PERTURBATIONS OF BI-CONTINUOUS SEMIGROUPS

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Preface

Abstract initial value problems (Cauchy problems) are usually studied via operator semigroups. In many cases, the well-developed theory of C_0-semigroups, i.e., one-parameter operator semigroups which are strongly continuous for the norm on a Banach space X, suffices and provides a powerful machinery to study such problems. The applications range from partial differential equations, Volterra integro-differential equations and dynamic boundary problems to delay equations. It seems that a linear (autonomous) evolution equation like

$$\begin{cases} u'(t) = Au(t) \\ u(0) = x, \end{cases} \tag{EE}$$

that is an equation that describes a system evolving from an initial state ”in time”, can be handled via the theory of C_0-semigroups. However, as the most trivial example shows this is not the case. Consider the left shift semigroup S on the space of bounded, continuous functions $C_b(\mathbb{R})$

$$S(t)f(s) := f(t+s).$$

It is clear that the orbit $t \mapsto S(t)f$ is continuous for the supremum norm, if and only if f is uniformly continuous. This shows that the shift semigroup does not fit into the framework of C_0-semigroups, nevertheless S clearly describes an evolution system on $\mathbb{R}$. The situation is not so bad either. If we replace the norm topology by the topology τ_c of uniform convergence on compact sets, then the orbits become τ_c-continuous. This leads us to weakening the notion of C_0-semigroups, thus allowing to consider such ”pathological” cases, these therefore turn out to be less unpleasant than thought previously. There are numerous generalisations of the theory of C_0-semigroups. Among these we find the approach of introducing new continuity notions, this method is we want to follow. Investigations on semigroups on locally convex spaces were started fairly long ago, see e.g., [14], [51], [52] and [64]. A nice exposition on the historical aspects can be found in [56]. To exploit the Banach space structure and the same time to introduce coarser topologies, the notion of bi-continuous semigroups was introduced recently by Kühnemund in [56]. The theory developed therein covers a large part of previously known results, and puts these concrete examples in an abstract framework. The general theory is then applicable in concrete cases.

Among the several examples of bi-continuous semigroups are the semigroups induced by jointly continuous flows [28, 29, 30], adjoint semigroup on dual spaces, implemented semigroups on Banach algebras [2, 3], Ornstein – Uhlenbeck semigroups [41], [56], Feller semigroups [61], infinite direct powers and evolution semigroups on $\mathrm{C_b}(\mathbb{R}_+, X)$ spaces. Some of them we will encounter later. Though bi-continuous semigroups are also useful, since many particular results can be proved via abstract reasonings, their worth, we hope, is proved by some results presented in this work.

As a counterpart of the theory of C_0-semigroups, Hille – Yosida type generation theorems and approximation theory were established for bi-continuous semigroup in [1], [9] and [54, 55, 56]. Our aim is to make a few initial steps towards the perturbation theory for such semigroups. The essence of such results is explained by the following. Assume that we are able to solve the above evolution equation (EE) via bi-continuous semigroups. We would like to add a term to the right hand side

$$u'(t) = (A + B)u(t), \tag{EE'}$$

and we expect that we can still find solutions. Moreover, it would be nice to obtain the solutions to this equation from the solutions of the original problem (EE). This is what we understand under *perturbation*. We will only consider *additive* perturbations as explained above, while multiplicative perturbations are among the future plans. The basic idea of the proofs of the perturbation result is quite simple. Let us sketch it briefly. If T is a bi-continuous semigroup with generator A, and B is the perturbing operator, we imagine that there exist the bi-continuous S semigroup generated by $A + B$. Then we define $U(s) = T(t - s)S(s)$ for some $t > 0$ and differentiate $U(s)$. By the definition of the generators and using the Leibnitz rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}U(s) = \frac{\mathrm{d}}{\mathrm{d}s}T(t-s)S(s) = -T(t-s)AS(s)+T(t-s)(A+B)S(s) = T(t-s)BS(s).$$

Integrating this from 0 to t, we arrive at

$$S(t) - T(t) = \int_0^t T(t - s)BS(s)\,\mathrm{d}s.$$

Similar formulas will serve as the starting points of our investigations. We apply fixed point arguments based on this formula.

Though we would like to elaborate an abstract theory, we are also deeply concerned with *applications*. Since we believe that an applicable theory is far more valuable, we shall put a great emphasis on this matter. The most important application in this work is the perturbation of the Ornstein – Uhlenbeck with a bounded non-linear drift operator. The approach we present here provides a purely functional analytic method to handle these Markov type transition semigroups.

The work is organised as follows. In Chapter 1, we introduce the bi-continuous semigroups and collect some of their important properties. We also refine the known results in order to be appropriate to our purposes. In Chapter 2, we present some examples of bi-continuous semigroups. General bi-continuous semigroups on spaces of bounded, continuous functions are investigated here. Further, the Ornstein – Uhlenbeck semigroups and the adjoint semigroups are introduced. The central part is then Chapter 3, where we establish perturbation results, and the rule of thumb is always the theory of C_0-semigroups. As a first step we prove a bounded perturbation theorem, and show its sharpness by instructive examples. Next we turn to Miyadera – Voigt type perturbations, the used method is basically a fixed point argument applied to abstract Volterra operators. The applications of the elaborated theory will come about in Chapter 4. Utilising the previously established results, we deal with homogenous and inhomogeneous abstract Cauchy problems, show the existence and positivity of transition semigroups corresponding to certain non-linear drifts.The closing chapter is an appendix which is not closely related to semigroup theory. There we recall some results from topological measure theory. References will be given always in the relevant chapters.

A few words should be said about notations. The symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are used as usual, and $\mathbb{N}$ contains 0. $\mathbb{K}$ always denotes $\mathbb{R}$ or $\mathbb{C}$. The linear spaces in all of the cases may considered over $\mathbb{K}$, this will never be specified explicitly. $\mathbb{R}_+$ is the half line of non-negative numbers. During this work, we will encounter several function spaces. Here, we briefly list them. For a set Ω and a normed space Y, let $\mathrm{B}(\Omega, Y)$ denote the space of bounded functions from Ω to Y. When Ω is a topological space, consider furthermore the following spaces: $\mathrm{C}(\Omega, Y)$ and $\mathrm{C_b}(\Omega, Y)$ is the space of continuous and bounded, continuous functions respectively, while $\mathrm{C}_c(\Omega, Y)$, $\mathrm{C}_0(\Omega, Y)$ are the subspaces of functions with compact support and the functions vanishing at infinity. The space of functions that are bounded on each compact subset of Ω is denoted by $\mathrm{B_{loc}}(\Omega, Y)$. When Ω carries a uniform structure, e.g., when it is metrisable, then $\mathrm{C_{ub}}(\Omega, Y)$ is the space of bounded, uniformly continuous functions, while $\mathrm{Lip_b}(\Omega, Y)$ stands for the space of bounded, Lipschitz continuous functions in case Ω is metrisable. If $\Omega = H$ is a Hilbert space, then $\mathrm{C}^k_{\mathrm{b}}(H, Y)$ denotes the space of k-times Fréchet differentiable functions with continuous and bounded derivatives up to the kth order. Similarly $\mathrm{C_{ub}}(H, Y)$ may have a superscript k, denoting the order of the derivative. In both cases $k = \infty$ may hold. For $\theta \in (0, 1)$ we write $\mathrm{C}^\theta_{\mathrm{ub}}(H, Y)$ for the space of uniformly Hölder continuous functions, while $\mathrm{C}^\theta_{\mathrm{b}}(H, Y)$ will be introduced in Section 1.2.a. The space of bounded, Borel measurable functions is $M_b(\Omega, Y)$. If a measure μ is given on Ω, then the Lebesgue spaces are defined and are denoted by $\mathrm{L}^p(\Omega, Y)$ or $\mathrm{L}^p(\Omega, \mu, Y)$ with $1 \leq p \leq \infty$. The space of locally integrable functions is denoted by $\mathrm{L}^1_{\mathrm{loc}}(\Omega, Y)$. For the Sobolev spaces, we apply the usual $\mathrm{W}^{k,p}(H, Y)$ notation. The symbols $c_0(Y)$ and $\ell^\infty(Y)$ stand for the Banach spaces of null and bounded sequences in Y respectively. When incidentally $Y = \mathbb{K}$, we omit Y from the notations above. If Y is not a Banach space but a locally convex space, then $\mathrm{C}^k(\Omega, Y)$ is defined similarly as above. For the space of bounded, Borel measures on Ω, we will write $\mathscr{M}(\Omega)$.

The closed ball in a normed space X centred at x with radius r is denoted by $B(x, r)$. We rarely make distinction between different norms, the meaning of a term as $\|\varphi\|$ should be always clear from the context. The dual pairing between the Banach space X and its dual X' and also the scalar product on a Hilbert space H will be denoted by $\langle \cdot, \cdot \rangle$. The space of continuous linear operators on X with values in Y is $\mathscr{L}(X, Y)$.

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Chapter 1

Bi-continuous semigroups

This chapter introduces the notion of *bi-continuous* semigroups. As we will see these semigroups are natural generalisations of C_0-semigroups. First, we recall important notions and results from operator theory for reference. Then the basic facts related to bi-continuous semigroups are introduced. Also the most important results are recalled here. These are mainly taken from [56], where the investigations on bi-continuous semigroups were started, therefore these results are presented without proofs, so the reader is referred to original articles.

§ 1.1 ONE-PARAMETER SEMIGROUPS

DEFINITION 1.1.1 — Under a *one-parameter semigroup* we understand a semigroup (indeed a monoid) homomorphism T of $\mathbb{R}_+$ into $\mathscr{L}(X)$

$$T : \mathbb{R}_+ \to \mathscr{L}(X),$$

where X is a Banach space. Unless otherwise stated, the term *semigroup* is understood implicitly as a one-parameter semigroup.

Semigroups which are continuous for the strong operator topology on $\mathscr{L}(X)$ arising from the norm on X bare the name of a *C_0-semigroup*. They are extensively studied and extremely successfully applied in many different fields; we refer the reader to [5] and [24], [31], [40], [43], [66], [68] and the references therein.

DEFINITION 1.1.2 — If we endow $\mathscr{L}(X)$ with a topology τ_{op}, and the semigroup $T :\mathbb{R}_+ \to (\mathscr{L}(X), \tau_{\mathrm{op}})$ is continuous, we call it a τ_{op}-*continuous semigroup*

When τ_{op} incidentally is a strong operator topology τ_{s} coming from a certain topology τ on X, then a τ_{s}-continuous semigroup is called τ-*strongly continuous*. Hence C_0-semigroups are, by definition, the norm-strongly continuous semigroups. Similar terminology will be applied for functions $F : I \to \mathscr{L}(X)$ defined on an interval $I \subseteq \mathbb{R}$.

Given a one-parameter semigroup T, which is strongly continuous for some topology, it is natural to ask for which points x the function $t \mapsto T(t)x$ is differentiable, also it is desirable to determine the derivative. The most trivial semigroups are exponential functions

$t \mapsto \mathrm{e}^{ta}$ ($a \in \mathbb{R}$), and one has $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{ta} = a\mathrm{e}^{ta}$. We see that in this trivial case there is a correspondence between the exponent and the derivative at 0. Thus in general situations it is also natural to look for the exponent and therefore to associate the *infinitesimal-generator* $(A, D(A))$ to a semigroup T:

$$Ax := \lim_{h\to+0} \frac{T(h)x - x}{h} \quad \text{, where } A \text{ is defined whenever the limit exists.}$$

In Banach spaces, for C_0-semigroups this notion turned out to be of extreme importance, which is, e.g., justified by the well-posedness of abstract Cauchy problems (cf. Section 4.1.a). However, we would like to weaken the strong continuity assumption of the semigroups considered. Thus in order to make this "definition" precise and then obtain nice properties of such generators we shall put further assumptions on both the semigroup and both the underlying space. The appropriate, and also convenient setting will be established in Section §1.2. Before that we recall some important results related also partly to C_0-semigroups.

a) Hille – Yosida operators. As established by the Hille – Yosida Theorem (cf. [31, Sec. II.3.]), densely defined Hille – Yosida operators are in the centre of interest in the theory of C_0-semigroups.

DEFINITION 1.1.3 — Let $(A, D(A))$ be a linear operator[1] on a Banach space X. By definition it is a *Hille – Yosida operator* of type ω, if there exist $\omega \subseteq \mathbb{R}, M \geq 0$, such that $(\omega, +\infty) \subseteq \varrho(A)$ and[2]

$$\|(\lambda - \omega)^n R(\lambda, A)^n\| \leq M \quad \text{for all } n \in \mathbb{N}, \lambda > \omega.$$

The importance of this notion is justified by the following theorem.

THEOREM 1.1.4 [Hille – Yosida] — *There is a one-to-one correspondence between densely defined Hille – Yosida operators and generators of C_0-semigroups.*

To a Hille – Yosida operator $(A, D(A))$ it is usual to associate the following spaces [31, Sec. II.5.]. First we define $X_0 = \overline{D(A)}$ and assume that $\omega < 0$. Then $A_0 := A|_{X_0}$ is densely defined on X_0, hence by the Hille – Yosida theorem it generates a C_0-semigroup T_0 on X_0. We equip $D(A)$ with the graph-norm and denote this normed space by $(X_1, \|\cdot\|_1)$, which is a Banach space by the closedness of A. Also we define the extrapolation space X_{-1} by first endowing X_0 with the norm $\|x\|_{-1} := \|A_0^{-1}x\|$ and then taking the completion. Observe that T_0 is continuous on $(X_0, \|\cdot\|_{-1})$ hence extends to X_{-1}. Further let T_1 be the restriction of T_0 to X_1; the respective generators are denoted by $(A_{-1}, D(A_{-1})$ and $(A_1, D(A_1))$. These two constructions can be repeated arbitrarily resulting in the Sobolev tower.

It is well known that on reflexive spaces being a Hille – Yosida operator suffices be a generator of a C_0-semigroup [5, Prop. 3.3.8].

THEOREM 1.1.5 — *If X is reflexive then every Hille – Yosida operator on X is already a generator of a C_0-semigroup.*

[1] It might seem a bit inconvenient and also redundant to write out the domain of an operator. Nevertheless, we rigorously do it because of two reasons. First, this is just usual. Second, when we are dealing with concrete operators, such as the Laplacian Δ, it is extremely important to emphasise its domain. Clearly our second motivation justifies the first.

[2] $\varrho(A)$ is the resolvent set of a linear operator A.

§ 1.2 Bi-continuous semigroups

As a generalisation of C_0-semigroups, the notion of *bi-continuous* semigroups were introduced by Kühnemund. The abstract theory developed in [56] covers a large part of previously known results, as the induced semigroups of Dorroh and Neuberger [28, 29, 30]. Also other important examples fit into this general setting as the Ornstein – Uhlenbeck semigroup or implemented semigroups [2, 3]. These will be discussed in more detail in the later sections. First, we specify the ingredients of the theory. We begin with the properties of the underlying space and study elementary but important consequences that later allow us to provide a fruitful theory of bi-continuous semigroups.

a) The topology. Throughout the forthcoming sections $(X, \|\cdot\|)$ or briefly X will denote a Banach space which is also endowed with a locally convex, Hausdorff topology τ. On the space (X, τ), we always assume the following (see [56, Sec. 1.1] for detailed discussion).

Assumption 1.2.1

i) τ is Hausdorff and coarser than the norm-topology.

ii) The locally convex space (X, τ) is sequentially complete on τ-closed, norm-bounded sets.

iii) The dual space $(X, \tau)'$ is norming[3] for $(X, \|\cdot\|)$, i.e.,

$$\|x\| = \sup_{\substack{\varphi \in (X,\tau)' \\ \|\varphi\| \le 1}} |\varphi(x)|.$$

The locally convex topology τ is determined by a directed family of seminorms $\mathcal{P}$, and for the sake of simplicity we shall assume that all seminorms $p \in \mathcal{P}$ satisfy $p(x) \le \|x\|$ for all $x \in X$.

Remarks. We point out that the assumption iii) may be replaced by the stronger but more natural condition that the linear functionals which are τ-continuous on norm-bounded sets form a norming family. Also one could assume that the set X° of bounded linear functionals that are sequentially τ-continuous on norm-bounded sets is norming. These are both satisfied in many of the examples in this work. Notice also that the norming property of $(X, \tau)'$ implies that it separates the points of X. Further, it suffices to require that the unit ball $B(0, 1) \subset X$ is sequentially τ-complete.

Examples.

i) If (Ω, υ) is an arbitrary Hausdorff topological space and $\mathbb{K}^\Omega$ denotes the set of all $\mathbb{K}$-valued functions ($\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$) on Ω, then it is possible to define functions $p_C : \mathbb{K}^\Omega \to [0, +\infty]$ for $C \subseteq \Omega$ by

$$p_C(f) := \sup_{x \in C} |f(x)|.$$

[3] Since τ is coarser than the norm topology, we have $(X, \tau)' \subseteq X'$.

Then the subset $\mathrm{C_b}(\Omega) \subseteq \mathbb{C}^\Omega$ of all bounded, continuous functions is indeed a Banach space with the supremum-norm $\|\cdot\| := p_\Omega$. Moreover we can define the *compact-open topology* (or *the topology of locally uniform convergence*) τ_c by the family of seminorms

$$\mathcal{P} := \{p_K : K \subseteq \Omega,\, K \text{ is compact}\}\,.$$

It is straightforward that when υ is completely regular then τ_c is Hausdorff, and when (Ω, ν) is a k_f-space[4] then $\mathrm{C_b}(\Omega)$ is τ_c-complete on closed and norm-bounded sets. The norming property is justified by the Dirac measures' being τ_c-continuous. Hence Assumption 1.2.1 is satisfied, when (Ω, υ) is such a nice topological space, particularly when it is locally compact or metrisable .

ii) Similarly as above, we may consider the space $\mathrm{C_b}(\Omega, Y)$ with Y a Banach space and $\tau = \tau_\mathrm{c}$ the compact-open topology.

iii) Let H be a Hilbert space and $\theta \in (0,1)$. For all $A \subseteq H$ define the functions

$$p_{A,\theta}(f) := \sup_{\substack{x,y\in A\\ x\neq y}} \frac{|f(x)-f(y)|}{\|x-y\|^\theta}.$$

Further let $[\cdot]_\theta := p_{H,\theta}$. Then the space of bounded θ-Hölder continuous functions on H is defined as

$$\begin{aligned}\mathrm{C}_\mathrm{b}^\theta(H) :=\Big\{ & f : f \in \mathrm{C_b}(H),\ [f]_\theta < +\infty,\\ & \forall \varepsilon > 0\ \forall K \subseteq H \text{ compact } \exists \delta(K,\varepsilon) > 0 \forall x \in H\ \forall h \in K\\ & (\|h\| < \delta(K,\varepsilon) \Rightarrow |f(x+h) - f(x)| \cdot \|h\|^{-\theta} < \varepsilon)\Big\}.\end{aligned}$$

We therefore obtain the Banach space $(\mathrm{C}_\mathrm{b}^\theta(H), \|\cdot\|_\theta)$ with the norm

$$\|\cdot\|_\theta := \|\cdot\| + [\,\cdot\,]_\theta.$$

We may also endow the linear space $\mathrm{C}_\mathrm{b}^\theta(H)$ with the locally convex topology ${\tau_\mathrm{c}}^\theta$ determined by the family of seminorms

$$\mathcal{P}_\theta := \{p_K + p_{K',\theta} :\ K', K \subseteq H \text{ compact and } \operatorname{diam}(K') > 0\}\,.$$

The locally convex space $(\mathrm{C}_\mathrm{b}^\theta(H), {\tau_\mathrm{c}}^\theta)$ satisfies condition i) of Assumption 1.2.1, while straightforward considerations show that also part ii) is satisfied. For $x, y \in H$ and $x \neq y$ we define

$$\phi_{\theta,x,y} := \frac{\delta_x - \delta_y}{\|x-y\|^\theta}.$$

It is then obvious that $\phi_{\theta,x,y}$ is a continuous linear functional on the Banach space $\mathrm{C}_\mathrm{b}^\theta(H)$, with $\|\phi_{\theta,x,y}\| \leq 1$, and one sees immediately that $\phi_{\theta,x,y}$ is also ${\tau_\mathrm{c}}^\theta$-continuous. Hence we can see that part iii) of Assumption 1.2.1 is satisfied.

iv) Similarly, one defines the space $\mathrm{C}_\mathrm{b}^\theta(H,H)$.

[4] A k_f-space is a topological space Ω on which a continuity of a function $f : \Omega \to \mathbb{R}$ depends only on the continuity of $f|_K$ for all $K \subseteq \Omega$ compact.

v) For a normed space Y the topological dual $X := Y'$ endowed with the operator norm is a Banach space and the weak*-topology $\sigma(X, Y)$ has the properties posed in Assumption 1.2.1. Moreover, it is possible to consider another topologies on X'. For example the Mackey topology $\mu(X, Y)$ satisfies the required properties (see [75, Sec. IV.3]). The sequential completeness follows from a theorem of Kreĭn asserting that an absolute convex, weak*-compact set is complete for the Mackey topology (see [75, Thm. 11.4] and Section §4.2).

Since

$$\|x\| = \sup\{p(x) : p \in \mathcal{P}\} \quad \text{for all } x \in X$$

by Assumption 1.2.1, we may state the following trivial but not less important fact.

PROPOSITION 1.2.2 — *The norm $\|\cdot\|$ is τ-lower-semicontinuous, and the unit ball $B(0,1)$ in X is τ-closed.*

As for measurability and integration, we may notice that this proposition yields immediately that for a τ-continuous function $F : \mathbb{R} \to X$ the map $t \mapsto \|F(t)\|$ is immediately lower semicontinuous hence (Borel) measurable. Further, if F is locally norm-bounded, then $\|F\| \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$.

We slightly improve the previous result in order to be able to take vector-valued Lebesgue integrals (Bochner integrals). Let $T : [0, t_0] \to \mathscr{L}(X)$ a norm-bounded τ-strongly-continuous function and take $f \in \mathrm{L}^1([0, t_0], X)$. We would like to define the integral of $T(t)f(t)$ on $[0, t_0]$. We proceed as follows. Since f is measurable there exists a sequence f_n of step functions with level sets closed intervals and converging pointwise to f. Then $T(t)f_n(t) \to T(t)f(t)$ in norm for all t. Also $p(T(t)f_n(t))$ is a piecewise continuous function hence it is measurable for all $p \in \mathcal{P}$. Since $\|T(t)f_n(t)\|$ is the pointwise supremum of these as p ranges over $\mathcal{P}$, we see that $\|T(t)f_n(t)\|$ and hence $\|T(t)f(t)\|$ is measurable. The integral

$$I_n := \int_0^{t_0} T(t) f_n(t)\,\mathrm{d}t$$

is interpreted as a piecewise τ-Riemann integral. We show that I_n is a norm Cauchy sequence. For $\phi \in (X, \tau)'$ and $\|\phi\| \leq 1$ we can write

$$|\langle I_n - I_m, \phi\rangle| \leq \int_0^{t_0} |\langle T(t)(f_n(t) - f_m(t)), \phi\rangle|\,\mathrm{d}t \leq \sup_{t \in [0,t_0]} \|T(t)\| \cdot \int_0^{t_0} \|f_n(t) - f_m(t)\|\,\mathrm{d}t,$$

hence $\|I_n - I_m\| \leq \varepsilon$, if $n, m \in \mathbb{N}$ are large enough. Thus I_n converges, and its limit I is the definition of

$$\int_0^{t_0} T(t) f(t)\,\mathrm{d}t := I.$$

It is straightforward that the value of the defined integral is independent of the choice of f_n. Moreover we also have that

$$\Big\|\int_0^{t_0} T(t)f(t)\,\mathrm{d}t\Big\| \leq \sup_{t\in[0,t_0]} \|T(t)\|\cdot\|f\|_1.$$

The facts appearing in this section will be used tacitly in the following.

b) Basic notions. We now recall the basic notions as introduced and studied in [54], [55] and [56]. We furthermore refer to Section §A.1 on mixed topologies.

DEFINITION 1.2.3 — A set of bounded linear operators $\mathcal{B} \subseteq \mathscr{L}(X)$ is said to be *bi-equicontinuous* (for the topology τ), if for every norm-bounded τ-null sequence x_n

$$\tau - \lim_{n\to\infty} Bx_n = 0$$

holds uniformly for $B \in \mathcal{B}$. We call a function $T : I \to \mathscr{L}(X)$ *bi-equicontinuous*, if $\mathrm{rg}\,(T)$ is bi-equicontinuous. A family of operators $\{T(s) : s \in \mathbb{R}_+\}$ is *locally-bi-equicontinuous*, if $\{T(s) : s \in [0,t]\}$ is bi-equicontinuous for all $t > 0$.

