), further, in Proposition 3.5 we shall show that due to some assumptions the solution $\omega$ of equation (3.1) is strictly positive hence we can divide equation (3.1) by $\omega$. By using the observation that the above equations are three types of differential equations we pose natural conditions on functions $a_i$, $b_i$, $f$, $g$, $h$ to ensure existence of weak solutions to the above system.

### 3.1.3 Assumptions

In what follows, $\xi$, $(\zeta_0, \zeta)$, $(\eta_0, \eta)$ refer to the variables $\omega$, $(u, Du)$ and $(\mathbf{p}, D\mathbf{p})$, respectively, further, $w$, $v_1$ and $v_2$ to the nonlocal dependence on $\omega$, $u$ and $\mathbf{p}$.

(A1) For fixed $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ functions $a_i \colon Q_T \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}$ $(i = 0, \ldots, n)$ have the Carathéodory property, i.e., they are measurable in $(t, x) \in Q_T$ for every $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ for a.a. $(t, x) \in Q_T$.

(A2) There exists a continuous function $c_1 \colon \mathbb{R} \to \mathbb{R}^+$ and bounded operators $\mathfrak{c}_1 \colon L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}^+$, $k_1 \colon L^\infty(Q_T) \times X_1 \times X_2 \to L^{q_1}(Q_T)$ such that

$$\begin{aligned}&|a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2)| \\ &\leq \mathfrak{c}_1(w, v_1, v_2) c_1(\xi) \left( |\zeta_0|^{p_1 - 1} + |\zeta|^{p_1 - 1} + |\eta_0|^{\frac{p_2}{q_1}} + |\eta|^{\frac{p_2}{q_1}} + [k_1(w, v_1, v_2)](t, x) \right),\end{aligned}$$

for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ $(i = 0, \ldots, n)$.

(A3) There exists a positive constant $C$ such that for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0, \zeta, \eta_0, \eta)$, $(\xi, \zeta_0, \tilde{\zeta}, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$

$$\begin{aligned}&\sum_{i=1}^n \left( a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) - a_i(t, x, \xi, \zeta_0, \tilde{\zeta}, \eta_0, \eta; w, v_1, v_2) \right) (\zeta_i - \tilde{\zeta}_i) \\ &\geq C \cdot |\zeta - \tilde{\zeta}|^{p_1}.\end{aligned}$$

(A4) There exist a constant $c_2 > 0$, a continuous function $\gamma \colon \mathbb{R} \to \mathbb{R}$ and bounded operators $\Gamma \colon L^\infty(Q_T) \to L^\infty(Q_T)$, $k_2 \colon X_1 \to L^1(Q_T)$ such that

$$\begin{aligned}&\sum_{i=0}^n a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) \zeta_i \\ &\geq c_2 \left( |\zeta_0|^{p_1} + |\zeta|^{p_1} \right) - \gamma(\xi) [\Gamma(w)](t, x) [k_2(v_1)](t, x)\end{aligned}$$

for a.a. $(t,x) \in Q_T$ and every $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$. Further,

$$\lim_{\|v_1\|_{X_1} \to +\infty} \frac{\|k_2(v_1)\|_{L^1(Q_T)}}{\|v_1\|_{X_1}^{p_1}} = 0. \tag{3.8}$$

(A5) If $(\omega_k)$ is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in $Q_T$ and $u_k \to u$ weakly in $X_1$, strongly in $L^{p_1}(Q_T)$, further, $\mathbf{p}_k \to \mathbf{p}$ strongly in $X_2$ then

$$a_i(\cdot, \omega_k, u_k, Du_k, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_k, \mathbf{p}_k) - a_i(\cdot, \omega_k, u_k, Du_k, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \to 0$$

in $L^{q_1}(Q_T)$.

(B1) For fixed $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ functions $b_i \colon Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}$ $(i = 0, \dots, n)$ have the Carathéodory property, i.e., they are measurable in $(t,x) \in Q_T$ for every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ for a.a. $(t,x) \in Q_T$.

(B2) There exist a continuous function $\hat{c}_1 \colon \mathbb{R} \to \mathbb{R}^+$ and bounded operators $\hat{\mathfrak{c}}_1 \colon L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}^+$, $\hat{k}_1 \colon L^\infty(Q_T) \times X_1 \times X_2 \to L^{q_2}(Q_T)$ such that

$$\begin{aligned} &|b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2)| \\ &\le \hat{\mathfrak{c}}_1(w, v_1, v_2) \hat{c}_1(\xi) \left( |\eta_0|^{p_2 - 1} + |\eta|^{p_2 - 1} + |\zeta_0|^{\frac{p_1}{q_2}} + [\hat{k}_1(w, v_1, v_2)](t,x) \right) \end{aligned}$$

for a.a. $(t,x) \in Q_T$ and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ $(i = 0, \dots, n)$.

(B3) There exists a constant $\hat{C} > 0$ such that for a.a. $(t,x) \in Q_T$, every $(\xi, \zeta_0, \eta_0, \eta)$, $(\xi, \zeta_0, \tilde{\eta}_0, \tilde{\eta}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$

$$\begin{aligned} &\sum_{i=0}^{n} \left( b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) - b_i(t, x, \xi, \zeta_0, \tilde{\eta}_0, \tilde{\eta}; w, v_1, v_2) \right) (\eta_i - \tilde{\eta}_i) \\ &\ge \hat{C} \cdot \left( |\eta_0 - \tilde{\eta}_0|^{p_2} + |\eta - \tilde{\eta}|^{p_2} \right). \end{aligned}$$

(B4) There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma} \colon \mathbb{R} \to \mathbb{R}$ and bounded operators $\hat{\Gamma} \colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\hat{k}_2 \colon X_2 \to L^1(Q_T)$ such that

$$\begin{aligned} &\sum_{i=0}^{n} b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) \eta_i \\ &\ge \hat{c}_2 \left( |\eta_0|^{p_2} + |\eta|^{p_2} \right) - \hat{\gamma}(\xi) [\hat{\Gamma}(w)](t,x) \left( |\zeta_0|^{p_1} + [\hat{k}_2(v_2)](t,x) \right) \end{aligned}$$

for a.a. $(t,x) \in Q_T$, and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$. Further,

$$\lim_{\|v_2\|_{X_2} \to \infty} \frac{\|\hat{k}_2(v_2)\|_{L^1(Q_T)}}{\|v_2\|_{X_2}^{p_2}} = 0. \tag{3.9}$$

(B5) If $(\omega_k)$ is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in $Q_T$ and $u_k \to u$ weakly in $X_1$, strongly in $L^{p_1}(Q_T)$, further, $\mathbf{p}_k \to \mathbf{p}$ weakly in $X_2$ then

$$b_i(\cdot, \omega_k, u_k, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_k, \mathbf{p}_k) - b_i(\cdot, \omega_k, u_k, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \to 0$$

in $L^{q_2}(Q_T)$.

(F1) For fixed $v_1 \in X_1$, function $f \colon Q_T \times \mathbb{R}^2 \times X_1 \to \mathbb{R}$ is a Carathéodory function, i.e., it is measurable in $(t, x) \in Q_T$ for every $(\xi, \zeta_0) \in \mathbb{R}^2$ and continuous in $(\xi, \zeta_0) \in \mathbb{R}^2$ for a.a. $(t, x) \in Q_T$. Further, there exists a bounded operator $\mathcal{K}_1 \colon X_1 \to \mathbb{R}^+$ such that

(i) for every bounded set $I \subset \mathbb{R}$ there is a continuous function $K_1 \colon \mathbb{R} \to \mathbb{R}^+$ satisfying $|K_1(\zeta_0)| \le d_1|\zeta_0|^{\frac{p_1}{q_2}} + d_2$ for every $\zeta_0 \in \mathbb{R}$, with some nonnegative constants $d_1$, $d_2$ (depending on $I$),

(ii) for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0), (\tilde{\xi}, \zeta_0) \in I \times \mathbb{R}$ and every $v_1 \in X_1$,

$$|f(t, x, \xi, \zeta_0; v_1) - f(t, x, \tilde{\xi}, \zeta_0; v_1)| \le \mathcal{K}_1(v_1)K_1(\zeta_0) \cdot |\xi - \tilde{\xi}|.$$

(F2) There exist a bounded operator $\mathcal{K}_2 \colon X_1 \to \mathbb{R}^+$ and a continuous function $K_2 \colon \mathbb{R} \to \mathbb{R}^+$ such that for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0)$, $(\xi, \tilde{\zeta}_0) \in \mathbb{R}^2$ and $v_1 \in X_1$,

$$|f(t, x, \xi, \zeta_0; v_1) - f(t, x, \xi, \tilde{\zeta}_0; v_1)| \le \mathcal{K}_2(v_1)K_2(\xi) \cdot |\zeta_0 - \tilde{\zeta}_0|.$$

(F3) There exists $\omega^* \in L^\infty(\Omega)$ such that for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0) \in \mathbb{R}^2$ and $v_1 \in X_1$,

$$(\xi - \omega^*(x)) \cdot f(t, x, \xi, \zeta_0; v_1) \le 0.$$

(F4) If $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $u_k \to u$ strongly in $L^{p_1}(Q_T)$ then

$$\lim_{k\to\infty} \|f(\cdot, \omega_k, u_k; u_k) - f(\cdot, \omega_k, u_k; u)\|_{L^1(Q_T)} = 0.$$

(G1) $G \in X_1^*$.

(H1) $H \in X_2^*$.

### 3.1.4 Weak formulation

If the above assumptions are satisfied we may define operators $A\colon L^\infty(Q_T)\times X_1\times X_2\to X_1^*$, $B\colon L^\infty(Q_T)\times X_1\times X_2\to X_2^*$ as follows:

$$\begin{aligned}
&[A(\omega,u,\mathbf{p}),v_1]\\
&:=\int_{Q_T}\sum_{i=1}^n a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})\\
&\qquad\times D_i v_1(t,x)\,dt\,dx\\
&\quad+\int_{Q_T} a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})\\
&\qquad\times v_1(t,x)\,dt\,dx,
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
&[B(\omega,u,\mathbf{p}),v_2]\\
&:=\int_{Q_T}\sum_{i=1}^n b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})\\
&\qquad\times D_i v_2(t,x)\,dt\,dx\\
&\quad+\int_{Q_T} b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})v_2(t,x)\,dt\,dx,
\end{aligned}\tag{3.11}$$

for $v_i\in X_i$ $(i=1,2)$. In addition, let us introduce the operator of differentiation $L: D(L)\to X_1^*$ by the formula

$$D(L)=\{u\in X_1\colon D_t u\in X_1^*,\ u(0)=0\},\qquad Lu=D_t u.\tag{3.12}$$

By using the operators above and functionals given in (G1), (H1) we define the weak form of system (3.5)–(3.7) as

$$\omega(t,x)=\omega_0(x)+\int_0^t f(s,x,\omega(s,x),u(s,x);u)\,ds\ \text{ a.e. in } Q_T\tag{3.13}$$

$$Lu+A(\omega,u,\mathbf{p})=G\tag{3.14}$$

$$B(\omega,u,\mathbf{p})=H.\tag{3.15}$$

Note that in (3.15) there is a "hidden" initial condition $u(0)=0$ which is given in the domain of $L$. One obtains the above weak forms by using Green's formula as it was explained in Section 2.1.3. If the boundary condition is homogeneous Neumann type then $V_i=W^{1,p_i}(\Omega)$ and if in case of homogeneous Dirichlet boundary condition then $V_i=W_0^{1,p_i}(\Omega)$. Further, if we have a partition, for example in one dimension with homogenous Dirichlet and Neumann boundary conditions, as in model (3.1)–(3.4), then $V_i=\{v\in W^{1,p_i}(0,1): v(0)=0, D_x v(1)=0\}$.

## 3.2 Weak solutions in $(0, T)$

Science is a differential equation. Religion is a boundary condition.

Alan Mathison Turing

### 3.2.1 Existence

In this section we prove

**Theorem 3.1.** *Suppose that conditions (A1)–(A5), (B1)–(B5), (F1)–(F4), (G1), (H1) are fulfilled. Then for every $\omega_0 \in L^\infty(\Omega)$ there exists a solution $\omega \in L^\infty(Q_T)$, $u \in D(L)$, $\mathbf{p} \in L^{p_2}(0, T; V_2)$ of problem* (3.13)–(3.15).

Before the proof we formulate some statements related to the solvability of the above equations (3.13)–(3.15).

**Proposition 3.2.** *Assume that conditions (F1), (F3) are satisfied. Then for every fixed $u \in X_1$ and $\omega_0 \in L^\infty(\Omega)$ there exists a unique solution $\omega \in L^\infty(Q_T)$ of the integral equation* (3.13). *Further, the solution $u$ satisfies estimate $\|\omega\|_{L^\infty(Q_T)} \leq \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$.*

*Proof.* Let us make an observation that we shall use many times. Namely, from (F3) and the continuity of $f$ in variable $\xi$ it follows $f(t, x, \omega^*(x), \zeta_0; v_1) = 0$ for a.a. $(t, x) \in Q_T$, every $\zeta_0 \in \mathbb{R}$ and $v_1 \in X_1$. Assume that $\omega$ is a solution of (3.13) for some fixed $u \in X_1$. Then it is continuous in variable $t$ (moreover, it is absolutely continuous). Now fix a point $x \in \Omega$. If $\omega(t_0, x) > \omega^*(x)$ for some $t_0 \in (0, T)$ then $\omega(t, x) > \omega^*(x)$ for all $t \in [t_0, t_0 + \varepsilon]$ where $\varepsilon$ is sufficiently small. Then by condition (F3) it follows $f(t, x, \omega(t, x), u(t, x); v_1) \leq 0$ whence

$$
\begin{aligned}
\omega(t, x) = \omega_0(x) &+ \int_0^t f(s, x, \omega(s, x), u(s, x); v_1)\, ds \\
&= \omega_0(x) + \int_0^{t_0} f(s, x, \omega(s, x), u(s, x); v_1)\, ds \\
&\quad + \int_{t_0}^t f(s, x, \omega(s, x), u(s, x); v_1)\, ds \\
&\leq \omega_0(x) + \int_0^{t_0} f(s, x, \omega(s, x), u(s, x); v_1)\, ds \\
&= \omega(t_0, x),
\end{aligned}
$$

that is, $\omega$ is decreasing in variable $t$. Similarly to this, if $\omega(t_0, x) < \omega^*(x)$ for some $t_0 > 0$ then $\omega$ is locally increasing in $t$. Now it is easily seen that $\omega(t, x) \in [\omega^*(x), \omega_0(x)]$

(or $[\omega_0(x), \omega^*(x)]$) for a.a. $(t,x) \in Q_T$ thus $|\omega(t,x)| \le |\omega_0(x)| + |\omega^*(x)|$ for a.a. $(t,x) \in Q_T$ hence $\|\omega\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$.

