SET-THEORETICAL METHODS IN REAL ANALYSIS

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Introduction

The development of set theory created a new trend in mathematical research. On one hand it produced strong techniques such as transfinite recursion to solve long-standing open problems, and on the other hand the new theories enabled us to prove that it is impossible to answer certain questions within the framework of ZFC; that is the usual axioms of set theory. Proving these so called consistent and independent statements is a very active and rapidly growing area of mathematics, specifically of analysis as well. My dissertation presents a collection of my results of this type from the field of real analysis.

In Chapter 1 we deal with the problem of existence of measurable envelopes. This notion (which is also called hull or cover) originates from the theory of analytic sets. First we show that it is not true in general that with respect to an outer measure every subset of the measure space has a measurable envelope. Then we prove that in the case of Hausdorff measures in Euclidean spaces this question is independent of ZFC.

Chapter 2 considers measurability of Sierpiński sets in the plane. We show that a set with countable vertical sections and co-countable horizontal sections cannot be Hausdorff measurable. More surprisingly, we also prove that if we replace countable by negligible, then the existence of such a set is already independent.

In Chapter 3 we investigate the connection between the notions of $\mathcal{H}^{d_1}$-measurability and $\mathcal{H}^{d_2}$-measurability. We also consider the connections between classes of negligible sets, and the corresponding questions concerning products of Hausdorff measures. We give a (consistent) answer to a question of T. Keleti.

In Chapter 4 first we summarize the known (but still unpublished) results of M. Laczkovich concerning solvability cardinals for systems of difference equations. Then we prove that it is consistent that this cardinal is ω_2 in the case of Borel functions. This answers a question of Laczkovich. In addition we deal with the Baire 1 case as well.

Chapter 5 is about linear orders representable by (point-wise ordered) real Baire 1 functions. Due to a classical theorem such an order cannot contain an increasing or decreasing sequence of length ω_1. We investigate the fine structure of these representable orders and conjecture that the converse of the above theorem is consistent.

Finally, Chapter 6 considers increasing (with respect to the point-wise order) transfinite sequences of functions defined on metric spaces. We settle the continuous case and show that the answer in the Baire 1 case is independent even of $ZFC + \neg CH$.

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I am also greatly indebted to Péter Komjáth and István Juhász who supported me in developing my knowledge of set theory in various ways.

Chapter 1

Measurable Envelopes

In the present chapter (based on [El1]) we are dealing with the following definition, which was motivated by the theory of analytic sets.

Definition 1.0.1 Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. We call a set $H \subset X$ *small* (with respect to $\mathcal{A}$) if every subset of H belongs to $\mathcal{A}$. The σ-ideal of small subsets is denoted by $\mathcal{A}_0$. We say that $A \in \mathcal{A}$ is a *measurable envelope* of $H \subset X$ (with respect to $\mathcal{A}$) if $H \subset A$ and for every $B \in \mathcal{A}$ such that $H \subset B \subset A$ we have $A \setminus B \in \mathcal{A}_0$.

I have learned the terminology 'every subset of X has a measurable envelope' from D. Fremlin. Another usual expression is '$(X, \mathcal{A})$ admits covers' (see e.g. [Kec]), and 'measurable hull with respect to $\mathcal{A}$' is also used.

For example it is not hard to see that if $\mathcal{A}$ is the Borel, Lebesgue or Baire σ-algebra in $\mathbb{R}^n$, then $\mathcal{A}_0$ is the σ-ideal of countable, Lebesgue negligible and first category sets, respectively. One can also prove that with respect to the Lebesgue or Baire σ-algebra, every subset of $\mathbb{R}^n$ has a measurable envelope, while in the case of the Borel sets this is not true. What makes these notions interesting is a theorem of Szpilrajn-Marczewski, stating that if $\mathcal{A}$ is a σ-algebra on X such that every subset of X has a measurable envelope with respect to $\mathcal{A}$, then $\mathcal{A}$ is closed under the Souslin operation [Kec, 29.13].

So the problem of envelopes has been considered for various σ-algebras for a long time (see e.g. [Ma] and [Pa]).

In this chapter we investigate the case $\mathcal{A} = \mathcal{M}_\mu$, where μ is an outer measure on X and $\mathcal{M}_\mu$ is the σ-algebra of μ-measurable sets (in the sense of Carathéodory). First we show that the σ-finite case is easy. Therefore we turn to Hausdorff measures, which are probably the most natural examples of non-σ-finite measures. We prove, that this question cannot be answered in ZFC.

As an application we give a short proof of the known statement that the existence of an $\mathcal{H}^1$-measurable Sierpiński set (see the definition below) is consistent with ZFC (this is proven for the so called 'one dimensional measures' in [DP, 3.11]).

1.1 Measurable Envelopes with Respect to Outer Measures

If μ is a σ-finite outer measure on a set X, then it is easy to check that every subset of X has a measurable envelope (with respect to $\mathcal{M}_\mu$). It is also known [Fr1], as the following example shows, that σ-finiteness is essential.

Example 1.1.1 *Put $X = \omega_2 \times \omega_2$ and let ν be the outer measure on X that is 0 for countable subsets and 1 otherwise. Define*

$$\mu(H) = \sum_{\alpha \in \omega_2} [\nu(H \cap (\{\alpha\} \times \omega_2)) + \nu(H \cap (\omega_2 \times \{\alpha\}))];$$

that is, let $\mu(H)$ be the number of uncountable horizontal and vertical sections. We claim that $\omega_1 \times \omega_2$ has no measurable envelope.

Proof. One can easily see that μ is an outer measure, $H \in \mathcal{M}_\mu$ iff H is either countable or co-countable on each section, and $H \in (\mathcal{M}_\mu)_0$ iff H is countable on each section.

Suppose now that M is a measurable envelope of $\omega_1 \times \omega_2$. Clearly $M \cap (\{\alpha\} \times \omega_2)$ is countable for every $\alpha \in \omega_2 \setminus \omega_1$ and M is co-countable on each horizontal section. But this gives a contradiction, as for any $A \subset \omega_2 \setminus \omega_1$ of cardinal ω_1 we have that $M \cap (A \times \omega_2)$ is also of cardinal ω_1, but the projection to the second coordinate of this set is the whole ω_2. □

Remark The same construction on $\omega_1 \times \omega_1$ does not work. Indeed, one can easily show that for any $H \subset \omega_1 \times \omega_1$ the set

$$H \cup \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \alpha < \beta \text{ and } H \cap (\{\alpha\} \times \omega_1) \text{ is uncountable}\} \cup$$

$$\cup \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \beta < \alpha \text{ and } H \cap (\omega_1 \times \{\beta\}) \text{ is uncountable}\}$$

is a measurable envelope of H.

So the next question is open.

Question 1.1.2 *Is it true that if μ is an outer measure on a set X of cardinality ω_1, then every subset of X has a measurable envelope?*

1.2 Measurable Envelopes with Respect to Hausdorff Measures

As we have already mentioned, for σ-finite outer measures our problem is trivial. That is why it is natural to examine the case of Hausdorff measures, which are probably the most well-known non-σ-finite measures. Let $\mathcal{H}^d$ denote the d-dimensional Hausdorff measure in $\mathbb{R}^n$. If μ is an outer measure, then instead of 'with respect to $\mathcal{M}_\mu$' we shall sometimes say 'with respect to μ'.

First we state some useful lemmas. Let λ denote one-dimensional Lebesgue measure.

Lemma 1.2.1 *Let $B \subset \mathbb{R}^n$ be Borel such that $0 < \mathcal{H}^d(B) < \infty$. Then there exists a bijection f between B and the interval $I = [0, \mathcal{H}^d(B)]$ such that both f and its inverse preserve the Borel sets, measurable sets and zero sets, and which is measure preserving between the measure spaces $(B, \mathcal{H}^d)$ and (I, λ).*

Proof. [Kec, 12.B] shows that B is a standard Borel space, hence we can apply [Kec, 17.41]. □

Our next lemma reads as follows.

Lemma 1.2.2 *Let $n \in \mathbb{N}$, $d \geq 0$ and $H \subset \mathbb{R}^n$ be arbitrary. Then the following statements are equivalent:*

1. *H is $\mathcal{H}^d$-measurable,*

2. *$\mathcal{H}^d(B) = \mathcal{H}^d(B \cap H) + \mathcal{H}^d(B \cap H^C)$ for every $B \subset \mathbb{R}^n$ Borel set with $0 < \mathcal{H}^d(B) < \infty$,*

3. *$H \cap B$ is $\mathcal{H}^d$-measurable for every $B \subset \mathbb{R}^n$ Borel set with $0 < \mathcal{H}^d(B) < \infty$,*

4. *for every $B \subset \mathbb{R}^n$ Borel set with $0 < \mathcal{H}^d(B) < \infty$, we have $H \cap B = A \cup N$, where A is Borel and N is $\mathcal{H}^d$-negligible.*

Proof. Straightforward using the Borel regularity of Hausdorff measures. □

Remark In fact we could also assume that the above B is compact, but we shall not need this.

Now we are able to characterize $\mathcal{H}^d$-small sets. $\mathcal{H}^d$-negligible sets are clearly small with respect to $\mathcal{H}^d$, and one could think that the converse also holds. However, this is far from being true.

Statement 1.2.3 *Let $n \in \mathbb{N}$, $d \geq 0$ and $H \subset \mathbb{R}^n$ be arbitrary. Then H is small with respect to $\mathcal{H}^d$ iff each of its subsets is either $\mathcal{H}^d$-negligible, or of infinite $\mathcal{H}^d$-measure.*

Proof. It is enough to show that if $D \subset \mathbb{R}$ and $0 < \mathcal{H}^d(D) < \infty$, then D has a non-$\mathcal{H}^d$-measurable subset. Suppose on the contrary, that this is not true. Then in particular D itself is measurable. Choose a Borel set B containing D with the same $\mathcal{H}^d$-measure, and apply the previous lemma to B. We obtain that $D \cap B = D = A \cup N$ as in the lemma, in particular D contains a Borel set A of positive finite $\mathcal{H}^d$-measure. But then by Lemma 1.2.1 we can find a non-$\mathcal{H}^d$-measurable subset in A which contradicts our assumptions. □

Remark It can be proved (in ZFC) that non-negligible small sets do indeed exist. For such an example see e.g. [Fr2, 439G].

Now we turn to the problem of envelopes. If $d = 0$ or $d > n$ then every subset of $\mathbb{R}^n$ is $\mathcal{H}^d$-measurable, hence every subset has a measurable envelope. If $d = n$ then $\mathcal{H}^d$ is σ-finite, therefore we get the same conclusion. But in the remaining cases we have

Theorem 1.2.4 *The following statement is independent of ZFC: for all $n \in \mathbb{N}$ and $0 < d < n$ every subset of $\mathbb{R}^n$ has a measurable envelope with respect to $\mathcal{H}^d$.*

Before the proof we need several lemmas and a well-known definition.

Definition 1.2.5 $\mathrm{add}(\mathcal{N})$ is the minimal cardinal κ for which there are κ Lebesgue negligible sets A_α ($\alpha < \kappa$) in $\mathbb{R}$ such that $\cup_{\alpha<\kappa} A_\alpha$ is of positive outer measure.

The next lemma concerning $\mathrm{add}(\mathcal{N})$ is also well-known.

Lemma 1.2.6 *Suppose that H_α ($\alpha < \mu$) are Lebesgue measurable for some $\mu < \mathrm{add}(\mathcal{N})$. Then their union $\cup_{\alpha<\mu} H_\alpha$ is also Lebesgue measurable.*

Proof. We prove this by transfinite induction on μ. We have to show that $\cup_{\alpha<\mu} H_\alpha$ is Lebesgue measurable. Define $A_\alpha = H_\alpha \setminus \cup_{\beta<\alpha} H_\beta$ for every $\alpha < \mu$. The sets A_α are clearly disjoint, and they are Lebesgue measurable by the inductional hypothesis. Thus only countably many of them can be of positive measure. The union of this countable family is obviously measurable, and the union of the remaining sets is also measurable by $\mu < \mathrm{add}(\mathcal{N})$. This completes the proof, as $\cup_{\alpha<\mu} A_\alpha = \cup_{\alpha<\mu} H_\alpha$. □

The following lemma is essentially contained in [Fe, 2.5.10].

Lemma 1.2.7 *Let $0 < d < n$ and suppose $\mathrm{add}(\mathcal{N}) = 2^\omega$. Then there exists a disjoint family $\{M_\alpha : \alpha < 2^\omega\}$ of $\mathcal{H}^d$-measurable subsets of $\mathbb{R}^n$ of finite $\mathcal{H}^d$-measure, such that a set $H \subset \mathbb{R}^n$ is $\mathcal{H}^d$-measurable iff $H \cap M_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$.*

Proof. Let $\{B_\alpha : \alpha < 2^\omega\}$ be an enumeration of the Borel subsets of $\mathbb{R}^n$ of positive finite $\mathcal{H}^d$-measure, and put $M_\alpha = B_\alpha \setminus (\cup_{\beta<\alpha} B_\beta)$. These are clearly pairwise disjoint sets of finite $\mathcal{H}^d$-measure. Moreover, $\mathrm{add}(\mathcal{N}) = 2^\omega$ together with Lemma 1.2.6 and Lemma 1.2.1 applied to B_α give that M_α is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$. The other direction being trivial we only have to verify that if $H \subset \mathbb{R}^n$ is such that $H \cap M_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$, then H is itself $\mathcal{H}^d$-measurable. By Lemma 1.2.2 we only have to show that $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$. But $B_\alpha = \cup_{\beta\le\alpha} M_\beta$, therefore $H \cap B_\alpha = \cup_{\beta\le\alpha}(H \cap M_\alpha)$, thus Lemma 1.2.1 applied to B_α and Lemma 1.2.6 yield that $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable, which completes the proof. □

Definition 1.2.8 $\text{non}^*(\mathcal{N})$ is the minimal cardinal κ such that in every subset of the reals of positive outer Lebesgue measure we can find a subset of positive outer Lebesgue measure and of cardinal $\leq \kappa$.

$\text{cov}(\mathcal{N})$ is the minimal cardinal κ such that $\mathbb{R}$ can be covered by κ Lebesgue negligible sets.

Remark $\text{non}^*(\mathcal{N}) < \text{cov}(\mathcal{N})$ is consistent with ZFC as it holds in the so called 'random real model', see [LM, Lemma 8].

Now we can turn to the proof of Theorem 1.2.4.

Proof. First we show that $\text{add}(\mathcal{N}) = 2^\omega$ implies that for every $n \in \mathbb{N}$ and $0 < d < n$ every subset of $\mathbb{R}^n$ has a measurable envelope with respect to $\mathcal{H}^d$. (This proves that this statement is consistent, as $\text{add}(\mathcal{N}) = 2^\omega$ follows e.g. from CH or MA.) Fix n, d and $H \subset \mathbb{R}^n$. Let $\{M_\alpha : \alpha < 2^\omega\}$ be as in Lemma 1.2.7. As M_α is of finite measure for every $\alpha < 2^\omega$, we can find a $\mathcal{H}^d$-measurable set H_α such that $H \cap M_\alpha \subset H_\alpha \subset M_\alpha$ and $\mathcal{H}^d(H \cap M_\alpha) = \mathcal{H}^d(H_\alpha)$. We claim that $A = \cup_{\alpha<2^\omega} H_\alpha$ is a measurable envelope of H. Clearly A is $\mathcal{H}^d$-measurable by Lemma 1.2.7. Suppose $H \subset B \subset A$, B is $\mathcal{H}^d$-measurable and $C \subset A \setminus B$. We want to show that C is measurable, therefore it is sufficient to check that $C \cap M_\alpha$ is measurable for every $\alpha < 2^\omega$, which is obvious, as it is of $\mathcal{H}^d$-measure zero.

Next we prove that for $n = 2$ and $d = 1$ it is consistent that there exists a subset of the plane without a $\mathcal{H}^1$-measurable envelope. We assume $\text{non}^*(\mathcal{N}) < \text{cov}(\mathcal{N})$. One can easily find a set $A \subset \mathbb{R}$ of full outer measure and of cardinal $\text{non}^*(\mathcal{N})$, and we claim that $A \times \mathbb{R}$ has no $\mathcal{H}^1$-measurable envelope. Otherwise, if M is such an envelope, then it is (one dimensional) Lebesgue measurable on each vertical and horizontal line, therefore it is Lebesgue negligible on all vertical lines over $\mathbb{R} \setminus A$ and co-negligible on all horizontal lines. As $\text{non}^*(\mathcal{N}) < \text{cov}(\mathcal{N})$, $\mathbb{R} \setminus A$ is not negligible, hence we can choose a set $B \subset \mathbb{R} \setminus A$ of positive outer measure and of cardinal $\text{non}^*(\mathcal{N})$. Then the projection of the set $(B \times \mathbb{R}) \cap M$ to the second coordinate consists of $\text{non}^*(\mathcal{N})$ zero sets, on the other hand, it is the whole line, a contradiction. $\square$

Remark The second direction of this proof (the last paragraph, in which we show a set without a measurable envelope) is due to D. Fremlin [Fr1].

In fact, it is not much harder to see that $\text{non}^*(\mathcal{N}) < \text{cov}(\mathcal{N})$ implies the existence of subsets of $\mathbb{R}^n$ without $\mathcal{H}^d$-measurable envelopes for any $0 < d \leq [\frac{n}{2}]$ and $n \geq 2$. Indeed, we can replace $\mathbb{R} \times \mathbb{R}$ by the square of a d dimensional Cantor set in $\mathbb{R}^{[\frac{n}{2}]}$ of positive and finite $\mathcal{H}^d$-measure, and repeat the above argument.

However, we do not know the answer to the following question.

Question 1.2.9 *Is it consistent that there exists a subset of* $\mathbb{R}^n$ *without a* $\mathcal{H}^d$*-measurable envelope for* $n = 1$, $0 < d < 1$ *or for* $n \geq 2$, $[\frac{n}{2}] < d < n$*?*

Chapter 2

Hausdorff Measurable Sierpiński Sets

The well-known Sierpiński sets (see the definition below) are fundamental examples of non-measurable objects. Indeed, as the Fubini Theorem shows, they cannot be Lebesgue measurable. However, if we consider Hausdorff measures, which are natural generalizations of Lebesgue measure, we arrive at a surprising phenomenon.

In this chapter we investigate the existence of two kinds of Sierpiński sets measurable with respect to Hausdorff measures.

The content of the chapter can essentially be found in [El1].

Definition 2.0.1 A set $S \subset \mathbb{R}^2$ is a *Sierpiński set in the sense of measure* if S is (one dimensional) Lebesgue negligible on each vertical line, but co-negligible (that is, the complement of S is negligible) on each horizontal line. A set $S \subset \mathbb{R}^2$ is a *Sierpiński set in the sense of cardinality* if S is countable on each vertical line, but co-countable on each horizontal line.