DEFINITION 1.2.4 — A semigroup T is called a *bi-continuous semigroup* , if

i) it is τ-strongly continuous,

ii) T is locally bounded, i.e., $T|_{[0,t]}$ is a norm-bounded function for some (and in this case for all) $t > 0$,

iii) $\{T(t) : t \in \mathbb{R}_+\}$ is locally-bi-equicontinuous.

REMARKS. When T is a bi-continuous semigroup, it is exponentially bounded, i.e., there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq M\mathrm{e}^{\omega t} \quad \text{for all } t \geq 0. \tag{EXP}$$

Such semigroups will be called a semigroup of *type* ω. The *exponential growth bound* $\omega_0(T)$ of T is defined as the infimum of those $\omega \in \mathbb{R}$ appearing in (EXP). It may happen that $\omega_0(T) = -\infty$ as nilpotent semigroups show. On the other hand, $\omega_0(T) = \frac{1}{t}\log r(T(t))$ for all $t > 0$. This can be seen just as in the case of C_0-semigroups ([31, Prop. IV.2.2]). We call a semigroup *uniformly exponentially stable* , if $\omega_0(T) < 0$, in which case the orbits $t \mapsto T(t)x$ decay to 0 in the norm topology. It would be nice to find conditions implying stability with respect to τ.

Notice that instead of the local-bi-equicontinuity in Definition 1.2.4 it suffices to assume that $T|_{[0,t]}$ is bi-equicontinuous for some $t_0 > 0$. This follows easily, if we write $t \in \mathbb{R}_+$ in the form $t = nt_0 + t_1$, where $n \in \mathbb{N}$ and $t_1 \in [0,1)$ (see the proof of Theorem 3.2.2).

When τ coincides with the norm-topology, then the Banach – Steinhaus theorem (also known as the uniform boundedness principle) implies that a τ-strongly continuous function is locally bounded and locally-bi-equicontinuous. Thus C_0-semigroups are bi-continuous with respect to the norm topology. However, in the general situation, such implications are not known. In particular, when the Banach – Steinhaus theorem holds for τ (i.e.,

(X, τ) is barelled), then we get back to the framework of C_0-semigroups (see Theorem 1.2.12). Nevertheless, there are certain cases when τ-strong continuity of T and some not very strong assumptions immediately imply the other two properties posed in Definition 1.2.4 (see Theorem 2.2.5).

DEFINITION 1.2.5 — A subset $D \subseteq X$ is called *bi-dense*, if for all $x \in X$ there exists a norm-bounded sequence $x_n \in D$ with $x_n \xrightarrow{\tau} x$.

c) A Hille – Yosida theorem. The *inifinitesimal generator* $(A, D(A))$ of a bi-continuous semigroup T is defined in the following, natural way.

DEFINITION 1.2.6 — The domain of the *generator* A is

$$D(A) := \Big\{x : x \in X,\ \tau - \lim_{h\to 0} \tfrac{T(h)x-x}{h},\ \sup_{h\in(0,1]} \tfrac{\|T(h)x-x\|}{h} < +\infty\Big\},$$

while the values are given by

$$Ax := \tau - \lim_{h\to 0} \frac{T(h)x - x}{h}.$$

Alternatively, we could have defined the generator via Laplace transforms as in [5] for the case of C_0-semigroups. Taking the Laplace transform of T in the τ-strong sense, it is not hard to check that we obtain a pseudo-resolvent, which is indeed a resolvent of a Hille – Yosida operator, which coincides with our generator defined through differentiation (see [54, Sec. 1.2]). This argumentation provides one direction of the Hille – Yosida type generation theorem below. This theorem was proved by Kühnemund in [54, Thm. 1.28] and [56]. First we mention some important properties of generators (see [54, Sec. 1.2]).

THEOREM 1.2.7 — *Let T be a bi-continuous semigroup with generator $(A, D(A))$. Then the following hold.*

a) $(A, D(A))$ is bi-closed*, i.e., whenever $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$ and both are norm-bounded, then $y \in D(A)$ and $Ax = y$.*

b) For $x \in D(A)$ we have $T(t)x \in D(A)$ and $T(t)Ax = AT(t)x$ for all $t \geq 0$.

c) For $t > 0$ and $x \in X$ one has

$$\int_0^t T(s)x\,\mathrm{d}s \in D(A) \quad \textit{and} \quad A\int_0^t T(s)x\,\mathrm{d}s = T(t)x - x.$$

d) For $\lambda > \omega_0(T)$ one has $\lambda \in \varrho(A)$ (thus A is closed) and

$$R(\lambda, A)x = \int_0^{+\infty} \mathrm{e}^{-\lambda s} T(s)x\,\mathrm{d}s, \quad x \in X,$$

where the integral is a τ-improper integral.

e) For $\omega \in \mathbb{R}$ the semigroup $t \mapsto \mathrm{e}^{\omega t}T(t)$ is bi-continuous and its generator is $(A + \omega, D(A))$ (rescaling).

THEOREM 1.2.8 [Generation theorem] — *Let $(A, D(A))$ be a linear operator on the Banach space X. The following assertions are equivalent.*

i) $(A, D(A))$ is the generator of a bi-continuous semigroup T of type ω.

ii) $(A, D(A))$ is a Hille – Yosida operator of type ω, it is bi-densely defined and the family

$$\big\{(s-\alpha)^k R(s,A)^k : k \in \mathbb{N},\ s \geq \alpha\big\}$$

is bi-equicontinuous for all $\alpha > \omega$.

As shown by various examples (cf. Chapter 2) the generator of a bi-continuous semigroup is not necessarily densely defined unless it generates indeed a C_0-semigroup. However, there are certain family of spaces where the notion of bi-continuous semigroups, and therefore the above theorem reduces to the case of C_0-semigroups (see Theorem 1.2.12). We close this section with a useful result, which helps identifying the generator of a bi-continuous semigroup (see [54, Prop. 1.21] and compare with [31, Prop. II.1.7]).

PROPOSITION 1.2.9 — *Let $(A, D(A))$ be the generator of the bi-continuous semigroup T, and let $D \subseteq D(A)$ be a bi-dense set. Assume that D is invariant under T. Then it is a* bi-core *for $(A, D(A))$, i.e., for all $x \in D(A)$ there exist $x_n \in D$ such that $x_n \xrightarrow{\tau} x$, $Ax_n \xrightarrow{\tau} Ax$ and both sequences are norm-bounded.*

d) Approximation theorems. The first approximation theorems for bi-continuous semigroups appear in [55] and [56]. The two Trotter – Kato theorems were presented there, and as application the Chernoff formula and the Euler formula was also obtained (cf. [31, Sec. III.4-5]). The Euler formula was later reproved in more general situations in [9]. The presently strongest versions of the Trotter – Kato theorem for bi-continuous semigroups was obtained in [1] (cf. [31, Thm. III.4.9]). We quote this result here in the form we will apply later.

THEOREM 1.2.10 [Trotter – Kato] — *Let T and T_k, $k \in \mathbb{N}$ be bi-continuous semigroups with generators $(A, D(A))$ and $(A_k, D(A_k))$ respectively. Suppose that*

i) there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|T_k(t)\| \leq M\mathrm{e}^{\omega t}$, and

ii) for all $t_0 > 0$ and $x_n \in X$ norm-bounded τ-null sequence

$$\tau - \lim_{n\to\infty} T_k(t)x_n = 0 \quad \textit{uniformly for } t \in [0, t_0] \textit{ and } k \in \mathbb{N}$$

holds.

Let $\lambda > \omega$ and consider the following assertions.

a) $A_n x \xrightarrow{\tau} Ax$ with $A_n x$ norm-bounded for all x in a bi-core D of A such that $(\lambda - A)D$ is bi-dense in X.

b) For a bi-dense subset $D \subseteq X$ we have

$$R(\lambda, A_k)x \to R(\lambda, A)x \quad \textit{for all } x \in D.$$

c) For all $x \in X$

$$T_k(t)x \xrightarrow{\tau} T(t)x \quad \textit{as } k \to \infty$$

locally uniformly for $t \in \mathbb{R}_+$.

Then a) implies b), while the two last assertions are equivalent.

We also mention the *Euler formula* also known as the *Post – Widder Inversion Formula*.

THEOREM 1.2.11 [Euler formula, Post – Widder inversion formula] — *For a bi-continuous semigroup T and its generator we have*

$$T(t)x = \tau - \lim_{n\to\infty} \Big[\tfrac{n}{t} R\big(\tfrac{n}{t}, A\big)\Big]^n x$$

for all $x \in X$ and uniformly for t in compact intervals.

e) Further topological considerations. For our purposes it is necessary to refine the notions introduced so far. We also state some technical results here.

THEOREM 1.2.12 — *If (X, τ) is barelled, then in fact it is normable, i.e., the norm topology coincides with τ.*

PROOF — A barelled space is characterised by the fact that every lower semicontinuous seminorm is indeed continuous [75, Sec. II.7]. By Proposition 1.2.2 we have that $\|\cdot\|$ is a continuous seminorm, thus τ coincides with the norm topology in such spaces. ■

The following notion will be important in Section §3.1, when we extend a not densely-defined operator to the whole space X.

DEFINITION 1.2.13 — Let $D \subseteq X$ an arbitrary subset and let $\eta > 1$. We say that D is η-*bi-dense* in X for the topology τ, if for all $x \in X$ there exists a sequence $x_n \in D$ which converges to x in the topology τ and furthermore satisfies

$$\|x_n\| \le \eta \|x\| \quad \text{for all } n \in \mathbb{N}. \tag{ηD}$$

Clearly an η-bi-dense set contains 0, thus we could have relaxed this notion, but we will only encounter η-bi-dense subspaces. The following two results show that requiring η-bi-denseness instead of bi-denseness is not so restrictive.

REMARK 1.2.14 — Suppose that the topology τ is metrisable. Then any bi-dense set $D \subseteq X$ which contains 0 is η-bi-dense for arbitrary $\eta > 1$. This can be seen by means of a simple diagonal process. In fact, take any $0 \neq x \in X$ and suppose that

$$p_1 \le p_2 \le \cdots \le p_n, \ldots$$

is cofinal in $\mathcal{P}$. Let $n \in \mathbb{N}$ and choose $x_n \in D$ such that $p_n(x_n - x) \leq 1/n$ holds. We show that $x_n \xrightarrow{\tau} x$. Indeed, let $p \in \mathcal{P}$ and take $N \in \mathbb{N}$ such that $p \leq p_N$, and for a given $0 < \varepsilon < \eta - 1$ write

$$p(x_n - x) \leq p_N(x_n - x) \leq p_n(x_n - x) \leq \frac{1}{n} \leq \varepsilon$$

for all $n \geq N(\varepsilon)$, with a suitable $N(\varepsilon) \in \mathbb{N}$. This estimate together with $p(x) \leq \|x\|$ yields $p(x_n) \leq \varepsilon + \|x\|$, showing that $\|x_n\| \leq \eta\|x\|$ for arbitrary $\eta > 1$, for all $n \geq N(\eta)$, and that was to be proved.

The following result is simple modification of the result on bi-denseness in [56].

Proposition 1.2.15 — *Let T be a bi-continuous semigroup. Denote its generator by $(A, D(A))$. Then $D(A^k)$ is η-bi-dense in X for an appropriate $\eta > 1$.*

Proof — Take $x \in X$, then it is shown in [56, Sec. 1.2] that

$$\tau - \lim_{n\to\infty} n^k R(n, A)^k x = x.$$

Let $M > 0$ and $\omega \in \mathbb{R}$ for which T satisfies (EXP). Since $(A, D(A))$ is a Hille – Yosida operator by Theorem 1.2.8, we have

$$\|n^k R(n, A)^k x\| \leq \frac{Mn^k}{(n-\omega)^k}\|x\| \quad \text{for all } t \geq 0,$$

hence taking $x_n = n^k R(n, A)^k x$ and any $\eta > M$ we see that $x_n \in D(A^k)$ and $\|x_n\| \leq \eta\|x\|$ is satisfied for large $n \in \mathbb{N}$. ■

From this we may also infer that whenever T is quasi-contractive, i.e., the constant M in (EXP) can be taken 1, $D(A)$ is η-bi-dense for all $\eta > 1$ (cf. also renorming, Section 1.3.b).

We introduce certain Banach spaces that will be utilised later in Chapter 3 for the perturbation results.

Definition 1.2.16 — For $t_0 > 0$ define the space

$$\begin{aligned}\mathbf{X}_{t_0} := \big\{&T : [0, t_0] \to \mathscr{L}(X),\ \tau\text{-strongly continuous},\\ &\text{norm-bounded, and } \{T(t) : t \in [0, t_0]\} \text{ is bi-equicontinuous}\big\}.\end{aligned}$$

It is clear that $\mathbf{X}_{t_0}$ is a linear space.

Lemma 1.2.17 — *The space $\mathbf{X}_{t_0}$ is complete for the supremum norm*

$$\|T\| := \sup\{\|T(t)\| : t \in [0, t_0]\} \quad \textit{for } T \in \mathbf{X}_{t_0}.$$

Proof — We show that $\mathbf{X}_{t_0}$ is a closed subspace of the Banach space $\mathrm{B}([0, t_0], \mathscr{L}(X))$ of all bounded functions from the interval $[0, t_0]$ to the space of bounded linear operators $\mathscr{L}(X)$ endowed with the supremum norm. Let $T_n \in \mathbf{X}_{t_0}$ be convergent to $T \in \mathrm{B}([0, t_0], \mathscr{L}(X))$ and take any $x \in X$. Then $T_n(\cdot)x$ converges to $T(\cdot)x$ for τ uniformly for $t \in [0, t_0]$. Indeed, for any $p \in \mathcal{P}$ we have

$$p(T_n(t)x - T(t)x) \leq \|T_n(t)x - T(t)x\| \leq \|T_n - T\| \cdot \|x\| \to 0.$$

Hence, T is τ-strongly continuous. The norm-boundedness of T is trivial. In order to prove the bi-equicontinuity of the set $\{T(t) : t \in [0, t_0]\}$ we take a norm-bounded τ-null sequence x_n and $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $\|T_m - T\| < \varepsilon/2$. Then we obtain for $p \in \mathcal{P}$

$$p(T(t)x_n) \leq p((T(t) - T_m(t))x_n) + p(T_m(t)x_n) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for n sufficiently large and by the bi-equicontinuity of $\{T_m(t) : t \in [0, t_0]\}$. ∎

LEMMA 1.2.18 — *The Banach space $\mathbf{X}_{t_0}$ is even a Banach algebra.*

PROOF — Take $0 < t < t_0$ and $h_n \in \mathbb{R}$ converging to 0. For $S, T \in \mathbf{X}_{t_0}$ and $x \in X$ we write

$$\begin{aligned}&T(t+h_n)S(t+h_n)x - T(t)S(t)x = \\ &T(t+h_n)S(t+h_n)x - T(t+h_n)S(t)x + T(t+h_n)S(t)x - T(t)S(t)x.\end{aligned}$$

Clearly we have

$$T(t+h_n)S(t)x - T(t)S(t)x \xrightarrow{\tau} 0.$$

From the bi-equicontinuity of $\{T(t) : t \in [0, t_0]\}$ it follows also that

$$T(t+h_n)S(t+h_n)x - T(t+h_n)S(t)x = T(t+h_n)(S(t+h_n) - S(t))x \xrightarrow{\tau} 0.$$

The left (respectively the right) τ-strong continuity of $T \cdot S$ in the endpoints of $[0, t_0]$ can be proved analogously. The norm-boundedness of $t \mapsto T(t)S(t)$ is obvious. It remains to show the bi-equicontinuity of the family $\{T(t)S(t) : t \in [0, t_0]\}$. Suppose the contrary, i.e., there exists a norm-bounded τ-null sequence x_n, an $\varepsilon > 0$ and a seminorm $p \in \mathcal{P}$ such that for all $n \in \mathbb{N}$ there exists $t_n \in [0, t_0]$ satisfying

$$p(T(t_n)S(t_n)x_n) > \varepsilon.$$

This leads us to a contradiction since $S(t_n)x_n$ is norm-bounded and τ-convergent to 0 (by the bi-equicontinuity of S), hence by means of the bi-equicontinuity of T we conclude that

$$p(T(t_n)S(t_n)x_n) \to 0,$$

completing the proof. ∎

The following technical lemma shows how we can differentiate τ-strongly continuous functions.

LEMMA 1.2.19 — *Let $g : [0, t_0] \to X$ be continuously τ-differentiable with norm-bounded derivative g'. Further let $F : [0, t_0] \to \mathscr{L}(X)$ be a strongly τ-differentiable function on* $\operatorname{rg}(g)$*, which is also τ-bi-equicontinuous on X. Then the function $F \cdot g$ is τ-differentiable and $(Fg)' = F'g + Fg'$.*

PROOF — Let F and g be as in the assertion. The usual trick gives:

$$\begin{aligned}&\tfrac{1}{h}(F(t+h)g(t+h) - F(t)g(t)) = \\ &\quad = \tfrac{1}{h}[F(t+h)(g(t+h) - g(t)) + (F(t+h) - F(t))g(t)] = \\ &\quad = F(t+h)[\tfrac{1}{h}(g(t+h) - g(t)) - g'(t)] + F(t+h)g'(t) + \\ &\quad\quad + \tfrac{1}{h}(F(t+h) - F(t))g(t) \xrightarrow{\tau} F(t)g'(t) + F'(t)g(t)\end{aligned}$$

as $h \to 0$ by the assumptions on F and g. ∎

The bi-equicontinuity of the operator family $\{T(t) : t \in [0, t_0]\}$ appearing in Definition 1.2.4 is a crucial assumption, for example to establish the results in Theorem 1.2.7. However, for unbounded perturbations, we shall require more as introduced in the following. Clearly, if T is locally-bi-equicontinuous function, then each of the operators $T(t)$ has to be sequentially τ-continuous on norm bounded sets. The following definition adjusts this notion of equicontinuity to our needs.

DEFINITION 1.2.20 — Let B be norm-bounded operator. It is called *local*, if for all $p \in \mathcal{P}$ and $\varepsilon > 0$ there exists a constant $K_{p,\varepsilon} > 0$ and a seminorm $p' \in \mathcal{P}$ such that

$$p(Bx) \leq K_{p,\varepsilon} p'(x) + \varepsilon \|x\|, \quad \text{for all } x \in X.$$

We also need the extension of this notion to the case of "uniformly local", operator valued functions.

DEFINITION 1.2.21 — A bounded function $F : [0, t_0] \to \mathscr{L}(X)$ (or a family of bounded operators) is said to be *local*, if for all $p \in \mathcal{P}$ and $\varepsilon > 0$ there exists a constant $K_{F,p,\varepsilon} > 0$ and a seminorm $p' \in \mathcal{P}$ such that

$$p(F(t)x) \leq K_{F,p,\varepsilon} p'(x) + \varepsilon \|x\|, \quad \text{for all } t \in [0, t_0] \text{ and } x \in X.$$

We denote the set of local functions in $\mathbf{X}_{t_0}$ by $\mathbf{X}_{t_0}^{\text{loc}}$.

Note that local operator is τ-continuous on norm-bounded sets, while for operator valued function we get the τ-equicontinuity on norm-bounded sets. Thus a local function $F : [0, t_0] \to \mathscr{L}(X)$ is always bi-equicontinuous.

LEMMA 1.2.22 — *The local part $\mathbf{X}_{t_0}^{\text{loc}}$ is a closed subspace of $\mathbf{X}_{t_0}$.*

PROOF — Suppose that $F_n \in \mathbf{X}_{t_0}$ converges to F in the norm of $\mathbf{X}_{t_0}$. Let $p \in \mathcal{P}$ and $\varepsilon > 0$, then for any $x \in X$, and for a sufficiently large fixed $n \in \mathbb{N}$

$$\begin{aligned} p(F(t)x) &\leq p(F(t)x - F_n(t)x) + p(F_n(t)x) \leq \|F_n(t)x - F(t)x\| + p(F_n(t)x) \leq \\ &\leq \|F_n - F\| \cdot \|x\| + p(F_n(t)x) \leq (\|F_n - F\| + \varepsilon/2)\, \|x\| + K_{F,p,\varepsilon/2} p'(x) \end{aligned}$$

we see that F is local. ∎

We finish this section by a statement on the resolvent of the generator of a bi-continuous semigroup. This should be contrasted with the relevant assertion of the generation Theorem 1.2.8.

LEMMA 1.2.23 — *Let T be a bi-continuous semigroup with generator $(A, D(A))$. Then for all $\alpha, \beta > \omega$ the set*

$$\{R(\lambda, A) : \lambda \in [\alpha, \beta]\}$$

is local.

PROOF — We take the Laplace transform of T in the topology τ. For $\lambda \in [\alpha, \beta]$, $\varepsilon > 0$

and $p \in \mathcal{P}$, we choose $\varepsilon' > 0$ sufficiently small and obtain the estimate for small $\varepsilon' > 0$

$$p(R(\lambda, A)x) \leq \int_0^{+\infty} \mathrm{e}^{-\lambda s} p(T(s)x)\,\mathrm{d}s =$$
$$= \int_0^t \mathrm{e}^{-\lambda s} p(T(s)x)\,\mathrm{d}s + \int_t^{+\infty} \mathrm{e}^{-\lambda s} p(T(s)x)\,\mathrm{d}s \leq$$
$$= \int_0^t \mathrm{e}^{-\lambda s} M_{p,\varepsilon'} p'(x)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\lambda s} \varepsilon' \|x\|\,\mathrm{d}s + \int_t^{+\infty} \mathrm{e}^{-\lambda s} M \mathrm{e}^{\omega s} \|x\|\,\mathrm{d}s =$$
$$= \frac{\mathrm{e}^{-\lambda t} - 1}{\lambda} (M_{p,\varepsilon} p'(x) + \varepsilon' \|x\|) + M \frac{\mathrm{e}^{(\omega - \lambda)t}}{\omega - \lambda} \|x\| \leq M'_{p,\varepsilon} p'(x) + \varepsilon \|x\|$$

by taking $t > 0$ large enough and with appropriate constant $M'_{p,\varepsilon}$ and a seminorm $p' \in \mathcal{P}$. ■

§ 1.3 BASIC CONSTRUCTIONS

In this section, we are concerned with constructing new bi-continuous semigroups from given ones. We look at similarity transformations, renorming techniques, direct products and the adjoint of bi-continuous semigroups. These constructions will turn out to be useful later.

a) Similar semigroups. Let $(X, \|\cdot\|)$, τ and $(Y, \|\cdot\|)$, $\tilde{\tau}$ satisfy Assumption 1.2.1 and T be a bi-continuous semigroup on the space $(X, \|\cdot\|)$ equipped with τ. If $B \in \mathscr{L}(X, Y)$ is also invertible, it is possible to define the semigroup S on Y by a similarity transformation $S(t) := BT(t)B^{-1}$. Then S is indeed a semigroup which is locally norm-bounded.
Further, suppose that B and B^{-1} are both sequentially τ-continuous on norm-bounded sets. Then S is a bi-continuous semigroup with generator $(BAB^{-1}, BD(A))$. The strong continuity, the semigroup property and the norm-boundedness is trivial. The bi-equicontinuity is also obvious. Taking $t_0 > 0$ and $y_n \xrightarrow{\tilde{\tau}} 0$ with $y_n \in X$ norm bounded, we have $B^{-1}y_n$ norm-bounded and τ-convergent to 0. Thus $T(t)B^{-1}y_n \xrightarrow{\tau} 0$ uniformly on $[0, t_0]$. As in Lemma 1.2.18, we see that $BT(t)B^{-1}y_n \xrightarrow{\tilde{\tau}} 0$ uniformly on $[0, t_0]$. One can simply compute the generator of S and arrive to the conclusion that it is indeed $(BAB^{-1}, BD(A))$.

b) Renorming. We have seen that quasi-contractive semigroups[5] have the pleasant property that their generator has η-bi-dense domain for all $\eta > 1$. For bounded C_0-semigroups it is possible to find an equivalent norm on X such that the semigroup becomes contractive. In our case, we have to be careful since taking an equivalent norm although does not destroy the topological structure, but the geometrical properties may

[5] The constant M in (EXP) can be taken 1.

be lost. Thus the renormed space may not satisfy the assumptions on (X, τ). Let T be a bi-continuous semigroup, we may assume that it is bounded by rescaling (cf. Theorem 1.2.7). Define a new norm by

$$|||x||| := \sup_{t \in \mathbb{R}_+} \|T(t)x\|.$$

This is indeed a norm, which is equivalent to $\|\cdot\|$ by

$$\|x\| \leq |||x||| \leq M\|x\|,$$

where M is obtained from the boundedness of T. For this new norm T becomes contractive. We have to show that Assumption 1.2.1 is still satisfied. Only the norming property is not absolutely obvious. For $\varphi \in X'$ denote the functional norm by $|||\varphi|||$. By definition

$$|||x||| = \sup_{t \in \mathbb{R}_+} \|T(t)x\| = \sup_{t \in \mathbb{R}_+} \sup_{\substack{\varphi \in (X,\tau)' \\ \|\varphi\| \leq 1}} |\langle T(t)x, \varphi\rangle| \leq \sup_{t \in \mathbb{R}_+} \sup_{\substack{\varphi \in (X,\tau)' \\ |||\varphi||| \leq 1}} |\langle T(t)x, \varphi\rangle| \leq |||x|||,$$

thus equality holds in all places. Clearly T is a bi-continuous semigroup for this new norm and the topology τ.

c) Direct product semigroups. Let T and S be bi-continuous semigroups on the spaces (X, τ') and (Y, τ''). It is then possible to consider the direct product semigroup on the space $(\mathcal{X}, \tau)$, where $\mathcal{X} := X \times Y$ and

$$\begin{aligned} \|(x,y)\| &:= \|x\| + \|y\| \quad \text{for all } x \in X, y \in Y \text{ and} \\ \tau &:= \tau' \times \tau'' \quad \text{is the product topology.} \end{aligned}$$

We observe that the space (X, τ) satisfies Assumption 1.2.1 with these defintions. This is indeed obvious, as long as we notice that $(\mathcal{X}, \tau)' = (X, \tau')' \times (Y, \tau'')'$.