Now let us define a function $\tilde{f}\colon Q_T \times \mathbb{R}^2 \times X_1 \to \mathbb{R}$ by

$$\tilde{f}(t,x,\xi,\zeta_0;v_1) = \begin{cases} f(t,x,\xi,\zeta_0;v_1), & \text{if } |\xi| \le c_{\omega_0,\omega^*}, \\ f(t,x,c_{\omega_0,\omega^*},\zeta_0;v_1), & \text{if } \xi \ge c_{\omega_0,\omega^*}, \\ f(t,x,-c_{\omega_0,\omega^*},\zeta_0;v_1), & \text{if } \xi \le -c_{\omega_0,\omega^*}, \end{cases}$$

with the constant $c_{\omega_0,\omega^*} = \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$ and consider the following problem instead of (3.13):

$$\omega(t,x) = \omega_0(x) + \int_0^t \tilde{f}(s,x,\omega(s,x),u(s,x);u)\,ds, \quad \text{for a.a. } (t,x) \in Q_T. \tag{3.16}$$

Obviously $\tilde{f}$ also fulfils condition (F2), (F3), further, by choosing interval $I = [-c_{\omega_0,\omega^*}, c_{\omega_0,\omega^*}]$ in condition (F1) then with some functions $\mathcal{K}_1, K_1$ it follows

$$|\tilde{f}(t,x,\xi,\zeta_0;v_1) - \tilde{f}(t,x,\tilde{\xi},\zeta_0;v_1)| \le \mathcal{K}_1(v_1)K_1(\zeta_0) \cdot |\xi - \tilde{\xi}|$$

for a.a. $(t,x) \in Q_T$, every $\xi, \tilde{\xi}, \zeta_0 \in \mathbb{R}$, $v_1 \in X_1$. Indeed, $f$ was extended as a constant function outside of $I$. This means that function $\tilde{f}$ satisfies condition (F1) globally. Clearly, if problem (3.16) has got a solution $\omega$ then $\|\omega\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$. Since $\tilde{f}$ equals with $f$ on interval $I$, every solution of (3.16) is a solution of (3.13) and converse. From the above arguments we conclude that it is sufficient to show that the problem (3.16) has a unique solution $\omega \in L^\infty(Q_T)$. In other words, we may assume that condition (F1) is fulfilled by function $f$, globally in $\xi$.

**Existence.** We use the method of successive approximation. Fix $u \in X_1$. Let $\omega_0(t,x) := \omega_0(x)$ ($(t,x) \in Q_T$) and define $w_k(t,x)$ as follows:

$$\omega_{k+1}(t,x) := \omega_0(x) + \int_0^t f(s,x,\omega_k(s,x),u(s,x);u)\,ds. \tag{3.17}$$

Now fix a point $x \in \Omega$. We show that

$$|\omega_{k+1}(t,x) - \omega_k(t,x)| \le c_{\omega_0,\omega^*} \cdot c_{x,u}^{k+1} \frac{t^{\frac{k+1}{p_2}}}{[(k+1)!]^{\frac{1}{p_2}}} \tag{3.18}$$

with the above defined $c_{\omega_0,\omega^*} = \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$ and with a suitable constant

$c_{x,u} > 0$. We proceed by induction on $k$. For $k = 0$ we have

$$
\begin{aligned}
&|\omega_1(t,x) - \omega_0(t,x)| \\
&= \left| \int_0^t f(s,x,\omega_0(x),u(s,x);u)\,ds \right| \\
&= \left| \int_0^t \left( f(s,x,\omega_0(x),u(s,x);u) - f(s,x,\omega^*(x),u(s,x);u) \right) ds \right| \\
&\le \int_0^t |\mathcal{K}_1(u)K_1(u(s,x))| \cdot |\omega_0(x) - \omega^*(x)|\,ds \\
&\le \left( \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)} \right) \cdot \int_0^t |\mathcal{K}_1(u)K_1(u(s,x))|\,ds.
\end{aligned}
$$

By using condition (F1), Hölder's inequality and the fact that $u \in X_1$ it follows

$$
\begin{aligned}
&\int_0^t |\mathcal{K}_1(u)K_1(u(s,x))|\,ds \\
&\le \left( \int_0^T |\mathcal{K}_1(u)K_1(u(s,x))|^{q_2}\,ds \right)^{\frac{1}{q_2}} \cdot \left( \int_0^t 1^{p_2} \right)^{\frac{1}{p_2}} \\
&\le \left( \int_0^t \left( d_1|u(s,x)|^{\frac{p_1}{q_2}} + d_2 \right)^{q_2} ds \right)^{\frac{1}{q_2}} \cdot |\mathcal{K}_1(u)| \cdot t^{\frac{1}{p_2}} \\
&\le \text{const} \cdot \left( \int_0^T (|u(s,x)|^{p_1} + 1)\,ds \right)^{\frac{1}{q_2}} \cdot |\mathcal{K}_1(u)| \cdot t^{\frac{1}{p_2}} \\
&= c_{x,u} \cdot t^{\frac{1}{p_2}}.
\end{aligned}
\tag{3.19}
$$

The above two estimates yield (3.18) for $k = 0$.

Now let us suppose that estimate (3.18) holds for $k-1$. Then condition (F1), the assumption of induction and (3.19) imply

$$
\begin{aligned}
&|\omega_{k+1}(t,x) - \omega_k(t,x)| \\
&\le \int_0^t |f(s,x,\omega_k(s,x),u(s,x);u) - f(s,x,\omega_{k-1}(s,x),u(s,x);u)|\,ds \\
&\le \int_0^t |\mathcal{K}_1(u)K_1(u(s,x))| \cdot |\omega_k(s,x) - \omega_{k-1}(s,x)|\,ds \\
&\le \int_0^t \left( |\mathcal{K}_1(u)K_1(u(s,x))| \cdot c_{\omega_0,\omega^*} \cdot c_{x,u}^k \cdot \frac{s^{\frac{k}{p_2}}}{(k!)^{\frac{1}{p_2}}} \right) ds \\
&\le c_{\omega_0,\omega^*} c_{x,u}^k \cdot \left( \int_0^T |\mathcal{K}_1(u)K_1(u(s,x))|^{q_2}\,ds \right)^{\frac{1}{q_2}} \cdot \left( \int_0^t \frac{s^k}{k!}\,ds \right)^{\frac{1}{p_2}} \\
&\le c_{\omega_0,\omega^*} c_{x,u}^{k+1} \cdot \frac{t^{\frac{k+1}{p_2}}}{[(k+1)!]^{\frac{1}{p_2}}}.
\end{aligned}
$$

The induction is complete. Estimate (3.18) yields

$$
|\omega_{k+l}(t,x) - \omega_k(t,x)| \le \sum_{i=k+1}^{k+l} c_{\omega_0,\omega^*} \cdot c_{x,u}^i \frac{T^{\frac{i}{p_2}}}{(i!)^{\frac{1}{p_2}}} \to 0
$$

as $k, l \to \infty$ for a.a. $(t,x) \in Q_T$. Whence $(\omega_k(t,x))$ is a Cauchy sequence for a.a. $(t,x) \in Q_T$, therefore it is convergent to some $\omega(t,x)$, $\omega_k \to \omega$ a.e. in $Q_T$, moreover, $\omega_k(\cdot,x) \to \omega(\cdot,x)$ in $L^\infty(0,T)$ for a.a. $x \in \Omega$. We show that $\omega$ is a solution of equation (3.13). It is clear that left hand side of the recurrence (3.17) converges to $\omega$ a.e. in $Q_T$ thus it suffices to show that the right hand side of (3.17) a.e. tends to the right hand side of equation (3.13). But this is true since

$$\begin{aligned}
&\left| \int_0^t (f(s,x,\omega(s,x),u(s,x);u) - f(s,x,\omega_k(s,x),u(s,x);u))\, ds \right| \\
&\leq \int_0^t |\mathcal{K}_1(u) K_1(u(s,x))| \cdot |\omega(s,x) - \omega_k(s,x)|\, ds \\
&\leq \int_0^T |\mathcal{K}_1(u) K_1(u(s,x))|\, ds \cdot \|\omega(\cdot,x) - \omega_k(\cdot,x)\|_{L^\infty(0,T)} \\
&\leq c_{x,u} \cdot T^{\frac{1}{p_2}} \cdot \|\omega(\cdot,x) - \omega_k(\cdot,x)\|_{L^\infty(0,T)} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

**Uniqueness.** Assume that $\omega, \tilde{\omega} \in L^\infty(Q_T)$ are solutions of (3.13). Then by (F1)

$$\begin{aligned}
&|\omega(t,x) - \tilde{\omega}(t,x)| \\
&\leq \int_0^t |f(s,x,\omega(s,x),u(s,x);u) - f(s,x,\tilde{\omega}(s,x),u(s,x);u)|\, ds \\
&\leq \int_0^t |\mathcal{K}_1(u) K_1(u(s,x))| \cdot |\omega(s,x) - \tilde{\omega}(s,x)|\, ds \\
&\leq \|\mathcal{K}_1(u) K_1(u(\cdot,x))\|_{L^{q_2}(Q_T)} \cdot \left( \int_0^t |\omega(s,x) - \tilde{\omega}(s,x)|^{p_2}\, ds \right)^{\frac{1}{p_2}}
\end{aligned}$$

hence

$$|\omega(t,x) - \tilde{\omega}(t,x)|^{p_2} \leq c_{x,u}^{p_2} \cdot \int_0^t |\omega(s,x) - \tilde{\omega}(s,x)|^{p_2}\, ds.$$

Gronwall's lemma yields $|\omega(t,x) - \tilde{\omega}(t,x)| = 0$ for a.a. $(t,x) \in Q_T$, i.e., $\omega = \tilde{\omega}$. $\square$

**Proposition 3.3.** *Assume (F1)–(F4) and let $(u_k) \subset X_1$, further, for every $k \in \mathbb{N}$ let $\omega_k$ be the solution of* (3.13) *corresponding to $u = u_k$. If $u_k \to u$ in $L^{p_1}(Q_T)$ then $\omega_k \to \omega$ a.e. in $Q_T$ where $\omega$ is the solution of* (3.13) *corresponding to $u$.*

*Proof.* The strong convergence of $(u_k)$ in $L^{p_1}(Q_T)$ implies $u_k(\cdot,x) \to u(\cdot,x)$ in $L^{p_1}(0,T)$ for a.a. $x \in \Omega$ (for a suitable subsequence). Fix such a point $x \in \Omega$. By Proposition 3.2 $(\omega_k)$ is bounded in $L^\infty(Q_T)$. Further,

$$\begin{aligned}
&|\omega_k(t,x) - \omega(t,x)| \\
&\leq \int_0^t |f(s,x,\omega_k(s,x),u_k(s,x);u_k) - f(s,x,\omega_k(s,x),u_k(s,x);u)|\, ds \\
&\quad + \int_0^t |f(s,x,\omega_k(s,x),u_k(s,x);u) - f(s,x,\omega(s,x),u(s,x);u)|\, ds.
\end{aligned}$$

The first integral converges to 0 for a.a. $x \in \Omega$ by condition (F4) (for a subsequence). In what follows, we show that the second integral tends to 0 as well. Indeed, by conditions (F1)–(F2),

$$\begin{aligned}
&\int_0^t |f(s,x,\omega_k(s,x),u_k(s,x);u) - f(s,x,\omega(s,x),u(s,x);u)|\,ds \\
&\le \int_0^t |\mathcal{K}_1(u)K_1(u_k(s,x))| \cdot |\omega_k(s,x) - \omega(s,x)|\,ds \\
&\quad + \int_0^t |\mathcal{K}_2(u)K_2(\omega(s,x))| \cdot |u_k(s,x) - u(s,x)|\,ds \\
&\le \left(\int_0^t |\mathcal{K}_1(u)K_1(u_k(s,x))|^{q_2}\,ds\right)^{\frac{1}{q_2}} \cdot \left(\int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2}\right)^{\frac{1}{p_2}} ds \\
&\quad + \|\mathcal{K}_2(u)K_2(\omega(\cdot,x))\|_{L^\infty(0,T)} \cdot \int_0^T |u_k(s,x) - u(s,x)|\,ds.
\end{aligned}$$

By choosing $u = u_k$ and $t = T$ in estimate (3.19) and by using the convergence of $u_k(\cdot,x)$ in $L^{p_1}(0,T)$ we conclude that the first term containing $u_k$ on the right hand side of the above inequality is bounded. In addition, the continuity of function $K_2$ implies that $\|\mathcal{K}_2(u)K_2(\omega(\cdot,x))\|_{L^\infty(0,T)}$ is finite. From the above arguments it follows

$$\begin{aligned}
&|\omega_k(t,x) - \omega(t,x)|^{p_2} \\
&\le \text{const} \cdot \int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2}\,ds + \text{const} \cdot \|u_k(\cdot,x) - u(\cdot,x)\|^{p_2}_{L^1(0,T)} + r(u_k,\omega_k)
\end{aligned}$$

where the remainder term $r(u_k,\omega_k)$ tends to 0 as $k \to \infty$. Thus Gronwall's lemma yields

$$|\omega_k(t,x) - \omega(t,x)|^{p_2} \le \text{const} \cdot \Big(\|u_k(\cdot,x) - u(\cdot,x)\|^{p_2}_{L^1(0,T)} + r(u_k,\omega_k)\Big)$$

where the right hand side tends to 0 as $k \to \infty$ which immediately implies the desired a.e. convergence of $(\omega_k)$ (for a subsequence, which is sufficient due to the "subsequence trick"). □

*Remark* 3.4. Since $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and convergent a.e. in $Q_T$, Lebesgue's theorem implies its strong convergence in $L^\alpha(Q_T)$ for arbitrary $1 \le \alpha < \infty$.

**Proposition 3.5.** *Suppose conditions (F1)–(F3), further, $|w_0| > 0$ a.e. in $\Omega$ and $\omega_0 \cdot \omega^* \ge 0$ (that is, they have the same sign). Then for the solution $\omega$ of* (3.13), *$|\omega(t,x)| > 0$ holds for a.a. $(t,x) \in Q_T$.*

*Proof.* Fix a point $x \in \Omega$. Without loss of generality we may assume that $\omega_0(x) > 0$. First suppose $\omega^*(x) > 0$. In the proof of Proposition 3.2 we have shown that $\omega(t,x) \in [\omega^*(x), \omega_0(x)]$ (or $\omega(t,x) \in [\omega_0(x), \omega^*(x)]$) for a.a. $t \in [0,T]$, consequently,

$\omega(t,x) \geq \min(\omega^*(x), \omega_0(x)) > 0$. Now suppose that $\omega^*(x) = 0$. Define $t^* := \inf\{t > 0 : \omega(t,x) = 0\}$. Then $\omega(t,x) > 0$ for every $t < t^*$. By using conditions (F1), (F3) it follows that for $\xi > \omega^*(x) = 0$, $\zeta_0 \in \mathbb{R}$, $f(t,x,\xi,\zeta_0) \geq -K_1(\zeta_0)\xi$. Then for a.a. $t \in (0,t^*)$,

$$\omega'(t,x) = f(t,x,\omega(t,x),u(t,x);u) \geq -\mathcal{K}_1(u)K_1(u(t,x))\omega(t,x).$$

(Note that $\omega$ is absolutely continuous in variable $t$ thus for a.a. $(t,x) \in Q_T$ there exists $\omega'(t,x)$.) Due to the definition of $t^*$ we may divide by $\omega(t,x)$ which yields

$$\frac{\omega'(t,x)}{\omega(t,x)} \geq -\mathcal{K}_1(u)K_1(u(t,x)).$$

Observe that the left hand side of the previous inequality equals to $(\log \omega(t,x))'$ thus by integrating the inequality in $(0,t)$ we obtain

$$\log \omega(t,x) - \log \omega_0(x) \geq -\int_0^t \mathcal{K}_1(u)K_1(u(s,x))\,ds.$$

By taking the exponential of both sides it follows

$$\omega(t,x) \geq \omega_0(x) \cdot e^{-\int_0^t \mathcal{K}_1(u)K_1(u(s,x))\,ds}.$$

The above estimate implies $\omega(t,x) > 0$ a.e. in $[0,T]$. The case $\omega_0(x) < 0$ can be treated similarly. $\square$

*Remark* 3.6. This proposition shows that if $|\omega_0|$ is positive a.e. in $Q_T$, further, $\omega_0$ and $\omega^*$ has the same sign in a.e. $Q_T$, then for the solution $\omega$ of (3.13), $\frac{1}{\omega}$ is a.e. finite. Consequently, operator $A$ and $B$ might depend on terms which contain $\frac{1}{\omega}$. The above proof also shows that if the modulus of the initial value $\omega_0$ is a.e. greater than a positive constant, further, $|\omega^*|$ is greater then a positive lower bound, or $K_1$ is bounded, then the absolute value of the solution $\omega$ of equation (3.13) is also greater than a positive constant a.e. in $Q_T$ thus $\frac{1}{\omega} \in L^\infty(Q_T)$.