2.1 Sierpiński Sets in the Sense of Cardinality

As all subsets of the plane are $\mathcal{H}^d$-measurable for $d = 0$ or $d > 2$, the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of cardinality for these d is equivalent to the existence of Sierpiński sets in the sense of cardinality, which is known to be equivalent to CH [Tr]. The following theorem answers the question for the other d.

Theorem 2.1.1 *For $0 < d \leq 2$ there exists no $\mathcal{H}^d$-measurable Sierpiński set in the sense of cardinality.*

Proof. For $d = 2$ the statement is obvious by the Fubini Theorem. Let $0 < d < 2$ and $C_{d/2}$ be a symmetric self-similar Cantor set in $[0, 1]$ of dimension $d/2$.

Lemma 2.1.2 *There exists $0 < c < \infty$, such that*

$$\mathcal{H}^d|C_{d/2} \times C_{d/2} = c\ (\mathcal{H}^{d/2}|C_{d/2} \times \mathcal{H}^{d/2}|C_{d/2}).$$

Proof. Since $K = C_{d/2} \times C_{d/2}$ is also self-similar, it is easy to see that $0 < \mathcal{H}^d(K) < \infty$, therefore if we let

$$c = \frac{\mathcal{H}^d(K)}{(\mathcal{H}^{d/2}(C_{d/2}))^2},$$

then $0 < c < \infty$ and the two above outer measures agree on K. By the self-similarity of K they also agree on the basic open sets, and as any open set is the disjoint union of countably many of these, the outer measures agree on all open sets. Hence (by finiteness) on all Borel sets as well, from which the lemma follows. □

Now we can complete the proof of Theorem 2.1.1 as follows. Note that if S is an $\mathcal{H}^d$-measurable Sierpiński set in the sense of cardinality, then $S \cap K$ is a $\mathcal{H}^d$-measurable set, which is countable on each vertical section of K and co-countable on each horizontal section of K. But this gives a contradiction, once we apply the previous lemma and the Fubini Theorem. □

2.2 Sierpiński Sets in the Sense of Measure

The question concerning Sierpiński sets in the sense of measure is more complicated. Just as above, for $d = 0$ or $d > 2$ $\mathcal{H}^d$-measurability is not really a restriction, and we know that the existence of Sierpiński sets in the sense of measure is independent of ZFC [La1, Theorem 2]. For $d = 2$ no such set can be $\mathcal{H}^d$-measurable by the Fubini Theorem, while in the remaining cases we have the following.

Theorem 2.2.1 *For $0 < d < 2$ the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of measure is independent of ZFC.*

Remark While preparing a paper containing the above results I realized, that the consistency of the existence of an $\mathcal{H}^d$-measurable Sierpiński sets in the sense of measure is known, when $d = 1$. This is proven for the so called 'one dimensional measures' in [DP, 3.11]).

Proof. On one hand, for example $\text{non}^*(\mathcal{N}) < \text{cov}(\mathcal{N})$ implies that there are no Sierpiński sets of any kind [La1, Theorem 2].

On the other hand, we assume $\text{add}(\mathcal{N}) = 2^\omega$ and prove the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of measure, separately for all $0 < d < 1$, $d = 1$ and all $1 < d < 2$.

If $d = 1$, then our statement is a consequence of [DP, 3.10], but we present another proof here. By [La1, Theorem 2] and $\text{add}(\mathcal{N}) = 2^\omega$ we can find a Sierpiński set in the sense of measure, and by 1.2.4 this set has a $\mathcal{H}^d$-measurable envelope. It is not hard to check that this envelope possesses the required properties.

Now let $0 < d < 1$. Enumerate the Borel subsets of $\mathbb{R}^2$ of positive finite $\mathcal{H}^d$-measure as $\{B_\alpha : \alpha < 2^\omega\}$ and also $\mathbb{R}$ as $\{x_\alpha : \alpha < 2^\omega\}$. We can assume that S is a Sierpiński set in the sense of measure, such that the cardinality of every vertical section is less than 2^ω (the proof in [La1] provides such a set). Then put

$$S_1 = S \cup \bigcup_{\alpha < 2^\omega} [B_\alpha \setminus (\bigcup_{\beta < \alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}))].$$

S_1 is a Sierpiński set in the sense of measure as its horizontal sections contain the horizontal sections of S, the vertical section over x is still of Lebesgue measure zero, since $\text{add}(\mathcal{N}) = 2^\omega$, and $x = x_\alpha$ for some $\alpha < 2^\omega$ so this section is increased only in the first α steps, and always by a set of finite $\mathcal{H}^d$-measure, therefore of zero Lebesgue measure. What remains to check is that S_1 is $\mathcal{H}^d$-measurable. By Lemma 1.2.2 we only have to prove that $S_1 \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$. We show this by induction on α as follows. Put

$$A_\alpha = \bigcup_{\beta < \alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}).$$

Then

$$S_1 \cap B_\alpha = S_1 \cap [(B_\alpha \setminus A_\alpha) \cup (B_\alpha \cap A_\alpha)] =$$

$$[S_1 \cap (B_\alpha \setminus A_\alpha)] \cup [S_1 \cap (B_\alpha \cap A_\alpha)] = [B_\alpha \setminus A_\alpha] \cup [S_1 \cap (B_\alpha \cap A_\alpha)] =$$

$$[B_\alpha \setminus A_\alpha] \cup [\bigcup_{\beta<\alpha} (B_\alpha \cap B_\beta \cap S_1)] \cup [\bigcup_{\beta<\alpha} (B_\alpha \cap (\{x_\beta\} \times \mathbb{R}) \cap S_1)].$$

Now we apply Lemma 1.2.1 to B_α in view of $add(\mathcal{N}) = 2^\omega$. Then the first expression in the last line is clearly $\mathcal{H}^d$-measurable, and the same holds for the second expression by our inductional hypotheses. In order to see this for the last one we note that $B_\alpha \cap (\{x_\beta\} \times \mathbb{R}) \cap S$ is of cardinal less than 2^ω for every $\beta < \alpha$, thus $\mathcal{H}^d$-negligible, but when we construct $B_\alpha \cap (\{x_\beta\} \times \mathbb{R}) \cap S_1$ out of this set, we increase it only in the first β steps, and always by a $\mathcal{H}^d$-measurable set. This completes the proof of the case $0 < d < 1$.

Finally, let $1 < d < 2$. As above, let $\{B_\alpha : \alpha < 2^\omega\} = \{B \subset \mathbb{R} : B \text{ Borel}, 0 < \mathcal{H}^d(B) < \infty\}$ and also $\{x_\alpha : \alpha < 2^\omega\} = \mathbb{R}$. Put

$$D_\alpha = B_\alpha \setminus [\bigcup_{\beta<\alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}) \cup (\mathbb{R} \times \{x_\beta : \beta < \alpha\})]$$

for every $\alpha < 2^\omega$. Since $D_\alpha \subset B_\alpha$, D_α is (two dimensional) Lebesgue negligible, therefore $\{x \in \mathbb{R} : \lambda((\{x\} \times \mathbb{R}) \cap D_\alpha) > 0\}$ is Lebesgue negligible, thus contained in a Borel set N_α of Lebesgue measure zero. Define

$$S_1 = S \cup [\bigcup_{\alpha<2^\omega} (D_\alpha \setminus (N_\alpha \times \mathbb{R}))] \setminus [\bigcup_{\alpha<2^\omega} (D_\alpha \cap (N_\alpha \times \mathbb{R}))].$$

First we check that S_1 is a Sierpiński set in the sense of measure. If $\mathbb{R} \times \{x\}$ is a horizontal line, then $x = x_\alpha$ for some $\alpha < 2^\omega$. D_ξ and $\mathbb{R} \times \{x_\alpha\}$ are disjoint for every $\xi > \alpha$, therefore our set is not modified after the first α steps. Hence

$$[S \setminus S_1] \cap (\mathbb{R} \times \{x_\alpha\}) \subset [\bigcup_{\xi\leq\alpha} (D_\xi \cap (N_\xi \times \mathbb{R}))] \cap (\mathbb{R} \times \{x_\alpha\}),$$

which is Lebesgue negligible on this horizontal line by $add(\mathcal{N}) = 2^\omega$. Similarly, on a vertical line $\{x_\alpha\} \times \mathbb{R}$

$$S_1 \cap (\{x_\alpha\} \times \mathbb{R}) \subset [S \cup \bigcup_{\xi\leq\alpha} (D_\xi \setminus (N_\xi \times \mathbb{R}))] \cap (\{x_\alpha\} \times \mathbb{R}),$$

which is again a zero set. What remains to show is the $\mathcal{H}^d$-measurability of S_1. It is again sufficient to prove by induction on α that $S_1 \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$. Just like above, we apply Lemma 1.2.1 to B_α.

$$S_1 \cap B_\alpha =$$

$$[S_1 \cap D_\alpha] \cup [S_1 \cap B_\alpha \cap (\bigcup_{\beta<\alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}) \cup (\mathbb{R} \times \{x_\beta : \beta < \alpha\}))],$$

where the expression in the first brackets equals $D_\alpha \setminus (N_\alpha \times \mathbb{R})$, which is clearly $\mathcal{H}^d$-measurable, while the second expression equals

$$B_\alpha \cap ([\bigcup_{\beta<\alpha} (B_\beta \cap S_1)] \cup [(\{x_\beta : \beta < \alpha\} \times \mathbb{R}) \cap S_1] \cup [(\mathbb{R} \times \{x_\beta : \beta < \alpha\}) \cap S_1]),$$

from which the $\mathcal{H}^d$-measurability follows, since $B_\beta \cap S_1$ is $\mathcal{H}^d$-measurable for every $\beta < \alpha$ by the inductional hypothesis, while $\{x_\beta : \beta < \alpha\} \times \mathbb{R}$ and $\mathbb{R} \times \{x_\beta : \beta < \alpha\}$ are of $\mathcal{H}^d$-measure zero in B_α. □

However, we do not know the answer to the following.

Question 2.2.2 *Let $0 < d < 2$. Is it consistent, that there exist Sierpiński sets in the sense of measure, but there exists no $\mathcal{H}^d$-measurable one?*

Chapter 3

The Classes of Negligible and Measurable Sets with Respect to Hausdorff Measures

In our next chapter we present some (still unpublished) results demonstrating the strange behaviour of the classes of negligible and measurable sets with respect to Hausdorff measures. The topic was motivated by the following question of T. Keleti [Kel1] : Does $\mathcal{H}^{d_1}$-measurability imply $\mathcal{H}^{d_2}$-measurability, or vice versa? We give (a consistent) answer to this question in the last section of the chapter.

3.1 Relations Between Classes of Negligible Sets

In the first section we investigate the possible inclusions between the classes of negligible sets with respect to geometric measures. For the sake of completeness we formulate the following simple statement.

Statement 3.1.1 *Let $0 \leq d_1 < d_2 \leq n$ and $H \subset \mathbb{R}^n$. Then $\mathcal{H}^{d_1}(H) = 0$ implies $\mathcal{H}^{d_2}(H) = 0$, but the converse is not true in general.*

Proof. Trivial. □

Next we turn to the case of products of Hausdorff measures, as they turn out to exhibit an interesting behaviour. For the sake of simplicity we only consider the following three outer measures in $\mathbb{R}^2$: $\mathcal{H}^{1/2} \times \mathcal{H}^1$, $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ and $\mathcal{H}^{3/2}$. Just as one would expect, these three measures coincide in many cases. (E.g. the restrictions of these measures agree on nice sets of finite measure.) However, the following holds.

Theorem 3.1.2 *There is no inclusion between the classes of $\mathcal{H}^{1/2} \times \mathcal{H}^1$-negligible, $\mathcal{H}^1 \times \mathcal{H}^{1/2}$-negligible and $\mathcal{H}^{3/2}$-negligible sets in $\mathbb{R}^2$.*

Proof. We remark here, that all our examples will be in addition compact.

Let $C_d \subset \mathbb{R}$ be a d-dimensional self-similar symmetric Cantor set, and let us denote $[0, 1]$ by I.

Clearly, $I \times C_{1/2}$ is $\mathcal{H}^{1/2} \times \mathcal{H}^1$-negligible but not $\mathcal{H}^1 \times \mathcal{H}^{1/2}$-negligible.

In order to show that $C_{3/4} \times C_{3/4}$ is $\mathcal{H}^{1/2} \times \mathcal{H}^1$-negligible but not $\mathcal{H}^{3/2}$-negligible it is enough to note that this set is a self-similar set of dimension $3/2$, thus its $\mathcal{H}^{3/2}$-measure is positive (and finite).

As our last example we show that $D = \{(x, x) : x \in I\}$ is $\mathcal{H}^{3/2}$-negligible but not $\mathcal{H}^{1/2} \times \mathcal{H}^1$-negligible. By the symmetric role played by $\mathcal{H}^{1/2} \times \mathcal{H}^1$ and $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ this will complete the proof. Obviously $\mathcal{H}^{3/2}(D) = 0$. Now we prove that $(\mathcal{H}^{1/2} \times \mathcal{H}^1)(D) = \infty$. Suppose on the contrary that $D \subset \cup_{i=1}^{\infty}(A_i \times B_i)$ such that $\sum_{i=1}^{\infty} \mathcal{H}^{1/2}(A_i) \times \mathcal{H}^1(B_i) < \infty$. Then for every $i \in \mathbb{N}$ we have either $\mathcal{H}^{1/2}(A_i) < \infty$ or $\mathcal{H}^1(B_i) = 0$. In both cases either $\mathcal{H}^1(A_i) = 0$ or $\mathcal{H}^1(B_i) = 0$ holds. Thus $\mathcal{H}^1(D \cap (A_i \times B_i)) = 0$, which contradicts $\mathcal{H}^1(D) > 0$. $\square$

3.2 Relations Between Classes of Measurable Sets

Now we answer (consistently) the above mentioned question of T. Keleti.

Theorem 3.2.1 *Let $0 < d_1 < d_2 \leq n$. Then $\mathcal{H}^{d_2}$-measurability does not imply $\mathcal{H}^{d_1}$-measurability in $\mathbb{R}^n$. If $\mathrm{add}(\mathcal{N}) = 2^\omega$ holds (e.g. under CH or MA), then $\mathcal{H}^{d_1}$-measurability does not imply $\mathcal{H}^{d_2}$-measurability in $\mathbb{R}^n$.*

Proof. Let $C \subset \mathbb{R}^n$ be compact such that $0 < \mathcal{H}^{d_1}(C) < \infty$. Then by Lemma 1.2.1 there exists a set $H \subset C$ that is not $\mathcal{H}^{d_1}$-measurable. But clearly $\mathcal{H}^{d_2}(H) = 0$, so the first statement of the theorem is verified.

To prove the second half, let $C \subset \mathbb{R}^n$ be compact such that $0 < \mathcal{H}^{d_2}(C) < \infty$, and let $\{B_\alpha : \alpha < 2^\omega\}$ be an enumeration of the Borel subsets of $\mathbb{R}^n$ of positive finite $\mathcal{H}^{d_1}$-measure, and $M_\alpha = B_\alpha \setminus (\cup_{\beta<\alpha} B_\beta)$ as in Lemma 1.2.7.

Let us also enumerate the ordered pairs (A, N) where $A \subset C$ is Borel and $N \subset C$ is $\mathcal{H}^{d_2}$-negligible Borel as $\{(A_\alpha, N_\alpha) : \alpha < 2^\omega\}$.

Now we define a transfinite sequence $\{x_\alpha : \alpha < 2^\omega\}$ of points of C as follows.

I. If $C \cap (\cup_{\beta>\alpha} M_\beta) \not\subset A_\alpha \cup N_\alpha \cup \{x_\beta : \beta < \alpha\}$, then let

$$x_\alpha \in (C \cap (\cup_{\beta>\alpha} M_\beta)) \setminus (A_\alpha \cup N_\alpha \cup \{x_\beta : \beta < \alpha\}).$$

In this case we say that x_α is of Type I.

II. Suppose now that $C \cap (\cup_{\beta>\alpha} M_\beta) \subset A_\alpha \cup N_\alpha \cup \{x_\beta : \beta < \alpha\}$. Let us apply Lemma 1.2.1 to C and $\mathcal{H}^{d_2}$. By $\mathrm{add}(\mathcal{N}) = 2^\omega$ we have $\mathcal{H}^{d_2}(\{x_\beta : \beta < \alpha\}) = 0$. Thus M_β $(\beta \leq \alpha)$, N_α and $\{x_\beta : \beta < \alpha\}$ cannot cover C (we apply $\mathrm{add}(\mathcal{N}) = 2^\omega$ again). But M_α $(\alpha < 2^\omega)$ clearly cover C, so $C \cap (\cup_{\beta>\alpha} M_\beta) \subset N_\alpha \cup \{x_\beta : \beta < \alpha\}$ cannot hold. Therefore we can choose $x_\alpha \in C \cap (\cup_{\beta>\alpha} M_\beta)$ such that

$$x_\alpha \in A_\alpha \setminus (N_\alpha \cup \{x_\beta : \beta < \alpha\}).$$

In this case we say that x_α is of Type II.

Now define

$$H = \{x_\alpha : x_\alpha \text{ is of Type I}\}.$$

First we check that H is $\mathcal{H}^{d_1}$-measurable. By Lemma 1.2.7 it is enough to show that $H \cap M_\alpha$ is $\mathcal{H}^{d_1}$-measurable for every $\alpha < 2^\omega$. But $x_\alpha \in \cup_{\beta>\alpha} M_\beta$ for every $\alpha < 2^\omega$, therefore $|H \cap M_\alpha| \leq |\alpha| < 2^\omega$, so Lemma 1.2.1 applied to B_α and $\mathrm{add}(\mathcal{N}) = 2^\omega$ yield that $H \cap M_\alpha$ is $\mathcal{H}^{d_1}$-measurable.

In order to prove that H is not $\mathcal{H}^{d_2}$-measurable, first note that if it were $\mathcal{H}^{d_2}$-measurable, then by Lemma 1.2.2 we could find a pair (A_α, N_α) such that $A_\alpha \subset H \subset A_\alpha \cup N_\alpha$. But x_α is either of Type I, in which case $x_\alpha \in H$ and $x_\alpha \notin A_\alpha \cup N_\alpha$, or of

Type II, in which case $x_\alpha \notin H$ and $x_\alpha \in A_\alpha$. These are both absurd, so the proof is complete. $\square$

Question 3.2.2 *Is this true in ZFC?*

Next we consider the classes of measurable sets with respect to the above measures $\mathcal{H}^{1/2} \times \mathcal{H}^1$, $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ and $\mathcal{H}^{3/2}$.