On this product space we consider the direct product semigroup $\mathcal{T}$ by

$$\mathcal{T}(t) := \begin{pmatrix} T(t) & 0 \\ 0 & S(t) \end{pmatrix}.$$

It is straightforward that $\mathcal{T}$ is a bi-continuous semigroup on $(\mathcal{X}, \tau)$ and its generator is the direct product of the two generators $(A, D(A))$ and $(B, D(B))$ with *diagonal domain*

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for all } (x,y) \in D(\mathcal{A}) = D(A) \times D(B).$$

d) Adjoint of a bi-continuous semigroup. Let X° be the set of all norm-bounded linear functionals which are τ-sequentially continuous on norm-bounded sets of X. We endow this linear space with the operator norm inherited from X'. We claim that it is indeed a Banach space, i.e., it is a closed linear subspace of X'. Take $\varphi_n \in X^\circ$ with $\|\varphi_n - \varphi\| \to 0$, where $\varphi \in X'$. We have to show that $\varphi \in X^\circ$. To this end, consider a norm-bounded τ-null sequence x_n. Then $\varphi(x_n) \to 0$ follows from

$$\begin{aligned} |\varphi(x_n)| &\leq |\varphi(x_n) - \varphi_k(x_n)| + |\varphi_k(x_n)| \leq \\ &\leq K\|\varphi - \varphi_k\| + |\varphi_k(x - x_n)| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

first by taking $k \in \mathbb{N}$ sufficiently large and then for fixed k using the continuity assumptions on φ_k.

Now, consider the Banach space X° equipped additionally with the weak topology $\tau^\circ := \sigma(X^\circ, X)$. We check Assumption 1.2.1. τ° is Hausdorff since X separates the points of X°. For all $x \in X$ the operator norm of $x \in (X^\circ)'$ coincides with $\|x\|$, since X° is norming for X, and clearly $X = (X^\circ, \tau^\circ)'$. Hence

$$\|\varphi\| = \sup_{\substack{x\in X\\ \|x\|\le 1}} |\varphi(x)|,$$

thus we see that $(X^\circ, \tau^\circ)'$ is norming for X°. Trivially τ° is coarser than the norm-topology on X°. It remains to check only the τ°-sequential completeness on closed, norm-bounded sets , but this generally may fail to hold. At this point we have to put a condition on our spaces.

HYPOTHESIS 1.3.1 Suppose that $X^\circ \cap B(0,1)$ is sequentially complete for $\sigma(X^\circ, X)$.

Later we shall give an example of such spaces. Now, we established the framework for considering bi-continuous semigroups.

Clearly a linear operator $B \in \mathscr{L}(X)$ which is also τ-sequentially continuous on norm-bounded sets admits an adjoint $B' \in \mathscr{L}(X')$ which leaves X° invariant. Take now a bi-continuous semigroup T on (X, τ). Then the operators $T(t)^\circ$ form a semigroup T°, which is τ°-strongly continuous by definition. The exponential boundedness of T° is trivial. To establish the local bi-equicontinuity, we assume the following.

HYPOTHESIS 1.3.2 Every $\varphi_n \in X^\circ$ norm-bounded τ°-null sequence is τ-sequentially equicontinuous on norm bounded sets.

To see the τ°-bi-equicontinuity of T°, take a norm-bounded τ°-null sequence and $x \in X$. For $t_0 > 0$ one has the τ-compactness of $\{T(t)x : \ t \in [0, t_0]\}$, thus by the equicontinuity of φ_n we have

$$[T^\circ(t)\varphi_n](x) = \varphi_n(T(t)x) \to 0$$

uniformly on $[0, t_0]$, which is the τ°-bi-equicontinuity of T°.

EXAMPLE 1.3.3 — Among our most important examples is always $X = \mathrm{C_b}(\Omega)$, with Ω Polish space. As usually, we consider it with the compact-open topology τ_{c}. It is proved in Section A.2.b that in this case $X^\circ = \mathscr{M}(\Omega)$. It was known to Aleksandrov [4] that $\mathscr{M}(\Omega)$ is $\sigma(\mathscr{M}(\Omega), \mathrm{C_b}(\Omega))$-sequentially complete, thus Hypothesis 1.3.1 is fulfilled. On the other hand, Hypothesis 1.3.2 is justified by Prokhorov's theorem (see Theorem A.2.5), since a $\{0, \ \varphi_n : \ n \in \mathbb{N}\}$ is weak*-compact hence is equicontinuous (cf. Theorem A.2.4). These considerations show that it is always possible to take the *dual* T° of a bi-continuous semigroup T on $\mathrm{C_b}(\Omega)$.

§ 1.4 POSITIVITY

In this section, we observe the positivity of bi-continuous semigroups. The notion of positivity turned out to be extremely useful for C_0-semigroups, for example in spectral theory.

We take X a Banach lattice and denote its positive cone by X_+. We shall assume that $X_+ \cap B(0,1)$ is sequentially τ-closed. This ensures that for norm-bounded, τ continuous functions $[0,t_0] \ni t \mapsto f(t) \in X_+$, the Riemann integral $\int_0^{t_0} f(t)\,\mathrm{d}t$ stays in the positive cone. If Y is another Banach lattice with positive cone Y_+, a linear operator $B : X \to Y$ is said to be *positive*, if $BX_+ \subseteq Y_+$ (in notation $B \geq 0$).

a) Positive bi-continuous semigroups. Similarly to the case of C_0-semigroups, for bi-continuous semigroups there is also a characterisation of positivity in terms of the resolvent of the generator. The proof is fairly obvious and just mimics the C_0 case.

THEOREM 1.4.1 — *Let T be a bi-continuous semigroup and denote its generator by $(A, D(A))$. The following are equivalent*

i) T is a positive semigroup, i.e., $T(t) \geq 0$ for all $t \geq 0$.

ii) There exists an $\alpha \in \mathbb{R}$ such that for all $\lambda > \alpha$ the resolvent $R(\lambda, A)$ exist and is positive.

PROOF — Assume first the positivity of T. Then taking the Laplace transform of T for $\lambda > \omega$

$$R(\lambda, A)x = \int_0^{+\infty} \mathrm{e}^{-\lambda t} T(t)x\,\mathrm{d}t$$

for all $x \in X$, the integral exists as a τ-Riemannian integral. Thus when $x \in X_+$, by the closedness of the positive cone we obtain the positivity of $R(\lambda, A)x$.

For the converse, we use the Euler formula, i.e., Theorem 1.2.11

$$T(t)x = \tau - \lim_{n\to\infty} \left[\tfrac{n}{t} R\big(\tfrac{n}{t}, A\big)\right]^n x$$

Again when $x \in X_+$, then as before we have $T(t)x \in X_+$. ∎

THEOREM 1.4.2 — *A bi-continuous semigroup T is positive, if and only if the corresponding C_0-semigroup T_0 is positive on X_0.*

PROOF — Follows by bi-denseness of X_0 and τ-closedness of X_+. ∎

§ 1.5 SPECTRAL PROPERTIES

For C_0-semigroups investigations on spectra are crucial for analysing long term behaviour or regularity properties of the semigroup. Being sufficient information on the spectrum of the generator at hand allows us to conclude many qualitative properties of the semigroup. In doing so, the spectral mapping theorem plays an essential role. We believe that further spectral observations on bi-continuous semigroups might help to better understand the spectral mapping theorem for C_0-semigroups, and perhaps identifying the missing parts in the approximate point spectrum (see Section 2.5.c). In this section, we only give two basic results on the point spectrum of a bi-continuous semigroup T and its generator $(A, D(A))$.

a) The point spectrum. The relation between the point spectrum of the bi-continuous semigroup T and the corresponding C_0-semigroup T_0 (and respectively their generators $(A, D(A))$ and $(A_0, D(A_0))$) is easy to settle.

THEOREM 1.5.1 — *The point spectrum of A and A_0 coincide. Similarly, $T(t)$ and $T_0(t)$ have the same eigenvalues.*

PROOF — The first assertion regarding the generators is trivial, since any eigenvector of A must lie in X_0.

Obviously, any eigenvector of $T_0(t)$ is an eigenvector of $T(t)$, so the inclusion

$$\sigma_{\mathrm{p}}(T_0(t)) \subseteq \sigma_{\mathrm{p}}(T(t)),$$

is not in question. Now, take $\lambda \in \sigma_{\mathrm{p}}(T(t))$ with eigenvector $x \in X$. We have to find $x' \in X_0$ such that $T(t)x' = \lambda x'$. Define

$$x'_s = \int_0^s T(u)x \,\mathrm{d}u.$$

It is then an element of $D(A)$ (cf. Theorem 1.2.7), and one can write

$$T(t)x'_s = T(t)\int_0^s T(u)x \,\mathrm{d}u = \int_0^s T(u)T(t)x \,\mathrm{d}u = \lambda \int_0^s T(u)x \,\mathrm{d}u = \lambda x'_s.$$

If $x'_s \neq 0$, then it is an eigenvector for the eigenvalue λ. Suppose that $x'_s = 0$ for all $s \geq 0$, i.e., for all $\varphi \in (X, \tau)'$ it holds that

$$\langle x'_s, \varphi\rangle = \int_0^s \langle T(s)x, \varphi\rangle \,\mathrm{d}s = 0, \quad \text{for all } s \geq 0.$$

This certainly would mean $x = 0$, which is not the case since $T(s)x$ is not constant 0. Hence for some $s > 0$ we have x'_s as the desired eigenvector. ∎

Referring to the spectral mapping theorem for the point spectrum in the C_0-case (cf. [31, Sec. IV.3.7]) gives immediately the following result. This can also be deduced directly, as in the case of C_0-semigroups, and thereafter one can conclude Theorem 1.5.1.

COROLLARY 1.5.2 — *The spectral mapping theorem for the point spectrum holds for arbitrary bi-continuous semigroups, that is*

$$\sigma_{\mathrm{p}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{p}}(A)}$$

b) Adjoint semigroups. We observe the relation between the generator $(A, D(A))$ of T and the generator of T°, which is denoted by $(A^\circ, D(A^\circ))$. For sufficiently large $\lambda \in \mathbb{R}$ the resolvents $R(\lambda, A)$ and $R(\lambda, A^\circ)$ are obtained via Laplace transformations:

$$R(\lambda, A)x = \int_0^{+\infty} \mathrm{e}^{\lambda t}T(t)x \,\mathrm{d}t,$$

where the integral is τ-convergent and the integrand is norm-bounded (λ large enough!). From this we infer that

$$\langle R(\lambda, A)x, \varphi\rangle = \int_0^{+\infty} \mathrm{e}^{\lambda t}\langle T(t)x, \varphi\rangle \,\mathrm{d}t = \int_0^{+\infty} \mathrm{e}^{\lambda t}\langle x, T^{\circ}(t)\varphi\rangle \,\mathrm{d}t = \langle x, R(\lambda, A^{\circ})\varphi\rangle$$

for $\varphi \in X^{\circ}$. Furthermore, by Theorem 1.2.8 we have that $R(\lambda, A)$ leaves X° invariant, thus

$$R(\lambda, A)^{\circ} = R(\lambda, A^{\circ}).$$

This equality, as one expects, yields the following result on the relation between point spectra of A and A°. For our purposes, it suffices this weak formulation. The stronger result, however, $\sigma(A) \subseteq \sigma(A^{\circ})$ or even the equality would be interesting.

THEOREM 1.5.3 — *Let* $\lambda \in \sigma_{\mathrm{p}}(A)$ *then* $\lambda \in \sigma(A^{\circ})$.

PROOF — We take $\mu \in \mathbb{R}$ sufficiently large such that $R(\mu, A)^{\circ} = R(\mu, A^{\circ})$. Then $\lambda \in \sigma(A^{\circ})$, if and only if $(\lambda - \mu)^{-1} \in \sigma(R(\mu, A^{\circ}))$. Let $0 \neq x \in X$ be an eigenvector of $R(\mu, A)$ for the eigenvalue $\lambda - \mu$. Then $(\lambda - \mu) - R(\lambda, A^{\circ})$ can not be surjective, since otherwise

$$0 = \langle(\lambda - \mu)x - R(\mu, A)x, \varphi\rangle = \langle x, ((\lambda - \mu) - R(\mu, A^{\circ}))\varphi\rangle = \langle x, \psi\rangle,$$

for all $\varphi \in X^{\circ}$ hence for all $\psi \in X^{\circ}$. This would mean $x = 0$, as X° separates X, which contradicts our assumption. ■

Chapter 2

Examples of bi-continuous semigroups

In this section, we present some examples of bi-continuous semigroups. We point out that the given examples, in most cases, do not belong to the C_0 class. First, we deal with adjoint semigroups, more precisely, semigroups on dual spaces. For a concise description of adjoint semigroups, we refer to [81]. In the second example are the semigroups on spaces of bounded, continuous functions. Among these we find the semigroups induced by non-linear flows. Such semigroups were studied by Dorroh and Neuberger in a series of papers [28], [29] and [30]. Also implemented semigroups on Banach algebras [2, 3] fit in the framework of bi-continuous semigroups [56, Sec. 3.4].

As possibly the most important examples of bi-continuous semigroups in this work, Ornstein – Uhlenbeck semigroups are also presented here. These can be also regarded as semigroups on the space of bounded, continuous functions, however we consider it also on the space of bounded, Hölder continuous functions as introduced in Section 2.3.b. The Ornstein – Uhlenbeck semigroups correspond to certain transition processes and are still subject of intensive research. The exact references will be given in the relevant section.

§ 2.1 Adjoint semigroups

It is shown in [56, Sec. 3.5] that the adjoint of a C_0-semigroup is bi-continuous with respect to the weak*-topology. We would like to study weak*-bi-continuous semigroups and show that in many cases they are just simply the adjoint of a C_0-semigroup. First, we remark that, if T is a weak*-continuous semigroup on X' with generator $(A, D(A))$ and X is invariant under the operators $T(t)'$, then T is the adjoint of a C_0-semigroup S. This holds since all weakly-continuous semigroupsare strongly continuous for the norm on X, and $T'|_X$ is weakly continuous (see [31, Thm. I.5.8]). To proceed, we need the following lemma.

Lemma 2.1.1 — *If X is a separable Banach space, then for every $K \subseteq X'$ weak*-compact set the topology $\sigma(X', X)|_K$ is metrisable. In particular, the unit ball $B(0, 1)$ of X' is metrisable for the weak*-topology.*

From this we infer that a $\sigma(X', X)$-bi-continuous semigroup T has the property that $T(t)$ is $\sigma(X', X)$-continuous on norm-bounded sets of X'.

LEMMA 2.1.2 — *Let $C \in \mathscr{L}(X')$ and suppose that C is weak*-continuous on norm-bounded sets. Then there exists $D \in \mathscr{L}(X)$ such that $D' = C$.*

PROOF — It is enough to show that $C' \in \mathscr{L}(X'')$ leaves X invariant. This follows if we prove that C has a weak*-adjoint

$$C'^{w^*} : X \to X$$

which, in this case, coincides with $C'|_X$. For $x \in X$ consider the linear form

$$\varphi_x(\cdot) := \langle C\cdot, x\rangle.$$

Then the restriction $\varphi_x|_B$ to the unit ball B in X' is weak*-continuous, so $\ker \varphi_x \cap B$ is weak*-closed. From the Kreĭn – Šmullian Theorem ([75, Cor. IV.6.4]) it follows that $\ker \varphi_x$ is weak*-closed, hence φ_x is weak*-continuous. Therefore C has a weak*-adjoint C'^{w^*} given by $C'^{w^*} x = \varphi_x$. ∎

An immediate consequence of this lemma and the previous remarks is the characterisation of weak*-bi-continuous semigroups.

THEOREM 2.1.3 — *Let T be a weak*-bi-continuous semigroup in X'. Suppose that either*

a) X is separable, or

b) T is local.

Then there exists a C_0-semigroup S on X with $S(t)' = T(t)$ for all $t \geq 0$.

§ 2.2 SEMIGROUPS ON $\mathrm{C_b}(\Omega)$

We are concerned with semigroups on $\mathrm{C_b}(\Omega)$ in this section. First we give an example of a space $\mathrm{C_b}(\Omega)$ that admits non-local bi-continuous semigroups. Later we will restrict ourself to the case when Ω is a Polish space, and see that the class of local and bi-continuous semigroups coincide.

In the example below we implement the usual set theoretical notation, thus ω_0 stands for the set of natural numbers as an ordinal number.

EXAMPLE 2.2.1 — Let $\Omega = \omega_1$ the first uncountable ordinal number and υ be the order topology. Since any $\alpha \in \omega_1$ has cofinality less than ω_0, we see that Ω is first countable. It is not hard to check that it is also normal. Suppose that $f_n \xrightarrow{\tau_c} 0$. We claim that there exists $\alpha \in \omega_1$ such that $f_n \to 0$ uniformly on $[\alpha, \omega_1)$. Suppose the contrary, i.e., for all $\alpha < \omega_1$ there exists $k \in \mathbb{N}$, $k > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ and $x \in [\alpha, \omega_1)$ with $|f_n(x)| > 1/k$. For all $\alpha \in \omega_1$ we have $k_\alpha \in \mathbb{N}$ and we may assume that $k_{\alpha_\xi} = k$ for a cofinal sequence $\alpha_\xi \in \omega_1$. By induction we choose a sequence

$$x_{\alpha_{\xi_1}} < x_{\alpha_{\xi_1}} \dot{+} 1 < x_{\alpha_{\xi_2}} < x_{\alpha_{\xi_2}} \dot{+} 1 < \cdots < x_{\alpha_{\xi_j}} < x_{\alpha_{\xi_j}} \dot{+} 1 < \cdots$$

with $f_{n_j}(x_{\alpha_{\xi_j}}) > 1/k$. Since $K := \{\lim_{j\to\infty} x_{\alpha_{\xi_j}},\ x_{\alpha_{\xi_j}} : j \in \mathbb{N}\}$ is compact and

$$\sup_{y\in K} |f_{n_j}(y)| \geq \tfrac{1}{k} \quad \text{for all } j \in \mathbb{N},$$

we arrived to a contradiction. Thus we have the existence of $\alpha \in \omega_1$ as asserted above. Now, consider the family $\{[\xi, \omega_1) : \xi > \alpha\}$, which has the finite intersection property and thus by compactness possesses an accumulation point $x \in \beta\Omega$. All f_n extends to the Stone – Čech compactification $\beta\Omega$ and $|f_n(y)| < \varepsilon$ for all $y \in [\alpha, \omega_1)$ if $n \geq N$. Take $n \in \mathbb{N}$, $n \geq N$. By the continuity of f_n on $\beta\Omega$ we have a neighbourhood U of x such that for all $y \in U$

$$|f_n(y) - f_n(x)| \leq \varepsilon.$$

There exist $\xi \in (\alpha, \omega_1)$ with $\emptyset \neq U \cap [\xi, \omega_1) \ni z$, so

$$|f_n(x)| \leq |f_n(x) - f_n(z)| + |f_n(z)| \leq \varepsilon + \varepsilon.$$

Thus $f_n(x) \to 0$, which shows that δ_x is τ_{c}-sequentially-continuous (on norm-bounded sets). However, it is clear that this is not τ_{c}-continuous on norm-bounded sets.

Consider now the C_0-semigroup T generated by the bounded operator $A := \mathbf{1} \otimes \delta_x$. Since A is idempotent the semigroup T takes the form

$$T(t) = I - A + \mathrm{e}^t A.$$

This semigroup is bi-continuous but not local since otherwise it would be τ_{c}-continuous on norm bounded sets.

From now on we assume that Ω is a Polish space. In Appendix A.2.b, we identify the space X° of linear functionals that are sequentially τ_{c}-continuous on norm-bounded sets of $\mathrm{C_b}(\Omega)$. It turns out that $X^\circ = \mathscr{M}(\Omega) \subseteq \mathscr{M}(\beta\Omega)$.

After these preparations, consider a norm-bounded operator T on $\mathrm{C_b}(\Omega)$ which is also τ_{c}-sequentially continuous on norm-bounded sets. By the above characterisation of $\mathscr{M}(\Omega)$, its adjoint $T' \in \mathscr{L}(\mathscr{M}(\beta\Omega))$ leaves $\mathscr{M}(\Omega)$ invariant (cf. Section 1.3.d).

LEMMA 2.2.2 — *Let $T : \mathbb{R}_+ \to \mathscr{L}(\mathrm{C_b}(\Omega))$ be a τ_{c}-strongly continuous function and $\mathcal{K} \subseteq \mathscr{M}(\Omega)$ a weak*-compact set. Then the map*

$$[0, +\infty) \times \mathcal{K} \ni (t, \nu) \mapsto T'(t)\nu \in \mathscr{M}(\beta\Omega)$$

is continuous if we take the weak-topology $\sigma(\mathscr{M}(\beta\Omega), \mathrm{C_b}(\Omega))$ on $\mathcal{K}$ and on $\mathscr{M}(\beta\Omega)$.*

PROOF — Let $t \in [0, +\infty)$ and $\nu \in \mathcal{K}$ fixed. Then

$$\begin{aligned} &|\langle f, T'(t)\nu - T'(t')\nu'\rangle| = |\langle T(t)f, \nu\rangle - \langle T(t')f, \nu'\rangle| \leq \\ &\quad |\langle T(t)f, \nu - \nu'\rangle| + |\langle T(t)f - T(t')f, \nu'\rangle| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{for all } f \in \mathrm{C_b}(\Omega). \end{aligned}$$

The last inequality follows by the continuity assumptions and by Prokhorov's theorem (see Theorem A.2.5) for $t' \in U$ and $\nu' \in V$ with appropriate neighbourhoods U, V of t and ν. ∎

LEMMA 2.2.3 — *Let $T:[0,+\infty)\to\mathscr{L}(\mathrm{C_b}(\Omega))$ be a τ_c-strongly continuous function. For a weak*-compact set $\mathcal{K}\subseteq\mathscr{M}(\Omega)$ and $t_0>0$ the set of measures*

$$\{T'(t)\nu : t\in[0,t_0],\ \nu\in\mathcal{K}\}$$

is $\sigma(\mathscr{M}(\beta\Omega),\mathrm{C_b}(\Omega))$-compact.

PROOF — It is a straightforward consequence of Lemma 2.2.2 and the fact that the set $[0,t_0]\times\mathcal{K}$ is compact for the product topology. ∎

LEMMA 2.2.4 — *Let $T:\mathbb{R}_+\to\mathscr{L}(\mathrm{C_b}(\Omega))$ be a τ_c-strongly continuous semigroup consisting of operators that are τ_c-continuous on norm-bounded sets. For a norm-bounded, weak*-compact set $\mathcal{K}\subseteq\mathscr{M}(\Omega)$ and $t_0>0$ the set of measures*

$$\{T'(t)\nu : t\in[0,t_0],\ \nu\in\mathcal{K}\}$$

is tight.