**Proposition 3.7.** *Assume conditions (A1)–(A5). Then for every fixed $\omega \in L^\infty(Q_T)$, $\mathbf{p} \in X_2$ and $G \in X_1^*$ there exists a solution $u \in D(L)$ of problem $Lu + A(\omega,u,\mathbf{p}) = G$.*

*Proof.* The proof follows from Theorem 2.1 and Theorem 1.65, since for fixed $\omega \in L^\infty(Q_T)$ and $\mathbf{p} \in X_2$ conditions (A1)–(A5) are the same conditions as (A1)–(A5) in Section 2.1.2 thus operator $A(\omega,\cdot,\mathbf{p})\colon X_1 \to X_1^*$ is bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$. $\square$

**Proposition 3.8.** *Suppose that (B1)–(B5) hold. Then for every fixed $\omega \in L^\infty(Q_T)$, $u \in X_1$ and $H \in X_2^*$ there exists a solution $\mathbf{p} \in X_2$ of problem $B(\omega,u,\mathbf{p}) = H$.*

*Proof.* We show that for fixed $\omega \in L^\infty(Q_T)$, $u \in X_1$ operator $B(\omega, u, \cdot)\colon X_2 \to X_2^*$ is bounded, demicontinuous, pseudomonotone and coercive. Then Theorem 1.54 implies the existence of solutions to equation $B(\omega, u, \mathbf{p}) = H$ for every $H \in X_2^*$. The boundedness, demicontinuity and coerciveness follows by the same arguments as in the proof of Theorem 2.1, since for fixed $\omega \in L^\infty(Q_T)$, $v_1 \in X_1$ assumptions (B1)–(B4) are the same as conditions (A1)–(A4) in Section 2.1.2. Now fix $\omega \in L^\infty(Q_T)$, $u \in X_1$. Introduce operator $\hat{B}_{\mathbf{p}}\colon X_2 \to X_2^*$ for fixed $v_2 \in X_2$ by

$$
\begin{aligned}
&[\hat{B}_{v_2}(\mathbf{p}), z_2] \\
&\quad := \int_{Q_T} \sum_{i=1}^n b_i(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); \omega, u, v_2) D_i z_2(t,x)\, dt\, dx \\
&\qquad + \int_{Q_T} b_0(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); \omega, u, v_2) z_2(t,x)\, dt\, dx
\end{aligned}
$$

where $z_2 \in X_2$. To verify the pseudomonotonicity suppose that $\mathbf{p}_k \to \mathbf{p}$ weakly in $X_2$ and

$$\limsup_{k\to\infty} [B(\omega, u, \mathbf{p}_k), \mathbf{p}_k - \mathbf{p}] \le 0.$$

Condition (B5) implies that

$$b_i(\cdot, \omega, u, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}_k) - b_i(\cdot, \omega, u, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \to 0$$

in $L^{q_2}(Q_T)$ thus

$$B(\omega, u, \mathbf{p}_k) - \hat{B}_{\mathbf{p}}(\mathbf{p}_k) \to 0 \ \text{ in } \ X_2^* \ \text{ and} \tag{3.20}$$

$$\lim_{k\to\infty} [B(\omega, u, \mathbf{p}_k) - \hat{B}_{\mathbf{p}}(\mathbf{p}_k), \mathbf{p}_k - \mathbf{p}] = 0. \tag{3.21}$$

From Theorem 1.54 it follows that for fixed $\mathbf{p} \in X_2$ operator $\hat{B}_{\mathbf{p}}$ is pseudomonotone (since then conditions (B1)–(B4) are the same as (i)–(iv) in Section 1.6). So that

$$\lim_{k\to\infty} [\hat{B}_{\mathbf{p}}(\mathbf{p}_k), \mathbf{p}_k - \mathbf{p}] = 0 \ \text{ and } \ \hat{B}_{\mathbf{p}}(\mathbf{p}_k) \to \hat{B}_{\mathbf{p}}(\mathbf{p}) = B(\omega, u, \mathbf{p}).$$

Hence by (3.20), (3.21) we conclude

$$
\begin{gathered}
\lim_{k\to\infty} [B(\omega, u, \mathbf{p}_k), \mathbf{p}_k - \mathbf{p}] = 0 \ \text{ and} \\
B(\omega, u, \mathbf{p}_k) \to B(\omega, u, \mathbf{p})
\end{gathered}
$$

which means the pseudomonotonicity of operator $B(\omega, u, \cdot)$. □

*Proof of Theorem 3.1.* We define sequences of approximate solutions of problem (3.13)–(3.15) and we show the boundedness of these sequences. Then the weak limits of suitable chosen weakly convergent subsequences will be the solutions. For simplicity, in the proof we omit the variable $(t, x)$ of functions $a_i$, $b_i$ if it is not confusing.

**Step 1: approximation.** Define sequences $(\omega_k), (u_k), (\mathbf{p}_k)$ as follows. Let $\omega_0(t,x) \equiv u_0(t,x) \equiv \mathbf{p}_0(t,x) \equiv 0$ $((t,x) \in Q_T)$ and for $k = 0, 1, \dots$ let $\omega_{k+1}, u_{k+1}, \mathbf{p}_{k+1}$ be a solutions of the equations :

$$\omega_{k+1}(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega_{k+1}(s,x), u_k(s,x); u_k)\, ds \tag{3.22}$$

$$Lu_{k+1} + A(\omega_k, u_{k+1}, \mathbf{p}_k) = G \tag{3.23}$$

$$B(\omega_k, u_k, \mathbf{p}_{k+1}) = H. \tag{3.24}$$

By Propositions 3.2, 3.7, 3.8 there exist solutions $\omega_{k+1} \in L^\infty(Q_T)$, $u_{k+1} \in X_1$, $\mathbf{p}_{k+1} \in X_2$ so we obtain the sequences $(\omega_k) \subset L^\infty(Q_T), (u_k) \subset X_1, (\mathbf{p}_k) \subset X_2$.

**Step 2: boundedness.** We show that the above defined sequences are bounded. By Proposition 3.2, for fixed $\omega_0 \in L^\infty(\Omega)$ the solution of equation (3.22) satisfies estimate $\|\omega_{k+1}\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$ thus $(\omega_k)$ is bounded in $L^\infty(Q_T)$

Now by choosing the test function $v = u_{k+1}$ in (3.23), further, by using condition (A4) and the monotonicity of operator $L$ one obtains

$$\begin{aligned}
[G, u_{k+1}] &= [Lu_{k+1}, u_{k+1}] + [A(\omega_k, u_{k+1}, \mathbf{p}_k), u_{k+1}] \\
&\ge \int_{Q_T} (c_2|u_{k+1}|^{p_1} + c_2|Du_{k+1}|^{p_1} - \gamma(\omega_k)\Gamma(\omega_k)k_2(u_{k+1})) \\
&\ge c_2\|u_{k+1}\|_{X_1} \left( \|u_{k+1}\|_{X_1}^{p_1-1} - \|\gamma(\omega_k)\Gamma(\omega_k)\|_{L^\infty(Q_T)} \cdot \frac{\|k_2(u_{k+1})\|_{L^1(Q_T)}}{\|u_{k+1}\|_{X_1}} \right).
\end{aligned}$$

Thus by the boundedness of $(\omega_k)$ we conclude for some $K > 0$ that

$$\|u_{k+1}\|_{X_1}^{p_1-1} \left( 1 - K \cdot \frac{\|k_2(u_{k+1})\|_{L^1(Q_T)}}{\|u_{k+1}\|_{X_1}^{p_1}} \right) \le \text{const.}$$

Now (3.8) implies the boundedness of $(u_k)$ in $X_1$.

The boundedness of $(\mathbf{p}_k)$ in $X_2$ follows by similar arguments as above by using condition (B4) and the boundedness of the sequences $(\omega_k)$, $(u_k)$.

We need also the boundedness of the sequence $(Lu_k)$ in $X_1^*$. To this end, we use Hölder's inequality and obtain for arbitrary $v \in X_1$

$$\begin{aligned}
&|[A(\omega_k, u_{k+1}, \mathbf{p}_k), v]| \\
&\le \left( \sum_{i=0}^n \|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \right) \cdot \|v\|_{X_1}.
\end{aligned}$$

From condition (A2) it follows for all $i$

$$\begin{aligned}
&\|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \\
&\le \text{const} \cdot c_1(\omega_k)\mathfrak{c}_1(\omega_k, u_{k+1}, \mathbf{p}_k) \left( \|u_{k+1}\|_{X_1}^{p_1} + \|\mathbf{p}_k\|_{X_2}^{p_2} + \|k_1(\omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \right).
\end{aligned}$$

Therefore, by the boundedness of the sequences $(\omega_k), (u_k), (\mathbf{p}_k)$ and the boundedness of operators $c_1, \mathfrak{c}_1, k_2$ we conclude

$$|[Lu_{k+1}, v]| = |[A(\omega_k, u_{k+1}, \mathbf{p}_k) + G, v]| \le \text{const} \cdot \|v\|_{X_1}$$

so $(Lu_k)$ is a bounded sequence in $X_1^*$.

**Step 3: convergence.** Due to the boundedness of the sequences $(u_k), (Lu_k), (\mathbf{p}_k)$ (in reflexive Banach spaces) each has a weakly convergent subsequence, further, by applying Corollary 1.48 it follows that there exist subsequences (which will be denoted, for simplicity, as the original sequences) and functions $u \in X_1$, $\mathbf{p} \in X_2$ such that

$$u_k \to u \text{ weakly in } X_1, \text{ strongly in } L^{p_1}(Q_T), \text{ a.e. in } Q_T;$$
$$\mathbf{p}_k \to \mathbf{p} \text{ weakly in } X_2.$$

In what follows, we show that $\omega, u, \mathbf{p}$ are solutions of problem (3.13)–(3.15).

Since $u_k \to u$ in $L^{p_1}(Q_T)$, further, $\omega_{k+1}$ is the solution of equation (3.22), by Proposition 3.3 it follows that $\omega_k \to \omega$ a.e. in $Q_T$ for some $\omega \in L^\infty(Q_T)$ such that functions $\omega, u$ satisfy the integral equation (3.13).

Now let us consider equation (3.24). We show that $\mathbf{p}_k \to \mathbf{p}$ in $X_2$. To this end, let us introduce operator $\tilde{B}\colon L^\infty(Q_T) \times X_1 \times X_2 \times L^\infty(Q_T) \times X_1 \times X_2 \to X_2^*$ by

$$\begin{aligned}
&[\tilde{B}(\omega, u, \mathbf{p}; w, v_1, v_2), z_2] \\
&:= \int_{Q_T} \sum_{i=1}^n b_i(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); w, v_1, v_2) D_i z_2(t,x)\, dt\, dx \\
&\quad + \int_{Q_T} b_0(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); w, v_1, v_2) z_2(t,x)\, dt\, dx
\end{aligned}$$

for $z_2 \in X_2$. Observe $B(\omega, u, \mathbf{p}) = \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p})$. Condition (B3) yields

$$[\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \ge \hat{C} \cdot \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2}^{p_2}. \quad (3.25)$$

On the left hand side of the above inequality we have the following decomposition:

$$\begin{aligned}
&[\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \\
&= [\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1} - \mathbf{p}] \\
&\quad + [\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1} - \mathbf{p}] \qquad (3.26) \\
&\quad + [\tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \\
&\quad - [\tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}].
\end{aligned}$$

Now we show that each term on the right hand side tends to 0 which implies by (3.25) the strong convergence of $(\mathbf{p}_k)$. Clearly, $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) = H$, further, $\mathbf{p}_{k+1} \to \mathbf{p}$ weakly in $X_2$ which yield the convergence of the first and the last term. In addition, it is easily seen that condition (B5) implies the convergence of the second term on the right hand side of (3.26). In order to verify the convergence of the third term, observe that

$$\begin{aligned} &|[\tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}) - \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}]| \\ &\le \sum_{i=0}^{n} \|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p})\|_{L^{q_2}(Q_T)} \\ &\quad \times \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2} \end{aligned} \tag{3.27}$$

and by condition (B2) it follows

$$\begin{aligned} &|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p})|^{q_2} \\ &\le \text{const} \cdot |\hat{\mathbf{c}}_1(\omega, u, \mathbf{p})|^{q_2} \cdot (|\hat{c}_1(\omega_k)|^{q_2} + |\hat{c}_1(\omega)|^{q_2}) \\ &\quad \times \left(|\mathbf{p}|^{p_2} + |D\mathbf{p}|^{p_2} + |u_k|^{p_1} + |u|^{p_1} + |\hat{k}_1(\omega, u, \mathbf{p})|^{q_2}\right). \end{aligned} \tag{3.28}$$

The boundedness of $(\omega_k)$ in $L^\infty(Q_T)$ and the convergence of $(u_k)$ in $L^{p_1}(Q_T)$ implies the equi-integrability of the left hand side of the above inequality. In addition, the left hand side a.e. converges to 0, therefore by Vitali's theorem it converges in $L^1(Q_T)$ to the zero function. Thus (because of the boundedness of $(\mathbf{p}_k)$) the right hand side of (3.27) tends to 0. Hence all terms on the right hand side of equation (3.26) converges to 0 so we have shown that $\mathbf{p}_{k+1} \to \mathbf{p}$ in $X_2$.

Now we show that $B(\omega_k, u_k, \mathbf{p}_{k+1}) \to B(\omega, u, \mathbf{p})$ weakly in $X_2^*$. Then from recurrence (3.24) we obtain $B(\omega, u, \mathbf{p}) = H$, i.e., $\omega, u, \mathbf{p}$ are solutions of problem (3.15). Consider the decomposition

$$\begin{aligned} &\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}) - B(\omega, u, \mathbf{p}) \\ &= (B(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) - \tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p})) \\ &\quad + (\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p})). \end{aligned} \tag{3.29}$$

Observe that the second term on the right hand side converges to zero by Vitali's theorem, one may use similar estimates as (3.27), (3.28). Further, the first term tends to 0 by condition (B5).