Theorem 3.2.3 *Measurability with respect to $\mathcal{H}^{1/2} \times \mathcal{H}^1$ does not imply measurability with respect to $\mathcal{H}^{3/2}$ in $\mathbb{R}^2$.*

Proof. We have already seen that $C_{3/4} \times C_{3/4}$ is $\mathcal{H}^{1/2} \times \mathcal{H}^1$-negligible, but $0 < \mathcal{H}^{3/2}(C_{3/4} \times C_{3/4}) < \infty$. Therefore every subset of this set is $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measurable, and so it is enough to find a subset that is not $\mathcal{H}^{3/2}$-measurable. But this is obvious by Lemma 1.2.1. $\square$

Theorem 3.2.4 *Measurability with respect to $\mathcal{H}^{1/2} \times \mathcal{H}^1$ does not imply measurability with respect to $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ in $\mathbb{R}^2$.*

Proof. We can repeat the above argument with $I \times C_{1/2}$. The only difference is that we replace Lemma 1.2.1 by [Kec, 17.41] and apply it to the restriction of $\mathcal{H}^{1/2} \times \mathcal{H}^1$ to the σ-algebra of Borel subsets of $I \times C_{1/2}$, which is clearly a continuous finite Borel measure. $\square$

These last two theorems can be strengthened as follows. However, the following is not a *ZFC* theorem.

Theorem 3.2.5 *It is consistent, that there exists a product set $A \times \mathbb{R} \subset \mathbb{R}^2$ that is $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measurable (that is, A is $\mathcal{H}^{1/2}$-measurable), but neither $\mathcal{H}^1 \times \mathcal{H}^{1/2}$-measurable nor $\mathcal{H}^{3/2}$-measurable.*

Proof. We shall prove, that if $A \subset I$ is $\mathcal{H}^{1/2}$-measurable but not Lebesgue measurable, then $A \times \mathbb{R}$ possesses the required properties. This is sufficient, as the existence of such a set is consistent by Theorem 3.2.1.

The fact, that $A \times \mathbb{R}$ is $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measurable is obvious.

Clearly, $0 < (\mathcal{H}^1 \times \mathcal{H}^{1/2})(I \times C_{1/2}) < \infty$. By [Fe, 2.10.45] we also have $0 < (\mathcal{H}^{3/2})(I \times C_{1/2}) < \infty$. As this set is compact, it is measurable with respect to all our measures, so it is sufficient to show that $A \times \mathbb{R}$ is not measurable in $I \times C_{1/2}$ with respect to $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ and $\mathcal{H}^{3/2}$.

Clearly $(A \times \mathbb{R}) \cap (I \times C_{1/2}) = A \times C_{1/2}$.

Just as above, the restriction of $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ or $\mathcal{H}^{3/2}$ to the Borel σ-algebra of $I \times C_{1/2}$ is a continuous finite Borel measure, so [Kec, 17.41] (and the Borel regularity of our outer measures) yield that it is enough to prove that $A \times C_{1/2} \neq B \cup N$, where B is Borel and N is a null-set (with respect to $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ or $\mathcal{H}^{3/2}$).

Suppose on the contrary, that this equation holds. Then $A \times C_{1/2} = (\mathrm{pr}_1 B \times C_{1/2}) \cup (N \setminus (\mathrm{pr}_1 B \times C_{1/2}))$, and the two terms on the right hand side are disjoint. (pr_1 is the projection on the first coordinate.) Therefore $N \setminus (\mathrm{pr}_1 B \times C_{1/2}) = (A \setminus \mathrm{pr}_1 B) \times C_{1/2}$ and so $(A \setminus \mathrm{pr}_1 B) \times C_{1/2} \subset N$ is a null-set.

In the case of $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ this immediately gives $\mathcal{H}^1(A \setminus \mathrm{pr}_1 B) = 0$. To get the same in the case of $\mathcal{H}^{3/2}$ we refer to [Fe, 2.10.45] again. But $\mathrm{pr}_1 B \subset A$ is an analytic set, hence Lebesgue measurable [Kec, 29.16], therefore A itself is also Lebesgue measurable, which contradicts our assumptions. □

Question 3.2.6 *Is this true in ZFC?*

To complete the above list of theorems, what remains to prove is that measurability with respect to $\mathcal{H}^{3/2}$ does not imply measurability with respect to $\mathcal{H}^{1/2} \times \mathcal{H}^1$ in $\mathbb{R}^2$. As above, the natural way would be to find a subset of $D = \{(x, x) : x \in I\}$ that is not $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measurable. Surprisingly enough, this is not possible.

Statement 3.2.7 *Every subset of $D = \{(x, x) : x \in I\}$ is $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measurable.*

Proof. It is enough to show that every subset is either of measure 0 or of measure ∞.

Let $H \subset D$ and suppose $(\mathcal{H}^{1/2} \times \mathcal{H}^1)(H) < \infty$. Then we can find a sequence $A_i \times B_i \subset \mathbb{R}^2$ such that $H \subset \cup_{i=1}^{\infty}(A_i \times B_i)$ and $\sum_{i=1}^{\infty} \mathcal{H}^{1/2}(A_i) \times \mathcal{H}^1(B_i) < \infty$. But $H \subset \cup_{i=1}^{\infty}(S_i \times S_i)$ and $\sum_{i=1}^{\infty} \mathcal{H}^{1/2}(S_i) \times \mathcal{H}^1(S_i) < \infty$ hold as well, where $S_i = A_i \cap B_i$.

Then for every i, either $\mathcal{H}^1(S_i) = 0$ or $\mathcal{H}^{1/2}(S_i) < \infty$. This latter also implies $\mathcal{H}^1(S_i) = 0$, so $\mathcal{H}^{1/2}(S_i) \times \mathcal{H}^1(S_i) = 0$, hence H is of $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measure zero. □

The above behaviour is unexpected indeed, as every non-$\mathcal{H}^d$-zero compact (or even analytic) subset of $\mathbb{R}^n$ contains a compact set of positive finite $\mathcal{H}^d$-measure [Fe, 2.10.48].

In the last theorem of the chapter we show that the above attempt to prove that the above two kinds of measurabilities are not related was hopeless: surprisingly we can prove a positive result here.

Theorem 3.2.8 *Measurability with respect to $\mathcal{H}^{3/2}$ implies measurability with respect to $\mathcal{H}^{1/2} \times \mathcal{H}^1$ in $\mathbb{R}^2$.*

Proof. Suppose that $H \subset \mathbb{R}^2$ is not $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measurable. We have to prove that it is also not $\mathcal{H}^{3/2}$-measurable. Put $\mu = \mathcal{H}^{1/2} \times \mathcal{H}^1$. As H is not μ-measurable, we can find a set $S \subset \mathbb{R}^2$ such that

$$\mu(S) < \mu(S \cap H) + \mu(S \cap H^C).$$

Clearly $0 < \mu(S) < \infty$. Choose $\varepsilon > 0$ such that $\mu(S \cap H) + \mu(S \cap H^C) - \mu(S) > \varepsilon$. As Hausdorff measures are Borel regular, we can choose two sequences A_i and B_i of Borel sets in $\mathbb{R}$ such that $S \subset \cup_{i=1}^{\infty} A_i \times B_i$ and $\sum_{i=1}^{\infty} \mathcal{H}^{1/2}(A_i) \times \mathcal{H}^1(B_i) < \mu(S) + \varepsilon$. By replacing each rectangle $A_i \times B_i$ by finitely many smaller rectangles (by a recursion on i), we may assume that our rectangles $A_i \times B_i$ are pairwise disjoint. We claim that $H \cap (A_i \times B_i)$ is not μ-measurable for some i. Otherwise, for every i we have

$$\mu(A_i \times B_i) = \mu((A_i \times B_i) \cap H) + \mu((A_i \times B_i) \cap H^C)$$

so

$$\sum_{i=1}^{\infty} \mu(A_i \times B_i) = \sum_{i=1}^{\infty} \mu((A_i \times B_i) \cap H) + \sum_{i=1}^{\infty} \mu((A_i \times B_i) \cap H^C)$$

and then

$$\mu(\cup_{i=1}^{\infty}(A_i \times B_i)) = \mu(\cup_{i=1}^{\infty}(A_i \times B_i) \cap H) + \mu(\cup_{i=1}^{\infty}(A_i \times B_i) \cap H^C).$$

But

$$\mu(S) + \varepsilon > \mu(\cup_{i=1}^{\infty}(A_i \times B_i)) = \mu(\cup_{i=1}^{\infty}(A_i \times B_i) \cap H) + \mu(\cup_{i=1}^{\infty}(A_i \times B_i) \cap H^C) \geq$$

$$\geq \mu(S \cap H) + \mu(S \cap H^C),$$

contradicting the choice of ε.

So let $A = A_i$ and $B = B_i$ be Borel sets such that $H \cap (A \times B)$ is not μ-measurable. Clearly, $0 < \mu(A \times B) < \infty$, so $0 < \mathcal{H}^{1/2}(A) < \infty$ and $0 < \mathcal{H}^1(B) < \infty$. By [Fe, 2.10.45] $0 < \mathcal{H}^{3/2}(A \times B) < \infty$, therefore the two outer measures $\mathcal{H}^{1/2} \times \mathcal{H}^1$ and $\mathcal{H}^{3/2}$ restricted to the Borel subsets of $A \times B$ are constant multiples of each other (apply [Kec, 17.41] to $A \times B$, which is a standard Borel space by [Kec, 12.B]). But both outer measures are Borel regular, so they are constant multiples of each other in $A \times B$ as outer measures as well. Thus $H \cap (A \times B)$ is also not $\mathcal{H}^{3/2}$-measurable, so H is not $\mathcal{H}^{3/2}$-measurable, which completes the proof. $\square$

Chapter 4

Solvability Cardinals and Systems of Difference Equations

This chapter is devoted to the theory of solvability cardinals (see the definitions below). As most results of the area have not been published yet, we present a brief summary here. Everything mentioned in this summary (together with the basic definitions related to solvability cardinals) is due to M. Laczkovich [La3]. Section 4.1 and 4.2 contain results by the author.

Definition 4.0.1 A *difference operator* is a mapping $D : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ of the form

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i),$$

where a_i and b_i are real numbers; that is D assigns to every function f a linear combination of certain translates of f.

Difference operators show up in various branches of analysis. They were probably first defined in Fourier analysis, but they are also related to the theory of generalized derivatives, to the notions of symmetric continuity and differentiability, to the so called Difference Property and to group-algebras as well.

Definition 4.0.2 A *difference equation* is a functional equation

$$Df = g,$$

where g is a given function and f is the unknown, while a *system of difference equations* is

$$D_i f = g_i \quad (i \in I),$$

where I is an arbitrary set of indices.

A linear algebraic argument shows that such a system is solvable iff each of its finite subsystems is solvable [La2]. However, if we are interested e.g. in bounded solutions, then this result is no longer true. This motivates the following.

Definition 4.0.3 Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The *solvability cardinal of* $\mathcal{F}$ is the minimal cardinal $\kappa(\mathcal{F})$ with the property that if every subsystem of size less than $\kappa(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$, then the whole system has a solution in $\mathcal{F}$.

The following results were proved by M. Laczkovich [La3].

1. $\kappa(\mathbb{R}^{\mathbb{R}}) = \omega$ (this is a reformulation of the statement preceding the last definition).

2. $\kappa(\{f : |f| \leq 1\}) = \omega$, but $\kappa(\{f : f \text{ is bounded}\}) = \omega_1$.

3. $\kappa(\{f : f \text{ is a trigonometric polynomial}\}) = \omega_1$, but $\kappa(\{f : f \text{ is a polynomial}\}) = 3$.

4. $\kappa(C(\mathbb{R})) = \omega_1$, moreover, $\{f : f \text{ is a trigonometric polynomial}\} \subset \mathcal{F} \subset C(\mathbb{R})$ implies $\kappa(\mathcal{F}) = \omega_1$.

Definition 4.0.4 $\mathrm{cof}(\mathcal{N})$ is the minimal possible cardinal of a family $\mathcal{C}$ of Lebesgue negligible subsets of the reals such that every negligible set is covered by a member of $\mathcal{C}$.

(As the negligible Borel sets form such a family, $\mathrm{cof}(\mathcal{N}) \leq 2^{\omega}$.)

And so the last result of Laczkovich we mention reads as follows.

5. If $\mathcal{M}$ is the set of Lebesgue measurable real functions, then $(\mathrm{cf}(\mathrm{non}(\mathcal{N})))^+ \leq \kappa(\mathcal{M}) \leq (\mathrm{cof}(\mathcal{N}))^+$. Consequently, CH implies that $\kappa(\mathcal{M}) = \omega_2$.

Finally, the following question was posed by Laczkovich: What can we say about $\kappa(\{f : f \text{ is Borel}\})$? He noted, that he only expects a consistent answer. We give such an answer in the next section.

4.1 Borel Solutions to Systems of Difference Equations

First we prove an auxiliary lemma.

Lemma 4.1.1 *There exist non-empty perfect subsets $\{P_\alpha : \alpha < 2^\omega\}$ of $\mathbb{R}$ and real numbers $\{p_\alpha : \alpha < 2^\omega\}$ such that*

$$(P_\alpha + G_{\alpha+1}) \cap (P_\beta + G_{\beta+1}) = \emptyset \quad (\alpha \neq \beta),$$

and for every $\alpha < 2^\omega$

$$(P_\alpha + g_1) \cap (P_\alpha + g_2) = \emptyset \quad (g_1, g_2 \in G_{\alpha+1}, g_1 \neq g_2),$$

where G_α is the additive subgroup of $\mathbb{R}$ generated by $\{p_\beta : \beta < \alpha\}$.

Proof. Let $P \subset \mathbb{R}$ be a non-empty perfect set that is linearly independent over the rationals [vN]. We can choose $P_\alpha \subset P$ and $p_\alpha \in P$ $(\alpha < 2^\omega)$ such that $P_\alpha \cap P_\beta = \emptyset$ for every $\alpha \neq \beta$ and such that $p_\alpha \notin P_\beta$ for every $\alpha, \beta < 2^\omega$. It is a straightforward calculation to check that all the requirements are fulfilled. □

Theorem 4.1.2 $\kappa(\{f : f$ *is Borel*$\}) \geq \omega_2$.

Proof. For every $\alpha < \omega_1$ let $B_\alpha \subset P_\alpha$ be a Borel set of class α (that is not of any smaller class). Define $A_\alpha = B_\alpha + G_\alpha$ and also a system of difference equations:

$$\Delta_{p_\alpha} f = \Delta_{p_\alpha}\Big(\sum_{\beta<\omega_1} \chi_{A_\beta}\Big) \quad (\alpha < \omega_1),$$

where $(\Delta_h f)(x) = f(x+h) - f(x)$ is what we originally called difference operator. We claim, that every countable subsystem of this system has Borel solution, but the whole system does not.

To prove the first statement we have to check that for every $\alpha < \omega_1$ the first α equations have a common Borel solution. We show that the Borel function

$$\sum_{\beta \leq \alpha} \chi_{A_\beta}$$

will do. If $\gamma < \beta$, then A_β is periodic mod p_γ, so $\Delta_{p_\gamma} \chi_{A_\beta} = 0$. Therefore, in view of the properties required in Lemma 4.1.1, we obtain that for $\gamma < \alpha$

$$\Delta_{p_\gamma}(\sum_{\beta < \omega_1} \chi_{A_\beta}) = \Delta_{p_\gamma}(\sum_{\beta \leq \alpha} \chi_{A_\beta}),$$

which proves this part of the theorem.

In order to show that the whole system has no Borel solution it is sufficient to check that the functions on the right hand side of the equations are of unbounded Baire class. But this is not hard to see, as $\Delta_{p_\alpha}(\sum_{\beta < \omega_1} \chi_{A_\beta})$ restricted to P_α equals $-\chi_{B_\alpha}$. □

Corollary 4.1.3 *It is consistent, that $\kappa(\{f : f \text{ is Borel}\}) = \omega_2 = (2^\omega)^+$.*

Proof. The cardinal of the set of difference operators, as well as that of the set of Borel functions is 2^ω. Therefore the cardinal of the set of difference equations is also 2^ω, from which we get $\kappa(\{f : f \text{ is Borel}\}) \leq (2^\omega)^+$. Thus CH; that is $\omega_2 = (2^\omega)^+$ implies our statement. □

Question 4.1.4 *What can be said in ZFC?*

Remark 1) By the above theorem we can always assume that every subsystem of cardinal at most ω_1 of the system we consider has a Borel solution. Let us assign to every such subsystem the minimal $\alpha < \omega_1$ for which it has a Baire α solution. We claim, that the set of these α is bounded in ω_1. Indeed, if we could find subsystems of size at most ω_1 with arbitrarily large α, then the union of ω_1-many appropriate subsystems would itself be a subsystem of cardinal ω_1 without a Borel solution.

So if every subsystem of cardinal at most ω_1 of a system has a Borel solution, then there exists an $\alpha < \omega_1$ such that every such subsystem has a Baire α solution.

Consequently, in order to prove $\kappa(\{f : f \text{ is Borel}\}) = \omega_2$ it would be sufficient to prove $\kappa(\{f : f \text{ is Baire } \alpha\}) \leq \omega_2$ for every $\alpha < \omega_1$.

2) For $2 \leq \alpha < \omega_1$ the proof of the last theorem probably gives $\kappa(\{f : f \text{ is Baire } \alpha\}) \geq \omega_2$. However, we still need an appropriate notion of rank for Baire α functions, which we were able to neither define, nor find in the literature so far.

This remark shows why we are interested in solvability cardinals of Baire α functions. The simplest case, namely $C(\mathbb{R})$ is solved already. So we take one step further.

4.2 Baire 1 Solutions to Systems of Difference Equations

As opposed to the case $2 \leq \alpha < \omega_1$, we conjecture that $\kappa(\{f : f \text{ is Baire } 1\}) = \omega_1$. However, we can prove this in some special cases only. What makes these cases interesting is that in almost all proofs and examples in this area, every difference operator D is of the form $D = \Delta_h$; that is $(Df)(x) = (\Delta_h f)(x) = f(x+h) - f(x)$. In particular, we shall show that the proofs of the previous section do not work in the Baire 1 case.

Theorem 4.2.1 *Let $D_i f = g_i$ $(i \in I)$ be a system of difference equations, and suppose that for every $i \in I$ there exists $h_i \in \mathbb{R}$ such that $D_i = \Delta_{h_i}$. Then if every countable subsystem has a Baire 1 solution, then the whole system has one as well.*

Proof. First suppose that the group G generated by $\{h_i : i \in I\}$ is not dense in $\mathbb{R}$. It is well-known, that this implies that $G = \mathbb{Z}d$ for some $d \in \mathbb{R}$, so G is countable. This yields that there are only countably many distinct left hand sides of our equations. But every pair of equations is solvable, so for a given left hand side there is only one possible right hand side. Therefore we have only countably many distinct equations, and thus the system is solvable by the assumption.