PROOF — The assertion follows immediately from Lemma 2.2.3, taking into account the arguments preceding Lemma 2.2.2 concerning the invariance of $\mathscr{M}(\Omega)$. ∎

In Appendix §A.2 it is shown that a linear operator that is τ_c-sequentially continuous, is also τ_c-continuous on norm bounded sets. Combining this with the previous lemma we obtain:

THEOREM 2.2.5 — *Let $T:\mathbb{R}_+\to\mathscr{L}(\mathrm{C_b}(\Omega))$ be τ_c-strongly continuous and norm-bounded. Suppose that $T(t)$ takes norm-bounded τ_c-null sequences into τ_c-null sequences. Then $T|_{[0,t_0]}$ is local for all $t_0>0$, in particular it is bi-equicontinuous.*

PROOF — Let $\varepsilon>0$, $t_0>0$ and $K\subseteq\Omega$ be a compact set. Take a compact set $K'\subseteq\Omega$ such that $|T'\delta_x|(\Omega\setminus K')\le\varepsilon$ for all $t\in[0,t_0]$ and $x\in K$. Such compact set exists by Lemma 2.2.4. We then obtain

$$\begin{aligned}\sup_{x\in K}|T(t)f(x)| &= \sup_{x\in K}\left|\int_\Omega f\,\mathrm{d}T'(t)\delta_x\right| \le\\ &\le \sup_{x\in K}\int_{K'}|f|\,\mathrm{d}|T'(t)\delta_x| + \sup_{x\in K}\int_{\Omega\setminus K'}|f|\,\mathrm{d}|T'(t)\delta_x| \le\\ &\le \sup_{t\in[0,t_0]}\|T(t)\|\cdot\sup_{x\in K'}|f(x)| + \varepsilon\|f\|,\end{aligned}$$

which is the assertion. ∎

We conclude this section by an important result on bi-continuous semigroups, which is partly just the reformulation of the latter theorem. We refer to Appendix A.2.b for the definition of β_0.

THEOREM 2.2.6 — *Let T be a bi-continuous semigroup on $\mathrm{C_b}(\Omega)$. Then it is local and locally-equicontinuous for β_0. Conversely, every β_0-strongly continuous and locally β_0-equicontinuous semigroup is bi-continuous.*

PROOF — The last assertion follows from Appendix §A.2. ∎

§ 2.3 ORNSTEIN – UHLENBECK SEMIGROUPS

In this section, we define the Ornstein – Uhlenbeck semigroups P on an infinite dimensional, separable Hilbert space H. These semigroups arise from stochastic differential equations and Ornstein – Uhlenbeck processes, and are in strong connection with Kolmogorov equations [22, 23], [53]. There are many different approaches to these semigroups. It is possible to define them on the space $M_b(H)$ of bounded, measurable functions on H. First of all, we mention the case when Ornstein – Uhlenbeck semigroups are of C_0 class. In certain cases, there exists an invariant measure for P, i.e., a Borel probability measure μ on H, such that

$$\int_H f \,\mathrm{d}\mu = \int_H P(t) f \,\mathrm{d}\mu$$

holds for all $t \geq 0$. In this situation, one defines the $\mathrm{L}^2(H, \mu)$ space and extends the semigroup P defined on $M_b(H)$ to the whole space. The advantage is then that P becomes a C_0-semigroup with respect to the L^2-norm. There are however a couple of drawbacks of this method. The existence of invariant measures is not always established, and it is more natural to consider the semigroup P on spaces of continuous functions. This approach is carried out by several authors, see e.g., [59], [60], [63] and [74]. Rhandi [74] applies perturbation techniques and obtains further transition semigroups on $\mathrm{L}^2(H, \mu)$. In [63], the generator of P is identified on $\mathrm{L}^p(\mathbb{R}^n, \mu)$, and it is indeed a non-trivial result using several deep and serious tools.

Another possibility is to work on $\mathrm{C}_{\mathrm{ub}}(H)$ or $\mathrm{C}_{\mathrm{b}}(H)$. It is shown in [84] (see also [27], [44],[42]) that the Ornstein – Uhlenbeck semigroup is not of C_0 class in all non-trivial cases (even when H is finite dimensional). Again there are two directions. First, one tries to work on the subspace of strong-continuity. Da Prato and his school ([10], [16, 17, 18, 19] and [21]) follows this approach, defines the generator L via Laplace transformation and resolvent, then proves the dissipativity of L. This results in a contractive C_0-semigroup on the closure of $D(L)$, and the powerful theory of C_0-semigroups is applicable. However, one should keep in mind that the domain of L is in general unknown apart form abstract characterisations. Nevertheless, it is also possible to obtain perturbation or approximation results. Secondly, one can relax the strong continuity assumption of P, considering various new topologies on $\mathrm{C}_{\mathrm{ub}}(H)$ [69] or on $\mathrm{C}_{\mathrm{b}}(H)$ [41]. Priola [69, 70, 71, 72] uses the notion of weakly-integrable semigroups of Jefferies [46, 47], and utilises the topology of pointwise convergence τ_{p} instead of τ_{c}, and defines the so-called π-semigroups. As explained well before, this method of introducing new topologies is what we would like to follow. It is proved by Kühnemund in [54, Sec. 3.3] that the Ornstein – Uhlenbeck semigroup is bi-continuous with respect to τ_{c}. In view of our results in Section §A.2, this restates the ones obtained by of Goldys and Kocan in [41], whereas they also work on spaces of polynomially increasing functions. We also consider the Ornstein – Uhlenbeck semigroup on $\mathrm{C}^{\theta}_{\mathrm{b}}(H)$. Similar results were presented by Da Prato [19] on $\mathrm{C}^{\theta}_{\mathrm{ub}}(H)$ using dissipativity arguments as described above. Finally, we remark that is also possible to consider Ornstein – Uhlenbeck semigroups on proper subsets of Hilbert spaces. Such results were obtained, e.g., by Da Prato, Goldys and Zabczyk [20], and

Priola [69, 71, 72] in the π-semigroup setting. The bi-continuous semigroup approach is also appropriate for such investigations and is a part of future research.

a) On the space $\mathrm{C_b}(H)$**.** Let $(A, D(A))$ be the generator of a $\mathrm{C_0}$-semigroup S on a separable Hilbert space H. Further let Q be a bounded, positive, self-adjoint operator Q, and define

$$Q(t) = \int_0^t S(s)QS^*(s)\,\mathrm{d}s,$$

where the integral is understood in the strong sense. Assume additionally that

$$Q(t) \text{ is of trace class for all } t \geq 0.$$

This also ensures that

$$\operatorname{Tr} Q(t) \to 0 \quad \text{as } t \to 0. \tag{2.1}$$

For all $x \in H$ and $t \geq 0$ there exists a Gaussian measure $\mathcal{N}(S(t)x, Q(t))$ with mean $S(t)x$ and covariance operator $Q(t)$ (see [16], [58] and [79]), i.e.,

$$\begin{aligned}
\langle S(t)x, y\rangle &= \int_H \langle \cdot, y\rangle\,\mathrm{d}\mathcal{N}(S(t)x, Q(t)) \quad \text{and} \\
\langle Q(t)y, y\rangle &= \int_H |\langle \cdot - S(t)x, y\rangle|^2\,\mathrm{d}\mathcal{N}(S(t)x, Q(t))
\end{aligned}$$

for all $x, y \in H$. Let $f \in M_b(H)$ and define

$$\begin{aligned}
[P(0)f](x) &:= f(x) \\
[P(t)f](x) &:= \int_H f\,\mathrm{d}\mathcal{N}(S(t)x, Q(t)), \quad t > 0.
\end{aligned} \tag{2.2}$$

We can rewrite this formula for $t > 0$ as follows

$$[P(t)f](x) := \int_H f(\cdot + S(t)x)\,\mathrm{d}\mathcal{N}(0, Q(t)).$$

Clearly $P(t)$ is a linear operator. It is well known that P is a semigroup on $M_b(H)$; it is called the *Ornstein – Uhlenbeck semigroup*.

It is not hard to show that $\mathrm{C_b}(H) \subseteq M_b(H)$ is invariant under $P(t)$ for each $t \geq 0$. Take $t \geq 0$ and $f \in \mathrm{C_b}(H)$ and let $x_n \to x \in H$, then $S(t)x_n \to S(t)x$ and

$$|[P(t)f](x) - [P(t)]f(x_n)| \leq \int_H |f(\cdot + S(t)x) - f(\cdot + S(t)x_n)|\,\mathrm{d}\mathcal{N}(0, Q(t)) \to 0,$$

since the integrand is bounded and converges to 0 pointwise, hence Lebesgue's convergence theorem applies. Note that $P(t)$ is contractive for all $t \geq 0$.

Now, we show that P is indeed a bi-continuous semigroup with respect to the compact-open topology τ_c. By Theorem 2.2.6 it is enough to show that $t \mapsto P(t)f$ is τ_c-continuous and $P(t) \in \mathscr{L}(\mathrm{C_b}(H))$ is sequentially τ_c-continuous on norm-bounded sets. So take $f_n \in \mathrm{C_b}(H)$ a norm-bounded sequence with $f_n \xrightarrow{\tau_c} 0$. Let $\varepsilon > 0$ and $K \subseteq H$ be an arbitrary compact set. By the regularity of $\mathcal{N}(0, Q(t))$ we have $K' \subseteq H$ compact such that $\mathcal{N}(0, Q(t))(H \setminus K')$. Then

$$\sup_{x\in K} |[P(t)]f_n(x)| \leq \sup_{x\in K} \int_H |f_n(\cdot + S(t)x)| \,\mathrm{d}\mathcal{N}(0, Q(t)) =$$

$$= \sup_{x\in K} \int_{K'} |f_n(\cdot + S(t)x)| \,\mathrm{d}\mathcal{N}(0, Q(t)) + \sup_{x\in K} \int_{H\setminus K'} |f_n(\cdot + S(t)x)| \,\mathrm{d}\mathcal{N}(0, Q(t)) \leq$$

$$\leq \sup_{x\in K+S(t)K'} |f_n(x)| + \varepsilon \sup_{n\in\mathbb{N}} \|f_n\| \leq C\varepsilon,$$

when n is large, since $K + S(t)K'$ is compact. Next we show the τ_c-strong continuity of $P(t)$. So take $f \in \mathrm{C_b}(H)$ and $K \subseteq H$ compact set. Define $K' := \bigcup\{P(t)K : t \in [0, 1]\}$, which is also compact. Then similarly as in Heine's theorem for uniformly continuous functions, one can show that there exists a $\delta := \delta(K', \varepsilon)$ such that for all $y \in K'$ we have $|f(y+z) - f(y)| \leq \varepsilon$ whenever $\|z\| \leq \delta$. Further $t \mapsto S(t)$ is uniformly strongly continuous on K. Hence, if $t > 0$ is sufficiently small $\|S(t)x - x\| \leq \delta/2$, thus $|f(x + S(t)x - x + z) - f(x)| \leq \varepsilon$ for all $x \in K$, $z \in B(0, \delta/2)$ and t sufficiently small. Thus with $B := B(0, \delta/2)$ we can write

$$\sup_{x\in K} |[P(t)f](x) - f(x)| \leq \sup_{x\in K} \int_H |f(\cdot + S(t)x) - f(x)| \,\mathrm{d}\mathcal{N}(0, Q(t)) =$$

$$= \sup_{x\in K} \int_B |f(\cdot + S(t)x) - f(x)| \,\mathrm{d}\mathcal{N}(0, Q(t)) +$$

$$+ \sup_{x\in K} \int_{H\setminus B} |f(\cdot + S(t)x) - f(x)| \,\mathrm{d}\mathcal{N}(0, Q(t)) \leq$$

$$\leq \varepsilon + \frac{8\mathrm{Tr}\, Q(t)}{\delta^2}\|f\|.$$

The last estimate on the integral outside the ball $B(0, \delta/2)$ holds by the Chebyshev inequality. By (2.1) we obtain that right hand side is small whenever $t \leq t_0$, this proves the τ_c-strong continuity of $P(t)$. Therefore we arrive at:

THEOREM 2.3.1 — *The Ornstein – Uhlenbeck semigroup is a local bi-continuous semigroup on* $\mathrm{C_b}(H)$.

b) On the space $\mathrm{C}^\theta_\mathrm{b}(H)$. We show that the Ornstein – Uhlenbeck semigroup leaves the space $\mathrm{C}^\theta_\mathrm{b}(H)$ invariant and it is also bi-continuous for the topology $\tau_c{}^\theta$. The proofs are similar and strongly rely on the case of $\mathrm{C_b}(H)$. These results appear in [32].

PROPOSITION 2.3.2 — *Let $f \in \mathrm{C}^\theta_\mathrm{b}(H)$ then for all $t \geq 0$ the function $P(t)f$ belongs to $\mathrm{C}^\theta_\mathrm{b}(H)$.*

PROOF — For arbitrary $x, x' \in H$ we have the following estimate

$$\begin{aligned} |[P(t)f](x) - [P(t)f](x')| &\leq \int_H |f(\cdot + S(t)x) - f(\cdot + S(t)x')| \,\mathrm{d}\mathcal{N}(0,Q(t)) \leq \\ &\leq \int_H [f]_\theta \cdot \|S(t)x - S(t)x'\|^\theta \,\mathrm{d}\mathcal{N}(0,Q(t)) \leq [f]_\theta \cdot \|S(t)\|^\theta \cdot \|x-x'\|^\theta. \end{aligned} \tag{2.3}$$

Let $K \subseteq H$ be a compact set. For a given $\varepsilon > 0$ take $K_1 \subseteq H$ compact in accordance with the tightness of the family $\{\mathcal{N}(0,Q(t)) : t \in [0,t_0]\}$ (cf. Lemma 2.2.4). Define moreover

$$K_2 := \{x : \exists t \in [0,t_0],\ x \in S(t)K + K_1\}.$$

Then K_2 is also compact, and we can write

$$\begin{aligned} |[P(t)f](x) - [P(t)f](x')| &\leq \int_{K_1} |f(\cdot + S(t)x) - f(\cdot + S(t)x')| \,\mathrm{d}\mathcal{N}(0,Q(t)) + \\ &+ \int_{H\setminus K_1} |f(\cdot + S(t)x) - f(\cdot + S(t)x')| \,\mathrm{d}\mathcal{N}(0,Q(t)) \leq \\ &\leq \int_{K_1} \varepsilon \cdot \|S(t)x - S(t)x'\|^\theta \,\mathrm{d}\mathcal{N}(0,Q(t)) + \int_{H\setminus K_1} [f]_\theta \cdot \|S(t)x - S(t)x'\|^\theta \,\mathrm{d}\mathcal{N}(0,Q(t)) \leq \\ &\leq \varepsilon \cdot \|S(t)\|^\theta \cdot \|x-x'\|^\theta + \varepsilon \cdot [f]_\theta \cdot \|S(t)\|^\theta \cdot \|x-x'\|^\theta, \end{aligned} \tag{2.4}$$

whenever $\|x - x'\| \leq \delta(\varepsilon)$. The equations (2.3) and (2.4) together show the invariance of $\mathrm{C}_\mathrm{b}^\theta(H)$ under $P(t)$. ∎

THEOREM 2.3.3 — *The Ornstein – Uhlenbeck semigroup defined by* (2.2) *is a bi-continuous semigroup on* $\mathrm{C}_\mathrm{b}^\theta(H)$.

PROOF — By similar reasonings as in (2.4), we obtain that

$$\begin{aligned} |[P(t)f](x) - [P(t)f](x')| &\leq \int_{K_1} |f(\cdot + S(t)x) - f(\cdot + S(t)x')| \,\mathrm{d}\mathcal{N}(0,Q(t)) + \\ &+ \int_{H\setminus K_1} |f(\cdot + S(t)x) - f(\cdot + S(t)x')| \,\mathrm{d}\mathcal{N}(0,Q(t)) \leq \\ &\leq \int_{K_1} p_{K_2,\theta}(f) \cdot \|S(t)x - S(t)x'\|^\theta \,\mathrm{d}\mathcal{N}(0,Q(t)) + \\ &+ \int_{H\setminus K_1} [f]_\theta \cdot \|S(t)x - S(t)x'\|^\theta \,\mathrm{d}\mathcal{N}(0,Q(t)) \leq \\ &\leq (p_{K_2,\theta}(f) + \varepsilon \cdot [f]_\theta) \cdot \|S(t)\|^\theta \cdot \|x-x'\|^\theta. \end{aligned} \tag{2.5}$$

The locality of $P(t)$ with respect to τ_c and (2.5) together establishes the locality and in particular the bi-equicontinuity of the family $\{P(t) :\ t \in [0,t_0]\}$ also for the topology τ_c^θ.

From (2.3) and from the contractivity of $P(t)$ on $\mathrm{C_b}(H)$, it follows that the $t \mapsto P(t)$ is also locally norm-bounded on $\mathrm{C}^\theta_\mathrm{b}(H)$. In particular, it is contractive if S is contractive. We show that the orbits $t \mapsto P(t)f$ are τ^θ_c-strongly continuous for all $f \in \mathrm{C}^\theta_\mathrm{b}(H)$. The τ_c-strong continuity was shown already in Section §2.3, thus it suffices to show the continuity of the orbits $t \mapsto P(t)f$ for all seminorms $p_{K,\theta}$ with $K \subseteq H$ compact. Take $f \in \mathrm{C}^\theta_\mathrm{b}(H)$ arbitrary and let $K \subseteq H$ compact. Then as above we have the compact sets K_1, $K_2 \subseteq H$. Define the compact set $K_3 := K \cup K_1 \cup K_2$. For $\varepsilon > 0$ let $\delta(f, K_3, \varepsilon)$ sufficiently small obtained from the Hölder continuity of f (see example iii) on page 4) . Further take $\delta > 0$ and $t' > 0$ such that

$$\|y + S(t)x - x\| \leq \delta(f, K_3, \varepsilon) \quad \text{for all } \|y\| \leq \delta,\, t \in [0, t'] \text{ and } x \in K_3.$$

Thus we obtain for arbitrary $x, x' \in K$

$$\begin{aligned}
&|[P(t)f](x) - f(x) - [P(t)f](x') + f(x')| \leq \\
&\quad \leq \int_H |f(\cdot + S(t)x) - f(x) - f(\cdot + S(t)x') + f(x')| \,\mathrm{d}\mathcal{N}(0, Q(t)) = \\
&\quad = \int_{B(0,\delta)\cap K_1} |f(\cdot + S(t)x) - f(x) - f(\cdot + S(t)x') + f(x')| \,\mathrm{d}\mathcal{N}(0, Q(t)) + \\
&\qquad + \int_{K_1 \backslash B(0,\delta)} |f(\cdot + S(t)x) - f(x) - f(\cdot + S(t)x') + f(x')| \,\mathrm{d}\mathcal{N}(0, Q(t)) + \\
&\qquad + \int_{H \backslash K_1} |f(\cdot + S(t)x) - f(x) - f(\cdot + S(t)x') + f(x')| \,\mathrm{d}\mathcal{N}(0, Q(t)) \leq \\
&\quad \leq I(t, x, x') + [f]_\theta(\|S(t)\|^\theta + 1)\|x - x'\|^\theta \frac{\operatorname{Tr} Q(t)}{\delta^2} + \varepsilon [f]_\theta(\|S(t)\|^\theta + 1)\|x - x'\|^\theta.
\end{aligned}$$

Now, there are two possibilities. When $\|x - x'\| \geq \delta(f, K_3, \varepsilon)$ then we have

$$\begin{aligned}
I(t, x, x') &:= \int_{B(0,\delta)\cap K_1} |f(\cdot + S(t)x) - f(x) - f(\cdot + S(t)x') + f(x')| \,\mathrm{d}\mathcal{N}(0, Q(t)) \leq \\
&\leq \int_{B(0,\delta)\cap K_1} |f(\cdot + S(t)x) - f(x)| + |f(\cdot + S(t)x') + f(x')| \,\mathrm{d}\mathcal{N}(0, Q(t)) \leq \\
&\leq 2\varepsilon \delta(f, K_3, \varepsilon)^\theta \leq 2\varepsilon \|x - x'\|^\theta.
\end{aligned}$$

On the other hand, when $\|x - x'\| \leq \delta(f, K_3, \varepsilon)$ we obtain

$$\begin{aligned}
I(t, x, x') &:= \int_{B(0,\delta)\cap K_1} |f(\cdot + S(t)x) - f(x) - f(\cdot + S(t)x') + f(x')| \,\mathrm{d}\mathcal{N}(0, Q(t)) \leq \\
&\leq \int_{B(0,\delta)\cap K_1} |f(\cdot + S(t)x) - f(\cdot + S(t)x')| + |f(x') - f(x)| \,\mathrm{d}\mathcal{N}(0, Q(t)) \leq \\
&\leq \varepsilon \|S(t)x - S(t)x'\|^\theta + \varepsilon \|x - x'\|^\theta \leq \varepsilon(\|S(t)\|^\theta + 1)\|x - x'\|^\theta.
\end{aligned}$$

Combining these estimates, we see that for sufficiently small $t'' > 0$ one has

$$|(P(t)f)(x) - f(x) - (P(t)f)(x') + f(x')| \leq \varepsilon \|x - x'\|^\theta$$

for all $x, x' \in K$ and $t \in [0, t'']$. This finishes the proof. ∎

§ 2.4 THE HEAT SEMIGROUP

We would like to deal with the heat semigroup generated by the Dirichlet Laplacian on unbounded domains $\Omega \subseteq \mathbb{R}^N$. The existence of an analytic C_0-semigroup is known on $\mathrm{C}_0(\Omega)$. We would like to drop the "boundary condition at infinity", and show the existence of bi-continuous semigroups and therefore establish the well-posedness of the corresponding parabolic problem (cf. Section 4.1.a).

The result presented in this section, was obtained during the joint work with A. Bátkai, D. Mugnolo and S. Piazzera as a side result. Let $\Omega \subseteq \mathbb{R}^N$ be non-empty, possibly unbounded open set. The following result is well-known and is taken in the present form from [5, Sec. 6.1].

THEOREM 2.4.1 — *The Dirichlet Laplacian Δ_2 considered on $\mathrm{L}^2(\Omega)$ in the variational sense*[1] *generates a positive, analytic, C_0-semigroup T_2. Moreover, the semigroup T_2 is given by a kernel $k_t(x, y)$ satisfying*

$$0 \leq k_t(x, y) \leq g_t(x, y) \quad \textit{for all } t \geq 0, x, y \in \Omega,$$

where $g_t(x, y)$ denotes the heat kernel on $\mathbb{R}^N$

$$g_t(x, y) := \frac{1}{(4\pi t)^{N/2}} \mathrm{e}^{\frac{|x-y|^2}{4t}}.$$

We recall the following result from [6], which shows that on nice domains Ω it is also possible to consider the Dirichlet Laplacian and it generates a C_0-semigroup on $\mathrm{C}_0(\Omega)$. Moreover, there is consistency between this semigroups and the one on $\mathrm{L}^2(\Omega)$.

THEOREM 2.4.2 — *If Ω is Wiener-regular*[2]*, then the Dirichlet Laplacian Δ_0 generates a C_0-semigroup T_0 on $\mathrm{C}_0(\Omega)$. Further,*

$$T_0(t)f = T_2(t)f \quad \textit{for } t \geq 0 \textit{ and } f \in \mathrm{C}_0(\Omega) \cap \mathrm{L}_2(\Omega).$$

The Dirichlet Laplacian Δ_0 is understood as the maximal distributional Laplacian on $\mathrm{C}_0(\Omega)$, i.e.,

$$D(\Delta_0) := \{f : \ f \in \mathrm{C}_0(\Omega),\ \Delta f \in \mathrm{C}_0(\Omega)\}.$$

These two results will serve as a tool to construct the heat semigroup on a large subspace of $\mathrm{C}_\mathrm{b}(\Omega)$.

[1] Obtained via the representation of closed symmetric forms.

[2] Wiener regularity is a geometric property, which is possessed by a domain Ω, if and only if the Dirichlet Laplacian generates an analytic semigroup on $\mathrm{C}_0(\Omega)$ (see [6]).

Define the Banach space

$$\mathrm{C}_{\mathrm{b},0}(\bar{\Omega}) := \left\{f : \ f \in \mathrm{C}_{\mathrm{b}}(\bar{\Omega}), \ f|_{\partial\Omega} = 0\right\}$$

endowed with the supremum norm and also with the subspace topology τ_{c} inherited from $\mathrm{C}_{\mathrm{b}}(\bar{\Omega})$. With the kernel $k_t(x,y)$ it is possible to define a semigroup T on $\mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ by

$$T(t)f(x) := \int\limits_{\Omega} f(y)k_t(x,y)\,\mathrm{d}y, \quad t > 0.$$

It is not hard to see that it is indeed a semigroup. From Theorem 2.4.2 we see that $T(t)f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ for all $f \in \mathrm{C}_c(\Omega)$. Let $f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ and take $\varepsilon > 0$, $K \subseteq \bar{\Omega}$ compact and $t_0 > 0$. Then we can write

$$\begin{aligned}\sup_{x\in K}|T(t)f(x)| &\leq \sup_{x\in K}\int\limits_{\Omega}|f(y)|k_t(x,y)\,\mathrm{d}y \leq \sup_{x\in K}\int\limits_{\Omega}|f(y)|g_t(x,y)\,\mathrm{d}y \leq \\ &= \sup_{x\in K}\int\limits_{\Omega\cap K'}|f(y)|g_t(x,y)\,\mathrm{d}y + \sup_{x\in K}\int\limits_{\Omega\setminus K'}|f(y)|g_t(x,y)\,\mathrm{d}y \leq p_{\bar{\Omega}\cap K'}(f) + \varepsilon\|f\|,\end{aligned}$$

for all $t \in (0, t_0]$, with an appropriate $K' \subseteq \mathbb{R}^N$ compact. This shows that $T(t) : \mathrm{C}_{\mathrm{b},0}(\bar{\Omega}) \to \mathrm{C}_{\mathrm{b}}(\bar{\Omega})$ is τ_{c}-continuous on norm bounded sets for all $t > 0$, thus $T(t)$ must leave $\mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ invariant. Also it is immediate that the semigroup T is local in the sense of Definition 1.2.21. Clearly $T(t)$ is a contraction for all $t \geq 0$. We show that $t \mapsto T(t)f$ is τ_{c}-continuous for all $f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$. Note that $\mathrm{C}_c(\Omega)$ is τ_{c}-bi-dense in $\mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ and that for $f \in \mathrm{C}_c(\Omega)$ the orbit $t \mapsto T(t)f$ is norm-continuous by Theorem 2.4.2. For $f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ take f_n norm-bounded and τ_{c}-convergent to f. Let $K \subseteq \bar{\Omega}$ compact and $\varepsilon > 0$. Then

$$p_K(T(t)f - f) \leq p_K(T(t)f - T(t)f_n) + p_K(T(t)f_n - f_n) + p_K(f_n - f) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},$$

by taking $n \in \mathbb{N}$ sufficiently large and making use of the local bi-equicontinuity of $T(t)$ and the norm continuity of $t \mapsto T(t)f_n$. We summarise the previous results in the following theorem.