Consequently, the right hand side of (3.29) converges to 0 thus

$$B(\omega_k, u_k, \mathbf{p}_{k+1}) - B(\omega, u, \mathbf{p}) \to 0 \ \text{ weakly in } \ X_2^*.$$

In the case of equation (3.23) we apply similar arguments as above. We introduce operator $\tilde{A}\colon L^\infty(Q_T) \times X_1 \times X_2 \times L^\infty(Q_T) \times X_1 \times X_2 \to X_1^*$ by

$$
\begin{aligned}
&[\tilde{A}(\omega, u, \mathbf{p}; w, v_1, v_2), z_2] \\
&:= \int_{Q_T} \sum_{i=1}^n a_i(t, x, \omega(t,x), u(t,x), Du(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); w, v_1, v_2) D_i z_2(t,x)\, dt\, dx \\
&\quad + \int_{Q_T} a_0(t, x, \omega(t,x), u(t,x), Du(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); w, v_1, v_2) z_2(t,x)\, dt\, dx
\end{aligned}
$$

for $z_1 \in X_1$. Note that $A(\omega, u, \mathbf{p}) = \tilde{A}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p})$. We have already shown the fact that $Lu_{k+1} \to Lu$ weakly in $X_1^*$ thus it remains to verify that

$$
\tilde{A}(\omega_k, u_{k+1}, \mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k) \to \tilde{A}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}) = A(\omega, u, \mathbf{p})
$$

weakly in $X_1^*$ then recurrence (3.23) yields (3.14). To this end, we show that $u_k \to u$ strongly in $X_1$. Since it is already shown that $u_k \to u$ in $L^{p_1}(Q_T)$ it suffices to show that $Du_k \to Du$ in $L^{p_1}(Q_T)$. Now by the monotonicity of operator $L$,

$$
\begin{aligned}
&[Lu_{k+1} - Lu, u_{k+1} - u] \\
&\quad + [\tilde{A}(\omega_k, u_{k+1}, \mathbf{p}_k; \omega, u, \mathbf{p}) - \tilde{A}(\omega_k, u, \mathbf{p}_k; \omega, u, \mathbf{p}); u_{k+1} - u] \\
&\geq \sum_{i=1}^n \int_{Q_T} \big[(a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \\
&\qquad\qquad - a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p})) \times (D_i u_{k+1} - D_i u)\big] \\
&\quad + \sum_{i=1}^n \int_{Q_T} \big[(a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \\
&\qquad\qquad - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \times (D_i u_{k+1} - D_i u)\big] \\
&\quad + \int_{Q_T} (a_0(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \\
&\qquad\qquad - a_0(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \times (u_{k+1} - u)\big].
\end{aligned}
\tag{3.30}
$$

Observe that by condition (A3) the first term on the right hand side of the above inequality is greater than $C \cdot \|Du_{k+1} - Du_k\|^{p_1}_{L^{p_1}(Q_T)}$. We show that the left hand side and the second, third integrals on the right hand side converge to 0, then the convergence of $(Du_k)$ in $L^{p_1}(Q_T)$ immediately follows. Consider the decomposition

$$
\begin{aligned}
&[Lu_{k+1} - Lu, u_{k+1} - u] + [\tilde{A}(\omega_k, u_{k+1}, \mathbf{p}_k, \omega, u, \mathbf{p}) - \tilde{A}(\omega_k, u, \mathbf{p}_k; \omega, u, \mathbf{p}), u_{k+1} - u] \\
&= [Lu_{k+1} + \tilde{A}(\omega_k, u_{k+1}, \mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k), u_{k+1} - u] - [Lu, u_{k+1} - u] \\
&\quad + [\tilde{A}(\omega_k, u_{k+1}, \mathbf{p}_k; \omega, u, \mathbf{p}) - \tilde{A}(\omega_k, u_{k+1}, \mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k), u_{k+1} - u] \\
&\quad + [\tilde{A}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}) - \tilde{A}(\omega_k, u, \mathbf{p}_k; \omega, u, \mathbf{p}), u_{k+1} - u] \\
&\quad - [\tilde{A}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}), u_{k+1} - u].
\end{aligned}
$$

The first term on the right hand side equals to $[G, u_{k+1}-u]$ because of the recurrence (3.23). By the weak convergence of $(u_k)$ the first two and the fifth terms tend to 0 as $k \to \infty$. Condition (A5) implies the convergence to 0 of the third term (similarly to the case of the decomposition (3.26)). Finally, by condition (A2), the a.e. convergence of $(\omega_k)$, the strong convergence of $(\mathbf{p}_k)$, it is easy to see (similarly to the case of operator $B$, see (3.26)) that the fourth term also tends to 0. Further, the above arguments imply that the left hand side of (3.30) tends to 0. Now turn to the integrals of the right hand side of (3.30). Clearly,

$$(a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k); \omega, u, \mathbf{p}) \to 0$$

a.e. in $Q_T$, further,

$$\begin{aligned}
&|a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p})|^{q_1} \\
&\le \text{const} \cdot |\mathfrak{c}_1(\omega, u, \mathbf{p}) c_1(\omega_k)| \\
&\quad \times (|u_{k+1}|^{p_1} + |u|^{p_1} + |Du|^{p_1} + |\mathbf{p}_k|^{p_2} + |D\mathbf{p}_k|^{p_2} + |k_1(\omega, u, \mathbf{p})|^{q_1})
\end{aligned}$$

where the right hand side converges in $L^1(Q_T)$. Hence by Vitali's theorem the second integral on the right hand side of (3.30) tends to 0. In order to verify the convergence of the last integral on the right hand side of (3.30), we use Hölder's inequality and condition (A2) and we conclude

$$\begin{aligned}
&\left| \int_{Q_T} (a_0(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p}) \right. \\
&\qquad\qquad \left. - a_0(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p})(u_{k+1} - u) \right| \\
&\le \text{const} \cdot \|\mathfrak{c}_1(\omega, u, \mathbf{p}) c_1(\omega_k)\|_{L^\infty(Q_T)} \\
&\quad \times \left( \|u_{k+1}\|_{X_1}^{\frac{p_1}{q_1}} + \|u\|_{X_1}^{\frac{p_1}{q_1}} + \|\mathbf{p}_k\|_{X_2}^{\frac{p_2}{q_1}} + \|k_1(\omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \right) \|u_{k+1} - u\|_{L^{p_1}(Q_T)}.
\end{aligned}$$

By the strong convergence of $(\mathbf{p}_k)$ in $X_2$ and $(u_k)$ in $L^{p_1}(Q_T)$ and by the boundedness of $(u_k)$ in $X_1$ it follows that the right hand side tends to 0.

Now the weak convergence $A(\omega_k, u_{k+1}, \mathbf{p}_k) \to A(\omega, u, \mathbf{p})$ in $X_1^*$ follows easily by condition (A2), by the strong convergences of the sequences and by Vitali's theorem (the same way as in the case of operator $B$). So we have shown that $\omega, u, \mathbf{p}$ are solutions of problem (3.14).

Summarizing, we have verified that $\omega, u, \mathbf{p}$ are solutions of system (3.13)–(3.15), the proof of the theorem is complete. □

### 3.2.2 Examples

We show some examples for functions satisfying conditions (A1)–(A5), (B1)–(B5). Let functions $a_i, b_i$ have the form

$$\begin{aligned} &a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) \\ &= [\pi(w)](t,x)[\varphi(v_1)](t,x)[\psi(v_2)](t,x)P(\xi)Q(\eta_0,\eta)\zeta_i|\zeta|^{p_1-2} \\ &\quad + [\tilde\pi(w)](t,x)[\tilde\varphi(v_1)](t,x)\tilde P(\xi)\zeta_i|\zeta|^{r_1-1}, \text{ if } i\neq 0, \end{aligned} \tag{3.31}$$

$$\begin{aligned} &a_0(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) \\ &= [\pi(w)](t,x)[\varphi(v_1)](t,x)[\psi(v_2)](t,x)P(\xi)Q(\eta_0,\eta)\zeta_0|\zeta_0|^{p_1-2} \\ &\quad + [\tilde\pi_0(w)](t,x)[\tilde\varphi_0(v_1)](t,x)\tilde P_0(\xi)\zeta_0|\zeta_0|^{r_1-1}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) \\ &= [\kappa(w)](t,x)[\lambda(v_1)](t,x)[\vartheta(v_2)](t,x)R(\xi)S(\zeta_0)\eta_i|(\eta_0,\eta)|^{p_2-2} \\ &\quad + [\tilde\kappa(w)](t,x)[\tilde\vartheta(v_2)](t,x)\tilde R(\xi)\eta_i|(\eta_0,\eta)|^{r_2-1}, \ i=0,\dots,n, \end{aligned} \tag{3.33}$$

where $1\le r_i < p_i - 1$ $(i=1,2)$ and the following hold.

(E1) a) Operators $\pi\colon L^\infty(Q_T)\to L^\infty(Q_T)$, $\varphi\colon L^{p_1}(Q_T)\to L^\infty(Q_T)$, $\psi\colon X_2\to L^\infty(Q_T)$ are bounded, $\varphi$ and $\psi$ are continuous, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k\to\omega$ a.e. in $Q_T$ then $\pi(\omega_k)\to\pi(\omega)$ in $L^\infty(Q_T)$. In addition, $P\in C(\mathbb{R})$, $Q\in C(\mathbb{R}^{n+1})\cap L^\infty(\mathbb{R}^{n+1})$ and there exists a positive lower bound for the values of $\pi,\varphi,\psi,P,Q$.

b) Operators $\tilde\pi,\tilde\pi_0\colon L^\infty(Q_T)\to L^\infty(Q_T)$, $\tilde\varphi,\tilde\varphi_0\colon L^{p_1}(Q_T)\to L^{\frac{p_1-1}{p_1-r_1-1}}(Q_T)$ are bounded, $\tilde\varphi$ and $\tilde\varphi_0$ are continuous, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k\to\omega$ a.e. in $Q_T$ then $\tilde\pi(\omega_k)\to\tilde\pi(\omega)$ and $\tilde\pi_0(\omega_k)\to\tilde\pi_0(\omega)$ in $L^\infty(Q_T)$. In addition, $\tilde P,\tilde P_0\in C(\mathbb{R})$, operators $\tilde\pi,\tilde\varphi$ and function $\tilde P$ are nonnegative and

$$\lim_{\|v_1\|_{X_1}\to+\infty}\frac{\displaystyle\int_{Q_T}|\tilde\varphi_0(v_1)|^{\frac{p_1-1}{p_1-r_1-1}}}{\|v_1\|_{X_1}^{p_1}}=0.$$

(E2) a) Operators $\kappa\colon L^\infty(Q_T)\to L^\infty(Q_T)$, $\lambda\colon L^{p_1}(Q_T)\to L^\infty(Q_T)$, $\vartheta\colon L^{p_2}(Q_T)\to L^\infty(Q_T)$ are bounded, $\lambda$ and $\vartheta$ are continuous, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k\to\omega$ a.e. in $Q_T$ then $\kappa(\omega_k)\to\kappa(\omega)$ in $L^\infty(Q_T)$. In addition, $R\in C(\mathbb{R})$, $S\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$ and there exists a positive lower bound for the values of $\kappa,\lambda,\vartheta,R,S$.

b) Operators $\tilde\kappa\colon L^\infty(Q_T)\to L^\infty(Q_T)$ and $\tilde\vartheta\colon L^{p_2}(Q_T)\to L^{\frac{p_2-1}{p_2-r_2-1}}(Q_T)$ are bounded, $\tilde\vartheta$ is continuous, function $\tilde R\in C(\mathbb{R})$, further, if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\tilde{\kappa}(\omega_k) \to \tilde{\kappa}(\omega)$ in $L^\infty(Q_T)$. In addition, operators $\tilde{\kappa}, \tilde{\vartheta}$ and function $\tilde{R} \in C(\mathbb{R})$ are nonnegative and

$$\lim_{\|v_2\|_{X_2}\to+\infty} \frac{\displaystyle\int_{Q_T} |\tilde{\vartheta}(v_2)|^{\frac{p_2-1}{p_2-r_2-1}}}{\|v_2\|_{X_2}^{p_2}} = 0.$$

**Proposition 3.9.** *Assume that (E1)-(E2) hold, then functions* (3.31)–(3.33) *fulfil conditions (A1)–(A5), (B1)–(B5).*

By using Young's and Hölder's inequality it is not difficult to prove the above statement. One may use the same arguments as in Section 2.2.3 since the above conditions are analogous to the assumptions there.

Operators $\pi, \tilde{\pi}, \tilde{\pi}_0, \kappa, \tilde{\kappa}$ may have the form $[\pi(w)](t,x) = \displaystyle\int_{Q_t} |w|^\beta$, where $1 \le \beta$. Further, operators $\varphi, \lambda$ may have one of the forms

$$[\varphi(v)](t,x) = \Phi\left(\int_{Q_t} |v|^\beta\right) \quad \text{or} \quad \Phi\left(\int_{Q_t} dv\right)$$

where $1 \le \beta \le p_1$, $d \in L^{q_1}(Q_T)$, $\Phi \in C(\mathbb{R})$ and $\Phi \ge \text{const} > 0$. Similarly, $\psi$ may be written in the form

$$[\psi(v)](t,x) = \Psi\left(\int_{Q_t} |v|^\beta + |Dv|^\beta\right) \quad \text{or} \quad \Psi\left(\int_{Q_t} d_1 v + d_2|Dv|\right)$$

where $1 \le \beta \le p_2$, $d_1, d_2 \in L^{q_2}(Q_T)$, $\Psi \in C(\mathbb{R})$ and $\Psi \ge \text{const} > 0$ For $\tilde{\varphi}$ consider

$$[\tilde{\varphi}(v)](t,x) = \begin{cases} \tilde{\Phi}\left(\displaystyle\int_0^t d(s,x)v(s,x)\,ds\right), \\ \tilde{\Phi}\left(\displaystyle\int_\Omega d(t,x)v(t,x)\,dx\right), \quad \text{or} \\ \tilde{\Phi}\left(\left[\displaystyle\int_0^t |v(s,x)|^\beta ds\right]^{\frac{1}{\beta}}\right) \end{cases}$$

where $d \in L^\infty(Q_T)$, $1 \le \beta \le p_1$, $\tilde{\Phi} \in C(\mathbb{R})$, $\tilde{\Phi} \ge 0$ and $|\tilde{\Phi}(\tau)| \le \text{const} \cdot |\tau|^{p_1-r_1-1}$. In the case of $\tilde{\varphi}_0$ one has similar examples as for $\tilde{\varphi}$ above, except $\tilde{\Phi}$ does not have to be nonnegative.

For operators $\vartheta, \tilde{\vartheta}$ we may consider similar examples as for $\varphi, \tilde{\varphi}$ above, by replacing exponents $p_1$ with $p_2$ and $r_1$ with $r_2$.

It is not difficult to show that the above operators fulfil conditions (E1)–(E2), one can show it by similar arguments as for the examples in Section 2.2.3.

As an example for function $f$ consider, e.g.,

$$f(t,x,\xi,\zeta_0;v) = -[\varphi(v)](t,x)f_1(t,x)f_2(\zeta_0)(\xi - \omega^*(x))$$

where $\varphi\colon L^{p_1}(Q_T) \to L^\infty(Q_T)$ is bounded and nonnegative, further, $f_1 \in L^\infty(Q_T)$, $f_2\colon \mathbb{R} \to \mathbb{R}$ are nonnegative, Lipschitz continuous and $f_2(\zeta_0) \le \text{const} \cdot |\zeta_0|^{\frac{p_1}{q_2}}$.

## 3.3 Solutions in $(0, \infty)$

Mathematics is the art of giving the same name to different things.