Now we may assume that G is dense. We can find a sequence $i_n \in I$ $(n \in \mathbb{N})$ such that the group generated by $\{h_{i_n} : n \in \mathbb{N}\}$ is also dense. By assumption, there exists a common Baire 1 solution f^* to the equations $\Delta_{h_{i_n}} f = g_{i_n}$ $(n \in \mathbb{N})$. We claim, that f^* is a solution to the whole system.

So let $i \in I$ be arbitrary. We have to prove $\Delta_{h_i} f^* = g_i$. As $\{h_i\} \cup \{h_{i_n} : n \in \mathbb{N}\}$ is also countable, we can choose a Baire 1 function f_1 such that

$$\Delta_{h_i} f_1 = g_i \text{ and } \Delta_{h_{i_n}} f_1 = g_{i_n} \quad (n \in \mathbb{N}).$$

Put $f_2 = f^* - f_1$. Then for every n

$$\Delta_{h_{i_n}} f_2 = \Delta_{h_{i_n}}(f^* - f_1) = \Delta_{h_{i_n}} f^* - \Delta_{h_{i_n}} f_1 = g_{i_n} - g_{i_n} = 0,$$

thus f_2 is periodic mod h_{i_n} for every n. Therefore it is also periodic mod elements of the group generated by $\{h_{i_n} : n \in \mathbb{N}\}$, which is dense in $\mathbb{R}$. Hence f_2 must be a constant function c, otherwise it would attain two distinct values on dense sets, so it would have no point of continuity, which is absurd as f_2 is Baire 1 [Kec, 24.15].

Thus

$$\Delta_{h_i} f^* = \Delta_{h_i}(f_1 + (f^* - f_1)) = \Delta_{h_i}(f_1 + f_2) = \Delta_{h_i} f_1 + \Delta_{h_i} f_2 = g_i + \Delta_{h_i} c = g_i + 0 = g_i,$$

which completes the proof. □

In the rest of our chapter we take a small step towards proving $\kappa(\{f : f \text{ is Baire } 1\}) = \omega_1$. We investigate the special case when every difference operator consists of at most two terms. First we need two lemmas.

Lemma 4.2.2 *Let $a, b \in \mathbb{R} \setminus \{0\}$. The solutions to the equation*

$$f(x + b) - af(x) = 0$$

are the functions of the form

$$f(x) = \varphi(x)(|a|^{1/b})^x,$$

where φ is an arbitrary function periodic mod b if $a > 0$, and an arbitrary function anti-periodic mod b if $a < 0$ (that is $\varphi(x + b) = -\varphi(x)$ for every $x \in \mathbb{R}$).

In addition, f is Baire 1 iff φ is Baire 1.

Proof. Straightforward calculations. □

Lemma 4.2.3 *Let $a_1, a_2, b_1, b_2 \in \mathbb{R}\setminus\{0\}$. Suppose that the equations $f(x+b_1)-a_1f(x) = 0$ and $f(x+b_2)-a_2f(x) = 0$ have a common Baire 1 solution which is not identically zero. Then $|a_1|^{1/b_1} = |a_2|^{1/b_2}$.*

Proof. Suppose that this is not true. Then by the previous lemma there exist two Baire 1 functions φ_1 and φ_2 such that $\varphi_1(x)(|a_1|^{1/b_1})^x = \varphi_2(x)(|a_2|^{1/b_2})^x$, where φ_1 and φ_2 are periodic (or anti-periodic) mod b_1 and b_2, respectively. We may assume that both functions are periodic, otherwise we could consider $\psi_i(x) = \varphi_i(2x)$ for $i = 1, 2$. We can also assume that $|a_1|^{1/b_1} < |a_2|^{1/b_2}$, and therefore

$$\varphi_1(x) = \varphi_2(x)c^x,$$

where $c > 1$. Finally, as φ_1 is not identically zero, we can also suppose (by applying an appropriate translation if needed) that $\varphi_1(0) \neq 0$.

Since φ_1 is periodic mod b_1, $\varphi_1(nb_1) \neq 0$ for every $n \in \mathbb{Z}$. As $c^x \neq 0$, this yields that $\varphi_2(nb_1) \neq 0$ for every $n \in \mathbb{Z}$, but φ_2 is periodic mod b_2, so $\varphi_2(nb_1 + kb_2) \neq 0$ for every $n, k \in \mathbb{Z}$. Consequently, $\varphi_1(nb_1 + kb_2) \neq 0$ for every $n, k \in \mathbb{Z}$.

Let us now suppose that $b_1/b_2 \in \mathbb{Q}$. Then φ_1 and φ_2 are periodic mod a common value p. But it is easy to see that this is impossible, since $c^x \neq 1$ when $x \neq 0$.

Therefore $b_1/b_2 \notin \mathbb{Q}$. Then it is well-known that for every (non-singular) interval $I \subset \mathbb{R}$ there exist integers $n, k \in \mathbb{Z}$ with k arbitrarily large such that $nb_1 + kb_2 \in I$. By substituting kb_2 into the above equation we get $\varphi_1(kb_2) = \varphi_2(kb_2)c^{kb_2}$ for every $k \in \mathbb{Z}$, thus $\varphi_1(kb_2) = \varphi_2(0)c^{kb_2}$ for every $k \in \mathbb{Z}$. Therefore $\varphi_1(nb_1 + kb_2) = \varphi_2(0)c^{kb_2}$ for every $n, k \in \mathbb{Z}$, which yields that φ_1 is unbounded on I. As I was arbitrary, φ_1 is unbounded on every subinterval of $\mathbb{R}$. But φ_1 is Baire 1, so it has a point of continuity, hence it must be bounded on some interval, a contradiction. $\square$

Remark The above lemma is closely related to the well-known statements that the identity function is not the sum of two measurable periodic functions, though it is surprisingly the sum of two periodic functions (see e.g. [LR] or [Kel2]). Indeed, taking logarithm in the above proof gives something very similar to this. The only problem is that our functions can vanish at certain points, and so in our case the argument above was more convenient.

Theorem 4.2.4 *Let $D_i f = g_i$ $(i \in I)$ be a system of difference equations, and suppose that every difference operator consists of at most two terms; that is for every $i \in I$ the equation is of the form*

$$a_i^{(1)} f(x + b_i^{(1)}) + a_i^{(2)} f(x + b_i^{(2)}) = g_i.$$

Then if every countable subsystem has a Baire 1 solution, then the whole system has one as well.

Proof. If any of the equations consists of a single term, then it has a unique solution, so we are clearly done. Thus by applying a translation and then by multiplying by a real number we may assume that every equation is of the form

$$f(x + b_i) - a_i f(x) = g_i.$$

Suppose first that for some $i_1, i_2 \in I$ we have $|a_{i_1}|^{1/b_{i_1}} \neq |a_{i_2}|^{1/b_{i_2}}$. Then it easily follows from Lemma 4.2.3 that the two corresponding equations have a unique common Baire 1 solution. This clearly solves the whole system, as every triple of equations is solvable.

So we can assume that there exists a $c > 0$ such that $|a_i|^{1/b_i} = c$ for every $i \in I$. If we now replace f by $f(x)/c^x$ (and modify g_i appropriately), then our equations will be of the form $f(x + b_i) \pm f(x) = g_i$. And then we can finish the proof by repeating the argument of the proof of Theorem 4.2.4 with a modification; we choose two sets b_{i_n} separately for the two types of equations. $\square$

Chapter 5

Linearly Ordered Sets of Real Baire 1 Functions

The set-theoretic nature of the next chapter is two-fold. On one hand, the vast majority of the results are based on techniques of Set Theory. On the other hand, when concluding the chapter (Theorem 5.5.4 and Question 5.5.5) we consider consistency results.

The problem we are dealing with (basically following [El2]) can be formulated as follows.

Any set $\mathcal{F}$ of real valued functions defined on an arbitrary set X is partially ordered by the pointwise order; that is $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. In other words put $f < g$ iff $f(x) \leq g(x)$ for all $x \in X$ and $f(x) \neq g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible order types of the linearly ordered (or simply 'ordered' from now on) subsets of this partially ordered set, which is the same as to characterize the ordered sets that are similar to an ordered subset of $\mathcal{F}$. Here two ordered sets are said to be similar iff there exists an order preserving bijection between them, and such a bijection from an ordered set onto an ordered subset of $\mathcal{F}$ is often referred to as a 'representation' of the ordered set. We sometimes say that the set is represented 'on X'. An ordered set similar to a representable one is also representable, so we can talk about 'representable order types' as well.

Since the functions in an ordered set are somehow 'above each other', one could think that this ordered set must be similar to a subset of the real line. As we shall see this is far from being true.

The problem of finding long sequences in $\mathcal{F}$; that is representing big ordinals has been studied for a long time. It was Miklós Laczkovich who posed the question how one can characterize the representable ordered sets, particularly in the case when $X = \mathbb{R}$ and $\mathcal{F}$ is the set of Baire 1 functions. What makes this problem interesting is that the corresponding questions about continuous (that is Baire 0) and Baire α functions ($\alpha > 1$) are completely solved. In the continuous case an ordered set is representable iff it is similar to a subset of $\mathbb{R}$ (an easy exercise), and for $\alpha > 1$ the question has turned out to be independent of ZFC [Ko].

The known facts about the case $\alpha = 1$ are the followings. The first is a classical theorem of Kuratowski asserting that there is no increasing or decreasing sequence of length ω_1 of real Baire 1 functions [Kur, §24. III.2']; that is ω_1 is not representable (in the sequel representable will always mean representable by real Baire 1 functions). The other is Péter Komjáth's Theorem stating that no Souslin line is representable [Ko]. (A Souslin line is a non-separable ordered set that does not contain an uncountable family of pairwise disjoint open intervals; that is ccc but not separable. The existence of Souslin lines is independent of ZFC [Je, Theorems 48,50].)

The main goal of this chapter is to present a few constructions of representable ordered sets which show that Kuratowski's Theorem is 'not too far' from being a characterization. In Section 2 we prove that certain operations result representable order types, and then in Section 3 and 4 we show that everything is representable that can be built up by certain steps, like forming countable products or replacing points by ordered sets.

We would also like to point out that if we restrict ourselves to the case of characteristic functions, we arrive at the problem of families of sets linearly ordered by inclusion. Indeed, $\chi_A < \chi_B$ iff $A \subsetneqq B$. The case of real Baire 1 functions corresponds to the problem of representing ordered sets by ambiguous subsets of the real line. (A set is called ambiguous iff it is F_σ and G_δ at the same time.) It is not hard to check that almost everything proved in this chapter is valid for this case as well, moreover, a kind of characterization of ordered sets that are representable by ambiguous sets is given in the last section.

For a topological space X the set of order types representable by real valued Baire 1 functions is denoted by $\mathcal{R}(X)$. The set of order types representable by ambiguous subsets is denoted by $\mathcal{R}_0(X)$.

5.1 Preliminaries

We shall frequently use the following simple lemma.

Lemma 5.1.1

(i) Let X and Y be metric spaces, $f : X \to \mathbb{R}$ Baire 1 and $g : Y \to X$ continuous. Then $f \circ g : Y \to \mathbb{R}$ is Baire 1.

(ii) Let X be a metric space and $X_n \subset X$ $(n \in \mathbb{N})$ F_σ sets such that $X = \bigcup_{n=1}^{\infty} X_n$. If $f : X \to \mathbb{R}$ is relatively Baire 1 on each X_n $(n \in \mathbb{N})$ then f is Baire 1.

Let us first consider the following question, which shall be a useful tool in the sequel. Which Polish spaces are equivalent to the real line in the sense that the same ordered sets can be represented on them? We shall ignore the countable metric spaces as it is easy to see that if an order type is representable on such a space then it is similar to a subset of the real line. Denote by C the Cantor set.

Theorem 5.1.2 $R(X) = R(C) = R(\mathbb{R})$ *for any σ-compact uncountable metric space X.*

Proof. It is obviously enough to prove the first equality. Let X be compact for the time being, then a classical theorem asserts that there exists a continuous surjection $F : C \to X$ [Kur, §41, VI.3a]. If $\{f_\alpha : \alpha \in \Gamma\}$ is an ordered set of Baire 1 functions defined on X, one can easily verify that $\{f_\alpha \circ F : \alpha \in \Gamma\}$ is also ordered, similar to the former ordered set as a consequence of the surjectivity of F and consists of Baire 1 functions defined on C by lemma 5.1.1.

In the general case $X = \cup_{n=1}^{\infty} X_n$ where $X_n \subset X$ is compact and let again be $\{f_\alpha : \alpha \in \Gamma\}$ an ordered set of Baire 1 functions on X. We shall show that this set is representable on the interval $[0,1]$ and therefore on C as well, since $[0,1]$ is a compact metric space and we can apply what we have proven in the previous case.

Fix a set $H_n \subset (\frac{1}{n}, \frac{1}{n+1})$ for each $n \in \mathbb{N}$ homeomorphic to the Cantor set and also a homeomorphism $g_n : H_n \to C$. We can choose furthermore continuous surjections $F_n : C \to X_n$ $(n \in \mathbb{N})$ since X_n is a compact metric space. Now we represent the set in the following way. For each $\alpha \in \Gamma$ let

$$g_\alpha = \begin{cases} f_\alpha \circ F_n \circ g_n & \text{on } H_n \quad (n \in \mathbb{N}) \\ 0 & \text{on } [0,1] \setminus \cup_{n=1}^{\infty} H_n. \end{cases}$$

Indeed, the map $g_\alpha \mapsto f_\alpha$ $(\alpha \in \Gamma)$ turns out to be a similarity as $F_n \circ g_n$ is surjective and moreover in view of Lemma 5.1.1 it is straightforward to verify that g_α is a Baire 1 function on $[0,1]$ for each $\alpha \in \Gamma$.

In order to check the opposite direction let $\{f_\alpha : \alpha \in \Gamma\}$ be an ordered set of Baire 1 functions on the Cantor set. According to a classical theorem every uncountable compact metric space contains a subspace homeomorphic to C [Kur, §36, V.1], which easily generalizes to the case of uncountable σ-compact metric spaces since if $X = \cup_{n=1}^{\infty} X_n$, X_n compact, then at least one X_n is uncountable. We can therefore fix a homeomorphism $h : C \to Y \subset X$ and for $\alpha \in \Gamma$ let

$$g_\alpha = \begin{cases} f_\alpha \circ h^{-1} & \text{on } Y \\ 0 & \text{on } X \setminus Y. \end{cases}$$

One can easily prove in the above manner that this is an ordered set of Baire 1 functions similar to the above one. □

As the next corollary shows, the above theorem implies the surprising fact that all the complicated ordered sets represented in the following sections are also representable by functions of connected graphs.

Corollary 5.1.3 *A representable ordered set is also representable by Darboux Baire 1 functions and consequently by Baire 1 functions of connected graphs.*

Proof. It is well-known that the graph of a Baire 1 function is connected iff it is Darboux [Br, II.1.1]. By the previous theorem we can assume that the set is represented on the Cantor set. It is not hard to extend the representing functions by a common continuous function to the complement of the Cantor set which makes the representing functions Darboux and Baire 1 by Lemma 5.1.1. □

Next we show that there are at most two distinct possible sets $\mathcal{R}(X)$ for all uncountable Polish spaces X.

Theorem 5.1.4 $R(X) = R(\mathbb{R} \setminus \mathbb{Q})$ *for any non-*σ*-compact Polish space* X*.*

Proof. We apply the argument of Theorem 5.1.2. In one direction we use that every Polish space is the continuous image of the irrationals [Kur, §36, II.1], while in the other direction we apply Hurewicz's Theorem [Kec, Theorem 7.10] asserting that every non-σ-compact Polish space contains a homeomorphic copy of the irrationals as a closed subspace. □

This leaves the question open whether all uncountable Polish spaces are equivalent or not.

Question 5.1.5 *Does* $R(C) = R(\mathbb{R} \setminus \mathbb{Q})$ *hold?*

Remark In order to give an affirmative answer it would be enough to prove that every ordered set of Baire 1 functions on the irrationals can be represented by Baire 1 functions on the reals. Indeed, on one hand every uncountable Polish space contains a subset which is homeomorphic to the Cantor set [Kur, §36, V.1], and on the other hand every Polish space is the continuous image of $\mathbb{R} \setminus \mathbb{Q}$ hence the above argument works.

Moreover, it can be shown that a Baire 1 function defined on the irrationals can be extended to the reals as a Baire 1 function, but so far we were unable to do this in an order preserving way.

5.2 Operations on Representable Ordered Sets

Now we investigate whether the class of representable sets are closed under certain operations. We shall make use of these operations when constructing complicated representable ordered sets.

Definition 5.2.1 For an arbitrary ordered set X we call $X \times \{0,1\}$ with the lexicographical order the *duplication of* X.

Question 5.2.2 *Is it true that the duplication of a representable set is also representable?*

In most cases this question can be replaced by the following statement.

Statement 5.2.3 *Let X be an ordered set such that the duplication of X is representable. Then so is the ordered set obtained by replacing every $x \in X$ by a representable set Y_x; that is $\{(x,y) : x \in X, y \in Y_x\}$ with the lexicographical order.*

Proof. First we replace the points of the real line by uncountable closed sets in the following way. Let $P : [0,1] \to [0,1]^2$ be a Peano curve, that is a continuous surjection, and let P_1 be its first coordinate function. Then $P_1 : [0,1] \to [0,1]$ is also a continuous surjection, moreover the preimages $P_1^{-1}(\{c\})$ are uncountable closed sets for all $c \in [0,1]$. In virtue of Theorem 5.1.2 we may assume that the duplication of X is represented on $[0,1]$ by the pairs of functions $f_x < g_x$ $(x \in X)$. If we consider the functions $f_x \circ P_1$ and $g_x \circ P_1$ we obtain a similar ordered set of Baire 1 functions, but in the latter set any two distinct elements differ on an uncountable closed sets, for if f_x and g_x attained different values at c_x then $f_x \circ P_1$ and $g_x \circ P_1$ differ on $P_1^{-1}(\{c_x\})$. Since this is a compact metric space we may assume that Y_x is represented on it. By composing with a increasing homeomorphism between $\mathbb{R}$ and the interval $(f_x(c_x), g_x(c_x))$ we also can assume that the functions representing Y_x only attain values between $f_x(c_x)$ and $g_x(c_x)$.