THEOREM 2.4.3 — *The semigroup T defined by*

$$[T(t)f](x) := \int\limits_{\Omega} f(y)k_t(x,y)\,\mathrm{d}y$$

on $\mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ is a bi-continuous semigroup.

For $t > 0$ one has $T(t)f \in \mathrm{C}_{\mathrm{ub}}(\bar{\Omega})$. This shows that the semigroup T is not a C_0-semigroup.

Now we identify the generator $(A, D(A))$ of T. Since on $\mathrm{C}_0(\Omega)$ the semigroup T is strongly continuous for the norm, we have that $D(\Delta_0) \subseteq D(A)$. Clearly $D(\Delta_0)$ is invariant under $T(t)$. Since $D(\Delta_0)$ is norm-dense in $\mathrm{C}_0(\Omega)$ and $\mathrm{C}_0(\Omega)$ is bi-dense in $\mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$ it follows that $D(\Delta_0)$ is bi-dense in $\mathrm{C}_{\mathrm{b},0}(\bar{\Omega})$. By Proposition 1.2.9 we obtain that $D(\Delta_0)$ is bi-core for $(A, D(A))$. Define

$$D(\Delta) := \left\{f : \ f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega}), \ \Delta f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega})\right\},$$

where the Laplace operator Δ is understood in the distributional sense. It is not hard to see that $(\Delta, D(\Delta))$ is bi-closed (for the definition see Theorem 1.2.7). Since $\mathrm{C}_c^\infty(\Omega) \subseteq D(\Delta_0)$, and every $f \in D(\Delta)$ can be approximated in τ_c by a norm bounded sequence $f_n \in \mathrm{C}_c^\infty(\Omega)$ with $Af_n = \Delta_0 f_n \xrightarrow{\tau_c} \Delta f$, we see that $D(\Delta_0)$ is a bi-core for $(\Delta, D(\Delta))$. Moreover A coincides with Δ on $D(\Delta_0)$, hence we conclude the following theorem.

THEOREM 2.4.4 — *The generator of T is the distributional Laplacian with maximal domain, i.e., $(A, D(A)) = (\Delta, D(\Delta))$, with*

$$D(\Delta) := \{f : \ f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega}), \ \Delta f \in \mathrm{C}_{\mathrm{b},0}(\bar{\Omega})\}.$$

It would de nice to extend these results to other elliptic operators, and we believe that with sufficiently nice coefficients of our elliptic operator this can be easily done.

§ 2.5 FURTHER EXAMPLES

Without going into the details, we briefly collect some more examples of bi-continuous semigroups, such as semigroups induced by jointly continuous flows, implemented semigroups and infinite direct powers.

a) Induced semigroups. Let Ω be a Polish space. Consider a (semi)-flow

$$\Phi : \mathbb{R}_+ \times \Omega \to \Omega,$$

i.e., $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $t, s \geq 0$ and $x \in \Omega$. Assume that it is jointly continuous. Then Dorroh and Neuberger [28, 29] and [30] defines the induced linear flow (semigroup) T on $\mathrm{C}_\mathrm{b}(\Omega)$

$$[T(t)f](x) := f(\Phi(t, x)).$$

Using the joint continuity, they show that this is a β_0-strongly continuous semigroup which is locally β_0-equicontinuous (for the definition of β_0 see Section A.2.b). In our terminology, this means that T is a bi-continuous semigroup with respect to τ_c. Also the generator of T can be identified.

DEFINITION 2.5.1 — The *Lie generator* of a jointly continuous flow is the linear operator A on $\mathrm{C}_\mathrm{b}(\Omega)$ with

$$Af := \lim_{t \to 0} \frac{f(\Phi(t, x)) - f(x)}{t}, \quad \text{whenever the limit exists.}$$

THEOREM 2.5.2 — *For a linear operator $(A, D(A))$ on $\mathrm{C}_\mathrm{b}(\Omega)$ the following are equivalent.*

a) $(A, D(A))$ is the generator of a jointly continuous flow.

b) $(A, D(A))$ is a derivation[3] *and generates a bi-continuous semigroup with respect to τ_c induced by a jointly continuous flow.*

[3] $f, g \in D(A)$ implies that $f \cdot g \in D(A)$ and $A(f \cdot g) = Af \cdot g + f \cdot Ag$.

REMARK 2.5.3 — Note that, for example, the left shift semigroup S

$$[S(t)f](s) := f(t+s),$$

on $\mathrm{C_b}(\mathbb{R})$ or $\mathrm{C_b}(\mathbb{R}_+)$ is an induced semigroup according to this terminology. It is also clear that on $\mathrm{C_b}(\mathbb{R}_+, X)$ with X Banach space we have again the shift semigroup as a bi-continuous semigroup with respect to τ_c.

b) Implemented semigroups. Let T and S be C_0-semigroups on a Banach space X. Consider $\mathscr{L}(X)$ endowed also with the strong operator topology τ_{op}. It is obvious that Assumption 1.2.1 is satisfied. The *implemented semigroup* U is defined by

$$U(t) := L_{S(t)} R_{T(t)},$$

where L and R stand for the left and right multiplication respectively [2, 3] and [56, Sec. 3.4]. It is easy to see that U is a semigroup on $\mathscr{L}(X)$ which is bi-continuous with respect to τ_{op}. Further it is rarely of C_0 class, e.g., when S and T are uniformly continuous.

c) Infinite direct powers. Let T be a C_0-semigroup on the Banach space X with generator $(A, D(A))$. Consider $\ell^\infty(X)$ equipped with the supremum norm and the product topology τ_{p}. It is natural to define $\tilde{T}$ on $\ell^\infty(X)$ acting pointwise, i.e.,

$$\tilde{T}(t)(x_0, x_1, \dots, x_k, \dots) := (T(t)x_0, T(t)x_1, \dots, T(t)x_k, \dots).$$

Such construction was observed for investigating the spectral mapping theorem of C_0-semigroups. The *critical spectrum* is introduced in this setting [67]. Let us briefly summarise the ideas.

One knows that the spectral mapping theorem holds for the point spectrum and the residual spectrum for C_0-semigroups:

$$\sigma_{\mathrm{p}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{p}}(A)} \quad \text{and} \quad \sigma_{\mathrm{r}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{r}}(A)}.$$

For a complete spectral mapping theorem, it suffices always to show that it holds for the approximate point spectrum, since these three parts together amount to the whole spectrum. Then one tries to look at where the problematic part of the approximate point spectrum lies. In the above construction, one factorises with $c_0(X)$ and obtains the factor semigroup $\hat{T}$ on $\ell^\infty(X)/c_0(X)$. Similarly one defines $\tilde{A}$ and $\hat{A}$. Clearly the approximate spectral points of $T(t)$ and A are exactly the eigenvalues of $\hat{T}$ and $\hat{A}$ respectively. If $\hat{T}$ was a C_0-semigroup, then it would be possible to prove the spectral mapping theorem for the approximate point spectrum of T, since in this case $\hat{A}$ must be the generator of $\hat{T}$. But $\tilde{T}$ and $\hat{T}$ are not C_0-semigroups in general, in fact only when T is uniformly continuous, in which situation the spectral mapping theorem is indeed trivial. The idea would be then to consider T as a bi-continuous semigroup and refer to the spectral mapping theorem for the point spectrum of a bi-continuous semigroup (Corollary 1.5.2). It is easy to see that indeed $\tilde{T}$ is bi-continuous with respect to τ_{p}. But we encounter another problem: $c_0(X)$ is not a τ_{p}-closed subspace of $\ell^\infty(X)$, thus factorisation does not yield a bi-continuous semigroup. It would be nice to develop an approach using bi-continuous semigroups to identify the missing part in the approximate point spectrum.

We close this chapter by mentioning that Feller semigroups [1], [61], and evolutions semigroups [13], [31, Sec. VI.9b] on $\mathrm{C}_\mathrm{b}(\mathbb{R}_+, X)$ or $\mathrm{L}^\infty(\mathbb{R}_+, X)$ are also bi-continuous semigroups.

Chapter 3

Perturbation theory

In this chapter, we give some results on the perturbation of the generator of a bi-continuous semigroup T. These results are of the following type. Assume that $(A, D(A))$ generates a bi-continuous semigroup T. Putting some assumptions on a linear operator B, we would like to show that $(A+B, D(A))$ is still a generator. Perturbation theory for C_0-semigroups is well-established (cf. [31, Sec. III.1 – 3]), but for bi-continuous semigroups no results are known yet. Though there are developments in the perturbation theory of not C_0-semigroup, these either do not respect the underlying extra topology τ [48], [49], or do not take the Banach space structure of X into account [25, 26]. Thus for bi-continuous semigroups none of them is applicable.

In proving the perturbation results, the motivation will be the *variation of parameters formula* denoted by (IE) below. First we consider bounded perturbations, which in view of certain examples, is a non-trivial result. Later we turn to Miyadera – Voigt type perturbations and we combine perturbation techniques with approximations as well. These find a nice application to transition semigroups.

§ 3.1 Bounded perturbations

As a first step towards the perturbation theory for bi-continuous semigroups, we consider bounded perturbations $B \in \mathscr{L}(X)$. First, we show by instructive examples that the class of bi-continuous semigroups is not invariant under this operation. It will turn out, that an additional property of the perturbing operator B is needed. We will establish a positive result on bounded perturbations and investigate their properties. We also give a characterisation of semigroups that are ”linearly close” to a preliminary given bi-continuous semigroup.

a) Counterexamples. We show in this section that the perturbation of the generator of a bi-continuous semigroup by a norm-bounded operator does not generate a bi-continuous semigroup in general. In fact, it suffices to show that a norm-bounded operator is not necessarily the generator of a bi-continuous semigroup. We will use the fact that whenever

A generates a C_0-semigroup and a bi-continuous semigroup as well, then the two semigroups must coincide.

The first example deals with adjoint semigroups which are bi-continuous for the weak*-topology Section §2.1. Namely, we show that not every bounded operator on a dual space is a generator of a weak*-bi-continuous semigroup.

EXAMPLE 3.1.1 — Consider the Banach spaces

$$X := \mathrm{C}([0,1]), \quad X' = \mathscr{M}([0,1])$$

the space of all complex Borel measures on $[0,1]$. Define the following shift operator

$$(B\mu)(S) = \mu((S-3/4)\cap[0,1]) \quad \text{for all } \mu \in \mathscr{M}([0,1]),\, S \subseteq [0,1] \text{ Borel set.}$$

Then B is a norm-continuous operator on $\mathscr{M}([0,1])$, and clearly $B^2 = 0$. We show that B is not weak*-continuous. Indeed, consider the sequence of Dirac measures $\delta_{3/4-1/(n+1)}$, then this sequence obviously converges to $\delta_{3/4}$ in the weak*-topology $\sigma(X', X)$. Further, we observe that $B\delta_{3/4-1/(n+1)} = 0$, whereas we have $B\delta_{3/4} = \delta_0$, showing that B is not continuous for the topology $\sigma(X', X)$. The C_0-semigroup T generated by B is given by

$$T(t) = I + tB, \quad t \geq 0,$$

which shows that T is not bi-continuous for the weak*-topology $\sigma(X', X)$.

The following two examples are given on $\mathrm{C_b}(\mathbb{R})$. Both rely on the fact that there exist Borel measures on the Stone – Čech compactification $\beta\mathbb{R}$ of $\mathbb{R}$ with support disjoint from $\mathbb{R}$. The first example is in the spirit of the one given in Example 2.2.1.

EXAMPLE 3.1.2 — Let $X = \mathrm{C_b}(\mathbb{R})$ and take $x \in \beta\mathbb{R} \setminus \mathbb{R}$ and consider the linear operator $B = \mathbf{1} \otimes \delta_x$ on X (i.e., Bf is constant $f(x)$). Then B is a contractive projection. Now, we show that B is not continuous with respect to the compact-open topology τ_c. Let f_n be a sequence of continuous functions satisfying

$$f_n|_{[-n,n]} = 1 \quad \text{and} \quad \operatorname{supp} f_n \subseteq [-(n+1), n+1].$$

Then $f_n \to \mathbf{1}$ in the topology τ_c, but $Bf_n = 0$ since the function f_n vanishes outside a compact set of $\mathbb{R}$, while $B\mathbf{1} = \mathbf{1}$ (see the argumentations of Example 2.2.1). Hence, B is not continuous with respect to τ_c. As before, the C_0-semigroup T generated by B is given by the series

$$T(t) = \sum_{n=0}^{\infty} \frac{(Bt)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{t^n}{n!} B = I - B + \mathrm{e}^t B, \quad t \geq 0.$$

This implies that $T(t)$ is not sequentially τ_c-continuous on bounded sets unless $t = 0$, therefore B is not the generator of a bi-continuous semigroup with respect to τ_c.

EXAMPLE 3.1.3 — Again we work on $X = \mathrm{C_b}(\mathbb{R})$ and take a Banach-limit, i.e., a linear functional φ with

$$\|\varphi\| = 1, \quad \varphi(\mathbf{1}) = 1, \quad \varphi(f(\cdot + r)) = \varphi(f) \text{ for all } f \in X \text{ and } r \in \mathbb{R}.$$

We prove the existence of such a functional as follows. Define the bounded linear functionals φ_t for $t > 0$ in the following way

$$\varphi_t(f) := \frac{1}{2t} \int_{-t}^{t} f(s)\,\mathrm{d}s$$

Obviously $\varphi_t \in X'$ and $\|\varphi_t\| = 1$, hence the sequence φ_n is relatively weakly*-compact, thus it has a weak*-accumulation point $\varphi \in X'$. First we prove that φ has the following properties, $\varphi(\mathbf{1}) = 1$, $\|\varphi\| = 1$ and $\varphi(f(\cdot + r)) = \varphi(f)$ for all $r \in \mathbb{R}$. The first two statement is trivial from the definition of φ. To prove the third assertion, take an arbitrary $r \in \mathbb{R}$ and compute

$$\frac{1}{2n} \int_{-n}^{n} f(s+r)\,\mathrm{d}s = \frac{1}{2n} \int_{-n+r}^{n+r} f(s)\,\mathrm{d}s =$$
$$\frac{1}{2n} \int_{-n}^{+n} f(s)\,\mathrm{d}s + \frac{1}{2n} \int_{-n+r}^{-n} f(s)\,\mathrm{d}s + \frac{1}{2n} \int_{n}^{n+r} f(s)\,\mathrm{d}s.$$

Now, $f \in \mathrm{C_b}(\mathbb{R})$ and $\varepsilon > 0$. We show that

$$|\langle f - f(\cdot + r), \varphi\rangle| \leq \varepsilon,$$

this will establish the equality $\varphi(f) = \varphi(f(\cdot + r))$. We can write

$$\begin{aligned} |\langle f - f(\cdot + r), \varphi\rangle| &\leq |\langle f - f(\cdot + r), \varphi - \varphi_n\rangle| + |\langle f - f(\cdot + r), \varphi_n\rangle| \leq \\ &\leq |\langle f - f(\cdot + r), \varphi - \varphi_n\rangle| + \frac{1}{2n} \int_{-n+r}^{-n} |f(s)|\,\mathrm{d}s + \frac{1}{2n} \int_{n}^{n+r} |f(s)|\,\mathrm{d}s. \end{aligned} \tag{3.1}$$

Since the weak*-neighbourhood U of φ determined by $\varepsilon/2 > 0$ and $f - f(\cdot + r) \in \mathrm{C_b}(\mathbb{R})$ contains an infinite sequence φ_{n_k}, we can make the first term on the right hand side of (3.1) smaller than $\varepsilon/2$ such that the last two terms together become smaller than $\varepsilon/2$, because we have

$$\left| \frac{1}{2k} \int_{-k+r}^{-k} f(s)\,\mathrm{d}s + \frac{1}{2k} \int_{k}^{k+r} f(s)\,\mathrm{d}s \right| \leq \frac{r}{k}\|f\| \to 0,$$

as $k \to \infty$. Thus we have proved the translation invariance of φ. We remark that the above properties of φ mean that it is indeed an *invariant probability measure* on $\beta\mathbb{R}$ for the shift semigroup.

With this φ, we define the norm-bounded linear operator $B = \mathbf{1} \otimes \varphi$. As in Example 3.1.2, one shows that B is not τ_c-continuous on norm-bounded sets, hence B is not the generator of a bi-continuous semigroup for the topology τ_c. Now, we show that there exists a non-trivial bi-continuous semigroup S with generator $(A, D(A))$, for which $(A + B, D(A))$ is not the generator of a bi-continuous semigroup.

By Section 2.5.a, the translation group S on X is τ_c-bi-continuous. Denote its generator by $(A, D(A))$. In the sequel, we show that a bounded perturbation of S need not be bi-continuous. Indeed, we take as the perturbing operator the operator B and make use of the translation invariance of φ. Indeed, the translation invariance of φ implies that B commutes with the semigroup S. Consider the function

$$\mathbb{R} \ni t \mapsto T(t) = S(t)(I - B + \mathrm{e}^t B) = (I - B + \mathrm{e}^t B)S(t).$$

Then T obviously satisfies the semigroup property and is τ_c-strongly continuous. A straightforward computation shows that for all $x \in D(A)$ the orbits $t \mapsto T(t)x$ are τ_c-differentiable with derivative $(A + B)T(t)x$. If $(A + B, D(A))$ is the generator of a bi-continuous semigroup, then this semigroup has to coincide with T. However, T is not locally-bi-equicontinuous, which can be seen by taking the sequence of functions f_n as before in Example 3.1.2 (cf. also Section §2.2). Therefore $(A + B, D(A))$ can not be the generator of a bi-continuous semigroup.

b) Bounded perturbations. We now turn our attention to positive results concerning bounded perturbations of bi-continuous semigroups and prove that a bi-continuous semigroup can be perturbed by a bounded operator provided that the perturbing operator is also τ-continuous on norm-bounded sets. We carry out this investigation by means of abstract Volterra operators as in [31, Sec. III.1] and put all the necessary properties of a bi-continuous semigroup into the Banach space $\mathbf{X}_{t_0}$ on which this operator acts (see Definition 1.2.16). Later we will characterise all bounded perturbations of a bi-continuous semigroup. The applications will come forth in Section §4.2.

Take now a τ-bi-continuous semigroup T and suppose that $B \in \mathscr{L}(X)$ is also τ-sequentially continuous on norm-bounded sets. Lemma 1.2.17 and 1.2.18 enable us to define an abstract Volterra operator $\mathbf{V}$ associated to the semigroup T on the Banach space $\mathbf{X}_{t_0}$ in the following way. For $t \in [0, t_0]$ and $x \in X$ set

$$[\mathbf{V}S](t)x := \int_0^t T(t-s)BS(s)x \,\mathrm{d}s, \quad x \in X, \tag{3.2}$$

for any $S \in \mathbf{X}_{t_0}$. The integral exists in the τ topology since $s \mapsto T(t-s)BS(s)$ is τ-continuous by assumption and Lemma 1.2.18. We now prove that $\mathbf{V} \in \mathscr{L}(\mathbf{X}_{t_0})$ and compute its spectral radius $r(\mathbf{V})$.

Notice that $[\mathbf{V}S](t)$ for $t \in [0, t_0]$ is independent of the particular choice of $t_0 > 0$.

LEMMA 3.1.4 — *The linear operator defined in* (3.2) *maps* $\mathbf{X}_{t_0}$ *into itself, is bounded, and has spectral radius* $r(\mathbf{V}) = 0$.

PROOF — Take $S \in \mathbf{X}_{t_0}$, $t \in [0, t_0]$ and $x \in X$, and write

$$\|[\mathbf{V}S](t)x\| = \sup_{\substack{\phi \in (X,\tau)' \\ \|\phi\| \le 1}} \Big| \Big\langle \int_0^t T(t-s)BS(s)x \,\mathrm{d}s, \phi \Big\rangle \Big| \le$$

$$\le \sup_{\substack{\phi \in (X,\tau)' \\ \|\phi\| \le 1}} \int_0^t |\langle T(t-s)BS(s)x, \phi\rangle| \,\mathrm{d}s \le \big\| T|_{[0,t_0]} \big\| \cdot \|B\| \cdot \|S\| \cdot \|x\|,$$

which gives that $[\mathbf{V}S](t) \in \mathscr{L}(X)$.

Second, we show that $\{[\mathbf{V}S](t) : t \in [0, t_0]\}$ is bi-equicontinuous. To do this, consider a norm-bounded τ-null sequence x_n and take $p \in \mathcal{P}$ and $\varepsilon > 0$. Then

$$\begin{aligned} p([\mathbf{V}S](t)x_n) = p\Big(\int_0^t T(t-s)BS(s)x_n \,\mathrm{d}s\Big) \leq \\ \leq \int_0^t p(T(t-s)BS(s)x_n)\,\mathrm{d}s \leq t_0\varepsilon \end{aligned}$$

for n sufficiently large since $\{T(t-s)BS(s) : s \in [0, t_0]\}$ is bi-equicontinuous by Lemma 1.2.18. Clearly $[\mathbf{V}S](\cdot)$ is norm-bounded and τ-strongly continuous, hence $\mathbf{V}S \in \mathbf{X}_{t_0}$.

We compute the spectral radius of $\mathbf{V}$ (see [31, Sec. III.1.5]). For $S \in \mathbf{X}_{t_0}$, $x \in X$ and $t \in [0, t_0]$ we write

$$\|[\mathbf{V}S](t)x\| \leq \int_0^t \|T(t-s)BS(s)x\|\,\mathrm{d}s \leq \|T|_{[0,t_0]}\| \cdot \|B\| \cdot \|S\| \cdot \|x\| t.$$

From this by induction it follows that

$$\|\mathbf{V}^n S\| \leq \frac{\big(t_0 \cdot \big\|T|_{[0,t_0]}\big\| \cdot \|B\|\big)^n}{n!} \cdot \|S\| \quad \text{for } n \in \mathbb{N}, \tag{3.3}$$

therefore $r(\mathbf{V}) = 0$. ■

The previous lemma plays a key role in the following since it implies $1 \in \varrho(\mathbf{V})$. Furthermore, the resolvent of $\mathbf{V}$ in 1 is given by

$$R(1, \mathbf{V}) = \sum_{n=0}^{\infty} \mathbf{V}^n,$$

the series converging in the operator norm of $\mathscr{L}(\mathbf{X}_{t_0})$. We are now ready to prove our main theorem.

THEOREM 3.1.5 — *Let T be a bi-continuous semigroup with generator $(A, D(A))$ and suppose that $B \in \mathscr{L}(X)$ is τ-sequentially continuous on norm-bounded sets. Then $(A+B, D(A))$ is also the generator of a bi-continuous semigroup S. Moreover S is given by the Dyson – Phillips series*

$$S(t) = \sum_{n=0}^{\infty} T_n(t), \quad t \geq 0, \tag{3.4}$$

which is uniformly norm-convergent on compact intervals. Here, the Dyson – Phillips terms T_n are defined as

$$T_0(t) := T(t), \quad T_n(t) := \int_0^t T(t-s)BT_{n-1}(s)\,\mathrm{d}s \quad \textit{for } n > 0,$$

where the integral is understood in the τ-strong topology.