Jules Henri Poincaré

In the previous section we have proved existence of solutions for all finite time interval $(0, T)$. In what follows, we shall show existence of weak solutions in $(0, \infty)$. We write briefly $X_i^\infty = L^{p_i}_{\mathrm{loc}}(0, \infty; V_i)$ $(i = 1, 2)$ (this space was introduced in Section 2.3.1). In the following we suppose

(Vol) Functions $a_i \colon Q_\infty \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times L^\infty_{\mathrm{loc}}(Q_\infty) \times X_1^\infty \times X_2^\infty \to \mathbb{R}$, $b_i \colon Q_\infty \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty_{\mathrm{loc}}(Q_\infty) \times X_1^\infty \times X_2^\infty \to \mathbb{R}$ $(i = 0, \dots, n)$ and $f \colon Q_\infty \times \mathbb{R}^2 \times L^\infty_{\mathrm{loc}}(Q_\infty) \times X_1^\infty \to \mathbb{R}$ have the Volterra property, i.e., for every $0 < T < \infty$ the restrictions $a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2)|_{(0,T)}$, $b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2)|_{(0,T)}$ and $f(t, x, \xi, \zeta_0; w)|_{(0,T)}$ depend only on $(w|_{(0,T)}, v_1|_{(0,T)}, v_2|_{(0,T)})$.

Besides the Volterra property we assume that conditions (A1)–(A5), (B1)–(B5), (F1), (F2), (F4) hold for every $0 < T < \infty$ in the sense that their restrictions to $(0, T)$ (that can be defined by the Volterra property, see (Vol) above) satisfy these conditions (not necessarily with the same $c_1, k_1, c_2, k_2$ etc.). Further, (F3) holds with the same $\omega^*$, i.e.,

(F3*) There exists $\omega^* \in L^\infty(\Omega)$ such that for a.a. $(t, x) \in Q_\infty$, every $(\xi, \zeta_0) \in \mathbb{R}^2$ and $v_1 \in X_1$,

$$(\xi - \omega^*(x)) \cdot f(t, x, \xi, \zeta_0; v_1) \le 0.$$

Finally, let

(G1*) $\mathcal{G} \in L^{q_1}_{\mathrm{loc}}(0, \infty; V_1^*)$

(H1*) $\mathcal{H} \in L^{q_2}_{\mathrm{loc}}(0, \infty; V_2^*)$.

Now we may define the weak form of (3.5)–(3.7) in $(0, \infty)$. For fixed $0 < T < \infty$ we introduce operators $\mathcal{A}_T \colon L^\infty(Q_T) \times X_1 \times X_2 \to X_1^*$ and $\mathcal{B}_T \colon L^\infty(Q_T) \times X_1 \times X_2 \to X_2^*$, $L_T \colon D(L_T) \to L^{q_1}(0, T; V_1^*)$ by formulae (3.10)–(3.12). In addition, let $\mathcal{G}_T = \mathcal{G}|_{(0,T)} \in X_1^*, \mathcal{H}_T = \mathcal{H}|_{(0,T)} \in X_2^*$ for every $0 < T < \infty$. By the Volterra property there exists operators $\mathcal{A} \colon L^\infty_{\mathrm{loc}}(Q_\infty) \times X_1^\infty \times X_2^\infty \to L^{q_1}_{\mathrm{loc}}(0, \infty; V_1^*)$ and $\mathcal{B} \colon L^\infty_{\mathrm{loc}}(Q_\infty) \times X_1^\infty \times X_2^\infty \to L^{q_2}_{\mathrm{loc}}(0, \infty; V_2^*)$ such that $\mathcal{A}_T(\omega, u, \mathbf{p}) = \mathcal{A}(\omega, u, \mathbf{p})|_{(0,T)}$, $\mathcal{B}_T(\omega, u, \mathbf{p}) = \mathcal{B}(\omega, u, \mathbf{p})|_{(0,T)}$ for every $0 < T < \infty$ and $(\omega, u, \mathbf{p}) \in L^\infty_{\mathrm{loc}}(Q_\infty) \times L^{p_1}_{\mathrm{loc}}(0, \infty; V_1) \times L^{p_2}_{\mathrm{loc}}(0, \infty; V_2)$ We say that $\omega \in L^\infty_{\mathrm{loc}}(Q_\infty), u \in L^{p_1}_{\mathrm{loc}}(0, \infty; V_1), \mathbf{p} \in$

$L^{p_2}_{\text{loc}}(0,\infty;V_2)$ is a weak solution of (3.5)–(3.7) in $(0,\infty)$ if for all $0 < T < \infty$, $u|_{(0,T)} \in D(L_T)$ and (for the restrictions of the functions to $(0,T)$)

$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x);u)\,ds \quad (t,x) \in Q_T \tag{3.34}$$

$$L_T u + \mathcal{A}_T(\omega,u,\mathbf{p}) = G_T \tag{3.35}$$

$$\mathcal{B}_T(\omega,u,\mathbf{p}) = \mathcal{H}_T. \tag{3.36}$$

(As in the previous chapter we omit the notation $|_{(0,T)}$ if it is not confusing, since the operators and the norms contain the information about the space). Observe that (as for the problem of the previous chapter) the Volterra property ensures that if $\omega, u, \mathbf{p}$ is a solution in $(0,T)$ for some $T$ then these functions are solutions in $(0,\tilde{T})$ for all $\tilde{T} < T$.

**Theorem 3.10.** *Assume (Vol). Further, suppose that conditions (A1)–(A5), (B1)–(B5), (F1), (F2), (F4) hold for every* $0 < T < \infty$ *(in the above explained sense), and (F3*), (G1*), (H1*) are staisfied. Then there exist weak solutions* $\omega \in L^\infty(Q_\infty)$, $u \in L^{p_1}_{\text{loc}}(0,\infty;V_1)$, $\mathbf{p} \in L^{p_2}_{\text{loc}}(0,\infty;V_2)$ *to problem* (3.34)–(3.36).

*Proof.* The main idea is the same as in the proof of Theorem 2.12. By Theorem 3.1, for every $0 < T < \infty$ there exist solutions in $(0,T)$. Then the limit of some weakly convergent subsequences of the solutions which was choosen by a diagonal method will be a solution in $(0,\infty)$.

Let $(T_k)$ be a monotone increasing sequence of positive numbers such that $T_k \to +\infty$. Then by Theorem 3.1, for every $T_k$ there exists a solution of (3.34)-(3.35), i.e., there are $\omega_k \in L^\infty(Q_{T_k})$, $u_k \in L^{p_1}(0,T_k;V_1)$, $\mathbf{p}_k \in L^{p_2}(0,T_k;V_2)$ such that

$$\omega_k(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega_k(s,x),u_k(s,x);u_k)\,ds$$

$$L_{T_k} u_k + \mathcal{A}_{T_k}(\omega_k,u_k,\mathbf{p}_k) = \mathcal{G}_{T_k}$$

$$\mathcal{B}_{T_k}(\omega_k,u_k,\mathbf{p}_k) = \mathcal{H}_{T_k}.$$

By applying Proposition 3.2 it follows

$$\|\omega_k\|_{L^\infty(Q_{T_m})} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}. \tag{3.37}$$

Further, by following the proof of Theorem 3.1 (with $(0,T) = (0,T_m)$) one obtains for fixed $m \in \mathbb{N}$ the boundedness of the sequences $(u_k)$, $(L_{T_m} u_k)$ and $(\mathbf{p}_k)$ in spaces $X_1^{T_m}$, $(X_1^{T_m})^*$ and $X_2^{T_m}$, respectively.

Now let $m = 1$. Since $(u_k), (L_{T_1} u_k), (\mathbf{p}_k)$ are bounded sequences in reflexive Banach spaces $X_1^{T_1}, (X_1^{T_1})^*, X_2^{T_1}$, respectively, there exist subsequences $(u_{1,k}) \subset$

$(u_k), (\mathbf{p}_{1,k}) \subset (\mathbf{p}_k)$ and functions $u_{1,*} \in X_1^{T_1} \cap D(L_{T_1})$, $\mathbf{p}_{1,*} \in X_2^{T_1}$ such that

$$u_{1,k} \to u_{1,*} \text{ weakly in } X_1^{T_1},$$
$$L_{T_1} u_{1,k} \to L_{T_1} u_{1,*} \text{ weakly in } (X_1^{T_1})^*,$$
$$\mathbf{p}_{1,k} \to \mathbf{p}_{1,*} \text{ weakly in } X_2^{T_1}.$$

If $(u_{m-1,k})_{k \geq m-1}$ is already given then sequences $(u_{m-1,k})_{k \geq m-1}$, $(L_{T_{m-1}} u_{m-1,k})_{k \geq m-1}$, $(\mathbf{p}_{m-1,k})_{k \geq m-1}$ are bounded in reflexive spaces $X_1^{T_{m-1}}, (X_1^{T_{m-1}})^*, X_2^{T_{m-1}}$ thus there exist subsequences $(u_{m,k}) \subset (u_{m-1,k}), (\mathbf{p}_{m,k}) \subset (\mathbf{p}_{m-1,k})$ and functions $u_{m,*} \in X_1^{T_m} \cap D(L_{T_m}), \mathbf{p}_{m,*} \in X_2^{T_m}$ such that

$$u_{m,k} \to u_{m,*} \text{ weakly in } X_1^{T_m},$$
$$L_{T_m} u_{m,k} \to L_{T_m} u_{m,*} \text{ weakly in } (X_1^{T_m})^*,$$
$$\mathbf{p}_{m,k} \to \mathbf{p}_{m,*} \text{ weakly in } X_2^{T_m}.$$

It is clear that for each fixed $l < m$ the above weak convergences hold in $X_1^{T_l}$, $(X_1^{T_l})^*$, $X_2^{T_l}$, respectively, which yields $u_{m,*}|_{(0,T_l)} = u_{l,*}$ and $\mathbf{p}_{m,*}|_{(0,T_l)} = u_{l,*}$ for $l < m$. Consequently, there exist unique functions $u\colon (0,\infty) \to V_1, \mathbf{p}\colon (0,\infty) \to V_2$ such that $u|_{(0,T_m)} = u_{m,*}$, $\mathbf{p}|_{(0,T_m)} = \mathbf{p}_{m,*}$ and $u_{m,*} \in D(L_{T_m})$ for every $m \in \mathbb{N}$. This means that $u \in L^{p_1}_{\text{loc}}(0,\infty;V_1)$, $u|_{(0,T)} \in D(L_T)$ for every $0 < T < \infty$ and $\mathbf{p} \in L^{p_2}_{\text{loc}}(0,\infty;V_2)$. Consider the "diagonal" sequences $(u_k) = (u_{k,k}), (\mathbf{p}_k) = (\mathbf{p}_{k,k})$ and the corresponding sequence $(\omega_k)$. Observe that $u_k \to u$ weakly in $X_1^{T_m}$, $D_t u_k \to D_t u$ weakly in $(X_1^{T_m})^*$, $\mathbf{p}_k \to \mathbf{p}$ weakly in $X_2^{T_m}$ for each fixed $m$. Thus by Corollary 1.48 we may assume that $u_k \to u$ in $L^{p_1}(Q_{T_m})$. Then from Proposition 3.2, 3.3 it follows that for every $m$ there exists $\omega_{m,*} \in L^\infty(Q_{T_m})$ such that $(\omega_k) \to \omega_{m,*}$ a.e. in $Q_{T_m}$ and

$$\omega_{m,*}(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega_{m,*}(s,x),u_{m,*}(s,x);u_{m,*})\,ds \quad (t,x) \in Q_{T_m}.$$

Since for every fixed $u_{m,*}$ the solution of the above equation is unique, further, functions $(u_{m,*})$ are the restrictions of the function $u$ to $(0,T_m)$, we cocnclude that there exists a unique $\omega \in L^\infty_{\text{loc}}(Q_\infty)$ such that $\omega_{m,*} = \omega|_{(0,T_m)}$ for every $m$ and

$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x);u)\,ds \quad (t,x) \in Q_\infty.$$

By (3.37), $\omega \in L^\infty(Q_\infty)$. Now fix $m \in \mathbb{N}$. Then we may deduce from the above arguments that

$$\omega_k \to \omega \text{ a.e. in } Q_{T_m}$$
$$u_k \to u \text{ weakly in } X_1^{T_m}, \text{ strongly in } L^{p_1}(Q_{T_m}), \text{ a.e. in } Q_{T_m};$$
$$L_{T_m} u_k \to L_{T_m} u \text{ weakly in } (X_1^{T_m})^*;$$
$$\mathbf{p}_k \to \mathbf{p} \text{ weakly in } X_2^{T_m}.$$

By applying word for word Step 3 of the proof of Theorem 3.1, the above convergences imply that $u_k \to u$ strongly in $X_1^{T_m}$, $\mathbf{p}_k \to \mathbf{p}_{m,*}$ strongly in $X_2^{T_m}$ and

$$\begin{aligned} L_{T_m} u + \mathcal{A}_{T_m}(\omega, u, \mathbf{p}) &= \mathcal{G}_{T_m} \\ \mathcal{B}_{T_m}(\omega, u, \mathbf{p}) &= \mathcal{H}_{T_m}. \end{aligned}$$

This means that $\omega, u, \mathbf{p}$ are solutions in $(0,\infty)$ so the proof of the theorem is complete. $\square$

### 3.3.1 Boundedness

Now we show that under some further assumptions, the solutions, formulated in Theorem 3.10, are bounded (in appropriate norms) in the time interval $(0,\infty)$. First suppose

(A4*) There exist a constant $c_2 > 0$, a continuous function $\gamma \colon \mathbb{R} \to \mathbb{R}$ and bounded operators $\Gamma \colon L^\infty_{\mathrm{loc}}(Q_\infty) \to L^\infty(\Omega)$, $k_2 \colon X_1^\infty \to L^1(\Omega)$ of Volterra type such that

$$\begin{aligned} &\sum_{i=0}^n a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta; w, v_1, v_2)\zeta_i \\ &\geq c_2 \left(|\zeta_0|^{p_1} + |\zeta|^{p_1}\right) - \gamma(\xi)[\Gamma(w)](x)[k_2(v_1)](x) \end{aligned}$$

for a.a. $(t,x) \in Q_T$ and every $(\xi,\zeta_0,\zeta,\eta_0,\eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$. Further, for every $0 < T < \infty$ and $K > 0$ there is a constant $L > 0$ such that $\|\Gamma(w)\|_{L^\infty(\Omega)}, \|k_2(v_1)\|_{L^q(\Omega)} \leq K$ whenever $\|w|_{Q_T}\|_{L^\infty(Q_T)}, \|v_1|_{(0,T)}\|_{X_1} \leq L$. In addition, for every $0 < T < \infty$

$$\lim_{\|v_1\|_{X_1} \to +\infty} \frac{\|k_2(v_1)\|_{L^1(Q_T)}}{\|v_1\|_{X_1}^{p_1}} = 0.$$

Finally, there exist constants $\alpha_1 > 0, \varrho_1 < p_1$ and a continuous function $\chi_1 \colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} \chi_1(t) = 0$, further, if $v_1 \in L^{p_1}_{\mathrm{loc}}(0,\infty;V_1)$ and $D_t v_1 \in L^{q_1}_{\mathrm{loc}}(0,\infty;V_1^*)$ then for a.a. $t \in (0,\infty)$,

$$\int_\Omega |[k_2(v_1)](t,x)|\,dx \leq \alpha_1 \left[ \sup_{\tau\in[0,t]} \|v_1(\tau)\|^{\varrho_1}_{L^2(\Omega)} + \chi_1(t) \sup_{\tau\in[0,t]} \|v_1(\tau)\|^{p_1}_{L^2(\Omega)} + 1 \right].$$