Now we claim that the following representation will do. For $x \in X$ and $y \in Y_x$ let

$$h_{(x,y)} = \begin{cases} f_x \circ P_1 & \text{on } [0,1] \setminus P_1^{-1}(\{c_x\}) \\ \text{the function representing y} & \text{on } P_1^{-1}(\{c_x\}). \end{cases}$$

These functions are easily seen to be Baire 1 so what remains to show is that the representation is order preserving. In the first case $x_1 < x_2$ so $f_{x_1} < g_{x_2}$ hence

$$h_{(x_1,y_1)} < g_{x_1} \circ P_1 < f_{x_2} \circ P_1 < h_{(x_2,y_2)}.$$

Finally, in the second case $x_1 = x_2 = x$ and $y_1 < y_2$. Obviously $h_{(x,y_1)}$ and $h_{(x,y_2)}$ differ on $P_1^{-1}(\{c_x\})$ only, where they are defined according to the order of Y_x, therefore $h_{(x,y_1)} < h_{(x,y_2)}$. □

Statement 5.2.4 *Let X be an ordered set such that the duplication of X is representable. Then X^ω endowed with the lexicographical order is also representable.*

Proof. As in the previous proof we can represent the duplication of X such that for every $x \in X$ the representing functions $f_x,\ g_x : \mathbb{R} \to [0,1]$ are different constant functions on a suitable Cantor set C_x. Denote d_x the difference of these two values. In the next step, for every fixed $x_1 \in X$ let us represent the duplication of X on C_{x_1} in the same manner as above; that is for each $x_2 \in X$ let $f_{x_1,x_2},\ g_{x_1,x_2} : \mathbb{R} \to [0, \min(\frac{1}{2}, d_{x_1})]$ be zero outside C_{x_1} such that they are different constants on a suitable Cantor set $C_{x_1,x_2} \subset C_{x_1}$. Let d_{x_1,x_2} denote the difference of the two values. Then we proceed inductively and make sure that $0 \leq f_{x_1,\ldots,x_{n+1}},\ g_{x_1,\ldots,x_{n+1}} \leq \min(\frac{1}{2^n}, d_{x_1,\ldots,x_n})$. It is not hard to see that

$$(x_1, x_2, \ldots) \mapsto \sum_{n=1}^{\infty} f_{x_1,\ldots,x_n}$$

is the required representation, as the uniform limit of Baire 1 functions is Baire 1 itself [Kur, §31, VIII.2]. □

Remark Instead of using the same set X at each level, we can prove in exactly the same way that if the duplication of X_n is representable for every $n \in \mathbb{N}$ then so is $\prod_{n=1}^{\infty} X_n$, and more generally we can also use different sets at a level; that is we can correspond a set $X_{x_1,\ldots,x_n}$ to each $x_1, \ldots, x_n$.

However, we do not know the answer to the question concerning longer products. As a simple transfinite induction shows, the following two questions are equivalent.

Question 5.2.5 *Is it true that if the duplication of X is representable, then the duplication of X^ω is also representable? Or equivalently, is it true, that if the duplication of X is representable, then so is X^α for every $\alpha < \omega_1$?*

Corollary 5.2.6 *Suppose that the duplications of representable orders are also representable. Then X^α is representable for every representable X and $\alpha < \omega_1$.*

Proof. We prove this by induction on α. If $\alpha = \beta + 1$ then X^α is similar to $X^\beta \times X$. But X^β is representable by the inductional hypothesis, so is its duplication by our assumption, therefore we can apply Statement 5.2.3 and we are done.

If α is a limit ordinal, then $[0, \alpha)$ can be written as the disjoint union of $[\alpha_n, \alpha_{n+1})$ for a suitable sequence α_n $(n \in \mathbb{N})$. The interval $[\alpha_n, \alpha_{n+1})$ is similar to an ordinal $\beta_n < \alpha$, so X^α is similar to $\prod_{n=1}^\infty X^{\beta_n}$, and we are again done by the previous remark. □

Remark As above, we can generalize this result as well to $\prod_{\beta<\alpha} X_\beta$ and also to the case when at each level we correspond an arbitrary representable set to each point.

Next we pose another question.

Question 5.2.7 *Is it true that the completion (as an ordered set) of a representable ordered set is also representable?*

Definition 5.2.8 Let X and X_n $(n \in \mathbb{N})$ be ordered sets. We say that X is *a blend of the sets* X_n if there exist pairwise disjoint subsets $H_n \subset X$ $(n \in \mathbb{N})$ such that $X = \cup_{n=1}^\infty H_n$ and H_n is similar to X_n.

Statement 5.2.9 *Suppose that duplications and completions of representable sets are also representable. Then so is a blend X of the representable sets X_n.*

Proof. Let H_n be as in the definition. By the hypothesis the completion of $H_n \times \{0, 1\}$ is representable for each $n \in \mathbb{N}$ and we may assume that it is represented on the interval $(n, n + 1)$. Let $x \in X$; that is $x \in H_n$ for exactly one n, and let

$$f_x = \begin{cases} \text{the function representing } (x, 0) & \text{on } (n, n+1) \\ \text{the function representing} & \\ \sup\{(y, i) \in H_m \times \{0, 1\} : y \le x\} & \text{on } (m, m+1) \; if \; m \ne n \\ 0 & \text{elsewhere,} \end{cases}$$

where 'sup' means supremum according to the order of the completion of $H_m \times \{0,1\}$. f_x is Baire 1 as the usual argument shows so we only have to check that this latter set of functions is similar to the original one. Let $x, y \in X$, $x < y$ and $x \in H_k$, $y \in H_l$ for some k and l. If $k = l$ then $f_x < f_y$ is obvious while if $k \neq l$ then one can easily check that $f_x \leq f_y$ on $(k, k+1)$, $(l, l+1)$ and on the complement of their union, moreover $f_x \neq f_y$ on $(k, k+1)$ since f_y is not less here then the function representing $(x, 1)$. □

5.3 The First Construction

In the sequel we present a few constructions of representable sets which have such a rich structure in some sense that we may hope to be able to produce all the representable order types this way.

Definition 5.3.1 Let α be an ordinal number and $I = [0,1]$. We denote by I^α the set of transfinite sequences in I of length α with the lexicographical order (i.e. $I^\alpha = \{f : f : \alpha \to I\}$ and $f < g$ iff $f(\gamma) = g(\gamma)$ and $f(\beta) < g(\beta)$ for some β and every $\gamma < \beta$).

When $\alpha \geq \omega_1$, then due to Kuratowski's Theorem [Kur, §24, III.2'], I^α is not representable as it contains a subset of type ω_1. However the following holds.

Theorem 5.3.2 *I^α is representable for all $\alpha < \omega_1$.*

Proof. For $\alpha < \omega$ the assertion follows from Statement 5.2.3 by induction. Denote by $H = \prod_{n=0}^{\infty} [0,1]$ the Hilbert cube; that is the topological product of countably many copies of the closed unit interval. It is well-known that H is a compact metric space so it is sufficient to represent I^α on H. We show that this is possible even by characteristic functions, in other words there exists a system of ambiguous subsets of H which is of order type I^α when ordered by inclusion. First we define an order of type I^α on H. As $\alpha < \omega_1$ there exists a bijection $\varphi : \mathbb{N} \to \alpha$ so we can assign to each element $a = (a_1, a_2, \ldots) \in H$ a transfinite sequence $x = (a_{\varphi(n)} : n \in \mathbb{N})$. Since this is a bijection between H and I^α it induces an order of type I^α on H which we shall denote by $<_H$. We claim that the sets of the form $H_x = \{y \in H : y <_H x\}$ constitute a system of sets possessing all the properties we need. First of all $H_x \subsetneqq H_y$ iff $x <_H y$ thus $\{H_x : x \in H\}$ is of order type I^α. We still have to check that $H_x \subset H$ is ambiguous for all $x \in H$. First we show that it is F_σ. Indeed,

$$H_x = \bigcup_{\beta<\alpha} \left(\bigcap_{\gamma<\beta} \left\{(y_1, y_2, \dots) \in H : y_{\varphi^{-1}(\gamma)} = x_{\varphi^{-1}(\gamma)}\right\} \cap \left\{y_{\varphi^{-1}(\beta)} < x_{\varphi^{-1}(\beta)}\right\} \right)$$

so it is sufficient to check that the members of the union are F_σ sets, but this is obvious as they are intersections of certain closed sets and an open set.

Similarly $\{y \in H : x <_H y\}$ is also F_σ, and as $\{x\}$ is F_σ, H_x is the complement of an F_σ set hence G_δ. □

In view of Kuratowski's Theorem it is natural to ask whether every representable set can be embedded into I^α for a suitable $\alpha < \omega_1$. We show in two steps that this is not true.

Lemma 5.3.3 *$I^{\alpha+1}$ cannot be embedded into I^α for any $\alpha < \omega_1$.*

Proof. Suppose on the contrary that $f : I^{\alpha+1} \to I^\alpha$ is an order-preserving injection and let $f = (f_0, f_1, \dots, f_\beta, \dots)$ where $f_\beta : I^{\alpha+1} \to I$ $(\beta < \alpha)$ are the coordinate functions. As $f_0 : I^{\alpha+1} \to I$ is monotone, and for distinct values of $c \in I$ the convex hulls of the sets $f_0(\{x_0, \dots, x_\beta, \dots, x_\alpha : x_0 = c\})$ are non-overlapping intervals in I, all but countably many of them are singletons. Therefore we can fix a_0 such that $f_0((a_0, x_1, \dots, x_\beta, \dots, x_\alpha))$ is constant. Once we have already chosen a_γ for each $\gamma < \beta$ such that $f_\gamma((a_0, \dots, a_\gamma, x_{\gamma+1}, \dots, x_\alpha))$ is constant then as before for distinct values of x_β we obtain essentially pairwise disjoint image sets and thus we can fix $a_\beta \in I$ such that $f_\beta((a_0, \dots, a_\beta, x_{\beta+1}, \dots, x_\alpha))$ is constant. But then eventually we get

$$f\left((a_0, \dots, a_\beta, \dots, 0)\right) = f\left((a_0, \dots, a_\beta, \dots, 1)\right),$$

contradicting the injectivity of f. □

Statement 5.3.4 *There exists a representable set that is not embeddable into I^α for any $\alpha < \omega_1$.*

Proof. The duplication of the real line is representable as it is similar to a subset of I^2, hence if we replace ω_1 arbitrary points of $\mathbb{R}$ by the sets I^α $(\alpha < \omega_1)$ we obtain a representable set. In virtue of the previous lemma and Statement 5.2.3 this set possesses the required property. □

This negative result shows how to go on to find new representable sets by iteration.

Definition 5.3.5 Let $\mathcal{H}$ be an arbitrary set of ordered sets. We define an increasing transfinite sequence S_α $(\alpha \in On)$ of sets as follows.

Let $S_0 = \mathcal{H} \cup \{\emptyset\}$ and S_α be the set of ordered sets that can be obtained by replacing the points of a set $X \in \bigcup_{\beta<\alpha} S_\beta$ by sets $Y_x \in \bigcup_{\beta<\alpha} S_\beta$ $(x \in X)$.

Finally, let $\mathcal{S}(\mathcal{H})$ denote the set of order types of $\bigcup_{\alpha \in On} S_\alpha$.

Lemma 5.3.6 *$\mathcal{S}(\mathcal{H})$ is a set indeed as there exists an ordinal α such that $S_\beta = S_\alpha$ for every $\beta \geq \alpha$.*

Proof. Let κ be a infinite cardinal such that $|H| \leq \kappa$ for every $H \in \mathcal{H}$. A simple transfinite induction shows that $|X| \leq \kappa$ for all $X \in S_\alpha$ and $\alpha \in On$. We choose a cardinal μ of cofinality greater than κ (e.g. 2^κ), and claim that $\alpha = \mu$ will do.

First we show that $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. Choose $X \in S_\alpha$; that is $Y, Z_y \in \bigcup_{\beta<\alpha} S_\beta$ and fix $\beta, \beta_y < \alpha$ $(y \in Y)$ such that $Y \in S_\beta$ and $Z_y \in S_{\beta_y}$ $(y \in Y)$. The set $\{\beta\} \cup \{\beta_y : y \in Y\}$ is at most of power κ which is less then the cofinality of α thus we can find a $\beta^* < \alpha$ such that $\beta, \beta_y < \beta^*$ $(y \in Y)$. But then $X \in S_{\beta^*} \subset \bigcup_{\beta<\alpha} S_\beta$.

Secondly, we check by transfinite induction that $S_\beta = S_\alpha$ for all $\beta \geq \alpha$. Suppose $S_\gamma = S_\alpha$ for $\alpha \leq \gamma < \beta$ and let $X \in S_\beta$; that is $Y, Z_y \in \bigcup_{\gamma<\beta} S_\gamma$. However,

$$\bigcup_{\gamma<\beta} S_\gamma = \bigcup_{\gamma<\beta} S_\alpha = S_\alpha = \bigcup_{\delta<\alpha} S_\delta$$

which implies $X \in S_\alpha$ by repeating the above argument. □

Theorem 5.3.7 *If $\mathcal{H}$ is a set of ordered sets such that the duplications of the elements of $\mathcal{H}$ are representable, then the elements of $\mathcal{S}(\mathcal{H})$ are also representable.*

Proof. We prove by transfinite induction on α the seemingly stronger statement that even the duplications of elements of $\mathcal{S}(\mathcal{H})$ are representable. For $\alpha = 0$ this is just a reformulation of our assumption. Suppose now that the statement holds for all $\beta < \alpha$ and let $X \in S_\alpha$; that is $Y, Z_y \in \bigcup_{\beta<\alpha} S_\beta$. As $Z_y \in \bigcup_{\beta<\alpha} S_\beta$ $Z_y \times \{0,1\}$ is representable by the inductional hypothesis. Moreover if we replace the points of Y by the sets $Z_y \times \{0,1\}$ what we obtain is exactly the duplication of X, which therefore turns out to be representable as by the inductional hypothesis $Y \times \{0,1\}$ is representable and so we can apply Statement 5.2.3. □

Definition 5.3.8 If $\mathcal{H}$ is a set of ordered sets, then let

$$\mathcal{H}^\omega = \{Y : Y \subset X^\omega, X \in \mathcal{H}\},$$

and let $\mathcal{H}^*$ be the closure of $\mathcal{H}$ under the operations $X \mapsto X^\alpha$ $(\alpha < \omega_1)$. (This closure can be formed by a similar transfinite construction as $\mathcal{S}(\mathcal{H})$.)

Corollary 5.3.9 *If $\mathcal{H}$ is a set of ordered sets such that the duplications of the elements of $\mathcal{H}$ are representable, then the elements of $\mathcal{S}(\mathcal{H})^\omega$ are also representable. This holds even for $\mathcal{S}(\mathcal{H})^*$, assuming that the duplications of representable sets are representable.*

Remark 1) We could define similar notions with products instead of powers, or even with the more complex constructions mentioned in the remark following Statement 5.2.4, but in fact we would not get more, as in the case we are interested in, there are always at most continuum many sets involved, thus we can put them together (e.g. replace the points of $\mathbb{R}$ by them) to form a huge set X that contains each of them, and so the power of this set X contains subsets similar to all these above constructions.

2) If we begin our procedure of building large representable orders, we can start with some set of simple ordered sets, for example the ones representable by constants or even continuous functions. In both cases we have $\mathcal{H} = \{\mathbb{R}\}$. It is not hard to prove that we will not get too far this way as I^ω will not be in $\mathcal{S}(\mathcal{H})$. (The proof goes by transfinite induction. Note that any non-trivial subinterval of I^ω contains a copy of I^ω and that building up a set X by replacing each element y of a set Y by X_y is the

same as partitioning X into subintervals that are ordered similarly to Y such that each subinterval is similar to the corresponding X_y.) Therefore we prefer starting with the set of 'unboundedly wide trees', $\{I^\alpha : \alpha < \omega_1\}$.

3) According to the previous theorems $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})$ contains order types of representable duplication only, as the duplication of I^α is a subset of $I^{\alpha+1}$. However, $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\}) \neq \mathcal{R}(\mathbb{R})$ as every element of the former set contains a non-trivial subinterval that is similar to a subset of I^α for some α, while if X is as in the proof of Statement 5.3.4, then X^ω does not. Therefore $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^\omega$ is a strictly larger class of representable orders. This holds for $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^*$ as well, under the assumption about duplications.

It seems quite plausible that if we are allowed to replace points by arbitrarily large sets of the form I^α (of course $\alpha < \omega_1$), and allowed to form countable products, then we can build up every set not containing a sequence of length ω_1. Moreover it can be shown that $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^*$ is closed under duplication, completion and blends. (The definition of these notions for order types instead of ordered sets is obvious.) Together with Kuratowski's Theorem this motivates the following question.

Question 5.3.10 *Does either $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^\omega = \mathcal{R}(\mathbb{R})$ or $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^* = \mathcal{R}(\mathbb{R})$ hold?*

5.4 The Second Construction

Now we turn to another approach of the problem which results in a notion very similar to $\mathcal{S}(\mathcal{H})$.

Statement 5.4.1 *Let $\{f_\alpha : \alpha \in \Gamma\}$ be an ordered set of functions defined on a second countable topological space and possessing the Baire property. If any two functions differ on a set of second category then the ordered set is similar to a subset of the real line.*

Proof. Recall that an ordered set is similar to a subset of $\mathbb{R}$ iff it is separable and does not contain more than countably many pairs of consecutive elements.

First we prove separability. Let X be the second countable space and suppose for the time being that X is a Baire space; that is every non-empty open subset is of second category. Denote by B a countable base of the space not containing the empty set. We construct a countable dense subset M of $\{f_\alpha : \alpha \in \Gamma\}$ in the following way. If for $U, V \in B$ and $p, q \in \mathbb{Q}$ there exists $h \in \{f_\alpha : \alpha \in \Gamma\}$ such that $p < h$ on a residual subset of U and $h < q$ on a residual subset of V then we choose such an h. M is obviously countable and to verify that it is dense let (f, g) be an open interval of the ordered set. If this interval is empty then we are done so we may assume that there exists an element h_0 of the ordered set in the interval. Obviously

$$X(f < h_0) = \bigcup_{p \in \mathbb{Q}} X(f < p < h_0)$$

and

$$X(h_0 < g) = \bigcup_{q \in \mathbb{Q}} X(h_0 < q < g),$$

where the sets on the left hand side are by assumption of second category hence for some p and q $X(f < p < h_0)$ and $X(h_0 < q < g)$ are of second category as well. It is easy to see that a set of second category which also possesses the Baire property is residual in some non-empty open subset, moreover this open set can be chosen to be an element of B. As f, g and h_0 have the Baire property $X(f < p < h_0)$ and $X(h_0 < q < g)$ have it as well so we can find $U, V \in B$ in which these sets are residual respectively. But this means that for $U, V \in B$ and $p, q \in \mathbb{Q}$ there exists an element of the ordered set, namely h_0, satisfying all the conditions of the definition of M so there must be such an element $h \in M$ as well. We show that $h \in (f, g)$. X is a Baire space hence U is not of first category therefore there exists $x \in U$ for which $f(x) < p < h(x)$ and similarly $y \in V$ for which $h(y) < q < g(y)$. But this implies $f < h < g$ proving the separability.