PROOF — Let $t_0 > 0$ arbitrary, and define the abstract Volterra operator $\mathbf{V}_{t_0}$ on the space $\mathbf{X}_{t_0}$ as above. Further set

$$S_{t_0}(t) := \delta_t \left(R(1, \mathbf{V}_{t_0}) T|_{[0,t_0]} \right) \tag{3.5}$$

for each $t \in [0, t_0]$. From the definition of $\mathbf{V}_{t_0}$, it is immediate that one has

$$S_{t_0}(t)x = S_{t_0'}(t)x \quad \text{for all } t \le t_0' \le t_0 \text{ and } x \in X.$$

This enables us to define

$$S(t) := \begin{cases} I & \text{for } t = 0, \\ S_t(t) & \text{for } t > 0. \end{cases}$$

First, we show that S is a semigroup. From now on write simply T instead of $T|_{[0,t_0]}$. As a first step, we prove that

$$S_{t_0}(t+s) = S_{t_0}(t) S_{t_0}(s)$$

whenever $0 \le s, t \le s + t \le t_0$. Indeed, taking such positive numbers t and s, we have by definition that

$$S_{t_0}(t) S_{t_0}(s) = \sum_{n=0}^{\infty} \left[\mathbf{V}_{t_0}^n T \right](t) \cdot \sum_{n=0}^{\infty} \left[\mathbf{V}_{t_0}^n T \right](s).$$

Since the series converges in the operator norm, the Cauchy product yields

$$S_{t_0}(t) S_{t_0}(s) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\mathbf{V}_{t_0}^k T \right](t) \left[\mathbf{V}_{t_0}^{n-k} T \right](s).$$

Therefore it remains to show that

$$\sum_{k=0}^{n} \left[\mathbf{V}_{t_0}^k T \right](t) \left[\mathbf{V}_{t_0}^{n-k} T \right](s) = \left[\mathbf{V}_{t_0}^n T \right](t+s), \tag{3.6}$$

which obviously holds for $n = 0$. We proceed by induction, assume that (3.6) holds for some n, then

$$\begin{aligned}
\left[\mathbf{V}_{t_0}^{n+1} T \right](t+s)x &= \int_0^{t+s} T(t+s-u) B \left[\mathbf{V}_{t_0}^n T \right](u) x \, \mathrm{d}u = \\
&= \int_0^{s} T(s-u) B \left[\mathbf{V}_{t_0}^n T \right](u+t) x \, \mathrm{d}u + \int_s^{s+t} T(u) B \left[\mathbf{V}_{t_0}^n T \right](t+s-u) B T(u) x \, \mathrm{d}u = \\
&= \int_0^{s} T(s-u) B \left[\mathbf{V}_{t_0}^n T \right](u+t) x \, \mathrm{d}u + \int_0^{t} T(u+s) B \left[\mathbf{V}_{t_0}^n T \right](t-u) x \, \mathrm{d}u = \\
&= \int_0^{s} T(s-u) B \sum_{k=0}^{n} \left[\mathbf{V}_{t_0}^k T \right](u) \left[\mathbf{V}_{t_0}^{n-k} T \right](t) x \, \mathrm{d}u + T(s) \left[\mathbf{V}_{t_0}^{n+1} T \right](t) = \\
&= \sum_{k=0}^{n} \left[\mathbf{V}_{t_0}^{k+1} T \right](s) \left[\mathbf{V}_{t_0}^{n-k} T \right](t) + T(s) \left[\mathbf{V}_{t_0}^{n+1} T \right](t) = \sum_{k=0}^{n+1} \left[\mathbf{V}_{t_0}^k T \right](s) \left[\mathbf{V}_{t_0}^{n+1-k} T \right](t).
\end{aligned}$$

This completes the induction.

Since $S|_{[0,t_0]} = S_{t_0} \in \mathbf{X}_{t_0}$, we see immediately that S is a bi-continuous semigroup and from the definition it is straightforward that $T_n(t) = (\mathbf{V}_t^n T|_{[0,t]})(t)$ and hence (3.4) is satisfied. The uniform convergence on compact intervals follows from the continuity of δ_t.

We claim that the generator $(C, D(C))$ of S is $(A+B, D(A))$. To prove this, we first remark that $(A, D(A))$ is a Hille – Yosida operator, and therefore its bounded perturbation $(A+B, D(A))$ is also a Hille – Yosida operator (see e.g., [5, Thm. 3.5.5]), in particular its resolvent set is not empty. Now, take $x \in X$ arbitrary, then

$$\frac{[\mathbf{V}_{t_0}T](h)x}{h} = \frac{1}{h}\int_0^h T(h-s)BT(s)x\,\mathrm{d}s,$$

from which we deduce that

$$\frac{[\mathbf{V}_{t_0}T](h)x}{h} \xrightarrow{\tau} Bx. \tag{3.7}$$

Indeed, by the τ-continuity of the orbits $s \mapsto T(s)Bx$ one can choose $0 < \delta < t_0$ for a given $\varepsilon > 0$ such that

$$p(T(h-s)Bx - Bx) \le \frac{\varepsilon}{2}$$

whenever $h \in [0,\delta]$ and $s \in [0,h]$. Further, by taking δ possibly smaller, we obtain by the bi-equicontinuity of $\{T(s) : s \in [0,t_0]\}$ and the τ-continuity of $s \mapsto BT(s)x$ that

$$\begin{aligned}
&p\Big(\frac{1}{h}\int_0^h T(h-s)BT(s)x\,\mathrm{d}s - Bx\Big) \le \\
&\le \frac{1}{h}\int_0^h p(T(h-s)Bx - Bx)\,\mathrm{d}s + \frac{1}{h}\int_0^h p(T(h-s)(BT(s)x - Bx))\,\mathrm{d}s \le \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

For $x \in D(A)$ we have

$$\frac{S(h)x - x}{h} = \frac{T(h)x - x}{h} + \frac{[\mathbf{V}T](h)x}{h} + \sum_{n=2}^{\infty}\frac{[\mathbf{V}^n T](h)x}{h}. \tag{3.8}$$

Using (3.3) we see that

$$\Big\|\sum_{n=2}^{\infty}\frac{[\mathbf{V}^n T](h)x}{h}\Big\| \le h\sum_{n=2}^{\infty}\frac{h^{n-2}(\|T\|\cdot\|B\|)^n}{n!}\|x\| \le hC\|x\|$$

for all $x \in X$ and for some constant $C > 0$, which shows that the third term in (3.8) converges to zero in the topology τ as $h \to 0$. All the terms in (3.8) remain bounded as $h \to 0$. Putting (3.7) and (3.8) together we obtain that

$$\frac{S(h)x - x}{h} \xrightarrow{\tau} Ax + Bx$$

proving that $A+B \subseteq C$. This together with the above remark on the non-empty resolvent set of $A+B$ implies that $A+B = C$. ∎

c) Characterisation of bounded perturbations. A bounded perturbation of a bi-continuous semigroup T satisfies certain "smallness" properties with respect to T. These are presented here as a starting point. The conclusion will be that these, in fact, describe all bounded perturbations of a given bi-continuous semigroup.

COROLLARY 3.1.6 — *If T is a bi-continuous semigroup with generator $(A, D(A))$ and $B \in \mathscr{L}(X)$ is τ-sequentially continuous on norm-bounded sets, then the semigroup S generated by $(A + B, D(A))$ satisfies the integral equation*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x \,\mathrm{d}s, \tag{IE}$$

$$T(t)x = S(t)x - \int_0^t S(s)BT(t-s)x \,\mathrm{d}s \tag{IE'}$$

for all $x \in X$ and $t \geq 0$, where the integral is understood in the τ-Riemannian sense. As a consequence

$$\|T(t) - S(t)\| \leq tM$$

holds for all $t \in [0, t_0]$ and some constant $M > 0$.

PROOF — The equation (IE) is just a reformulation of (3.5). The integral equation (IE') can be deduced from (IE) applying the bounded perturbation theorem with the operator $-B$. For the third assertion, let $t_0 \geq 0$ be arbitrary and set

$$M := \|B\| \cdot \sup\{\|T(t)\| : t \in [0, t_0]\} \cdot \sup\{\|S(t)\| : t \in [0, t_0]\}.$$

Then the desired inequality immediately follows from (IE). ∎

COROLLARY 3.1.7 — *Let T be a bi-continuous semigroup with generator $(A, D(A))$ and $B \in \mathscr{L}(X)$ an operator which is sequentially τ-continuous on norm-bounded sets. Denote by S the semigroup generated by $(A+B, D(A))$. Then we have that the operators*

$$U(0) := 0, \quad \text{and} \quad U(t) := \frac{S(t) - T(t)}{t} \quad \text{for } t > 0$$

form a locally-bi-equicontinuous family.

PROOF — Let $\varepsilon > 0$ and $x_n \xrightarrow{\tau} 0$ be a norm-bounded sequence. We make use of (IE) and write for $p \in \mathcal{P}$

$$p(U(t)x_n) = \frac{1}{t} p(T(t)x_n - S(t)x_n) \leq \frac{1}{t} \int_0^t p(T(t-s)BS(s)x_n) \,\mathrm{d}s \leq \frac{1}{t} \cdot t\varepsilon = \varepsilon,$$

the last inequality being true for sufficiently large $n \in \mathbb{N}$ since the family of operators

$$\{T(t-s)BS(s) : s \in [0, t]\}$$

is bi-equicontinuous by Lemma 1.2.18. ∎

We show that these two corollaries together characterise bounded perturbations of a given bi-continuous semigroup T (cf. [31, Cor. III.3.12]).

THEOREM 3.1.8 — *Let T and S be bi-continuous semigroups with respective generators $(A, D(A))$ and $(C, D(C))$. Suppose that*

i) there exists $M > 0$ such that

$$\|T(t) - S(t)\| \leq tM$$

for all $t \in [0, 1]$, and that

ii) the family

$$U(0) := 0, \quad \text{and} \quad U(t) := \frac{S(t) - T(t)}{t} \quad \text{for } t > 0$$

is locally-bi-equicontinuous.

If $D(A) \cap D(C)$ is η-bi-dense for some $\eta > 1$ (see Definition 1.2.13), then there exists a bounded operator B which is sequentially τ-continuous on norm-bounded sets such that $C = A + B$.

PROOF — Let $x \in D(A) \cap D(C)$, then

$$U(1/n)x = n \cdot (S(1/n)x - x) + n \cdot (x - T(1/n)) \xrightarrow{\tau} Ax - Cx \tag{3.9}$$

as $n \to \infty$. We define

$$Bx := \tau - \lim_{n\to\infty} U(1/n)x$$

for $x \in D(A) \cap D(C)$. Now, let $x \in X$ be arbitrary, and take a norm-bounded sequence $x_n \in D(A) \cap D(C)$ which is τ-convergent to x and satisfies (ηD). Then, for arbitrary $p \in \mathcal{P}$ we conclude by ii) that

$$\begin{aligned} p(B(x_m - x_k)) &\leq \overline{\lim_{n\to\infty}}\, p\left(U(1/n)(x_m - x_k)\right) \leq \\ &\leq \overline{\lim_{n\to\infty}}\, p\left(U(1/n)(x_m - x)\right) + \overline{\lim_{n\to\infty}}\, p\left(U(1/n)(x - x_k)\right) \leq 2\varepsilon \end{aligned} \tag{3.10}$$

if $m, k \in \mathbb{N}$ are sufficiently large. Therefore, Bx_m is a norm-bounded, τ-Cauchy sequence, hence it is convergent. So we can extend B to the whole space X by

$$Bx := \tau - \lim_{m\to\infty} Bx_m.$$

We claim that B is a bounded operator. Indeed it follows from i) and Proposition 1.2.2 that

$$\|Bx\| \leq \overline{\lim_{m\to\infty}}\, \|Bx_m\| \leq \overline{\lim_{m\to\infty}}\, \overline{\lim_{n\to\infty}}\, \|U(1/n)x_m\| \leq \overline{\lim_{m\to\infty}}\, M\|x_m\| \leq M\eta\|x\|$$

for all $x \in X$ proving $B \in \mathscr{L}(X)$. From (3.10) it is straightforward that B is sequentially τ-continuous on norm-bounded sets.

We conclude the proof by showing that $A + B = C$. To this end, consider $x \in D(A) \cap D(C)$, and let $\bar{S}$ denote the semigroup generated by $(A + B, D(A))$. Then it is straightforward from (3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{S}(t)x - S(t)x\right) = 0$$

for all $t \geq 0$, taking the derivative in the τ topology. Also

$$\bar{S}(0)x - S(0)x = 0,$$

therefore

$$\big(\bar{S}(t) - S(t)\big)\, x = 0 \quad \text{for all } x \in D(A) \cap D(C).$$

Since $D(A) \cap D(C)$ is bi-dense in X, we obtain $\bar{S} = S$, hence $A + B = C$. ∎

§ 3.2 UNBOUNDED PERTURBATIONS

We deal with perturbations of bi-continuous semigroups and try to relax the continuity properties of the perturbing operator B. In the case of the bounded perturbation theorem, the operator B was everywhere defined and continuous. Here we would like to consider perturbing operators which are defined on the domain of the generator $D(A)$ and satisfy continuity assumptions with respect to the graph-norm $\|\cdot\|_A$ and the graph topology τ_A. This kind of results were obtained by Miyadera [65] and Voigt [85] for C_0-semigroups, in which case it suffices $B \in \mathscr{L}(X_1, X)$ (recall X_1 from Section 1.1.a). We prove similar results for bi-continuous semigroups. Such perturbations are useful, e.g., for perturbing second-order differential operators with first order terms.

A couple words about open questions fit perfectly here. It would be desirable to establish results that are known for C_0-semigroups as Desch – Schappacher perturbations. These are somehow "dual" to the Miyadera – Voigt type perturbations, as one assumes $B \in \mathscr{L}(X_{-1}, X)$. For the complete characterisation of semigroups S and T satisfying $\|T(t) - S(t)\| \leq Mt$ of Theorem 3.1.8 these perturbation results are needed. Also for applications, e.g., for dynamic boundary problems for the heat equation considered in Section §2.4, such perturbations may be necessary. To apply operator matrix techniques, multiplicative perturbations (i.e., instead of $A+B$ the operator AB) are more convenient, and no results are known in this direction.

a) Miyadera – Voigt type perturbations. In this section, we study perturbation of bi-continuous semigroups in the spirit as in [31, Sec. 3.3.c] for C_0-semigroups. We try to imitate the proof of the bounded perturbation theorem, whereas more subtle arguments are needed, because of the discontinuity of the perturbing operator B for the topology τ.

Let T be a bi-continuous semigroup with generator $(A, D(A))$. Assume that $D(A)$ is η-bi-dense. Recall that the topology τ is determined by the family of seminorms $\mathcal{P}$. We define a new locally convex topology τ_A as follows. It is determined by the family of seminorms

$$\mathcal{P}_A := \{p(\cdot) + p'(A\cdot) : \; p, p' \in \mathcal{P}\}.$$

Clearly τ_A is coarser than the graph norm $\|\cdot\|_A$.

Let $B \in \mathscr{L}((D(A), \tau_A), (X, \tau))$ continuous on $\|\cdot\|_A$-bounded sets, then for arbitrary $x \in D(A)$ the orbits

$$s \mapsto F(t-s)BT(s)x$$

are τ-continuous by a similar reasoning as in Lemma 1.2.18 for all $F \in \mathbf{X}_{t_0}$ with $t_0 > 0$. This allows us to make the following definition, in particular the Riemann integral in (3.11) makes sense and defines a linear operator $B(F,t)$ over $D(A)$.

DEFINITION 3.2.1 — We say that an operator $B : (D(A), \tau_A) \to (X, \tau)$ which is continuous on $\|\cdot\|_A$-bounded sets is *Miyadera – Voigt admissible* on Y, where Y is a closed subspace of $\mathbf{X}_{t_0}$, if there exists $t_0 > 0$ for which $T|_{[0,t_0]} \in Y$ and the following conditions are satisfied.

i) For all $x \in D(A)$ the maps

$$s \mapsto \|BT(s)x\|, \quad s \in [0, t_0]$$

are bounded.

ii) The operator

$$B(F,t)x := \int_0^t F(t-s)BT(s)x \, \mathrm{d}s \tag{3.11}$$

defined on $D(A)$ extends to linear operator $\bar{B}(F,t)$ on X which is τ-continuous on norm-bounded sets for all $t \in [0, t_0], F \in Y$. We require moreover that the operator $\bar{B}(F,t)$ is also norm-bounded.

iii) The abstract Volterra operator $\mathbf{V}_{t_0}$ on Y defined by

$$[\mathbf{V}_{t_0}F](t)x = \bar{B}(F,t)x, \quad \text{for all } x \in X, t \in [0, t_0]$$

is a bounded operator on Y and we have $\|\mathbf{V}_{t_0}\| < 1/\eta$.

Clearly, Miyadera – Voigt admissibility depends on the semigroup T. This, however should not cause any confusion in the sequel.

THEOREM 3.2.2 — *Let T be a bi-continuous semigroup and suppose that B is Miyadera – Voigt admissible on $\mathbf{X}_{t_0}$. In this case, $(A+B, D(A+B))$ generates a bi-continuous semigroup S. Moreover the semigroup S satisfies the following*

$$S(t)x = T(t)x + \int_0^t S(t-s)BT(s)x \, \mathrm{d}s \tag{IE}$$

for all $x \in D(A)$ and $t \geq 0$.

PROOF — In this proof we will denote the restriction $T|_{[0,t_0]}$ simply by T. We can define the abstract Volterra operator $\mathbf{V}_{t_0} \in \mathscr{L}(\mathbf{X}_{t_0})$. By assumptions $1 \in \varrho(\mathbf{V}_{t_0})$. Let $t > 0$ arbitrary, then we write $t = nt_0 + t_1$, where $n \in \mathbb{N}$, $t_1 \in [0, t_0)$ and define

$$S(t) = (\delta_{t_0}(R(1, \mathbf{V}_{t_0})T))^n \cdot \delta_{t_1}(R(1, \mathbf{V}_{t_0})T) \tag{3.12}$$

We show that $S : [0, +\infty) \to \mathscr{L}(X)$ is a semigroup. First of all let

$$0 \leq s, t \leq s + t \leq t_0$$

Analogously to the norm-bounded case (see the proof of Theorem 3.1.5), we can show that

$$S(t+s) = S(t)S(s) \tag{3.13}$$

Secondly, take any $t, s > 0$ and write these real numbers in the form

$$t = nt_0 + t_1, \quad s = mt_0 + t_2, \quad n, m \in \mathbb{N}, t_1, t_2 \in [0, t_0)$$

Then we have

$$\begin{aligned} S(t)S(s) = S(t_0)^n \cdot S(t_1) \cdot S(t_0)^m \cdot S(t_2) = \\ S(t_0)^n \cdot S(t_1) \cdot S(t_0) \cdot S(t_0)^{m-1} \cdot S(t_2) = \\ S(t_0)^n \cdot S(t_1) \cdot S(t_0 - t_1) \cdot S(t_1) \cdot S(t_0)^{m-1} \cdot S(t_2) = \dots \\ \dots = S(t_0)^n \cdot S(t_0)^m \cdot S(t_1) \cdot S(t_2), \end{aligned}$$

but

$$S(t_0)^n \cdot S(t_0)^m \cdot S(t_1) \cdot S(t_2) = \begin{cases} S(t_0)^{n+m} \cdot S(t_1 + t_2) & \text{if } t_1 + t_2 < t_0 \\ \\ S(t_0)^{n+m+1} \cdot S(t_2 - (t_0 - t_1)) & \text{if } t_1 + t_2 \geq t_0 \end{cases}$$

which in both cases equals $S(t+s)$ by definition. It is also clear that $S(0) = I$, hence S is a semigroup. Since we have

$$S|_{[0,t_0]} = R(1, \mathbf{V}_{t_0})T|_{[0,t_0]}$$

S is locally-bounded and $\{S(t) : t \in [0, t_0]\}$ is bi-equicontinuous. Let $t > 0$ be given, we show that $\{S(s) : s \in [0, t]\}$ is bi-equicontinuous. Indeed, let $m = \lfloor t/t_0 \rfloor$. Since

$$\{S(t_0)^k : k = 1, \dots, m\}$$

is bi-equicontinuous on norm-bounded sets, it is immediate that

$$\{S(t_0)^k : k = 1, \dots, m\} \cdot \{S(s) : s \in [0, t_0]\}$$

is also bi-equicontinuous (cf. Remarks on page 6). The τ-strongly continuity of S is also obvious from the semigroup property (3.13), since $S|_{[0,t_0]}$ is τ-strong continuous by definition. Thus we have proved that S is a bi-continuous semigroup. Also by definition

$$S(t)x = T(t)x + \int_0^t S(t-s)BT(s)x \,\mathrm{d}s$$

for all $x \in D(A)$ and $t \in [0, t_0]$, we prove this for arbitrary $t > 0$. Let us write t in the form $t = nt_0 + t_1$, $n \in \mathbb{N}$ and $t_1 \in [0, t_0)$. Then we can write for all $x \in D(A)$ that

$$\int_0^t S(t-s)BT(s)x \,\mathrm{d}s = \sum_{k=0}^{n-1} \int_{kt_0}^{(k+1)t_0} S(t-s)BT(s)x \,\mathrm{d}s + \int_{nt_0}^{t} S(t-s)BT(s)x \,\mathrm{d}s =$$

$$= \sum_{k=0}^{n-1} \int_0^{t_0} S(t - (s + kt_0))BT(s + kt_0))x \,\mathrm{d}s + \int_0^{t-nt_0} S(t - (s + nt_0))BT(s + nt_0)x \,\mathrm{d}s.$$

And we may continue by taking into account the definitions of S

$$\begin{aligned}
&\sum_{k=0}^{n-1} S(t-(k+1)t_0)\int\limits_0^{t_0} S(t_0-s)BT(s)T(kt_0)x\,\mathrm{d}s+\\
&\quad+\int\limits_0^{t-nt_0} S((t-nt_0)-s)BT(s)T(nt_0)x\,\mathrm{d}s=\\
&=\sum_{k=0}^{n-1} S(t-(k+1)t_0)\,[S(t_0)-T(t_0)]\,T(kt_0)x+[S(t-nt_0)-T(t-nt_0)]T(nt_0)x\\
&=S(t)x-S(t-t_0)T(t_0)x+\sum_{k=1}^{n-1} S(t-kt_0)T(kt_0)x-\\
&\quad-\sum_{k=1}^{n-1} S(t-(k+1)t_0)T((k+1)t_0)x+S(t-nt_0)T(nt_0)x-T(t)x=\\
&=[S(t)-T(t)]x
\end{aligned}$$

Now, we skip to proving that the generator of S is $(A+B,D(A))$. First of all we show that the resolvent set of $(A+B,D(A+B))$ is not empty. Let $\lambda\in\varrho(A)$. Taking the Laplace transform of T, we obtain for all $x\in D(A)$ that

$$R(\lambda,A)x=\sum_{k=0}^{\infty}\mathrm{e}^{-\lambda kt_0}\int\limits_0^{t_0}\mathrm{e}^{-\lambda t}T(t)T(kt_0)x\,\mathrm{d}t,$$

where the series and the integral converges in the topology τ and, in this case also in the topology τ_A (cf. Theorem 1.2.7). Hence we also have

$$\begin{aligned}
BR(\lambda,A)x&=\sum_{k=0}^{\infty}\mathrm{e}^{-\lambda kt_0}\int\limits_0^{t_0}\mathrm{e}^{-\lambda t}BT(t)T(kt_0)x\,\mathrm{d}t=\\
&=\sum_{k=0}^{\infty}\mathrm{e}^{-\lambda kt_0}[\mathbf{V}_{t_0}F_\lambda](t_0)T(kt_0)x,
\end{aligned}$$

for all $x\in D(A)$, where $F_\lambda=\mathrm{e}^{-\lambda(t_0-\cdot)}I$, $F_\lambda\in\mathbf{X}_{t_0}$. This yields the following norm estimate for $x\in D(A)$

$$\begin{aligned}
\|BR(\lambda,A)x\|&\le\sum_{k=0}^{\infty}\mathrm{e}^{-\lambda kt_0}\,\|[\mathbf{V}_{t_0}F_\lambda](t_0)T(kt_0)x\|\le\\
&\le\sum_{k=0}^{\infty}\mathrm{e}^{-\lambda kt_0}\|\mathbf{V}_{t_0}\|\cdot\|F_\lambda\|\cdot\|T(kt_0)\|\cdot\|x\|\le\\
&\le\|\mathbf{V}_{t_0}\|\cdot\|x\|+M\|\mathbf{V}_{t_0}\|\sum_{k=1}^{\infty}\mathrm{e}^{-\lambda kt_0+\omega kt_0}\|x\|\le\\
&\le\|\mathbf{V}_{t_0}\|\cdot\|x\|+\frac{M}{\eta}\cdot\frac{\mathrm{e}^{(\omega-\lambda)t_0}}{1-\mathrm{e}^{(\omega-\lambda)t_0}}\|x\|,
\end{aligned}$$

where $\omega \in \mathbb{R}$ and $M \in \mathbb{R}_+$ are chosen such that (EXP) is satisfied. Now, let $x \in X$ arbitrary and take any sequence $x_n \in D(A)$ appearing in Definition 1.2.13. In this case $BR(\lambda, A)x_n$ converges to $BR(\lambda, A)x$ in the topology τ, by the continuity assumption on B. So we have by Proposition 1.2.2 that

$$\begin{aligned}\|BR(\lambda, A)x\| &\le \limsup_{n\to\infty} \|BR(\lambda, A)x_n\| \le \\ &\le \limsup_{n\to\infty}\Big(|\mathbf{V}_{t_0}\| \cdot \|x_n\| + \frac{M}{\eta}\cdot\frac{\mathrm{e}^{(\omega-\lambda)t_0}}{1-\mathrm{e}^{(\omega-\lambda)t_0}}\|x_n\|\Big) \le \\ &\le \eta\|\mathbf{V}_{t_0}\| \cdot \|x\| + M\frac{\mathrm{e}^{(\omega-\lambda)t_0}}{1-\mathrm{e}^{(\omega-\lambda)t_0}}\|x\|\end{aligned}$$

From this we obtain that $\|BR(\lambda, A)\| < 1$ for sufficiently large λ. For such a λ we have

$$R(\lambda, A+B) = R(\lambda, A)\left(I - BR(\lambda, A)\right)^{-1}, \tag{3.14}$$

showing $\varrho(A+B) \neq \emptyset$. Now, take $x \in D(A)$ and $t < t_0$, then by (IE) we can write

$$\frac{S(t)x - x}{t} = \frac{T(t)x - x}{t} + \frac{1}{t}\int_0^t S(t-s)BT(s)x\,\mathrm{d}s$$

The first term converges to Ax in the topology τ by definition. By the τ-continuity of the orbits $s \mapsto S(s)Bx$ one can choose $0 < \delta < t_0$ for a given $\varepsilon > 0$ such that

$$p(S(t-s)Bx - Bx) \le \frac{\varepsilon}{2}$$

whenever $t \in [0, \delta]$ and $s \in [0, t]$. Further, by taking δ possibly a bit smaller we can write by the bi-equicontinuity of $\{S(s) : s \in [0, t_0]\}$ and the τ-continuity of $s \mapsto BT(s)x$ that

$$\begin{aligned}&p\Big(\frac{1}{t}\int_0^t S(t-s)BT(s)x\,\mathrm{d}s - Bx\Big) \le \\ &\le \frac{1}{t}\int_0^t p(S(t-s)Bx - Bx)\,\mathrm{d}s + \frac{1}{t}\int_0^t p(S(t-s)(BT(s)x - Bx))\,\mathrm{d}s \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

This shows that $A+B$ is a restriction of the generator C of S, that is $A+B = C$ indeed. Also, one could finish the proof by taking the Laplace transform of (IE), but this would require some more arguments concerning the interchanging of integrals. ∎

b) A sufficient condition. The conditions in the present form of the proved Miyadera – Voigt type result are not very natural, and therefore the result itself is not easily applicable. In this section, we give a sufficient condition on B and T such that B becomes Miyadera – Voigt admissible with respect to T. The result is quite similar to that of [85] (see also [31, Cor. III.3.16]). Nevertheless, we also have to put some further assumptions because of the topology τ. The proof is therefore more involved.