(B4*) There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma} \colon \mathbb{R} \to \mathbb{R}$ and operators $\hat{\Gamma} \colon L^\infty_{\mathrm{loc}}(Q_\infty) \to L^\infty(\Omega)$, $\hat{k}_2 \colon X_2^\infty \to L^1(\Omega)$ of Volterry type such that

$$\begin{aligned} &\sum_{i=0}^n b_i(t,x,\xi,\zeta_0,\eta_0,\eta; w, v_1, v_2)\eta_i \\ &\geq \hat{c}_2 \left(|\eta_0|^{p_2} + |\eta|^{p_2}\right) - \hat{\gamma}(\xi)[\hat{\Gamma}(w)](x) \Big(|\zeta_0|^{p_1} + [\hat{k}_2(v_2)](x)\Big) \end{aligned}$$

for a.a. $(t,x) \in Q_\infty$, and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty_{\text{loc}}(Q_\infty) \times X_1^\infty \times X_2^\infty$. Further, for every $0 < T < \infty$ and $K > 0$ there exists a constant $L > 0$ such that $\|\hat{\Gamma}(w)\|_{L^\infty(\Omega)}, \|\hat{k}_2(v_2)\|_{L^1(\Omega)} \le L$. In addition, for every $0 < T < \infty$

$$\lim_{\|v_2\|_{X_2} \to \infty} \frac{\|\hat{k}_2(v_2)\|_{L^1(Q_T)}}{\|v_2\|_{X_2}^{p_2}} = 0.$$

Finally, there exist constants $\alpha_2 > 0, \varrho_2 < p_2$ and a continuous function $\chi_2 \colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} \chi_2(t) = 0$, further, if $v_2 \in L^{p_2}_{\text{loc}}(0,\infty; V_2)$ then for a.a. $t \in (0,\infty)$,

$$\int_\Omega |[\hat{k}_2(v_2)](t,x)|\,dx \le \alpha_2 \left[ \operatorname*{ess\,sup}_{\tau\in[0,t]} \|v_2(\tau)\|_{V_2}^{\varrho_2} + \chi_2(t) \operatorname*{ess\,sup}_{\tau\in[0,t]} \|v_2(\tau)\|_{V_2}^{p_2} + 1 \right].$$

(G1**) There exists $t_*$ such that $\mathcal{G}|_{(t_*,\infty)} \in L^\infty(t_*, \infty; V_1^*)$.

(H1**) There exists $\hat{t}_*$ such that $\mathcal{H}|_{(\hat{t}_*,\infty)} \in L^\infty(\tilde{t}_*, \infty; V_2^*)$.

*Remark* 3.11. The suprema in condition (A4*) make sense since $v \in L^{p_1}_{\text{loc}}(0,\infty; V_1)$ and $D_t v \in L^{q_1}_{\text{loc}}(0,\infty; V_1^*)$ implies that $v \in C([0,\infty), L^2(\Omega))$.

**Theorem 3.12.** *Assume (Vol), further suppose that conditions (A1)–(A3), (A5), (B1)–(B3), (B5), (F1), (F2), (F4) are staisfied for every $0 < T < \infty$. In addition, (F3*), (A4*), (B4*), (G1**), (H1**) are fulfilled. Then for the solutions $\omega, u, \mathbf{p}$ of problem* (3.34)–(3.36)*, $\omega \in L^\infty(Q_\infty)$, $u \in L^\infty(0,\infty; L^2(\Omega))$, $\mathbf{p} \in L^\infty(0,\infty; V_2)$ hold.*

*Proof.* In Theorem 3.1 we have verified that $\omega \in L^\infty(Q_\infty)$ (which was a trivial consequence of (3.37)). In the following let $y(t) = \|u(t,\cdot)\|^2_{L^2(\Omega)}$. First note that $u \in C([0,T], L^2(\Omega))$ thus $y$ is continuous in $[0,T]$. We show that $y$ is bounded in $(0,\infty)$. Since $u$ is a solution of (3.35) for all $0 < T < \infty$, thus for arbitrary $t_* < T_1 < T_2 < \infty$,

$$\int_{T_1}^{T_2} \langle D_t u(t), u(t)\rangle\,dt + \int_{T_1}^{T_2} \langle [\mathcal{A}(\omega, u, \mathbf{p})](t), u(t)\rangle\,dt = \int_{T_1}^{T_2} \langle \mathcal{G}(t), u(t)\rangle\,dt. \tag{3.38}$$

On the right hand side the $\varepsilon > 0$-inequality yields

$$\begin{aligned}\int_{T_1}^{T_2} \langle \mathcal{G}(t), u(t)\rangle\,dt &\le \int_{T_1}^{T_2} \left( \frac{\varepsilon^{p_1}}{p_1} \|u(t)\|_{V_1}^{p_1} + \frac{1}{q_1 \varepsilon^{q_1}} \|[\mathcal{G}(t)\|_{V_1^*} \right) dt \\ &\le \int_{T_1}^{T_2} \left( \frac{\varepsilon^{p_1}}{p_1} \|u(t)\|_{V_1}^{p_1} + c(\varepsilon) \right) dt.\end{aligned}$$

By using Corollary 1.43, condition (A4*) on the left hand side of (3.38), further, by applying the above estimate with sufficiently small $\varepsilon$ on the right hand side, it follows (similarly as in the proof of Theorem 2.15)

$$\frac{1}{2}\left(y(T_1)-y(T_2)\right)+\frac{1}{2}c_2\int_{T_1}^{T_2} y^{\frac{p_1}{2}}\,dt$$
$$\le \text{const}\int_{T_1}^{T_2}\int_\Omega\left[\sup_{\tau\in[0,t]} y(\tau)^{\frac{\varrho_1}{2}}+\chi_1(t)\sup_{\tau\in[0,t]} y(\tau)^{\frac{p_1}{2}}+1\right]dx\,dt.$$

The above inequality implies the boundedness of $y$, one may prove it by contradiction, the same way as in Theorem 2.15.

It remains to show that $\mathbf{p}\in L^\infty(0,\infty;V_2)$. The proof goes the same way as in the previous part (moreover it is simpler since there is no derivative with respect to $t$), by using condition (B4*) and the boundedness of $\omega$, from $\mathcal{B}(\omega,u,\mathbf{p})=\mathcal{H}$ it follows

$$\|\mathbf{p}(t)\|_{V_2}^{p_2}$$
$$\le \text{const}\left(\|u(t)\|^2_{L^2(\Omega)}+\operatorname*{ess\,sup}_{\tau\in[0,t]}\|\mathbf{p}(\tau)\|_{V_2}^{\varrho_2}+\chi_2(t)\operatorname*{ess\,sup}_{\tau\in[0,t]}\|\mathbf{p}(\tau)\|_{V_2}^{p_2}+1\right). \tag{3.39}$$

We show that the above inequality implies $\mathbf{p}\in L^\infty(0,\infty;V_2)$. Since $\mathbf{p}$ is not necessarily continuous we may not apply the arguments of the proof of Theorem 2.15 word for word, we have to generalize it to measurable functions. Supposing that $\mathbf{p}$ is not bounded, the sequence $(\operatorname{ess\,sup}_{t\in[n,n+1]})(\|\mathbf{p}(t)\|_{V_2})_{n\in\mathbb{N}}$ has got a subsequence $(M_k)$ which tends to $+\infty$ increasingly. Denote by $A_k$ the intervals corresponding to $M_k$. Then for every $k$ there exists a measurable subset $B_k\subset A_k$ with positive measure such that $\|\mathbf{p}(t)\|_{V_2}>M_k-1$ a.e. in $B_k$. Now by integrating inequality (3.39) on $B_k$, the above estimates on $\mathbf{p}$ and the boundedness of $\|u(t)\|^2_{L^2(\Omega)}$ yield (similarly to (2.52))

$$(M_k-1)^{p_2}\lambda(B_k)\le d_3M_k^{\varrho_2}\lambda(B_k)+d_3M_k^{p_2}\lambda(B_k)\int_{B_k}\chi_2(t)\,dt+d_3\lambda(B_k)$$

where $\lambda(B_k)$ is the measure of $B_k$ and $\chi_2(t)\to 0$ as $t\to+\infty$. By using the fact that $\lambda(B_k)\le 1$ we may deduce $\int_{B_k}\chi_2(t)\,dt\to 0$ as $k\to\infty$. Since $M_k\to+\infty$, by the same arguments as at the end of the proof of Theorem 2.15 we may arrive to a contradiction. The proof of Theorem 3.12 is complete. □

### 3.3.2 Stabilization

In this section we consider a special case of problem (3.34)–(3.36), namely, let $p_1=p_2=p$ (thus $q_1=q_2=q$, $V_1=V_2=V$, $X^\infty=L^p_{\text{loc}}(0,\infty;V)$). In what follows, we prove stabilization of the solutions, that is, we show the convergence of solutions as $t\to\infty$ to some stationary solutions. We need some further assumptions:

(A2$^+$) For every $w \in L^\infty(Q_\infty), v_1 \in X^\infty \cap L^\infty(0,\infty;L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty;V)$, there exist a constant $c_{(w,v_1,v_2)} > 0$ and a function $k_{(w,v_1,v_2)} \in L^q(\Omega)$ such that

$$\begin{aligned}&|a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)|\\ &\leq c_{(w,v_1,v_2)}\left(|\zeta_0|^{p_1-1}+|\zeta|^{p_1-1}+|\eta_0|^{\frac{p_2}{q_1}}+|\eta|^{\frac{p_2}{q_1}}+k_{(w,v_1,v_2)}(x)\right),\end{aligned}$$

for a.a. $(t,x) \in Q_T$, every $(\xi,\zeta_0,\zeta,\eta_0,\eta) \in \mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ $(i=0,\dots,n)$.

(A6) There exist Carathéodory functions $a_{i,\infty}\colon \Omega\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to\mathbb{R}$ $(i=0,\dots,n)$ such that for a.a. $x\in\Omega$ and every $(\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, $\xi^*\in\mathbb{R}$, $w \in L^\infty(Q_\infty), v_1 \in X^\infty \cap L^\infty(0,\infty;L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty;V)$,

$$\lim_{\substack{t\to\infty\\ \xi\to\xi^*}} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) = a_{i,\infty}(x,\xi^*,\zeta_0,\zeta,\eta_0,\eta).$$

(B2$^+$) For every $w \in L^\infty(Q_\infty), v_1 \in X^\infty \cap L^\infty(0,\infty;L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty;V)$, there exist a constant $\hat{c}_{(w,v_1,v_2)} > 0$ and a function $\hat{k}_{(w,v_1,v_2)} \in L^q(\Omega)$ such that

$$\begin{aligned}&|b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2)|\\ &\leq \hat{c}_{(w,v_1,v_2)}\left(|\eta_0|^{p_2-1}+|\eta|^{p_2-1}+|\zeta_0|^{\frac{p_1}{q_2}}+\hat{k}_{(w,v_1,v_2)}(x)\right)\end{aligned}$$

for a.a. $(t,x)\in Q_T$ and every $(\xi,\zeta_0,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1}$ $(i=0,\dots,n)$.

(B6) There exist Carathéodory functions $b_{i,\infty}\colon \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1}\to\mathbb{R}$ $(i=0,\dots,n)$ such that for a.a. $x\in\Omega$ and every $(\zeta_0,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}$, $\xi^*\in\mathbb{R}$, $w \in L^\infty(Q_\infty), v_1 \in X^\infty \cap L^\infty(0,\infty;L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty;V)$,

$$\lim_{\substack{t\to\infty\\ \xi\to\xi^*}} b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) = b_{i,\infty}(x,\xi^*,\zeta_0,\eta_0,\eta).$$

(AB) There exists a positive constant $\mathcal{C}$ such that for a.a. $(t,x)\in Q_\infty$ and every $\xi\in\mathbb{R}$, $(\zeta_0,\zeta,\eta_0,\eta)$, $(\tilde{\zeta}_0,\tilde{\zeta},\tilde{\eta}_0,\tilde{\eta})\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, $w\in L^\infty(Q_\infty)$, $v_1,v_2\in X^\infty$,

$$\begin{aligned}&\sum_{i=0}^n\left(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)-a_i(t,x,\xi,\tilde{\zeta}_0,\tilde{\zeta},\tilde{\eta}_0,\tilde{\eta};w,v_1,v_2)\right)(\zeta_i-\tilde{\zeta}_i)\\ &\quad+\sum_{i=0}^n\left(b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2)-b_i(t,x,\xi,\tilde{\zeta}_0,\tilde{\eta}_0,\tilde{\eta};w,v_1,v_2)\right)(\eta_i-\tilde{\eta}_i)\\ &\geq \mathcal{C}\cdot\left(|\zeta_0-\tilde{\zeta}_0|^p+|\zeta-\tilde{\zeta}|^p+|\eta_0-\tilde{\eta}_0|^p+|\eta-\tilde{\eta}|^p\right)\\ &\quad-r(t,x,\zeta_0,\tilde{\zeta}_0,\eta_0,\tilde{\eta}_0;w,v_1,v_2).\end{aligned}$$

where $r\colon Q_\infty\times\mathbb{R}^2\times\mathbb{R}^2\times L^\infty(Q_\infty)\times X^\infty\times X^\infty\to\mathbb{R}$ such that for $w\in L^\infty(Q_\infty)$, $u,\tilde{u},\mathbf{p},\tilde{\mathbf{p}}\in L^\infty(0,\infty;L^2(\Omega))$,

$$\lim_{t\to\infty}\int_\Omega r(t,x,u(t,x),\tilde{u}(t,x),\mathbf{p}(t,x),\tilde{p}(t,x);w,u,\mathbf{p})\,dx=0.$$

(F5) For every fixed $v \in X^\infty \cap L^\infty(0,\infty; L^2(\Omega))$ there is a constant $m > 0$ such that

$$(\xi - \omega^*(x)) f(t, x, \xi, \zeta_0; v) \le -m(\xi - \omega^*(x))^2$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R}^2$.

(G2) There exists $\mathcal{G}_\infty \in V^*$ such that $\lim_{t\to\infty} \|\mathcal{G}(t) - \mathcal{G}_\infty\|_{V^*} = 0$.

(H2) There exists $\mathcal{H}_\infty \in V^*$ such that $\lim_{t\to\infty} \|\mathcal{H}(t) - \mathcal{H}_\infty\|_{V^*} = 0$.

*Remark* 3.13. In conditions (A6), (B6) by the convergence of measurable functions we mean the same as in Remark 2.17.

Now introduce $\mathcal{A}_\infty \colon L^\infty(\Omega) \times V \times V \to V^*$ and $\mathcal{B}_\infty \colon L^\infty(\Omega) \times V \times V \to V^*$ by

$$\begin{aligned}
\langle \mathcal{A}_\infty(\omega, u, \mathbf{p}), v \rangle &:= \int_\Omega \sum_{i=1}^n a_{i,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x)) D_i v(x)\, dx \\
&\quad + \int_\Omega a_{0,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x)) v(x)\, dx, \\
\langle \mathcal{B}_\infty(\omega, u, \mathbf{p}), v \rangle &:= \int_\Omega \sum_{i=1}^n b_{i,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x)) D_i v(x)\, dx \\
&\quad + \int_\Omega b_{0,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x)) v(x)\, dx,
\end{aligned}$$

for $v \in V$.