Let now $f_i < g_i$ $(i \in I)$ be distinct consecutive elements in the ordered set. Like above, for every $i \in I$

$$X(f_i < g_i) = \bigcup_{p \in \mathbb{Q}} X(f_i < p < g_i)$$

hence for a suitable p_i $X(f_i < p_i < g_i)$ is of second category and we can thus fix $U_i \in B$ in which this set is residual. We show that the map $i \mapsto (p_i, U_i)$ is injective which implies that I is countable. Indeed, if $i \neq i'$ and $(p_i, U_i) = (p_{i'}, U_{i'}) = (p, U)$ than, as U is of second category, we obtain that for some $x \in U$ $f_i(x) < p < g_i(x)$ and $f_{i'}(x) < p < g_{i'}(x)$ contradicting the consecutiveness of the pairs.

Finally, if X is not a Baire space than as a consequence of Banach's Union Theorem [Kur, §10, III] we can write it as $X = G \cup A$ where G is an open subset which is a Baire space as a subspace and A is of first category. If we consider the restrictions of the functions to G we obtain a similar ordered set as any two functions differ on a set of second category in X hence they can not coincide on G. In fact, by the same argument they differ in G on a set of second category and thus we can apply what we have proven in the previous case. □

This statement enables us to simplify the structure of a represented set X in the following way. Zorn's lemma implies that we can find a maximal subset of X in which every two elements differ on a set of second category. As this subset must be separable we can choose a countable dense subset M of it. The maximal intervals of $X \setminus M$ are of a simpler structure than X since any two elements of such an interval coincide on a residual set, moreover it follows from Kuratowski's Theorem that all elements of the interval coincide on a common residual set. We can thus go on and repeat this procedure inside this residual set. This motivates the following.

Definition 5.4.2 Let $\mathcal{H}$ be an arbitrary set of ordered sets. We call elements of $\mathcal{H}$ and the empty set sets of rank 0. For an ordinal α we say that an ordered set X is of rank at most α if there exists a countable subset $M \subset X$ such that all maximal intervals I of $X \setminus M$ are of rank at most β for some $\beta < \alpha$ where β may depend on I. The class of ordered sets of rank at most α is denoted by T_α.

Finally, let $\mathcal{T}(\mathcal{H})$ be the set of order types of $\bigcup_{\alpha \in On} T_\alpha$.

Lemma 5.4.3 *If X is a set of rank at most α then it is similar to a set obtained by replacing the points of $\mathbb{R}$ by elements of $\bigcup_{\beta<\alpha} T_\beta$.*

Proof. Let $M \subset X$ be the countable subset as in the definition. Recall that every countable ordered set can be embedded into $\mathbb{Q}$ and fix a $\varphi : M \to \mathbb{Q}$ order preserving injective map.

A maximal interval I of $X \setminus M$ splits M into two parts M_1 and M_2 in a natural way. Define

$$F(I) = \sup\{\varphi(x) : x \in M_1\},$$

where we may assume the supremum to be finite as we may attach a first and a last element to X which may also be elements of M. Now if I_1, I_2 and I_3 are distinct maximal intervals following each other in this order then we can find an element $x \in M$ between I_1 and I_2 and $y \in M$ between I_2 and I_3 therefore $F(I_1) < F(I_3)$ as $\varphi(x) < \varphi(y)$. Similarly, $F(I_1) = F(I_2)$ implies that there is exactly one $x \in M$ between I_1 and I_2. Consequently we can map X to the real line via φ and F in an order preserving way such that the preimage of a real number is one of the followings: the empty set, a single point, a maximal interval, a maximal interval plus an extra point to the left or right or two intervals and a point in between. But these sets are obviously elements of $\bigcup_{\beta<\alpha} T_\beta$ hence the lemma follows. □

Corollary 5.4.4 *If $\mathbb{R} \in \mathcal{H}$ then $\mathcal{T}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ thus $\mathcal{T}(\mathcal{H})$ is a set indeed.*

Corollary 5.4.5 *If the duplication of every element of rank 0 is representable then so is every element of $\mathcal{T}(\mathcal{H})$.*

Remark $\mathcal{T}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$ fails in general as the examples $\mathcal{H} = \{\mathbb{R}\}$ or $\mathcal{H} = \{X : X \subset I^\omega\}$ show, since in both cases $\mathcal{T}(\mathcal{H})$ is a subset of the order types of $\{X : X \subset I^\omega\}$.

However, the following question is open.

Question 5.4.6 *Does $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\}) = \mathcal{T}(\{I^\alpha : \alpha < \omega_1\})$ or $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^\omega = \mathcal{T}(\{I^\alpha : \alpha < \omega_1\})^\omega$ or $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^* = \mathcal{T}(\{I^\alpha : \alpha < \omega_1\})^*$ hold?*

5.5 Concluding Remarks

First we give a characterization of $\mathcal{R}_0(\mathbb{R})$, which in fact does not show too much about the structure of these orders. This is motivated by the way our constructions worked.

Theorem 5.5.1 *An ordered set X is representable by ambiguous sets iff there exists an order on a compact metric space such that certain initial segments are ambiguous and ordered similarly to X by inclusion.*

Proof. If we have such an order then of course the initial segments will do. Conversely, let $\{H_x : x \in X\}$ be a representation by ambiguous sets. Let

$$a \prec b \text{ iff } \exists x \in X \text{ such that } a \in H_x \text{ and } b \notin H_x.$$

One can easily see that this is a partial order on the compact metric space. By Zorn's lemma every partial order can be extended to an order, thus denote $\prec^*$ such an extension. We only have to show that H_x is an initial segment indeed of $\prec^*$ for each $x \in X$. So let $a \in H_x$, $b \prec^* a$ and show that $b \in H_x$. If this was not true then $b \notin H_x$, $a \in H_x$ and $b \prec^* a$ would hold, which contradicts the definition of $\prec^*$. □

Question 5.5.2 *Does $\mathcal{R}(\mathbb{R}) = \mathcal{R}_0(\mathbb{R})$ hold?*

One can show that this is equivalent to the following.

Question 5.5.3 *Suppose X is representable. Is it also representable by Baire 1 functions that attain irrational values only?*

To summarize our results we may say that the class of representable ordered sets seems to be quite close to the ones not containing sequences of length ω_1. Our last theorem asserts that one actually can not prove in ZFC that these two classes coincide.

Theorem 5.5.4 *The statement that a set is representable iff it does not contain a sequence of length ω_1 is not provable in ZFC.*

Proof. A Souslin line does not contain such a long increasing sequence otherwise $\{(x_\alpha, x_{\alpha+2}) : \alpha < \omega_1 \text{ is a limit ordinal}\}$ would be an uncountable system of pairwise disjoint non-empty open intervals. The case of decreasing sequences is similar. Therefore in view of Komjáth's Theorem and the independence of the existence of Souslin lines the theorem follows. □

Finally we pose a fundamental question.

Question 5.5.5 *Is it consistent with ZFC that an ordered set is representable iff it does not contain a sequence of length* ω_1*?*

Chapter 6

Transfinite Sequences of Functions on Metric Spaces

In this chapter our aim will be to investigate the possible lengths of the increasing or decreasing well-ordered sequences of functions with respect to the order defined in Chapter 5. As we have already mentioned, a classical theorem [Kur, §24.III, Theorem 2'] asserts that if $\mathcal{F}$ is the set of Baire 1 functions defined on a Polish space X, then there exists a monotone sequence of length ξ in $\mathcal{F}$ iff $\xi < \omega_1$. P. Komjáth [Ko] proved that the corresponding question concerning Baire α functions for $2 \leq \alpha < \omega_1$ is independent of *ZFC*.

In the present chapter we investigate what happens if we replace the Polish space X by an arbitrary metric space. (These results can be found in [EK].)

Section 6.1 considers chains of continuous functions. We show that for any metric space X, there exists a chain in $C(X, \mathbb{R})$ of order type ξ iff $|\xi| \leq d(X)$. Here, $d(X)$ denotes the density of the space X; that is

$$d(X) = \max(\min\{|D| : D \subset X \ \& \ \overline{D} = X\},\ \omega) \ .$$

In particular, for separable X, every well-ordered chain has countable length, just as for Polish spaces.

Section 6.2 considers chains of Baire 1 functions on separable metric spaces. Here, the situation is entirely different from the case of Polish spaces, since on some separable metric spaces, there are well-ordered chains of every order type less than ω_2. Furthermore, the existence of chains of type ω_2 and longer is independent of $ZFC + \neg CH$. Under MA, there are chains of all types less than $(2^\omega)^+$, whereas in the Cohen model, all chains have type less than ω_2.

6.1 Sequences of Continuous Functions

Lemma 6.1.1 *For any topological space X: If there is a well-ordered sequence of length ξ in $C(X,\mathbb{R})$, then $\xi < d(X)^+$.*

Proof. Let $\{f_\alpha : \alpha < \xi\}$ be an increasing sequence in $C(X,\mathbb{R})$, and let $D \subset X$ be a dense subset of X such that $d(X) = \max(|D|,\omega)$. By continuity, the $f_\alpha \restriction D$ are all distinct; so, for each $\alpha < \xi$, choose a $d_\alpha \in D$ such that $f_\alpha(d_\alpha) < f_{\alpha+1}(d_\alpha)$. For each $d \in D$ the set $E_d = \{\alpha : d_\alpha = d\}$ is countable, because every well-ordered subset of $\mathbb{R}$ is countable. Since $\xi = \bigcup_{d\in D} E_d$, we have $|\xi| \leq \max(|D|,\omega) = d(X)$. □

The converse implication is not true in general. For example, if X has the countable chain condition (ccc), then every well-ordered chain in $C(X,\mathbb{R})$ is countable (because $X \times \mathbb{R}$ is also ccc). However, the converse is true for metric spaces:

Lemma 6.1.2 *If (X,ϱ) is any metric space and $\prec$ is any total order of the cardinal $d(X)$, then there is a chain in $C(X,\mathbb{R})$ which is isomorphic to $\prec$.*

Proof. First, note that every countable total order is embeddable in $\mathbb{R}$, so if $d(X) = \omega$, then the result follows trivially using constant functions. In particular, we may assume that X is infinite, and then fix $D \subset X$ which is dense and of size $d(X)$. For each $n \in \omega$, let D_n be a subset of D which is maximal with respect to the property $\forall d,e \in D_n\,[d \neq e \to \varrho(d,e) \geq 2^{2-n}]$. Then $\bigcup_n D_n$ is also dense, so we may assume that $\bigcup_n D_n = D$. We may also assume that $\prec$ is a total order of the set D. Now, we shall produce $f_d \in C(X,\mathbb{R})$ for $d \in D$ such that $f_d < f_e$ whenever $d \prec e$.

For each n, if $c \in D_n$, define $\varphi^n_c(x) = \max(0,\ 2^{-n} - \varrho(x,c))$. For each $d \in D$, let $\psi^n_d = \sum\{\varphi^n_c : c \in D_n \ \&\ c \prec d\}$. Since every $x \in X$ has a neighborhood on which all but at most one of the φ_c^n vanish, we have $\psi_d^n \in C(X, [0, 2^{-n}])$, and $\psi_d^n \le \psi_e^n$ whenever $d \prec e$. Thus, if we let $f_d = \sum_{n<\omega} \psi_d^n$, we have $f_d \in C(X, [0, 2])$, and $f_d \le f_e$ whenever $d \prec e$. But also, if $d \in D_n$ and $d \prec e$, then $\psi_d^n(d) = 0 < 2^{-n} = \psi_e^n(d)$, so actually $f_d < f_e$ whenever $d \prec e$. □

Putting these lemmas together, we have:

Theorem 6.1.3 *Let (X, ϱ) be a metric space. Then there exists a well-ordered sequence of length ξ in $C(X, \mathbb{R})$ iff $\xi < d(X)^+$.*

Corollary 6.1.4 *A metric space (X, ϱ) is separable iff every well-ordered sequence in $C(X, \mathbb{R})$ is countable.*

6.2 Sequences of Baire 1 Functions

If we replace continuous functions by Baire 1 functions, then Corollary 6.1.4 becomes false, since on some separable metric spaces, we can get well-ordered sequences of every type less than ω_2. To prove this, we shall apply some basic facts about $\subset^*$ on $\mathcal{P}(\omega)$. As usual, for $x, y \subset \omega$, we say that $x \subseteq^* y$ iff $x \backslash y$ is finite. Then $x \subset^* y$ iff $x \backslash y$ is finite and $y \backslash x$ is infinite. This $\subset^*$ partially orders $\mathcal{P}(\omega)$.

Lemma 6.2.1 *If $X \subset \mathcal{P}(\omega)$ is a chain in the order $\subset^*$, then on X (viewed as a subset of the Cantor set $2^\omega \cong \mathcal{P}(\omega)$), there is a chain of Baire 1 functions which is isomorphic to $(X, \subset^*)$.*

Proof. Note that for each $x \in X$,

$$\{y \in X : y \subseteq^* x\} = \bigcup_{m \in \omega} \{y \in X : \forall n \ge m\, [y(n) \le x(n)]\} \quad ,$$

which is an F_σ set in X. Likewise, the sets $\{y \in X : y \supseteq^* x\}$, $\{y \in X : y \subset^* x\}$, and $\{y \in X : y \supset^* x\}$, are all F_σ sets in X, and hence also G_δ sets. It follows that if $f_x : X \to \{0, 1\}$ is the characteristic function of $\{y \in X : y \subset^* x\}$, then $f_x : X \to \mathbb{R}$ is a Baire 1 function. Then, $\{f_x : x \in X\}$ is the required chain. □

Lemma 6.2.2 *For any infinite cardinal κ, suppose that $(\mathcal{P}(\omega), \subset^*)$ contains a chain $\{x_\alpha : \alpha < \kappa\}$ (i.e., $\alpha < \beta \to x_\alpha \subset^* x_\beta\}$). Then $(\mathcal{P}(\omega), \subset^*)$ contains a chain X of size κ such that every ordinal $\xi < \kappa^+$ is embeddable into X.*

Proof. Let $S = \bigcup_{1 \le n < \omega} \kappa^n$. For $s = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \in S$, let $s^+ = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n + 1)$. Starting with the $x_{(\alpha)} = x_\alpha$, choose $x_s \in \mathcal{P}(\omega)$ by induction on length(s) so that $x_s = x_{s^\frown 0} \subset^* x_{s^\frown \alpha} \subset^* x_{s^\frown \beta} \subset^* x_{s^+}$ whenever $s \in S$ and $0 < \alpha < \beta < \kappa$. Let $X = \{x_s : s \in S\}$. Then, whenever $x, y \in X$ with $x \subset^* y$, the ordinal κ is embeddable in $(x, y) = \{z \in X : x \subset^* z \subset^* y\}$. From this, one easily proves by induction on $\xi < \kappa^+$ (using $\mathrm{cf}(\xi) \le \kappa$) that ξ is embeddable in each such interval (x, y). □

Since $\mathcal{P}(\omega)$ certainly contains a chain of type ω_1, these two lemmas yield:

Theorem 6.2.3 *There is a separable metric space X on which, for every $\xi < \omega_2$, there is a well-ordered chain of length ξ of Baire 1 functions.*

Remark This sharpening of my original version (I only proved that uncountable chains exist) is due to K. Kunen.

Under *CH*, this is best possible, since there will be only $2^\omega = \omega_1$ Baire 1 functions on a separable metric space, so there could not be a chain of length ω_2. Under $\neg CH$, the existence of longer chains of Baire 1 functions depends on the model of set theory. It is consistent with 2^ω being arbitrarily large that there is a chain in $(\mathcal{P}(\omega), \subset^*)$ of type 2^ω; for example, this is true under *MA* [vD]. In this case, there will be a separable X with well-ordered chains of all lengths less than $(2^\omega)^+$. However, in the Cohen model, where 2^ω can also be made arbitrarily large, we never get chains of type ω_2. We shall prove this by using the following lemma, which relates it to the rectangle problem:

Lemma 6.2.4 *Suppose that there is a separable metric space Y with an ω_2-chain of Borel subsets, $\{B_\alpha : \alpha < \omega_2\}$ (so, $\alpha < \beta \to B_\alpha \subsetneqq B_\beta$). Then in $\omega_2 \times \omega_2$, the well-order relation $<$ is in the σ-algebra generated by the set of all rectangles, $\{S \times T : S, T \in \mathcal{P}(\omega_2)\}$.*

Proof. Each B_α has some countable Borel class. Since there are only ω_1 classes, we may, by passing to a subsequence, assume that the classes are bounded. Say, each B_α is a Σ^0_μ set for some fixed $\mu < \omega_1$.

Let $J = \omega^\omega$, and let $A \subset Y \times J$ be a universal Σ^0_μ set; that is, A is Σ^0_μ in $Y \times J$ and every Σ^0_μ subset of Y is of the form $A^j = \{y : (y, j) \in A\}$ for some $j \in J$ [Kur, §31]. Now, for $\alpha, \beta < \omega_2$, fix $y_\alpha \in B_{\alpha+1} \backslash B_\alpha$, and fix $j_\beta \in J$ such that $A^{j_\beta} = B_\beta$. Then $\alpha < \beta$ iff $(y_\alpha, j_\beta) \in A$. Thus, $\{(y_\alpha, j_\beta) : \alpha < \beta < \omega_2\}$ is a Borel subset of $\{y_\alpha : \alpha < \omega_2\} \times \{j_\beta : \beta < \omega_2\}$, and is hence in the σ-algebra generated by open rectangles, so $<$, as a subset of $\omega_2 \times \omega_2$, is in the σ-algebra generated by rectangles. $\square$

Theorem 6.2.5 *Assume that $V[G]$ is an extension of V by $\geq \omega_2$ Cohen reals, where the ground model, V, satisfies CH. Then in $V[G]$, no separable metric space can have a chain of length ω_2 of Baire 1 functions.*

Proof. By [Kun], in $V[G]$, the well-order relation in $\omega_2 \times \omega_2$ is not in the σ-algebra generated by all rectangles. Now, suppose that $\{f_\alpha : \alpha < \omega_2\}$ is a chain of Baire one functions on the separable metric space X. Let $B_\alpha = \{(x, r) \in X \times \mathbb{R} : r \leq f_\alpha(x)\}$. Then the B_α form an ω_2-chain of Borel subsets of the separable metric space $X \times \mathbb{R}$, so we have a contradiction by Lemma 6.2.4. $\square$

Remark In my original version I used Q-sets [Ha] to show that long chains of Baire 1 functions exist, and proved the other direction in the above way (Lemma 6.2.4 and Theorem 6.2.5). Then K. Kunen replaced the Q-set argument by the stronger above one, using the $\subset^*$-idea.