THEOREM 3.2.3 — *Let T be a local bi-continuous semigroup, denote its generator by $(A, D(A))$ whose domain is η-bi-dense. And let $B : (D(A), \tau_A) \to (X, \tau)$ continuous on $\|\cdot\|_A$-bounded sets. Suppose that there exist $t_0 > 0$ and a real number $q < 1/\eta^2$ such that $s \mapsto \|BT(s)x\|$ is bounded on $[0, t_0]$, and*

$$\int_0^t \|BT(s)x\| \, \mathrm{d}s < q\|x\| \tag{3.15}$$

for all $t \in [0, t_0]$ and $x \in D(A)$. Assume furthermore that for all $\varepsilon > 0$ and for all $p \in \mathcal{P}$ there exist a seminorm $p' \in \mathcal{P}$ and a positive real number $M_{p,\varepsilon}$ such that

$$\int_0^{t_0} p(BT(s)x) \, \mathrm{d}s < M_{p,\varepsilon} p'(x) + \varepsilon\|x\| \tag{3.16}$$

for all $x \in D(A)$. Then B is Miyadera – Voigt admissible on the space $\mathbf{X}_{t_0}^{\mathrm{loc}}$.

PROOF — We show that $\mathbf{X}_{t_0}^{\mathrm{loc}}$ is invariant under $\mathbf{V}_{t_0}$ and simultaneously verify the three conditions of Definition 3.2.1, then apply the method of abstract Volterra operators to the space $\mathbf{X}_{t_0}^{\mathrm{loc}}$ and the restriction of $\mathbf{V}_{t_0}$ to $\mathbf{X}_{t_0}^{\mathrm{loc}}$. First of all, let $x \in X$ and $x_n \in D(A)$ be an arbitrary sequence which is τ-convergent to x and satisfies $\|x_n\| \leq K$ for some $K > 0$. Then we have for $p \in \mathcal{P}$ and $F \in \mathbf{X}_{t_0}^{\mathrm{loc}}$

$$\begin{aligned}
p(B(F,t)(x_n - x_m)) &= p\left(\int_0^t F(t-s)BT(s)(x_n - x_m)\,\mathrm{d}s\right) \leq \\
&\leq \int_0^t p(F(t-s)BT(s)(x_n - x_m))\,\mathrm{d}s \leq \\
&\leq K_{F,p,\varepsilon} \int_0^t p'(BT(s)(x_n - x_m)) + \varepsilon \int_0^t \|BT(s)(x_n - x_m)\|\,\mathrm{d}s \leq \\
&\leq K_{F,p,\varepsilon} \cdot M_{p',\varepsilon'} \cdot p''(x_n - x_m) + (\varepsilon' \cdot K_{F,p,\varepsilon} + \varepsilon q)\|x_n - x_m\| \leq \\
&\leq K_{F,p,\varepsilon} \cdot M_{p',\varepsilon'} \cdot p''(x_n - x_m) + 2K(\varepsilon' \cdot K_{F,p,\varepsilon} + \varepsilon q).
\end{aligned}$$

This shows that first taking $\varepsilon > 0$ very small and then choosing $\varepsilon' > 0$ appropriately small, $p(B(F,t)(x_n - x_m))$ can be made sufficiently small for large indices n and m, i.e., we see that $B(F,t)x_n$ is a τ-Cauchy sequence. As it is norm bounded, by sequential completeness we obtain that this sequence is convergent, and we denote its limit by $\bar{B}(F,t)x$. One sees immediately that the definition of $\bar{B}(F,t)x$ is indeed independent of the particular choice of the sequence x_n and further that $\bar{B}(F,t)$ is linear. It is also important to note that according to this definition of $\bar{B}(F,t)$, we have

$$p(\bar{B}(F,t)x) \leq K_{F,p,\varepsilon} \cdot M_{p',\varepsilon'} \cdot p''(x) + 2K(\varepsilon' \cdot K_{F,p,\varepsilon} + \varepsilon q),$$

and if we choose x_n in accordance with (ηD), we get

$$p(\bar{B}(F,t)x) \leq K_{F,p,\varepsilon} \cdot M_{p',\varepsilon'} \cdot p''(x) + 2\eta(\varepsilon' \cdot K_{F,p,\varepsilon} + \varepsilon q). \tag{3.17}$$

Now, we proceed by verifying the three conditions of Definition 3.2.1. Assumption i) appears in the statement of the theorem. As for assumption ii), take $\eta > 0$ appearing in Definition 3.2.1 and let $x \in X$ arbitrary. By the assumption there exists $x_n \in D(A)$ such that (ηD) holds. We can now estimate the norm as follows

$$\|B(F,t)x_n\| = \sup_{\substack{\phi\in(X,\tau)'\\ \|\phi\|\le 1}} \left|\left\langle \int_0^t F(t-s)BT(s)x_n \,\mathrm{d}s, \phi \right\rangle\right| \le$$
$$\le \sup_{\substack{\phi\in(X,\tau)'\\ \|\phi\|\le 1}} \int_0^t |\langle F(t-s)BT(s)x_n, \phi\rangle| \,\mathrm{d}s \le$$
$$\le \sup_{\substack{\phi\in(X,\tau)'\\ \|\phi\|\le 1}} \|F\|\|\phi\| \int_0^t \|BT(s)x_n\| \,\mathrm{d}s \le q\|F\|\cdot\|x_n\| \le \eta q\|F\|\cdot\|x\|$$

Hence $B(F,t)x_n$ is norm-bounded and τ-convergent to $\bar{B}(F,t,x)$, we have by Proposition 1.2.2 that $\|\bar{B}(F,t)x\| \le \eta q\|F\|\cdot\|x\|$, showing that $\bar{B}(F,t) \in \mathscr{L}(X)$. The locality of $\bar{B}(F,t)$ and thus the bi-equicontinuity of $\{\bar{B}(F,t) : t \in [0,t_0]\}$ follows from (3.17). It is easy to see the τ-strong continuity of $t \mapsto \bar{B}(F,t)$. Indeed, let $x \in D(A)$ arbitrary, then

$$p((B(F,t+h) - B(F,t))x) =$$
$$= p\Big(\int_0^{t+h} F(t+h-s)BT(s)x \,\mathrm{d}s - \int_0^t F(t-s)BT(s)x \,\mathrm{d}s\Big) \le$$
$$\le p\Big(\int_0^t (F(t+h-s) - F(t-s))BT(s)x \,\mathrm{d}s\Big) +$$
$$+ p\Big(\int_t^{t+h} F(t+h-s)BT(s)x \,\mathrm{d}s\Big) \le$$
$$\le \int_0^t p((F(t+h-s) - F(t-s))BT(s)x) \,\mathrm{d}s + \int_t^{t+h} p(F(t+h-s)BT(s)x) \,\mathrm{d}s \xrightarrow{\tau} 0$$

as $h \to 0$, since the first term is dominated by $\|F\|\cdot\|BT(s)x\|$ and the integrand $p((F(t+h-s) - F(t-s))BT(s)x)$ converges to 0 for all $s \in [0,t]$ hence Lebesgue's dominated convergence theorem applies, while the second term obviously converges to 0. The case when $h < 0$ can be handled similarly. For an arbitrary $x \in X$ let $\varepsilon > 0$ and $p \in \mathcal{P}$ and take a sequence $x_n \xrightarrow{\tau} x$ which is also norm bounded and satisfies the estimates in (ηD), then

$$p\left(\bar{B}(F,t+h)x - \bar{B}(F,t)x\right) =$$
$$p\left(\bar{B}(F,t+h)(x-x_n) + \bar{B}(F,t+h)x_n - \bar{B}(F,t)x_n + \bar{B}(F,t)(x_n - x)\right) \le$$
$$p\left(\bar{B}(F,t+h)(x-x_n)\right) + p\left(\bar{B}(F,t+h)x_n - \bar{B}(F,t)x_n\right) + p\left(\bar{B}(F,t)(x_n-x)\right) \le \varepsilon$$

holds for sufficiently small $h > 0$, which can be concluded by means of the bi-equicontinuity of the family $\{\bar{B}(F,t) : t \in [0,t_0]\}$ and choosing n sufficiently large.

Now, it is straightforward to conclude that $\bar{B}(F,t) \in \mathbf{X}^{\text{loc}}_{t_0}(X)$, and also that the Volterra operator $\mathbf{V}_{t_0}$ given by
$$(\mathbf{V}_{t_0}F)(t)x := \bar{B}(F,t)x$$
is a bounded operator on the Banach space $\mathbf{X}^{\text{loc}}_{t_0}$, moreover for the operator norm we have that
$$\|[\mathbf{V}_{t_0}F](t)x\| \leq \eta q \|F\| \cdot \|x\|,$$
hence $\|\mathbf{V}_{t_0}\| < 1/\eta$. ■

REMARK 3.2.4 — Theorem 3.2.3 yields that the perturbed semigroup S will be also local in the sense of Definition 1.2.21.

REMARK 3.2.5 — If T is a quasi-contractive, bi-continuous semigroup, i.e., it is possible to take $M = 1$ in (EXP), then $\eta > 1$ can be arbitrary, hence in Definition 3.2.1 $\|\mathbf{V}_{t_0}\| < 1/\eta$ can be replaced by $\|\mathbf{V}_{t_0}\| < 1$. As a consequence, in Theorem 3.2.3 it also suffices to require $q < 1$. These should be contrasted with the renorming introduced in Section 1.3.b, since renorming does not destroy the topologies nor the boundedness only the actual norm bounds may change in the renormed space.

We turn our attention to the properties of the perturbed semigroup. First we state a lemma similar to Lemma 1.2.19.

LEMMA 3.2.6 — *Let T and S be bi-continuous semigroups with generator $(A, D(A))$ and $(A+B, D(A))$ respectively. Then for all $x \in D(A)$ the maps*
$$s \mapsto T(t-s)S(s)x$$
are τ-differentiable for all $t > 0$, $s \in [0,t]$ and $x \in D(A)$. Further one has
$$\frac{\mathrm{d}}{\mathrm{d}s} T(t-s)S(s)x = T(t-s)BS(s)x \tag{3.18}$$
and $s \mapsto T(t-s)BS(s)x$ is τ-Riemann integrable.

PROOF — Let $s \in (0,t)$, if $|h|$ is sufficiently small, we can write
$$\begin{aligned}&\frac{1}{h}\left[T(t-(s+h))S(s+h)x - T(t-s)S(s)x\right] = \\ &\quad = \frac{1}{h}T(t-s-h)\left(S(s+h)x - S(s)\right) + \frac{1}{h}\left(T(t-s-h) - T(t-s)\right)S(s)x\end{aligned} \tag{3.19}$$

Notice that $1/h\,[S(s+h) - S(s)]\,x$ is norm-bounded. Indeed, we have
$$\frac{1}{h}\left[S(s+h) - S(s)\right]x = \frac{1}{h}\int_s^{s+h} S(u)(A+B)x \,\mathrm{d}u$$

in the topology τ, hence we obtain

$$\frac{1}{h}\|S(s+h)x - S(s)x\| \le \frac{1}{h}\int_s^{s+h} \|S(u)(A+B)x\|\,\mathrm{d}u \le \sup_{u\in[0,t]}\|S(u)\|\cdot\|(A+B)x\|.$$

Therefore the bi-equicontinuity of the family $\{T(s) : s \in [0,t]\}$ yields that the first term on the right hand side of (3.19) converges to $T(t-s)(A+B)S(s)x$ for τ as $h \to 0$. It is obvious that the second term goes to $-AT(t-s)S(s)x$ also for the topology τ. We have proved the assertions concerning the differentiability, the last assertion is an immediate consequence of these. ■

COROLLARY 3.2.7 — *Let S be a Miyadera–Voigt perturbation of T in the sense of Theorem 3.2.3. Then the bi-continuous semigroup satisfies the following*

a) For all $x \in D(A)$

$$T(t)x = S(t)x + \int_0^t T(t-s)BS(s)x\,\mathrm{d}s \tag{IE'}$$

b) There exist $M' \ge 1$ and $\omega' \in \mathbb{R}$ depending only on η, t_0 and T such that

$$\|S(t)\| \le M'\mathrm{e}^{\omega' t},$$

for all $t \ge 0$.

PROOF — a) We apply Lemma 3.2.6 for T and S, and obtain the desired equality by integrating (3.18).

b)Let M, ω satisfy (EXP). We use directly the definition of S

$$S|_{[0,t_0]} = R\left(1, \mathbf{V}_{t_0}\right) T|_{[0,t_0]}$$

We have $\|\mathbf{V}_{t_0}\| < 1/\eta$, therefore from the Neumann series we can estimate the norm of the resolvent

$$\|R(1,\mathbf{V}_{t_0})\| \le \frac{\eta}{\eta - 1}$$

Define

$$M' := \frac{M\eta}{\eta-1}\max_{s\in[0,t_0]}\mathrm{e}^{\omega s},$$

then putting the above estimates together yields

$$\|S(t)\| \le M'$$

for all $t \in [0, t_0]$, proving part b). ■

§ 3.3 APPROXIMATION

In this section we would like to formulate an approximation result for a sequence of bi-continuous semigroups which are Miyadera – Voigt admissible perturbations of a preliminary given bi-continuous semigroup T. The main tool will be Theorem 1.2.10, and the application of the result will appear in Section §4.5 as establishing the positivity of certain transition semigroups. In order to conform with the assumptions of Theorem 1.2.10 we shall require uniform estimates in (3.15) and (3.16) for B_n.

THEOREM 3.3.1 — *Let $(A, D(A))$ generate a local bi-continuous semigroup T, and let $(D(A))$ be η-bi-dense. Suppose that B and B_n satisfy the assumptions of Theorem 3.2.3. Suppose furthermore that there exist $t_0 > 0$ and $q < 1/\eta^2$ such that*

$$\int_0^{t_0} \|B_n T(s)x\| \,\mathrm{d}s \leq q\|x\| \quad \textit{for all } x \in D(A) \textit{ and } n \in \mathbb{N}, \tag{3.20}$$

and for all $\varepsilon > 0$ and $p \in \mathcal{P}$ there exists $M_{p,\varepsilon} > 0$ and $p' \in \mathcal{P}$ such that

$$\int_0^{t_0} p(B_n T(s)x) \,\mathrm{d}s \leq M_{p,\varepsilon} p'(x) + \varepsilon\|x\| \quad \textit{for all } x \in D(A) \textit{ and } n \in \mathbb{N}. \tag{3.21}$$

Assume also that $B_n x \xrightarrow{\tau} Bx$ for all $x \in D(A)$. Then for the semigroups S_n and S generated by $(A + B_n, D(A))$ and $(A + B, D(A))$ respectively we have that

$$S_n(t)x \xrightarrow{\tau} S(t)x \quad \textit{for all } x \in X$$

and uniformly for t in compact intervals of $\mathbb{R}_+$.

PROOF — To prove the assertion we check the assumptions of Theorem 1.2.10. To this end, we recall from the proof of Theorem 3.2.3 that the perturbed semigroups S_n and S are defined via Volterra operators $\mathbf{V}_n$ on Banach spaces $\mathbf{X}_n$. Note that from the assumptions of this theorem it follows that the Banach space on which the Volterra operator acts can be taken the same for all B_n and B. From Corollary 3.2.7 b) we obtain that there exists $\omega' \in \mathbb{R}$ such that

$$\|S_n(t)\| \leq M' \mathrm{e}^{\omega' t}$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Thus assumption i) of Theorem 1.2.10 is settled.

To show assumption ii) requires more arguments. Rewriting (3.17) yields

$$p([\mathbf{V}_n F](t)x) \leq K_{F,p,\varepsilon} \cdot M_{p',\varepsilon'} \cdot p''(x) + 2\eta(\varepsilon' K_{F,p,\varepsilon} + \varepsilon q)\|x\|$$

for all $t \in [0, t_0]$ and $n \in \mathbb{N}$. This gives that for all $\varepsilon > 0$ and $p \in \mathcal{P}$ there exist $M_{p,\varepsilon} > 0$ and $p' \in \mathcal{P}$ such that

$$p([\mathbf{V}_n F](t)x) \leq M_{p,\varepsilon} p'(x) + \varepsilon\|x\| \quad \text{for all } t \in [0, t_0] \text{ and } n \in \mathbb{N}.$$

From (3.12), we see that

$$S_n(t) = \sum_{k=0}^{\infty} \mathbf{V}_n^k T|_{[0,t_0]} \quad \text{for } t \in [0, t_0].$$

Take now $N \in \mathbb{N}$ sufficiently large, then $\sum_{k=N+1}^{\infty} \|\mathbf{V}_n^k\| < \varepsilon$ for all $n \in \mathbb{N}$. Again from (3.17) we conclude

$$p([\mathbf{V}_n^k T|_{[0,t_0]}](t)x) \leq M_{p,\varepsilon} p'(x) + \varepsilon \|x\| \quad \text{for all } t \in [0, t_0],\, n \in \mathbb{N} \text{ and } 0 \leq k \leq N$$

with suitable constant $M_{p,\varepsilon} > 0$ and $p' \in \mathcal{P}$. Thus we may write

$$\begin{aligned} p(S_n(t)x) &\leq \sum_{k=0}^{N} p([\mathbf{V}_n^k T|_{[0,t_0]}[(t)x) + \sum_{k=N+1}^{\infty} \|\mathbf{V}_n^k\| \cdot \|T|_{[0,t_0]}\| \cdot \|x\| \leq \\ &\leq N M_{p,\varepsilon} p'(x) + \varepsilon(1 + \|T|_{[0,t_0]}\|) \cdot \|x\|. \end{aligned}$$

This shows that ii) in Theorem 1.2.10 is fulfilled on the interval $[0, t_0]$, but then on each compact interval in $\mathbb{R}_+$. The proof is hence complete. ∎

Chapter 4

Applications of the results

This part is devoted to the applications of the abstract results obtained in the previous sections. First, we stay a bit conceptual, as we deal with well-posedness of evolution equations both in the homogenous and inhomogeneous case. In Section §4.2, we apply the bounded perturbation theorem in various situations. The most important applications will come forth in the subsequent sections, where we consider the perturbations of the Ornstein – Uhlenbeck semigroup with non-linear drifts. We apply our Miyadera – Voigt type perturbation theorem and obtain the existence of transition semigroups on $\mathrm{C_b}(H)$ and on $\mathrm{C}^\theta_\mathrm{b}(H)$. Finally, we concentrate on the positivity of the perturbed semigroups.

§ 4.1 Abstract Cauchy problems

To study initial value problems, it is usual to rewrite them in the form of abstract Cauchy problems. As an example, consider the following equation, which describes a population system on the half line $\mathbb{R}_+$ with diffusion, transport, death rate d and birth rate b:

$$\begin{cases} \partial_1 v(t,x) = \partial_2^2 v(t,x) + \partial_2 v(t,x) + b(t)\cdot v(t,x) - d(t)\cdot v(t,x) & \text{for } t \geq 0 \\ v(t,0) = 0 \\ v(0,x) = f(x). \end{cases} \tag{PE}$$

In order to solve this equation, we may rewrite it as follows. Let $X = \mathrm{C_b}(\mathbb{R}_+)$ and $A = \partial_2^2 + \partial_2 + M_b - M_d$, where M_b and M_d denote the multiplication with b and d respectively. The boundary conditions above is what we put in the domain of A

$$D(A) := \{g :\ g \in \mathrm{C_b}(\mathbb{R}_+),\ g(0) = 0,\ Ag \in \mathrm{C_b}(\mathbb{R}_+)\}.$$

We have then the abstract initial value problem in the Banach space $\mathrm{C_b}(\mathbb{R}_+)$

$$\begin{cases} u'(t) = Au(t) & \text{for } t \geq 0 \\ u(0) = f \in D(A). \end{cases} \tag{ACP}$$

The connection between the solution of the two systems is the following $(u(t))(x) = v(t,x)$. If we can show that (ACP) has a solution, this provides a solution to our original problem. This justifies the importance of investigating abstract Cauchy problems.

a) The homogenous case. Here, as in the case of C_0-semigroups we establish the connection between the generator property of A and the well-posedness of the abstract Cauchy problem (ACP). The proof is the straightforward generalisation of [31, Thm. II.6.7], though the result was not known explicitly. Therefore we present it here for the sake of completeness and as a reference. Let A be a linear operator. Consider the Cauchy problem:

$$\begin{cases} u'(t) = Au(t) & \text{for } t \geq 0 \\ u(0) = x \in D(A) \end{cases} \tag{ACP}$$

DEFINITION 4.1.1 — The initial value problem (ACP) is called *well-posed*, if

i) For all $x \in D(A)$ there exists a solution $u(t) := u(t,x)$ of (ACP), with $u \in \mathrm{B}_{\mathrm{loc}}(\mathbb{R}_+, X) \cap \mathrm{C}^1(\mathbb{R}_+, (X,\tau))$ and $u' \in \mathrm{B}_{\mathrm{loc}}(\mathbb{R}_+, X)$, where the differentiation is understood in the vector valued sense with respect to the topology τ.

ii) The solution of (ACP) is unique.

iii) The solution u of (ACP) depends continuously on the initial data x, i.e., if x_n is norm bounded and τ-converges to 0 then the solutions $u_n(t) := u_n(t, x_n)$ converge to 0 in the topology τ and uniformly on compact intervals $[0, t_0] \subseteq \mathbb{R}_+$.