**Theorem 3.14.** *Assume (Vol), further, suppose that conditions (A1)–(A3), (A5), (B1)–(B3), (B5), (F1), (F2), (F4) hold for every* $0 < T < \infty$. *In addition, (F3*), (A2+), (A4*), (B2+), (B4*), (A6), (B6), (AB), (F5), (G2), (H2) are satisfied. Then there exist unique* $u_\infty \in V, \mathbf{p}_\infty \in V$ *such that the solutions* $\omega, u, \mathbf{p}$ *of* (3.34)–(3.36) *possess the following convergence relations:*

$$\begin{gathered}
\|\omega(t,\cdot) - \omega^*\|_{L^\infty(\Omega)} \le \|\omega_0\|_{L^\infty(\Omega)} e^{-mt}, \\
u(t) \to u_\infty \text{ in } L^2(\Omega), \quad \int_{t-1}^{t+1} \|u(s) - u_\infty\|_V^p\, ds \to 0, \\
\int_{t-1}^{t+1} \|\mathbf{p}(s) - \mathbf{p}_\infty\|_V^p\, ds \to 0.
\end{gathered}$$

*In addition,*

$$\mathcal{A}_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = \mathcal{G}_\infty \tag{3.40}$$

$$\mathcal{B}_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = \mathcal{H}_\infty. \tag{3.41}$$

*Proof.* Let $\omega, u, \mathbf{p}$ be solution of (3.13)–(3.15) in $(0,\infty)$ then from Theorem 3.12 it follows $\omega \in L^\infty(Q_\infty)$, $u \in L^\infty(0,\infty; L^2(\Omega))$, $\mathbf{p} \in L^\infty(0,\infty; V_2)$.

First we show that $\omega(t,\cdot) \to \omega^*$ in $L^\infty(\Omega)$ as $t \to \infty$. Fix $x \in \Omega$ and assume $\omega_0(x) > \omega^*(x)$. By using similar arguments as in the proof of Proposition 3.5 we obtain $\omega(t,x) > \omega^*(x)$ for $t > 0$. Then condition (F5*) yields

$$f(t,x,\omega(t,x),u(t,x);u) \leq -m(\omega(t,x) - \omega^*(x)).$$

Since $\omega$ is absolutely continuous, it is a.e. differentiable in $Q_\infty$ so

$$\omega'(t,x) = f(t,x,\omega(t,x),u(t,x);u) \leq -m(\omega(t,x) - \omega^*(x)).$$

By the positivity of $\omega - \omega^*$ it follows

$$\frac{\omega'(t,x)}{\omega(t,x) - \omega^*(x)} \leq -m$$

hence

$$\omega(t,x) - \omega^*(x) \leq \omega_0(x)e^{-mt}.$$

When $\omega_0(x) < \omega^*(x)$ one has estimate

$$-\omega_0(x)e^{-mt} \leq \omega(t,x) - \omega^*(x)$$

thus

$$\|\omega(t,\cdot) - \omega^*\|_{L^\infty(\Omega)} \leq \|\omega_0\|_{L^\infty(\Omega)}e^{-mt}.$$

Before the proof of the other convergences we note the following. By fixing $w \in L_{\mathrm{loc}}(Q_\infty)$, $v_1, v_2 \in X^\infty$ in condition (A2),

$$|a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)| \leq \mathfrak{c}\cdot c_1(\xi)\left(|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1} + |\eta_0|^{\frac{p_2}{q_1}} + |\eta|^{\frac{p_2}{q_1}} + k_1(x)\right)$$

for a.a. $(t,x) \in Q_\infty$, every $(\zeta_0,\zeta), (\eta_0,\eta) \in \mathbb{R}^{n+1}$ with constant $\mathfrak{c} = \mathfrak{c}(w,v_1,v_2)$ and function $k_1 = k_1(w,v_1,v_2) \in L^q(\Omega)$. Now passing to the limit as $t \to \infty$ yields

$$|a_{i,\infty}(x,\xi,\zeta_0,\zeta,\eta_0,\eta)| \leq \mathfrak{c}\cdot c_1(\xi)\left(|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1} + |\eta_0|^{\frac{p_2}{q_1}} + |\eta|^{\frac{p_2}{q_1}} + k_1(x)\right)$$

so functions $a_{i,\infty}$ can be estimated similarly as functions $a_i$ in condition (A2). Functions $b_{i,\infty}$ can be estimated similarly.

Now we show that problem (3.40)–(3.41) has got a unique solution $u_\infty \in V, \mathbf{p}_\infty \in V$ for fixed $\omega^* \in L^\infty(\Omega)$. By using similar arguments as in the proof of Lemma 2.19 it follows that operator $(\mathcal{A}_\infty, \mathcal{B}_\infty)\colon V \times V \to (V \times V)^*$ is bounded, hemicontinuous, coercive and uniformly monotone. So that there exist unique $u_\infty, \mathbf{p}_\infty \in V$ satisfying (3.40)–(3.41) $\mathcal{A}_\infty(u_\infty) + \mathcal{B}_\infty(\mathbf{p}_\infty) = \mathcal{G}_\infty + \mathcal{H}_\infty$. Thus by choosing $u = 0$ and $\mathbf{p} = 0$ it follows that $u_\infty, \mathbf{p}_\infty$ are unique solutions of (3.40)–(3.41).

In order to show the desired convergences we prove an integral inequality for $u$ and $\mathbf{p}$. From equations (3.34)–(3.36) and (3.40)–(3.41) it follows

$$\begin{aligned}
&\langle \mathcal{G}(t)-\mathcal{G}_\infty, u(t)-u_\infty\rangle + \langle \mathcal{H}(t)-\mathcal{H}_\infty, \mathbf{p}(t)-p_\infty\rangle \\
&= \langle D_t(u(t)-u_\infty), u(t)-u_\infty\rangle \\
&\quad + \langle [\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - \mathcal{A}_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), u(t)-u_\infty\rangle \\
&\quad + \langle [\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - \mathcal{B}_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), \mathbf{p}(t)-\mathbf{p}_\infty\rangle
\end{aligned} \tag{3.42}$$

where for fixed $(w,v_1,v_2)\in L^\infty_{\mathrm{loc}}(Q_\infty)\times X^\infty\times X^\infty$ and $t>0$ operator $[\mathcal{A}_{(w,v_1,v_2)}](t)\colon L^\infty_{\mathrm{loc}}(Q_\infty)\times X^\infty\times X^\infty\to V^*$ is given by

$$\begin{aligned}
&\langle [\mathcal{A}_{(w,v_1,v_2)}(\omega,u,\mathbf{p})](t), z\rangle \\
&:= \int_\Omega \sum_{i=1}^n a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);w,v_1,v_2)D_i z(x)\,dx \\
&\quad + \int_\Omega a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);w,v_1,v_2)z(x)\,dx
\end{aligned}$$

with $z\in V$. Operator $\mathcal{B}_{(w,v_1,v_2)}$ is given in the same manner. Observe that the first term on the left hand side of the above equation equals to $\frac{1}{2}y'(t)$ where $y(t)=\int_\Omega (u(t)-u_\infty)^2$ (note that $y$ is bounded in $[0,\infty)$ by Theorem 3.12). Now consider the following decomposition on the right side of (3.42):

$$\begin{aligned}
&\langle [\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - \mathcal{A}_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), u(t)-u_\infty\rangle \\
&\quad + \langle [\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - B_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), \mathbf{p}(t)-\mathbf{p}_\infty\rangle \\
&= \langle [\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - [\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t), u(t)-u_\infty\rangle \\
&\quad + \langle [\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - [\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t), \mathbf{p}(t)-\mathbf{p}_\infty\rangle \\
&\quad + \langle [\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t) - \mathcal{A}_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), u(t)-u_\infty\rangle \\
&\quad + \langle [\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t) - \mathcal{B}_\infty(\omega^*,u_\infty,\mathbf{p}_\infty), \mathbf{p}(t)-\mathbf{p}_\infty\rangle.
\end{aligned} \tag{3.43}$$

By using the $\varepsilon$-inequality and condition (AB) on the right hand side of the above inequality we have

$$\begin{aligned}
&\langle [\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - [\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t), u(t)-u_\infty\rangle \\
&\quad + \langle [\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u,\mathbf{p})](t) - [\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t), \mathbf{p}(t)-\mathbf{p}_\infty\rangle \\
&\geq \mathcal{C}\cdot(\|u(t)-u_\infty\|_V^p + \|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p) \\
&\quad - \frac{\varepsilon^p}{p}\|u(t)-u_\infty\|_V^p - \frac{\varepsilon^p}{p}\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p \\
&\quad - \frac{1}{q\varepsilon^q}\|[\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t) - \mathcal{A}_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q \\
&\quad - \frac{1}{q\varepsilon^q}\|[\mathcal{B}_{(\omega,u,\mathbf{p})}(\omega,u_\infty,\mathbf{p}_\infty)](t) - \mathcal{B}_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q.
\end{aligned} \tag{3.44}$$

We show that last two terms on the right hand side of (3.44) converge to 0 as $t \to \infty$. Clearly,

$$\begin{aligned}
&\|[\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega, u_\infty, \mathbf{p}_\infty)](t) - \mathcal{A}_\infty(\omega^*, u_\infty, \mathbf{p}_\infty)\|_{V^*}^q \\
&\le \sum_{i=0}^n \int_\Omega |a_i(t, \omega(t), u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty; \omega, u, \mathbf{p}) - a_{i,\infty}(\omega^*, u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty)|^q.
\end{aligned}$$

The integrand on the right hand side of the above estimate is a.e. convergent in $\Omega$ as $t \to \infty$ by condition (A6) since $\omega(t,x) \to \omega^*(x)$ for a.a. $x \in \Omega$. Further, condition (A2) implis

$$\begin{aligned}
&|a_i(t, \cdot, \omega(t, \cdot), u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty; \omega, u, \mathbf{p}) - a_{i,\infty}(\omega^*, u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty)|^q \\
&\le \text{const} \cdot \big(\|c_1(\omega)\|_{L^\infty(Q_\infty)} + \|c_1(\omega^*)\|_{L^\infty(Q_\infty)}\big) \\
&\quad \times (|u_\infty|^p + |Du_\infty|^p + |\mathbf{p}_\infty|^p + |D\mathbf{p}_\infty|^p + \|k_1\|_{L^q(\Omega)})
\end{aligned}$$

where the right hand side is integrable in $L^1(\Omega)$ thus Lebesgue's theorem yields

$$\|[\mathcal{A}_{(\omega,u,\mathbf{p})}(\omega, u_\infty, \mathbf{p}_\infty)](t) - \mathcal{A}_\infty(\omega, u_\infty, \mathbf{p}_\infty)\|_{V^*}^q \to 0$$

as $t \to \infty$. The convergence of the last term in (3.44) can be proved similarly, by using (B2*), (B6*).

Finally, the left hand side of (3.42) may be estimated as follows

$$\begin{aligned}
&|\langle \mathcal{G}(t) - \mathcal{G}_\infty, u(t) - u_\infty\rangle| + |\langle \mathcal{H}(t) - \mathcal{H}_\infty, u(t) - u_\infty\rangle| \\
&\le \frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p + \frac{\varepsilon^p}{p}\|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \\
&\quad + \frac{1}{q\varepsilon^q}\|\mathcal{G}(t) - \mathcal{G}_\infty\|_V^p + \frac{1}{q\varepsilon^q}\|\mathcal{H}(t) - \mathcal{H}_\infty\|_V^p
\end{aligned} \tag{3.45}$$

Now, by choosing sufficiently small $\varepsilon$ in (3.45) and by using (3.43), (3.44) and the above convergences we obtain

$$y'(t) + \text{const} \cdot \|u(t) - u_\infty\|_V^p + \text{const} \cdot \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \le \phi(t) \tag{3.46}$$

where $\phi(t) \to 0$ as $t \to \infty$ and the constants are positive. By applying the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ it follows

$$y'(t) + \text{const} \cdot y(t)^{\frac{p}{2}} + \text{const} \cdot \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \le \phi(t)$$

Now by using the arguments of the proof of Theorem 2.18 one may show that this inequality implies $\lim\limits_{t\to\infty} y(t) = 0$.

By integrating (3.46) over $(T-1, T+1)$ we conclude

$$\begin{aligned}
&y(T+1) - y(T-1) + \text{const} \cdot \int_{T-1}^{T+1} \|u(t) - u_\infty\|_V^p \, dt \\
&\qquad + \text{const} \int_{T-1}^{T+1} \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \, dt \le \int_{T-1}^{T+1} \phi(t)\, dt.
\end{aligned}$$

Clearly the right hand side tends to 0 as $T \to \infty$, and by the convergence of $y(t)$,

$$y(T+1) - y(T-1) \to 0 \text{ as } T \to \infty.$$

which yields the desired convergences. The proof of stabilization is complete. $\square$

As in the previous chapter, we may give explicit convergence "speed". Suppose

(Est) There exist constants $k^* > 0$, $\beta > 1$ such that

$$\begin{aligned}
&\big\| a_i(t,\cdot,\omega(t,\cdot),u(\cdot),Du(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot);w,v_1,v_2) \\
&\qquad\qquad - a_{i,\infty}(\cdot,\omega^*(\cdot),u(\cdot),Du(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot))\big\|^q_{L^q(\Omega)} \\
&\le k^* t^{-\beta}, \\
&\qquad \big\| b_i(t,\cdot,\omega(t,\cdot),u(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot);w,v_1,v_2) \\
&\qquad\qquad\qquad - b_{i,\infty}(\cdot,\omega^*(t,\cdot),u(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot))\big\|^q_{V^*} \\
&\qquad \le k^* t^{-\beta},
\end{aligned}$$

for every $w \in L^\infty(Q_\infty)$, $u, \mathbf{p} \in V, v_1, v_2 \in L^\infty(0,\infty;L^2(\Omega))$ if $\omega(t,\cdot) \to \omega^*$ in $L^\infty(Q_\infty)$ $(i = 0,\ldots,n)$,

$$\int_\Omega |r(t,x,u(t,x),\tilde{u}(t,x),\mathbf{p}(t,x),\tilde{\mathbf{p}}(t,x);w,v_1,v_2)|\,dx \le k^* t^{-\beta}$$

for a.a. $t \in (0,\infty)$ and every $w \in L^\infty(Q_\infty)$, $u, \tilde{u}, \mathbf{p}, \tilde{\mathbf{p}}, v_1, v_2 \in L^\infty(0,\infty;L^2(\Omega))$,

$$\begin{aligned}
\|\mathcal{G}(t) - \mathcal{G}_\infty\|^q_{V^*} &\le k^* t^{-\beta}, \\
\|\mathcal{H}(t) - \mathcal{H}_\infty\|^q_{V^*} &\le k^* t^{-\beta}.
\end{aligned}$$

**Proposition 3.15.** *Assume (Vol), further, conditions (A1)–(A3), (A5), (B1)–(B3), (B5), (F1), (F2), (F4) hold for every $0 < T < \infty$. In addition, (F3*), (A2+), (A4*), (B2+), (B4*), (A6), (B6), (AB), (F5), (G2), (H2) are satisfied with further assumption (Est). Then for the solutions $u, u_\infty$ formulated in Theorem 3.14, $y(t) := \|u(t) - u_\infty\|_V^2$ and $z(t) := \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^2$ has the asymptotic property*

$$\int_t^\infty y(s)^\alpha\,ds + \int_t^\infty z(s)^\alpha\,ds \le \text{const} \cdot t^{\frac{1}{1-\alpha}}$$

*holds for $t > 0$ sufficiently large and for*

$$\alpha = \max\left\{\frac{p}{2}, 1 + \frac{1}{\beta - 1}\right\}.$$

*Proof.* Fix $\alpha$ as above. Then as in the proof of Proposition 2.21, assumptions (I)—(III) imply

$$y'(t) + \text{const} \cdot y(t)^{\frac{p}{2}} + \text{const} \cdot z(t)^{\frac{p}{2}} \le \text{const} \cdot t^{-\beta}.$$

By integrating on interval $(t, \infty)$ (with $t$ sufficiently large) it follows

$$\int_t^\infty (c^* \cdot y(s)^\alpha + c^* \cdot z(s)^\alpha)\, ds \le t^{-\beta+1} + y(t) \le t^{-\beta+1} + y(t) + z(t).$$

Now one proceeds as in the above mentioned proof and one may deduce a differential inequality which implies the desired estimate. □

### 3.3.3 Examples

Now we give some examples satisfying Theorem 3.10, 3.12, 3.14. By using arguments of Sections 2.2.3 and 2.3.4, one can easily see that the examples below satisfy each condition of the theorems.