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Summary

Problems in real analysis are often of set-theoretic nature. On one hand in some cases the question itself involves set theory, and on the other hand many questions can be answered using methods of set theory. This is the case for example when a statement turns out to be independent of ZFC. In my dissertation I collect my results of this type.

In Chapter 1 we deal with the problem of existence of measurable envelopes. We prove among others that in the case of Hausdorff measures in Euclidean spaces this question is independent of ZFC.

The main result of Chapter 2 is that the existence of Hausdorff measurable Sierpiński sets is also independent.

In Chapter 3 we investigate the connection between the classes of negligible (and also of measurable) sets with respect to certain geometric measures. We give a (consistent) answer to a question of T. Keleti.

Chapter 4 deals with solvability cardinals for systems of difference equations. We prove that it is consistent that this cardinal is ω_2 in the case of Borel functions. This answers a question of M. Laczkovich. In addition we deal with the Baire 1 case as well.

Chapter 5 is about linear orders representable by (point-wise ordered) real Baire 1 functions. Due to a classical theorem such an order cannot contain an increasing or decreasing sequence of length ω_1. We investigate the fine structure of these representable orders and conjecture that the converse of the above theorem is consistent.

Finally, Chapter 6 considers increasing (with respect to the point-wise order) transfinite sequences of functions defined on metric spaces. We settle the continuous case and show that the answer in the Baire 1 case is independent even of $ZFC + \neg CH$.

Magyar nyelvű összefoglalás

A valós függvénytanban igen gyakoriak egyfelől a halmazelméleti módszerekkel megválaszolható kérdések, ilyenek például a ZFC axiómarendszertől függetlennek bizonyuló állítások, másfelől pedig az önmagukban is halmazelméleti jellegű problémák. Disszertációm az ilyen típusú eredményeimet tartalmazza.

Az első fejezet a burok létezésének problémájával foglalkozik. Bebizonyítjuk többek között, hogy a Hausdorff mértékek esetén ez a kérdés független a ZFC axiómarendszertől.

A második fejezet fő eredménye, hogy Hausdorff mértékekre nézve mérhető Sierpiński halmaz létezése is független.

A harmadik fejezet néhány geometriai mérték szerinti nullhalmazok (ill. mérhető halmazok) osztályainak kapcsolatáról szól. Keleti Tamás egy kérdését is megválaszoljuk (konzisztencia erejéig).

A negyedik fejezetben differencia-egyenletrendszerek megoldhatósági számosságait vizsgáljuk. Bemutatjuk az ismert (de még publikálatlan) eredményeket, majd igazoljuk, hogy a Borel függvények esetében konzisztens, hogy ez a számosság ω_2. Ezzel Laczkovich Miklós egy kérdésére válaszolunk. Foglalkozunk emellett a Baire 1 esettel is.

Az ötödik fejezet a valós Baire 1 függvényekkel reprezentálható rendezésekről szól.

Végül a hatodik fejezetben többek között megmutatjuk, hogy független még $ZFC + \neg CH$-től is, hogy szeparábilis metrikus tereken milyen hosszú növő transzfinit sorozatokat lehet építeni Baire 1 függvényekből.

Theses

of the PhD Thesis

SET-THEORETICAL METHODS IN REAL ANALYSIS

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2002

The development of set theory created a new trend in mathematical research. On one hand it produced strong techniques such as transfinite recursion to solve long-standing open problems, and on the other hand the new theories enabled us to prove that it is impossible to answer certain questions within the framework of ZFC; that is the usual axioms of set theory. Proving these so called independent (that is something that can be neither proved, nor disproved) and consistent (something that cannot be disproved) statements is a very active and rapidly growing area of mathematics, specifically of analysis as well. The Continuum Hypothesis (CH), Martin's Axiom (MA) and forcing together with a lot of related techniques provide a large number of consistent theorems, which can be used to prove consistent statements in analysis as well.

My dissertation presents a collection of my results of this type from the field of real analysis, partly on the basis of [1], [2] and [3]. In the sequel chapters and theorems are numbered as in the dissertation.

1 The Existence of Measurable Envelopes

It was an error in a paper by Lebesgue that called attention to the fact that projections and continuous images of Borel sets are not necessarily Borel. Continuous images of Borel sets are now called *analytic sets* or *Souslin sets*. It can be shown that we obtain the same class by projections or by using the so called *Souslin operation* as well. The beautiful and involved theory of analytic sets has a large number of applications in diverse fields of mathematics. E.g. it can be useful in showing that a given set is measurable, and surprisingly it can also be used to prove that a set is Borel. For example the above proof of Lebesgue can be corrected this way.

The results concerning measurability are based on a nice theorem of Szpilrajn-Marczewski stating that if $\mathcal{A}$ is a σ-algebra such that every set has a so called *measurable envelope*, then $\mathcal{A}$ is closed under Souslin operation. The definition of measurable envelope reads as follows.

Definition Let $\mathcal{A}$ be a σ-algebra on a set X. We say that $Y \subset X$ is *small with respect to $\mathcal{A}$*, if every subset of it is in $\mathcal{A}$. A set $A \in \mathcal{A}$ is called a *measurable envelope* of a set $H \subset X$, if $H \subset A$, and A is minimal in the sense that if $B \in \mathcal{A}$ and $H \subset B \subset A$, then $A \setminus B$ is small with respect to $\mathcal{A}$.

So it is a fundamental problem to decide, which are the σ-algebras for which every subset has a measurable envelope. The question arises naturally: What happens if μ is an outer measure and $\mathcal{A}$ consists of the μ-measurable sets? It is not hard to see that if μ is σ-finite, then every subset has a measurable envelope. However, not every interesting outer measure is σ-finite. D. Fremlin gave the first (purely set-theoretic) example of an outer measure for which this does not hold. Perhaps the most natural examples of non-σ-finite measures are Hausdorff measures, well-known generalizations of Lebesgue measure, which are the basic objects of geometric measure theory. Let $\mathcal{H}^d$ stand for d-dimensional Hausdorff measure. The next theorem is the main result of Chapter 1.

Theorem 1.2.4 *The following statement is independent of ZFC: for all $n \in \mathbb{N}$ and $0 < d < n$ every subset of $\mathbb{R}^n$ has a measurable envelope with respect to $\mathcal{H}^d$.*

Most results of this chapter can be found in [1].

2 Measurable Sierpiński Sets

Fubini Theorem can fail for non-measurable sets. The following beautiful example was constructed by Sierpiński using CH. Let $\prec$ be an order of type ω_1 well-ordering $\mathbb{R}$, and let

$$S = \{(x, y) \in \mathbb{R}^2 : x \prec y\}.$$

Then S contains only countably many points of each horizontal line, but contains all but countably many points of each vertical line. Therefore Fubini Theorem fails for S, and so S cannot be Lebesgue measurable.

Definition A set $S \subset \mathbb{R}^2$ is a *Sierpiński set in the sense of measure* if S is (one dimensional) Lebesgue negligible on each vertical line, but co-negligible (that is the complement of S is negligible) on each horizontal line. A set $S \subset \mathbb{R}^2$ is a *Sierpiński set in the sense of cardinality* if S is countable on each vertical line, but co-countable on each horizontal line.

As Hausdorff measures are natural generalizations of Lebesgue measure it is an interesting question whether a Sierpiński set can be Hausdorff measurable. Just as one would expect the following holds.

Theorem 2.1.1 *For $0 < d \leq 2$ there exists no $\mathcal{H}^d$-measurable Sierpiński set in the sense* *of cardinality.*

However, the last theorem of the chapter is much more surprising.

Theorem 2.2.1 *For $0 < d < 2$ the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of measure is independent of ZFC.*

The content of this chapter can be found in [1].

3 Hausdorff Measurable and Hausdorff Null Sets

In this chapter we investigate the classes of measurable and negligible sets with respect to Hausdorff measures. One of the motivations is the following question of Tamás Keleti: Does $\mathcal{H}^{d_1}$-measurability imply $\mathcal{H}^{d_2}$-measurability, or vice versa? In the chapter we give (a consistent) answer to this question as well. Besides Hausdorff measures we also consider products of Hausdorff measures, as they turn out to exhibit an interesting behaviour. For the sake of simplicity we only consider the following three outer measures in $\mathbb{R}^2$: $\mathcal{H}^{1/2} \times \mathcal{H}^1$, $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ and $\mathcal{H}^{3/2}$. Just as one would expect, these three measures coincide in many cases.

First we deal with the classes of negligible sets. Our first statement is trivial.

Statement 3.1.1 *Let $0 \leq d_1 < d_2 \leq n$ and $H \subset \mathbb{R}^n$. Then $\mathcal{H}^{d_1}(H) = 0$ implies $\mathcal{H}^{d_2}(H) = 0$, but the converse is not true in general.*

However, the following surprising fact holds for products.

Theorem 3.1.2 *There is no inclusion between the classes of $\mathcal{H}^{1/2} \times \mathcal{H}^1$-negligible, $\mathcal{H}^1 \times \mathcal{H}^{1/2}$-negligible and $\mathcal{H}^{3/2}$-negligible sets in $\mathbb{R}^2$.*

We remark here, that all our examples can be chosen to be compact.

Then we turn to the classes of measurable sets. First we answer the question of Keleti.

Theorem 3.2.1 *Let $0 < d_1 < d_2 \leq n$. Then $\mathcal{H}^{d_2}$-measurability does not imply $\mathcal{H}^{d_1}$-measurability in $\mathbb{R}^n$. If $\operatorname{add}(\mathcal{N}) = 2^\omega$ holds (e.g. under CH or MA), then $\mathcal{H}^{d_1}$-measurability does not imply $\mathcal{H}^{d_2}$-measurability in $\mathbb{R}^n$.*

As far as our product measures are concerned, the next two theorems provide counter-examples to four out of the six possible implications. (In fact, as $\mathcal{H}^{1/2} \times \mathcal{H}^1$ and $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ play a symmetric role, essentially we have three pairs of implications.)

Theorem 3.2.3 *Measurability with respect to $\mathcal{H}^{1/2} \times \mathcal{H}^1$ does not imply measurability with respect to $\mathcal{H}^{3/2}$ in $\mathbb{R}^2$.*

Theorem 3.2.4 *Measurability with respect to $\mathcal{H}^{1/2} \times \mathcal{H}^1$ does not imply measurability with respect to $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ in $\mathbb{R}^2$.*

It is also possible to show that the $\mathcal{H}^{1/2} \times \mathcal{H}^1$-measurable sets in these theorems (only consistently this time) can be chosen to be product sets of the form $A \times \mathbb{R}$.

However, we fail to find counter-examples to the last pair of implications, since the following surprising theorem is valid.

Theorem 3.2.8 *Measurability with respect to $\mathcal{H}^{3/2}$ implies measurability with respect to $\mathcal{H}^{1/2} \times \mathcal{H}^1$ in $\mathbb{R}^2$.*

4 Solvability Cardinals and Systems of Difference Equations

A *difference operator* is a mapping $D : \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R}$ of the form

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i),$$

where a_i and b_i are real numbers; that is D assigns to every function f a linear combination of certain translates of f.

Difference operators show up in various branches of analysis. They were probably first defined in Fourier analysis, but they are also related to the theory of generalized derivatives, to the notions of symmetric continuity and differentiability, to the so called Difference Property and to group-algebras as well. The following definitions and results are due to Miklós Laczkovich.

Definition A system of equations

$$D_i f = g_i \quad (i \in I),$$

where I is an arbitrary set of indices, D_i is a difference operator and g_i is a given function (for every $i \in I$), and f is the unknown function is called a *system of difference equations.*

A linear algebraic argument shows that such a system is solvable iff each of its finite subsystems is solvable. However, if we are interested e.g. in bounded solutions, then this result is no longer true. This motivates the following.

Definition Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The *solvability cardinal of* $\mathcal{F}$ is the minimal cardinal $\kappa(\mathcal{F})$ with the property that if every subsystem of size less than $\kappa(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$, then the whole system has a solution in $\mathcal{F}$.

Laczkovich computed the solvability cardinal for most usual classes of functions, and these values are sometimes quite unexpected. E.g. for functions of absolute value at most 1 this cardinal equals ω, while for bounded functions it is ω_1. For trigonometric polynomials it is ω_1, but for polynomials it is only 3. In addition, for continuous functions we get ω_1, and for measurable ones this cardinal is consistently $(2^\omega)^+$.

Finally, the following question was posed by Laczkovich: What can we say about $\kappa(\{f : f \text{ is Borel}\})$? He noted, that he only expects a consistent answer. My main result in the chapter is the following.

Corollary 4.1.3 *It is consistent, that* $\kappa(\{f : f \textit{ is Borel}\}) = \omega_2 = (2^\omega)^+$.

In addition, I also hope to prove this in ZFC. In order to do this I verified that it would be sufficient to show that $\kappa(\{f : f \text{ is Baire } \alpha\}) \leq \omega_2$ holds for every $\alpha < \omega_1$. This leads us to the last topic of the chapter.

Th case $\alpha = 0$ is simply the case of continuous functions, so it is solved. Therefore we take one step further, and examine $\kappa(\{f : f \text{ is Baire } 1\})$. The following partial result suggests the conjecture $\kappa(\{f : f \text{ is Baire } 1\}) = \omega_1$.

Theorem 4.2.4 *Suppose that every difference operator in a system of difference equations consists of at most two terms. Then if every countable subsystem has a Baire 1 solution, then the whole system has one as well.*

What makes this special case interesting is that almost every difference operator in all the proof and examples in this area is of the form $(Df)(x) = f(x+h) - f(x)$. Therefore our theorem shows that the above conjecture cannot be disproved by the usual methods.

5 Chains of Baire 1 Functions with Respect to Point-Wise Order

In this chapter we consider the set of real Baire 1 functions endowed with point-wise partial order. According to a classical theorem of Kuratowski this set contains no uncountable increasing or decreasing transfinite sequence with respect to this order. M. Laczkovich posed the problem of characterizing the possible order types of the linearly (that is totally) ordered subsets of this partially ordered set of functions. What makes the Baire 1 case interesting is that the problem is already solved for most usual classes of functions, including the Baire 0 and the Baire α ($\alpha \geq 2$) classes. In this chapter I collected my results about the Baire 1 class.

First we prove a surprising theorem, which we apply a large number of times later. By *Polish space* we mean complete separable metric space, which is one of the basic notions of real analysis. If X is a Polish space, then $\mathcal{R}(X)$ denotes the set of possible order types of linearly ordered families of real valued Baire 1 functions defined on X. Two Polish spaces X and Y are said to be *equivalent* if $\mathcal{R}(X) = \mathcal{R}(Y)$.

Theorems 5.1.2 and 5.1.4 *There are at most two equivalence classes of (uncountable) Polish spaces.*

Then we present a list of results showing that our order types can be of extremely complicated structure. Roughly speaking we prove that $\mathcal{R}(\mathbb{R})$ is closed under countable operations. In fact we sometimes prove even more, as we can e.g. use the operation that replaces each point of an ordered set by another ordered set. All in all, these results suggest that the converse of Kuratowski's theorem might be true; that is an order type is in $\mathcal{R}(\mathbb{R})$ iff it does not contain a monotone sequence of length ω_1. But a theorem of Péter Komjáth yields the following.

Theorem 5.5.4 *This classification is not provable in ZFC.*

So we have to modify our conjecture appropriately.

Question 5.5.5 *Is this classification consistent?*

Most results of this chapter can be found in [2].

6 Increasing Transfinite Sequences of Functions

In this chapter we investigate how long a monotone (with respect to point-wise order) transfinite sequence of real valued functions defined on a metric space can be. First we deal with continuous functions. Let $d(X)$ denote the density of the space X; that is the minimal cardinal of a dense subspace.

Theorem 6.1.3 *Let X be a metric space. Then there exists a well-ordered sequence of length α of continuous functions defined on X iff $\alpha < d(X)^+$.*

Then we turn to the Baire 1 case. The above theorem of Kuratowski also yields that we can construct an increasing sequence of length α of real valued Baire 1 functions defined on a complete separable metric space iff $\alpha < \omega_1$. The examples of discrete metric spaces provide examples to show that the situation changes dramatically once we drop separability. However, the role of completeness is not so clear. We show that the possible length of sequences does indeed increase if we drop completeness, and also that an exact bound cannot be given in ZFC.

Theorem 6.2.3 *There is a separable metric space X on which, for every $\alpha < \omega_2$, there is a well-ordered chain of length α of Baire 1 functions.*

Under CH, this is best possible, since there will be only $2^\omega = \omega_1$ Baire 1 functions on a separable metric space, so there could not be a chain of length ω_2. But we show by the next two theorems that the question is independent even of $ZFC + \neg CH$.

Theorem *$MA + 2^\omega > \omega_2$ implies the existence of a separable metric space on which there exists a sequence of Baire 1 functions of length ω_2.*

Theorem 6.2.5 *If we add ω_2 Cohen reals to a model of ZFC, then in the resulting model no separable metric space can have a chain of length ω_2 of Baire 1 functions.*

The above form of Theorem 6.2.3 was proved by Kenneth Kunen. In my original version I only showed the existence of uncountable chains. A sharpened version (which is contained in my thesis) of the last but one theorem is also due to him. All these can be found in [3].

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Tézisek

a

SET-THEORETICAL METHODS IN REAL ANALYSIS

(Halmazelméleti módszerek a valós függvénytanban)

című PhD értekezéshez

Elekes Márton

2002

A halmazelmélet megjelenése új irányvonalat teremtett az analízis kutatásában. Egyfelől nagyon erős eszközök jelentek meg addig nyitott problémák megoldásához (ilyen például a transzfinit rekurzió), másfelől pedig lehetőség nyílt olyan típusú állítások igazolására, melyek azt mutatják, hogy bizonyos kérdésekre nem lehet a szokásos ZFC axiómarendszerben választ adni. Az ilyen úgynevezett konzisztens (tehát nem cáfolható), illetve független (se nem bizonyítható, se nem cáfolható) állítások vizsgálata ma is rendkívül aktív terület. A kontinuum-hipotézis (CH), a Martin-axióma (MA) és a forszolás a hozzájuk kapcsolódó rengeteg új technikával olyan konzisztens tételeket produkálnak, amelyeket aztán analízisbeli állítások konzisztenciájának igazolásához is fel lehet használni.