#### Case of Theorem 3.10

It is clear that examples (3.31)–(3.33) fulfil the conditions of the above theorem if operators $\pi, \tilde{\pi}, \tilde{\pi}_0, \kappa, \tilde{\kappa} \colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\varphi, \lambda \colon L^{p_1}_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\psi, \vartheta \colon L^{p_2}_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\tilde{\varphi}, \tilde{\varphi}_0 \colon L^{p_1}_{\text{loc}}(Q_\infty) \to L^{\frac{p_1-1}{p_1-r_1-1}}_{\text{loc}}(Q_\infty)$, $\tilde{\vartheta} \colon L^{p_2}_{\text{loc}}(Q_\infty) \to L^{\frac{p_2-1}{p_2-r_2-1}}_{\text{loc}}(Q_\infty)$ are of Volterra type and conditions (E1)–(E2) are satisfied for all finite $T > 0$. E.g., the operators given after Proposition 3.9 serve as examples for the above.

#### Case of Theorem 3.12

If some further assumptions are satisfied then example (3.31)–(3.33) fulfil the conditions of Theorem 3.12. Suppose that the conditions above hold, in addition

$$\begin{aligned}&\int_\Omega |[\tilde{\varphi}_0(v_1)](t,x)|^{\frac{p_1-1}{p_1-r_1-1}}\, dx \\ &\le \alpha_1 \left[ \operatorname*{ess\,sup}_{\tau\in[0,t]} \|v_1(\tau)\|^{\varrho_1}_{L^2(\Omega)} + \chi_1(t) \operatorname*{ess\,sup}_{\tau\in[0,t]} \|v_1(\tau)\|^{p_1}_{L^2(\Omega)} + 1 \right]\end{aligned}$$

for all $v_1 \in L^{p_1}_{\text{loc}}(Q_\infty)$ with some constants $\alpha_1 > 0, \varrho_1 < p_1$ and function $\chi_1 \colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} \chi_1(t) = 0$, further, similar condition holds for $\tilde{\vartheta}$ (by changing the indeces from 1 to 2, and $L^2(\Omega)$ to $V_2$). For example, operator $\tilde{\varphi}_0$ may have the form

$$[\tilde{\varphi}(v)](t,x) = \tilde{\Phi}\left(\int_\Omega d(t,x)v(t,x)\,dx\right),$$
$$[\tilde{\varphi}(v)](t,x) = \tilde{\Phi}\left(\int_\Omega |d(t,x)||v(t,x)|^\beta\,dx\right) \text{ or}$$
$$[\tilde{\varphi}(v)](t,x) = \chi_1(t)\tilde{\Phi}_0\left(\left[\int_\Omega |d(t,x)||v(t,x)|^2\,dx\right]^{\frac{1}{\beta}}\right),$$

where $d \in L^\infty(Q_\infty)$, $1 \le \beta \le 2$, $\tilde{\Phi}, \tilde{\Phi}_0, \chi_1 \in C(\mathbb{R})$ and $|\tilde{\Phi}(\tau)| \le \text{const} \cdot |\tau|^{p_1-\varrho_1-1}$, $|\tilde{\Phi}_0(\tau)| \le \text{const} \cdot |\tau|^{p_1-r_1-1}$, $\lim_{\tau\to\infty} \chi_1(\tau) = 0$.

**Case of Theorem 3.14**

Now consider for $i = 0, \ldots, n$ the following:

$$a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) = [\pi(w)](t,x)[\varphi(v_1)](t,x)[\psi(v_2)](t,x)P(\xi)\zeta_i|(\zeta_0,\zeta,\eta_0,\eta)|^{p-2}, \tag{3.47}$$

$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) = [\kappa(w)](t,x)[\lambda(v_1)](t,x)[\vartheta(v_2)](t,x)R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{p-2}. \tag{3.48}$$

Suppose

(E3) a) Operators $\pi\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\varphi,\psi\colon L^p_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$ are of Volterra type, further, for every $0 < T < \infty$, $\pi\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\varphi,\psi\colon L^p(Q_T) \to L^\infty(Q_T)$ are bounded, $\varphi$ and $\psi$ are continuous, and if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\pi(\omega_k) \to \pi(\omega)$ in $L^\infty(Q_T)$. In addition, $P \in C(\mathbb{R})$, and there exists a positive lower bound for the values of $\pi, \varphi, \psi, P$.

b) There exist $\pi_\infty, \varphi_\infty, \psi_\infty \in L^\infty(\Omega)$ such that for every $w \in L^\infty(Q_\infty), v_1 \in X^\infty \cap L^\infty(0,\infty;L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty;V)$,

$$\lim_{t\to\infty} \|[\pi(w)](t,\cdot) - \pi_\infty\|_{L^\infty(\Omega)} = 0,$$
$$\lim_{t\to\infty} \|[\varphi(v_1)](t,\cdot) - \varphi_\infty\|_{L^\infty(\Omega)} = 0,$$
$$\lim_{t\to\infty} \|[\psi(v_2)](t,\cdot) - \psi_\infty\|_{L^\infty(\Omega)} = 0.$$

(E4) a) Operators $\kappa\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\lambda,\vartheta\colon L^p_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$ are of Volterra type, further, for every $0 < T < \infty$, $\kappa\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\lambda,\vartheta\colon L^p(Q_T) \to L^\infty(Q_T)$ are bounded, $\lambda$ and $\vartheta$ are continuous, and if $(\omega_k)$ is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\kappa(\omega_k) \to \kappa(\omega)$ in $L^\infty(Q_T)$. In addition, $R \in C(\mathbb{R})$, and there exists a positive lower bound for the values of $\kappa, \lambda, \vartheta, R$.

b) There exist $\kappa_\infty, \lambda_\infty, \vartheta_\infty \in L^\infty(\Omega)$ such that for every $w \in L^\infty(Q_\infty), v_1 \in X^\infty \cap L^\infty(0,\infty; L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty; V)$

$$\lim_{t\to\infty} \|[\kappa(w)](t,\cdot) - \kappa_\infty\|_{L^\infty(\Omega)} = 0,$$
$$\lim_{t\to\infty} \|[\lambda(v_1)](t,\cdot) - \lambda_\infty\|_{L^\infty(\Omega)} = 0,$$
$$\lim_{t\to\infty} \|[\vartheta(v_2)](t,\cdot) - \vartheta_\infty\|_{L^\infty(\Omega)} = 0.$$

By using similar arguments as in Sections 2.2.3, 2.3.4 and Proposition 1.58 we obtain

**Proposition 3.16.** *Suppose* $2 \le p \le 4$ *and (E3)–(E4). Then the above* (3.47)–(3.48) *functions satisfy conditions (A1)–(A3), (A4*), (A5)–(A6), (A2+), (B1)–(B3), (B4*), (B5)–(B6), (B2+), (AB) with* $p_1 = p_2 = p$.

Consider

$$a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta; w, v_1, v_2) \\ = \zeta_i|(\zeta_0,\zeta)|^{p-2} + [\pi(w)](t,x)[\phi(v_1)](t,x)P(\xi)\zeta_i|(\zeta_0,\zeta,\eta_0,\eta)|^{r-2}, \tag{3.49}$$

$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta; w, v_1, v_2) \\ = \zeta_i|(\eta_0,\eta)|^{p-2} + [\kappa(w)](t,x)(t,x)[\vartheta(v_2)](t,x)R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{r-2} \tag{3.50}$$

where $2 \le r \le 4 < p$ and (E3)–(E4) hold then it is easy to see that these functions satisfy conditions (A1)–(A3), (A4*), (A5)–(A6), (B1)–(B6), (AB) with $p_1 = p_2 = p \ge \max\{2, r\}$. E.g., operators $\pi, \varphi$ may have the form

$$[\pi(w)](t,x) = \chi(t)\int_{Q_t} |w|^\beta + \pi_\infty(x)$$
$$[\varphi(v)](t,x) = \tilde\chi(t)\int_\Omega |d(t,x)||v(t,x)|^\beta\,dx + \varphi_\infty(x),$$

where $\lim_{t\to\infty}\chi(t) = 0$, $\lim_{t\to\infty}\tilde\chi(t) = 0$ and $d \in L^\infty(Q_\infty), 1 \le \beta \le 2$. The other operators may have similar form.

As an example for function $f$ consider, e.g.,

$$f(t,x,\xi,\zeta_0; v) = -(\xi - \omega^*(x))\int_\Omega |v(t,x)|^\beta\,dx$$

where $1 \le \beta \le 2$.

**Case of Proposition 3.15**

Now let functions $a_i, b_i$ have the form:

$$
\begin{aligned}
a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) &= P(\xi)\zeta_i|(\zeta_0,\zeta,\eta_0,\eta)|^{p-2},\\
a_0(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) &= P(\xi)\zeta_0|(\zeta_0,\zeta,\eta_0,\eta)|^{p-2}\\
&\quad + \varphi(t)\cdot\chi\left(\left[\int_\Omega a(t,\xi)|v_1(t,\xi)|^2 d\xi\right]^{\frac{1}{2}}\right),\\
b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) &= R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{p-2}\\
b_0(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2) &= R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{p-2}\\
&\quad + \tilde{\varphi}(t)\cdot\tilde{\chi}\left(\left[\int_\Omega b(t,\xi)|v_2(t,\xi)|^2 d\xi\right]^{\frac{1}{2}}\right)
\end{aligned}
$$

where $a, b \in L^\infty(\Omega)$, $\varphi, \tilde{\varphi}, \chi, \tilde{\chi} \colon [0,\infty) \to \mathbb{R}$ are nonnegative functions such that $\varphi(\tau), \tilde{\varphi}(\tau) \leq \text{const}\cdot\tau^{-\beta}$, $\chi(\tau), \tilde{\chi}(\tau) \leq \text{const}\cdot|\tau|^{p-1}$. By using the arguments of Section 2.2.3, 2.3.4 and Remark 2.23 one can show that the above functions fulfil the conditions of Proposition 3.15.

# Summary

> Every human activity, good or bad, except mathematics, must come to an end.
>
> Paul Erdős

In this dissertation, we study systems of nonlinear parabolic differential equations that may contain nonlocal dependence on the unknowns. Such problems may occur, e.g., in diffusion processes (heat or population) where the diffusion coefficient may depend on the unknowns in a nonlocal way. For example, in population dynamics the growing rate of a population may depend on the size of the population, mathematically, on the integral of the density.

The mathematical background of our investigation is the theory of operators of monotone type. We demonstrate and apply some methods of this theory to study two types of systems. The first type consists of parabolic equations and the second type contains three different types of equations: an ordinary, a parabolic and an elliptic one. The latter problem can be considered as a generalization of a fluid flow model in porous medium.

For both systems we show, under suitable conditions, existence of weak solutions in time interval $(0, T)$ where $0 < T \leq \infty$. In addition, we study the long-time behaviour of the solutions. Boundedness and stabilization, i.e., the convergence to equilibrium as $t \to \infty$ is shown. An estimate on the rate of this convergence is given. For a modified problem we prove existence of periodic solutions. Besides theoretical results, we illustrate our assertions with some examples.

The results on the first system are based on the works of the author's supervisor made on this topic. These results are applied to the second type of system. In this case the method of finding weak solutions, which is the so-called successive approximation, and the choice of the space of solutions is a new idea which differs from the usual tools concerning the topic of monotone operators.

The topic of further research may be some numerical investigations. For the second model these may be especially relevant since the successive approximation serves as a numerical method.

# Összefoglalás

> Minden, ami emberi, akár rossz, akár jó, előbb-utóbb véget ér. Kivéve a matematikát.
>
> Erdős Pál

E munkban nemlineáris differenciálegyenletek olyan rendszereivel foglalkozunk, amelyek az ismeretlen függvényektől nemlokális módon (azaz nem csak adott pontbeli értékeiktől) is függhetnek. Ilyen típusú problémák előfordulhatnak többek között olyan (hőterjedési vagy populációdinamikai) diffúziós folyamatokban, amelyekben a diffúziós együttható az ismeretlenektől nemlokálisan függ. Például egy populáció növekedési rátája függhet a populáció méretétől, azaz a sűrűségének integráljától.

Vizsgálataink fő matematikai eszköze a monoton típusú operátorok elmélete. Bemutatunk és alkalmazunk néhány módszert e témakörből két speciális nemlineáris differenciál-egyenletrendszer tanulmányozására. Az egyik csupa parabolikus egyenletből álló rendszer, a másik három különböző típusú egyenletből áll: egy közönséges, egy parabolikus és egy elliptikus differenciálegyenletből. Ez utóbbi probléma egy porózus közegbeli folyadékáramlási modell általánosításaként fogható fel.

Mindkét rendszer esetében megfelelő feltételek mellett belátjuk gyenge megoldás létezését véges és végtelen időintervallumon. Megvizsgáljuk a megoldások aszimptotikus tulajdonságait: a korlátosságot és a $t \to \infty$ esetén való stabilizációt, azaz egy stacionárius állapothoz való konvergenciát. A konvergencia sebességére becslést is adunk. Ezt követően módosítjuk a kiindulási problémát, hogy értelmezhessük periodikus megoldás fogalmát és igazoljuk létezésüket. Mindezek mellett eredményeinket példákkal egészítjük ki.

Az első típusú rendszer esetében eredményeink a szerző témavezetője által e témakörben kapott korábbi eredmények folytatásai. A másik rendszer esetében az alapterek irodalomban megszokottól eltérő megválasztása, illetve a szukcesszív approximáció módszerének alkalmazása lesz a vizsgálataink kulcsa.

További kutatás tárgyát képezheti az egyenletek numerikus szempontból való vizsgálata. Ez a második rendszer esetében különösen érdekes, hiszen a szukcesszív approximáció numerikus módszerként is használható.

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Do not count articles, weigh them.

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