Doktori értekezésemben elsősorban az ilyen típusú eredményeimet ismertetem, részben az [1], [2] és [3] dolgozataim alapján. A továbbiakban a fejezeteket és a tételeket a disszertációt követve számozzuk.

1. A mérhető burok problémája

Lebesgue egy hibás bizonyítása hívta fel a figyelmet arra a tényre, hogy Borel halmaz vetülete, illetve folytonos képe nem feltétlenül Borel. A Borel halmazok folytonos képeként előálló halmazokat *analitikus halmazoknak* vagy *Szuszlin-halmazoknak* nevezzük. Belátható, hogy vetületként, illetve az úgynevezett *Szuszlin-operáció* eredményeként ugyanezek a halmazok állnak elő. Ennek a halmazosztálynak azóta gyönyörű és hatalmas elmélete fejlődött ki, amelynek rengeteg alkalmazása van a matematika legkülönbözőbb területein. Használható többek között annak igazolására, hogy egy adott halmaz mérhető, sőt meglepő módon az is belátható így, hogy egy halmaz Borel. Így javítható például Lebesgue fenti tételének bizonyítása.

A mérhetőséggel kapcsolatos eredmények Szpilrajn-Marczewski egy szép tételén alapulnak, amely azt mondja ki, hogy ha egy $\mathcal{A}$ σ-algebrára nézve az alaphalmaz minden részhalmazának van úgynevezett *burka*, akkor a Szuszlin-operáció nem vezet ki $\mathcal{A}$-ból. A burkot a következőképpen definiáljuk:

Definíció. Legyen $\mathcal{A}$ egy σ-algebra egy X alaphalmazon. Azt mondjuk, hogy $Y \subset X$ *kis halmaz $\mathcal{A}$-ra nézve*, ha minden részhalmaza $\mathcal{A}$-beli. Egy $A \in \mathcal{A}$ halmazt pedig $H \subset X$ egy *burkának* nevezünk, ha $H \subset A$, és A minimális abban az értelemben, hogy

$B \in \mathcal{A}$, $H \subset B \subset A$ esetén $A \setminus B$ kis halmaz $\mathcal{A}$-ra nézve.

Tehát alapfeladat annak eldöntése, hogy mely σ-algebrák esetén létezik minden halmaznak burka. Természetes kérdés, mi a helyzet, ha μ egy külső mérték, $\mathcal{A}$ pedig a μ-mérhető halmazokból áll. Könnyű látni, hogy a σ-véges esetben mindig létezik minden részhalmaznak burka. D. Fremlin mutatott először (tisztán halmazelméleti) példát külső mértékre, amelyre ez nem igaz. Disszertációm első fejezetében a geometriai mértékelméletből jól ismert Hausdorff-mértékek esetét vizsgálom, melyek a nem-σ-véges mértékek talán legtermészetesebb példái. A d-dimenziós Hausdorff mértéket $\mathcal{H}^d$-vel jelöljük. A fejezet fő eredménye a következő:

1.2.4. Tétel. *Az az állítás, mely szerint „tetszőleges $n \in \mathbb{N}$-re és tetszőleges $0 < d < n$-re $\mathbb{R}^n$ minden részhalmazának van burka $\mathcal{H}^d$-re nézve", független ZFC-től.*

A fejezet eredményei nagyrészt [1]-ből származnak.

2. Mérhető Sierpiński-halmazok

A Fubini-tétel nem mérhető halmazokra nem marad érvényben. Erre Sierpiński konstruálta a következő gyönyörű páldát CH segítségével. Legyen a $\prec$ reláció $\mathbb{R}$ egy ω_1-típusú jólrendezése, és legyen

$$S = \{(x, y) \in \mathbb{R}^2 : x \prec y\}.$$

Ekkor S minden vízszintes egyenesnek csak megszámlálható sok pontját tartalmazza, viszont minden függőleges egyenest megszámlálható sok pont kivételével tartalmaz. Így a Fubini-tétel állítása S-re nem érvényes, tehát S nem Lebesgue-mérhető.

Definíció. Egy $S \subset \mathbb{R}^2$ halmazt *megszámlálható szekciójú Sierpiński-halmaznak* nevezünk, ha minden vízszintes egyenesnek csak megszámlálható sok pontját tartalmazza, viszont minden függőleges egyenest megszámlálható sok pont kivételével tartalmaz. Azt mondjuk, hogy S *nullmértékű szekciójú Sierpiński-halmaz*, ha minden vízszintes egyenesen (egy-dimenziós) nullmértékű, de minden függőleges egyenest nullmértékű halmaz kivételével tartalmaz.

Mivel a Hausdorff-mértékek a Lebesgue-mérték természetes általánosításai, érdekes kérdés, hogy egy Sierpiński-halmaz lehet-e Hausdorff-mérhető. A fejezet nem túl meglepő eredménye a következő.

2.1.1. Tétel. *$0 < d \leq 2$ esetén nem létezik $\mathcal{H}^d$-mérhető, megszámlálható szekciójú Sierpiński-halmaz.*

Annál meglepőbb viszont a következő.

2.2.1. Tétel. *$0 < d < 2$ esetén $\mathcal{H}^d$-mérhető, nullmértékű szekciójú Sierpiński-halmaz létezése független ZFC-től.*

A fejezet eredményei megtalálhatóak [1]-ben.

3. Hausdorff-mérhető és Hausdorff-null halmazok

Ebben a fejezetben a Hausdorff-mértékek szerint mérhető illetve nullmértékű halmazok osztályaival foglalkozunk. A témakör vizsgálatát részben Keleti Tamás egy kérdése indította el: „Mi a kapcsolat $\mathbb{R}^n$-ben a $\mathcal{H}^{d_1}$-mérhető és a $\mathcal{H}^{d_2}$-mérhető halmazok között?" Fejezetünkben (konzisztencia erejéig) erre a kérdésre is választ adunk. A Hausdorff-mértékek mellett Hausdorff-mértékek szorzatait is vizsgáljuk, mivel ezek tulajdonságai igen meglepőnek bizonyulnak. Az egyszerűség kedvéért csak a síkbeli $\mathcal{H}^{1/2} \times \mathcal{H}^1$, $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ és $\mathcal{H}^{3/2}$ mértékek esetére szorítkozunk, mely mértékek elvárásainknak megfelelően sok esetben egybeesnek.

Először a nullhalmazok osztályaival foglalkozunk. Triviális a következő.

3.1.1. Állítás. *Legyen $0 \leq d_1 < d_2 \leq n$. Ekkor a $\mathcal{H}^{d_1}$-nullhalmazok egyúttal $\mathcal{H}^{d_2}$-nullhalmazok is, viszont a megfordítás általában nem igaz.*

Azonban a szorzatok esetében meglepő jelenséget tapasztalunk.

3.1.2. Tétel. *A $\mathcal{H}^{1/2} \times \mathcal{H}^1$-null, $\mathcal{H}^1 \times \mathcal{H}^{1/2}$-null és $\mathcal{H}^{3/2}$-null halmazok osztályai között semmilyen tartalmazás nem áll fenn.*

Megjegyezzük, hogy a bizonyításban szereplő összes ellenpélda kompakt.

Ezután a mérhetőség problémájára térünk át. Keleti Tamás kérdésére válaszol a következő.

3.2.1. Tétel. *Legyen $0 < d_1 < d_2 \leq n$. Ekkor $\mathbb{R}^n$-ben a $\mathcal{H}^{d_2}$-mérhetőségből nem következik a $\mathcal{H}^{d_1}$-mérhetőség, és $\mathrm{add}(\mathcal{N}) = 2^\omega$ esetén (pl. CH vagy MA) a $\mathcal{H}^{d_1}$-mérhetőségből sem következik a $\mathcal{H}^{d_2}$-mérhetőség.*

Szorzatmértékeinkre rátérve a lehetséges hat implikációból (valójában $\mathcal{H}^{1/2} \times \mathcal{H}^1$ és $\mathcal{H}^1 \times \mathcal{H}^{1/2}$ szimmetrikus szerepe miatt ezek lényegében csak három párt jelentenek) négyről dönt a következő két tétel.

3.2.3. Tétel. *A $\mathcal{H}^{1/2} \times \mathcal{H}^1$-mérhetőségből nem következik a $\mathcal{H}^{3/2}$-mérhetőség.*

3.2.4. Tétel. *A $\mathcal{H}^{1/2} \times \mathcal{H}^1$-mérhetőségből nem következik a $\mathcal{H}^1 \times \mathcal{H}^{1/2}$-mérhetőség.*

Emellett azt is bizonyítjuk (igaz, ezt csak konzisztencia erejéig), hogy a két tételben a $\mathcal{H}^{1/2} \times \mathcal{H}^1$-mérhető halmazokat választhatjuk $A \times \mathbb{R}$ alakú hengerhalmazoknak is.

Azonban meglepő módon az utolsó két implikációra nem találunk ellenpéldát. Fennáll ugyanis a következő.

3.2.8. Tétel. *A $\mathcal{H}^{3/2}$-mérhető halmazok egyúttal $\mathcal{H}^1 \times \mathcal{H}^{1/2}$-mérhetőek is.*

4. Differencia-egyenletrendszerek megoldhatósági számosságai

Differencia-operátornak nevezünk egy $D : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ leképezést, ha

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i)$$

alakú valamely a_i és b_i valós számokkal, azaz a D lineáris operátor minden függvényhez bizonyos eltoltjainak lineáris kombinációját rendeli.

Differencia-operátorok az analízis számos területén felbukkannak. Valószínűleg a Fourier-sorokkal kapcsolatban definiálták őket először, de kapcsolatban vannak a szimmetrikus folytonosság és differenciálhatóság fogalmával, valamint a csoport-algabrák elméletével, és az úgynevezett differencia-tulajdonsággal is. A következő definíciók és eredmények Laczkovich Miklóstól származnak.

Definíció. Egy

$$D_i f = g_i \quad (i \in I)$$

egyenletrendszert, ahol I tetszőleges indexhalmaz, D_i differencia-operátor és g_i adott függvény (minden $i \in I$-re), valamint f az ismeretlen, *differencia-egyenletrendszernek* nevezünk.

Egy lineáris algebrai gondolatmenettel igazolható, hogy egy differencia-egyenletrendszer pontosan akkor megoldható, ha minden véges részrendszere megoldható. Azonban ha például csak korlátos megoldásokat keresünk, az analóg eredmény nem lesz igaz. Ez motiválja a következőt.

Definíció. Legyen $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ valós függvények egy osztálya. *$\mathcal{F}$ megoldhatósági számosságának* nevezzük, és $\kappa(\mathcal{F})$-fel jelöljük azt a minimális számosságot, amelyre fennáll, hogy amennyiben egy differencia-egyenletrendszer minden $\kappa(\mathcal{F})$-nál kisebb számosságú részrendszerének van $\mathcal{F}$-beli megoldása, akkor az egész rendszernek is van $\mathcal{F}$-beli megoldása.

Laczkovich a legtöbb szokásos függvényosztályra kiszámította a megoldhatósági számosságot, melynek értéke néha igen meglepő. Például legfeljebb 1 abszolútértékű függvényekre ω, de korlátos függvényekre ω_1. Trigonometrikus polinomokra ω_1, de polinomokra csupán 3. Emellett folytonos függvényekre ω_1, mérhetőekre pedig konzisztens, hogy $(2^\omega)^+$.

Szintén tőle származik a kérdés, hogy mit mondhatunk a Borel függvények osztályáról. Megjegyzi, hogy véleménye szerint csak konzisztens válasz várható. Ebben a fejezetben fő eredményem a következő.

4.1.3. Következmény. *Konzisztens, hogy $\kappa(\{f : f \text{ Borel}\}) = \omega_2$.*

Nem tartom azonban elképzelhetetlennek, hogy ez az eredmény ZFC-ben is bizonyítható. Igazoltam ugyanis, hogy $\kappa(\{f : f \text{ Borel}\}) = \omega_2$ bizonyításához elegendő lenne minden $\alpha < \omega_1$-re $\kappa(\{f : f \text{ Baire } \alpha\}) \leq \omega_2$-t ellenőrizni. Így jutunk a fejezet utolsó témájához.

$\alpha = 0$-re a fenti kérdés épp a folytonos függvényekre vonatkozik, így megoldott. Ezért az eggyel bonyolultabb esetet, vagyis $\kappa(\{f : f \text{ Baire } 1\})$-et vizsgálom. Ezzel kapcsolatban a következő részeredményem van, amely alapján $\kappa(\{f : f \text{ Baire } 1\}) = \omega_1$-et sejtem.

4.2.4. Tétel. *Tegyük fel, hogy egy differencia-egyenletrendszerben minden differencia-operátor legfeljebb kéttagú. Ekkor igaz, hogy ha minden megszámlálható részrendszernek van Baire 1 megoldása, akkor az egész rendszernek is van.*

Ennek a speciális esetnek az a jelentősége, hogy a témakörben szinte minden példában és bizonyításban $(Df)(x) = f(x + h) - f(x)$ alakú differencia-operátorok szerepelnek. Tételünk tehát azt mutatja, hogy a szokásos módszerekkel nem lehet sejtésünket cáfolni.

5. Baire 1 függvények láncai a pontonkénti rendezés szerint

Ebben a fejezetben a valós Baire 1 függvények halmazát tekintjük a pontonkénti parciális rendezéssel. Kuratowski klasszikus tétele szerint ez a halmaz nem tartalmaz nem megszámlálható növő vagy fogyó transzfinit sorozatot. Laczkovich Miklós vetette fel a kérdést, hogy milyen rendtípusúak lehetnek ennek a függvényhalmaznak a lineárisan (azaz teljesen) rendezett részhalmazai. (Azért éppen a Baire 1 eset vált érdekessé, mert ugyanez a kérdés a legtöbb szokásos függvényosztályra, beleértve a Baire 0 és a Baire α $(2 \leq \alpha)$ osztályokat, már eldöntött.) Ebben a fejezetben a Baire 1 esettel kapcsolatos eredményeimet gyűjtöm össze.

Előkészületként bebizonyítunk egy meglepő és önmagában is érdekes eredményt, mely később számtalanszor alkalmazásra kerül. Lengyel tér alatt teljes, szeparábilis metrikus teret értünk, ami a valós analízis egyik alapfogalma. Az X lengyel téren értelmezett, valós értékű Baire 1 függvényekből álló lineáris rendezések lehetséges rendtípusait $\mathcal{R}(X)$-szel jelöljük. X és Y lengyel terek *ekvivalensek*, ha $\mathcal{R}(X) = \mathcal{R}(Y)$.

5.1.2. és 5.1.4. Tétel. *A (nem megszámlálható) lengyel tereknek legfeljebb két ekvivalencia-osztálya van.*

Ezután olyan eredményeket ismertetünk, amelyek azt mutatják, hogy vizsgált rendtípusaink rendkívül bonyolult struktúrájúak lehetnek. Ezt nagy vonalakban úgy fogalmazhatnánk, hogy $\mathcal{R}(\mathbb{R})$ zárt a megszámlálható operációkra. Valójában ennél időnként még többet is bizonyítunk, hiszen például azt az operációt is használhatjuk, ami egy rendezett halmaz minden pontját egy-egy másik rendezett halmazzal helyettesíti. Mindezek alapján azt sejthetnénk, hogy Kuratowski fenti tétele megfordítható, azaz $\mathcal{R}(\mathbb{R})$ éppen azokból a rendtípusokból áll, melyek nem tartalmaznak ω_1 hosszú növő vagy fogyó sorozatot. Azonban Komjáth Péter egy tételéből adódik a következő.

5.5.4. Tétel. *A fenti klasszifikáció nem bizonyítható ZFC-ben.*

Így sejtésünk értelemszerűen gyengébb állításra vonatkozik.

5.5.5. Kérdés. *Konzisztens-e, hogy érvényes a fenti klasszifikáció?*

A fejezet nagyrészt [2]-ből származik.

6. Függvények növő transzfinit sorozatai

Ebben a fejezetben azt vizsgáljuk, hogy metrikus téren értelmezett, valós értékű függvényekből milyen hosszú (pontonkénti rendezésre nézve) növő vagy fogyó transzfinit sorozatot lehet építeni. Először folytonos függvényekkel foglalkozunk. $d(X)$-szel az X tér sűrűségét jelöljük, azaz minimális számosságú sűrű alterének számosságát.

6.1.3. Tétel. *Egy X metrikus téren pontosan akkor létezik folytonos függvényekből álló, α hosszúságú növő vagy fogyó transzfinit sorozat, ha $\alpha < d(X)^+$.*

Ezután rátérünk a Baire 1 függvényekre. Kuratowski fenti tételéből az is adódik, hogy teljes, szeparábilis metrikus téren pontosan akkor létezik α hosszúságú sorozat, ha $\alpha < \omega_1$. A diszkrét metrikus terek mutatják, hogy a szeparabilitás elhagyásával a helyzet drasztikusan megváltozik. Azonban a teljesség esetében ez nem világos. Megmutatjuk, hogy így is nő a sorozatok lehetséges hossza, viszont pontos korlát nem adható.

6.2.3. Tétel. *Létezik szeparábilis metrikus tér, amelyen minden $\alpha < \omega_2$ rendszámhoz van α hosszú Baire 1 függvényekből álló növő sorozat.*

Ha feltesszük CH-t, akkor ez a korlát nyilvánvalóan éles, hiszen ekkor csak $2^\omega < \omega_2$ darab Baire 1 függvény létezik. Azonban két tétel segítségével megmutatjuk, hogy kérdésünk még a $ZFC + \neg CH$ axiómarendszertől is független.

Tétel. *$MA + 2^\omega > \omega_2$ esetén létezik szeparábilis metrikus tér, amelyen van Baire 1 függvényekből álló ω_2 hosszú növő sorozat.*

6.2.5. Tétel. *Ha CH egy modelljéhez ω_2 Cohen-valóst adunk, akkor a kapott ZFC-modellben minden szeparábilis metrikus téren értelmezett, Baire 1 függvényekből álló növő vagy fogyó sorozat rövidebb mint ω_2.*

A 6.2.3. Tétel K. Kunentől származik. Én eredetileg csak nem megszámlálható sorozat létezését igazoltam. Ő bizonyította az utolsó előtti (számozatlan) tétel itt kimondottnál élesebb, disszertációmban leírt változatát is. Mindezek megtalálhatóak [3]-ban.

Irodalom

[1] M. Elekes, Measurable envelopes, Hausdorff measures and Sierpiński sets, benyújtva.
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[3] M. Elekes and K. Kunen, Transfinite sequences of continuous and Baire class 1 functions, benyújtva.