THEOREM 4.1.2 — *If* $(A, D(A))$ *generates a bi-continuous semigroup then the abstract Cauchy problem* (ACP) *is well-posed.*

PROOF — Suppose that $(A, D(A))$ generates a bi-continuous semigroup T. Take $x \in D(A)$ and define

$$u(t) := T(t)x.$$

Then, by definition of an infinitesimal generator and the properties of a bi-continuous semigroup one has $u \in \mathrm{C}^1(\mathbb{R}_+, (X,\tau))$, $u \in \mathrm{B}_{\mathrm{loc}}(\mathbb{R}_+, X)$ and also $u' \in \mathrm{B}_{\mathrm{loc}}(\mathbb{R}_+, X)$, hence there is no doubt that u is a solution of (ACP). Assume that $\tilde{u}$ is also a solution to our abstract Cauchy problem. Take $t > 0$ and define

$$U(s) := T(t-s)\tilde{u}(s) \quad \text{for } s \in [0,t].$$

We may differentiate this function U with respect to s by Lemma 1.2.19, thus we can write

$$U'(s) = -AT(t-s)\tilde{u}(s) + T(t-s)\tilde{u}'(s) = -AT(t-s)\tilde{u}(s) + T(t-s)A\tilde{u}(s),$$

which equals simply 0 since T and A in our case commute (cf. Theorem 1.2.7). Hence U is constant and therefore

$$T(t)x = U(0) = U(t) = \tilde{u}(t),$$

establishing the uniqueness. The continuous dependence is the reformulation of the bi-equicontinuity of T.

■

We proved only one direction of the usual well-posedness theorem, namely that the existence of a semigroup implies the well-posedness of (ACP). In applications, this is the important fact which is usually referred to. The converse, however, it is not known to the author in its general form. When τ is metrisable then it is relatively easy to settle. Also when some "locality" (compare with Definition 1.2.21) of the solutions are assumed, it seems to hold without any extra assumptions on τ. For a general, nice formulation, a substitute for the introduced notion of well-posedness may be necessary. Moreover, we remark that it is possible to study not just classical solutions but also *mild solutions* (integrated solutions, weak solutions). In doing so, well-posedness can also be proved (cf. [5, Sec. 3.1] and [31, Sec. II.6]).

b) Inhomogeneous Cauchy problems. We would like to add a term to our abstract Cauchy problem (ACP) resulting in the inhomogeneous problem

$$\begin{cases} u'(t) = Au(t) + f(t) & \text{for } t \geq 0 \\ u(0) = x \in D(A). \end{cases} \tag{iACP}$$

First we carry out some abstract considerations and apply the perturbation results of Chapter 3. To obtain the solution to (iACP) we use operator matrices (see Section 1.3.c and [31, Sec. VI.7]).

Let Y be either

1) $\mathrm{C_b}(\mathbb{R}_+, X)$ with the supremum norm and $\tau'' = \tau_{\mathrm{c}}$, or

2) $\mathrm{L}^1(\mathbb{R}_+, X)$ with the L^1-norm and τ'' the norm-topology[1].

In both cases, the left shift S is a bi-continuous semigroup on Y, denote the generator by $(C, D(C))$, in fact $C = \frac{\mathrm{d}}{\mathrm{d}s}$ is the differentiation with an appropriate domain $D(C)$. On the product space

$$\mathcal{X} := X \times Y$$

consider the direct product semigroup as in Section 1.3.c

$$\mathcal{T}(t) := \begin{pmatrix} T(t) & 0 \\ 0 & S(t) \end{pmatrix},$$

which is bi-continuous semigroup for the respective norm and topology. The generator is given by the formula

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(C).$$

Now, define

$$\mathcal{B} := \begin{pmatrix} 0 & \delta_0 \\ 0 & 0 \end{pmatrix},$$

with domain $D(\mathcal{B}) := \mathrm{C_b}(\mathbb{R}_+)$ in case 1), and $D(\mathcal{B}) := \mathrm{W}^{1,1}(\mathbb{R}_+)$ in case 2). In the first case $\mathcal{B} \in \mathscr{L}(\mathcal{X})$ and clearly $\mathcal{B}$ is continuous for the product topology τ. Thus by

[1]The Banach space $\mathrm{L}^1(\mathbb{R}_+, X)$ is the space of the Bochner integrable functions.

Theorem 3.1.5 there exists a bi-continuous semigroup $\mathcal{S}$ with generator $(\mathcal{A}+\mathcal{B}, D(\mathcal{A}))$ and the variation of parameters formula holds:

$$\mathcal{S}(t) = \mathcal{T}(t) + \int_0^t \mathcal{S}(t-s)\mathcal{B}\mathcal{T}(s)\,\mathrm{d}s.$$

In the second case, we show that $\mathcal{B}$ is Miyadera-Voigt admissible on $\mathbf{X}_1$ (see Definition 3.2.1). Clearly $\mathcal{B} : X \times D(C) \to X \times Y$ continuous. For $F \in \mathbf{X}_1$ and $t \in [0,1]$ define

$$B(F,t)\tbinom{x}{f} := \int_0^t F(t-s)\mathcal{B}\mathcal{T}(s)\tbinom{x}{f}\,\mathrm{d}s = \int_0^t F(t-s)\tbinom{[S(s)f](0)}{0}\,\mathrm{d}s = \int_0^t F(t-s)\tbinom{f(s)}{0}\,\mathrm{d}s.$$

Then

$$\|B(F,t)\tbinom{x}{f}\| \leq \|F\| \cdot \|f\|_1, \tag{4.1}$$

thus $B(F,t)$ extends to $[\mathbf{V}F](t) \in \mathscr{L}(\mathcal{X})$. It is also clear that $\mathbf{V}$ maps $\mathbf{X}_1$ into itself and it is norm-bounded with $\|\mathbf{V}\| \leq 1$. Take now $f \in Y$, then

$$\int_0^t F(t-s)\tbinom{f(s)}{0}\,\mathrm{d}s \tag{4.2}$$

exists as a Bochner integral (see page 5). Thus when $f_n \in D(C)$ converges to f in L^1, by (4.1) we must have that $B(F,t)\tbinom{x}{f_n}$ converges to the integral in (4.2). This means that

$$[\mathbf{V}F](t)\tbinom{x}{f} = \int_0^t F(t-s)\tbinom{f(s)}{0}\,\mathrm{d}s,$$

for all $t \in [0,1]$, $x \in X$ and $f \in Y$. Now, similarly as in the bounded perturbation theorem one can compute the spectral radius $r(\mathbf{V}) = 0$. This shows that the assumptions of Theorem 3.2.2 are fulfilled, hence there exists a bi-continuous semigroup $\mathcal{S}$ with generator $(\mathcal{A}+\mathcal{B}, D(\mathcal{A}))$.

Note that in both cases the perturbed semigroup $\mathcal{S}$ is given by the series

$$\mathcal{S}(t)\tbinom{x}{f} = \mathcal{T}(t)\tbinom{x}{f} + [\mathbf{V}\mathcal{T}](t)\tbinom{x}{f} + \sum_{n=2}^{\infty}[\mathbf{V}^n\mathcal{T}](t)\tbinom{x}{f}. \tag{4.3}$$

Taking into account that $\pi_2([\mathbf{V}T](t)\tbinom{x}{f}) = 0$ by the integral representation of $\mathbf{V}$, where π_2 is the projection of $\mathcal{X}$ to the second coordinate, we have that the infinite sum in (4.3) vanishes. We obtain therefore the following theorem.

THEOREM 4.1.3 — *Let T be a bi-continuous semigroup on X, and denote its generator by $(A, D(A))$. Further let S be the left shift semigroup on $Y := \mathrm{C_b}(\mathbb{R}_+, X)$ or on $Y := \mathrm{L}^1(\mathbb{R}_+, X)$. Then the operator*

$$\mathcal{S}(t) := \begin{pmatrix} T(t) & Q(t) \\ 0 & S(t) \end{pmatrix}$$

is a bi-continuous semigroup on the product space $\mathcal{X} = X \times Y$, where

$$Q(t)f := \int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

The generator of $\mathcal{T}$ is

$$\mathcal{A} = \begin{pmatrix} A & \delta_0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix}$$

with diagonal domain $D(A) \times D(C)$. Moreover for $x \in D(A)$ and $f \in D(C)$ the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \geq 0 \\ u(0) = x \end{cases} \tag{iACP}$$

has a unique solution $u(t)$.

PROOF — Only the last assertion was not shown before. Take $x \in D(A)$ and $f \in D(C)$ and define $u(t) = \pi_1(\mathcal{S}(t)\binom{x}{f})$, where π_1 is the projection from $\mathcal{X}$ to X. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\binom{u(t)}{S(t)f} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(t)\binom{x}{f} = \mathcal{A}\binom{u(t)}{S(t)f} = \binom{Au(t)+\delta_0(S(t)f)}{S(t)f'} = \binom{Au(t)+f(t)}{S(t)f'},$$

this shows that $u(t)$ solves (iACP). The uniqueness follows from the fact that all solutions u have to satisfy $u(t) = T(t)x + Q(t)f$, which can be proved by a simple calculation. ∎

This matrix technique could also allow another kinds of functions f appearing in (iACP). At this point, we obtain solutions when f is regular enough, i.e., $f \in D(C)$. It would be nice to relax this condition as in the case of C_0-semigroups [31, Cor. VI.7.8], replacing the time regularity of f by certain space regularity , e.g., considering $f : \mathbb{R}_+ \to D(A)$ continuous functions.

§ 4.2 BOUNDED PERTURBATIONS

We give some applications of the bounded perturbation theorem proved in Section §3.1. The examples presented here are rather of abstract flavour than real applications. First, we consider bi-continuous semigroups on dual spaces. Let T be a bi-continuous semigroup on X' for the weak*-topology. Denote its generator by $(A, D(A))$ and let $B \in \mathscr{L}(X)$. When Theorem 2.1.3 applies, i.e., when X is separable or when T is local, we have a C_0-semigroup S with $S' = T$. Thus applying the bounded perturbation theorem for C_0-semigroups (see [31, Sec. III.1] or [5, Cor. 3.5.6]) for S and B, we obtain that there exist a bi-continuous semigroup on X' with generator $(A + B', D(A))$. This is nevertheless true for any weak*-bi-continuous semigroup:

PROPOSITION 4.2.1 — *Let T be a bi-continuous semigroup on X' with respect to the weak*-topology with generator $(A, D(A))$, then for any $B \in \mathscr{L}(X)$ we have that $(A + B', D(A))$ is a generator of a bi-continuous semigroup on X' with respect $\sigma(X', X)$.*

PROOF — In view of Theorem 3.1.5, it suffices to show B' is $\sigma(X', X)$-continuous (on norm-bounded sets) for any operator $B \in \mathscr{L}(X)$. To this end, consider the $\sigma(X', X)$ open neighbourhood U of 0 in X' determined by the seminorms $p_i(\cdot) = |\langle x_i, \cdot \rangle|$, $x_i \in X$, $i = 1, \ldots n$ and the positive real number ε

$$U = \{x' : x' \in X', \ \ p_i(x') < \varepsilon, \ \ i = 1, \ldots, n\}.$$

Then we have to find a $\sigma(X', X)$-neighbourhood of 0 for which $B'V \subseteq U$ holds. However, it is obvious that the $\sigma(X', X)$-neighbourhood V determined by the seminorms $q_i(\cdot) = |\langle Bx_i, \cdot \rangle|$ and $\varepsilon > 0$ fulfills this requirement. ∎

Next, we prove a similar result for the Mackey topology $\mu(X', X)$, i.e., the topology determined by the family of seminorms

$$\left\{ \sup_{x \in K} |\langle x, x' \rangle| : K \subseteq X \text{ is weakly-compact, absolute convex} \right\}.$$

PROPOSITION 4.2.2 — *Let T be a bi-continuous semigroup with respect to the Mackey topology with generator $(A, D(A))$ and $B \in \mathscr{L}(X)$. Then $(A + B', D(A))$ is the generator of a bi-continuous semigroup on X' with respect to the Mackey topology .*

PROOF — We show first that the adjoint B' of an operator satisfying the above assumptions is also continuous for the Mackey topology. In Proposition 4.2.1 we saw that B' is weak*-continuous. Next, we take $K \subseteq X$ a weakly compact, absolutely convex set. Then $BK \subseteq X$ is also weakly compact and absolutely convex. For a given $\mu(X', X)$-neighbourhood U of 0

$$U := \{x' : x' \in X', \ \ p_i(x') < \varepsilon, \ \ i = 1, \ldots, n\},$$

with

$$p_i(x') = \sup\{|\langle x, x' \rangle| : x \in K_i\} \quad \text{and } \varepsilon > 0,$$

we have to find a $\mu(X', X)$-neighbourhood $V \subseteq X'$ of 0 such that $B'V \subseteq U$. But the neighbourhood V determined by the seminorms

$$q_i(x') = \sup\{|\langle x, x' \rangle| : x \in BK_i\}$$

and $\varepsilon > 0$ fulfills this requirement. We have seen that the assumptions of Theorem 3.1.5 are satisfied, therefore our statement is proved. ∎

Finally, we give some examples of τ_c-continuous operators.

REMARK 4.2.3 — Let (Ω, τ) be a locally compact, Hausdorff topological space, and suppose that $\Phi : \Omega \to \Omega$ is continuous. Then the linear operator B on $\mathrm{C_b}(\Omega)$ defined as

$$(Bf)(x) = f(\Phi(x)) \quad \text{for } x \in \Omega,$$

is continuous on $\mathrm{C_b}(\Omega)$ with respect to the compact-open topology τ_c. Hence Theorem 3.1.5 can be applied to these perturbations.

REMARK 4.2.4 — For a given function $f \in \mathrm{C_b}(\mathbb{R})$ the multiplication operator

$$V_f : \mathrm{C_b}(\mathbb{R}) \to \mathrm{C_b}(\mathbb{R}), \quad V_f(g) := f \cdot g,$$

is norm-bounded and τ_c-continuous on norm-bounded sets.

As an application, we consider the heat equation in $\mathrm{C_b}(\mathbb{R})$ with a bounded, continuous potential $V = V_f$:

$$\begin{cases} u'(t) = \Delta u(t) + V u(t) \quad \text{for all } t \geq 0, \\ u(0) = u_0, \quad u_0 \in \mathrm{C_b}(\mathbb{R}). \end{cases} \tag{HE}$$

Let

$$D(\Delta) := \{ f : f \in \mathrm{C_b}(\mathbb{R}),\ f'' \text{ exists}, f'' \in \mathrm{C_b}(\mathbb{R}) \}.$$

Then it is easy to see that $(\Delta, D(\Delta))$ is the generator of a bi-continuous semigroup (see Section §2.4). Indeed, it generates the Gaussian-semigroup

$$[P(t)f](x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{\frac{-|y-x|^2}{4t}} f(y)\,\mathrm{d}y \quad \text{for } f \in \mathrm{C_b}(\mathbb{R}).$$

In order to solve the abstract Cauchy problem corresponding to the equation (HE), it suffices to show that $(\Delta + V, D(\Delta))$ is the generator of a bi-continuous semigroup. This is, however, a straightforward consequence of Theorem 3.1.5 and Remark 4.2.4 above. We have therefore the well-posedness of (HE) by Theorem 4.1.2.

§ 4.3 TRANSITION SEMIGROUPS ON $\mathrm{C_b}(H)$

We turn our attention to the perturbation of the Ornstein – Uhlenbeck semigroup by a *non-linear drift operator*

$$Bf(x) := \langle F(x), \mathrm{D}f(x) \rangle, \quad f \in \mathrm{C_b}(H),$$

where $F \in \mathrm{C_b}(H,H)$. Similar results were obtained by several authors. When $F \in \mathrm{C_{ub}}(H,H)$ and an invariant measure μ exists for P, Rhandi proves the existence of transition semigroup R with generator $(\overline{L + B, D(L)})$ on $\mathrm{L}^2(H,\mu)$. Da Prato uses dissipativity methods to obtain a $\mathrm{C_0}$-semigroup on $\overline{D(L)}$ and applies the theory of $\mathrm{C_0}$-semigroups in the case $F \in \mathrm{C_{ub}}(H,H)$ [16, 17, 18, 19]. Approximation and stochastic methods were also applied [77]. Goldys and Kocan [41] considers semigroups with respect to certain mixed topologies and prove the existence of the perturbed transition semigroups when $F \in \mathrm{Lip_b}(H,H)$. In view of Section A.2.b, the semigroups obtained in [41] coincide with bi-continuous semigroups for the topology τ_c, thus our result considerably strengthens that of [41].

Recall the notations from Section §2.3. The Cameron – Martin formula provides the relation between Gaussian measures in Hilbert spaces [18]. If Q is positive, self-adjoint, trace-class operator, then the Gaussian measure $\mathcal{N}(0,Q)$ exists. Under $Q^{-1/2}$ we understand the pseudo-inverse of $Q^{1/2}$, i.e., for $y \in \operatorname{rg} Q^{1/2}$ we define $Q^{-1/2}(y) := x$, where x is of minimal norm in the pre-image $Q^{-1/2}(\{y\})$ of y.

THEOREM 4.3.1 [Cameron – Martin formula] — *Let Q be a positive, self-adjoint, trace class operator on the separable Hilbert space H, and suppose that $a \in \operatorname{rg} Q^{1/2}$. Then the Gaussian measures $\mathcal{N}(0,Q)$ and $\mathcal{N}(a,Q)$ are equivalent, and one has for the Radon – Nikodým derivative*

$$\frac{\mathrm{d}\mathcal{N}(a,Q)}{\mathrm{d}\mathcal{N}(0,Q)}(x) = d(a,x) \quad \textit{for a.a. } x \in H,$$

with

$$d(a,x) = \mathrm{e}^{-1/2\|Q^{-1/2}a\|^2 + \langle Q^{-1/2}x, Q^{-1/2}a\rangle}, \quad x \in H.$$

We now collect important properties of P from [16]. Denote by $\mathrm{D}f$ the derivative[2] of a Fréchet differentiable function $f \in \mathrm{C_b}(H)$.

In the sequel, we assume that

$$\operatorname{rg} S(t) \subseteq \operatorname{rg} Q^{1/2}(t), \tag{$*$}$$

and introduce the following function

$$\Lambda(t) := Q(t)^{-1/2} S(t) \quad \text{for } t > 0. \tag{4.4}$$

It is easy to see that $\Lambda(t)$ linear, and by the closed graph theorem it is bounded. The next result is basically proved in [16] as Proposition 2.3. for $f \in \mathrm{C_{ub}}(H)$. It is, however, valid for bounded, continuous functions as presented here.

LEMMA 4.3.2 — *For all $f \in \mathrm{C_b}(H)$ and $t > 0$ we have $P(t)f \in \mathrm{C_b^1}(H)$ (also $P(t)f \in \mathrm{C_{ub}^\infty}(H)$) and*

$$\big\langle [\mathrm{D}P(t)f](x), h \big\rangle = \int_H \big\langle Q(t)^{-1/2}\cdot, \Lambda(t)h \big\rangle\, f(S(t)x + \cdot)\, \mathrm{d}\mathcal{N}(0, Q(t)), \tag{4.5}$$

and hence

$$\big\|[\mathrm{D}P(t)f](x)\big\|^2 \le \|\Lambda(t)\|^2 \cdot \big|[P(t)f^2](x)\big| \quad \textit{for all } x \in H. \tag{4.6}$$

In particular,

$$\big\|[\mathrm{D}P(t)f](x)\big\| \le \|\Lambda(t)\| \cdot \|f\| \quad \textit{for all } x \in H. \tag{4.7}$$

PROOF — Let $x \in H$ and $t > 0$. By assumption ($*$) and Theorem 4.3.1 the measures $\mathcal{N}(S(t)x, Q(t))$ and $\mathcal{N}(0, Q(t))$ are equivalent. The Radon – Nikodým derivative is given by

$$d(S(t)x, y) := \frac{\mathrm{d}\mathcal{N}(S(t)x, Q(t))}{\mathrm{d}\mathcal{N}(0, Q(t))}(y) = \mathrm{e}^{-1/2\|Q(t)^{-1/2}S(t)x\|^2 + \langle Q(t)^{-1/2}y, Q(t)^{-1/2}S(t)x\rangle}$$

for almost all $y \in H$. Therefore we can write

$$[P(t)f](x) = \int_H f(\cdot)\mathrm{e}^{-1/2\|\Lambda(t)x\|^2 + \langle Q(t)^{-1/2}\cdot, \Lambda(t)x\rangle}\, \mathrm{d}\mathcal{N}(0, Q(t)).$$

[2]Note that D is a closed operator on $\mathrm{C_b}(H)$, thus it can be interchanged with integrals whenever both terms make sense.

Differentiating this equality and interchanging integration and derivation yields

$$\big\langle \mathrm{D}[P(t)f](x), h\big\rangle = \Big\langle \mathrm{D}\int_H \langle f(\cdot)\mathrm{e}^{-1/2\|\Lambda(t)x\|^2+\langle Q(t)^{-1/2}\cdot,\Lambda(t)x\rangle}\,\mathrm{d}\mathcal{N}(0,Q(t)), h\Big\rangle =$$

$$= \int_H f(\cdot)\left(\langle Q(t)^{-1/2}\cdot, \Lambda(t)h\rangle - \langle \Lambda(t)x, \Lambda(t)h\rangle\right) d(S(t)x,\cdot)\,\mathrm{d}\mathcal{N}(0,Q(t)) =$$

$$= \int_H f(\cdot)\langle Q(t)^{-1/2}(\cdot - S(t)x), \Lambda(t)h\rangle d(S(t)x,\cdot)\,\mathrm{d}\mathcal{N}(0,Q(t)) =$$

$$= \int_H f(\cdot + S(t)x)\langle Q(t)^{-1/2}\cdot, \Lambda(t)h\rangle\,\mathrm{d}\mathcal{N}(0,Q(t)).$$

This shows (4.5). For the other two assertions, we use the Hölder inequality and let h range over $B(0,1) \subseteq H$. $P(t)f \in \mathrm{C}^\infty_{\mathrm{ub}}(H)$ can be seen by repeating these arguments. ∎

The fact that for all $f \in \mathrm{C_b}(H)$ one has $P(t)f \in \mathrm{C}^\infty_{\mathrm{ub}}(H)$ is called the *strong Feller property*.

From now on, we assume the following:

$$\int_0^{t_0} \|\Lambda(s)\|\,\mathrm{d}s < +\infty \quad \text{for some } t_0 > 0. \tag{$**$}$$

The next result is similar to [16, Proposition 2.4].

LEMMA 4.3.3 — *For $\lambda > 0$ and $f \in \mathrm{C_b}(H)$ we have $R(\lambda, L)f \in \mathrm{C}^1_{\mathrm{b}}(H)$ (indeed $R(\lambda, L)f \in \mathrm{C}^\infty_{\mathrm{ub}}(H)$), and*

$$\big\|[\mathrm{D}R(\lambda, L)f](x)\big\| \le M \cdot \|f\|, \tag{4.8}$$

where M is independent of f. Moreover, for all $K \subseteq H$ compact set and $\varepsilon > 0$ there exist a constant $M_{K,\varepsilon} > 0$ and a compact set $K' \subseteq H$ such that

$$\sup_{x\in K}\big\|[\mathrm{D}R(\lambda, L)f](x)\big\| \le M_{K,\varepsilon} \cdot \sup_{x\in K'} |f(x)| + \varepsilon\|f\|. \tag{4.9}$$

PROOF — Let $t_0 > 0$ be as in $(**)$ and $0 < \lambda \in \varrho(L)$. Then

$$R(\lambda, L)f = \mathrm{e}^{-\lambda t_0}P(t_0)R(\lambda, L)f + \int_0^{t_0} \mathrm{e}^{-\lambda s}P(s)f\,\mathrm{d}s$$

for all $f \in \mathrm{C_b}(H)$. This shows that $R(\lambda, L)f \in \mathrm{C}^1_{\mathrm{b}}(H)$. By (4.6) we have

$$\big\|[\mathrm{D}R(\lambda, L)f](x)\big\| \le \mathrm{e}^{-\lambda t_0}\,\big\|[\mathrm{D}P(t_0)R(\lambda, L)f](x)\big\| + \int_0^{t_0} \mathrm{e}^{-\lambda s}\,\big\|[\mathrm{D}P(s)f](x)\big\|\,\mathrm{d}s \le$$

$$\le \mathrm{e}^{-\lambda t_0}\|\Lambda(t_0)\| \cdot \big|\big[P(t_0)(R(\lambda, L)f)^2\big](x)\big|^{1/2} + \int_0^{t_0} \|\Lambda(s)\| \cdot \big|[P(s)f^2](x)\big|^{1/2}\,\mathrm{d}s.$$

From this, we conclude

$$\big\|[\mathrm{D}R(\lambda,L)f](x)\big\| \le \Big(\mathrm{e}^{-\lambda t_0}\|\Lambda(t_0)\|\cdot\|R(\lambda,L)\| + \int_0^{t_0}\|\Lambda(s)\|\,\mathrm{d}s\Big)\cdot\|f\|, \tag{4.10}$$

showing (4.8). Further, by Lemma 1.2.23 we can write

$$\begin{aligned}
\sup_{x\in K}\big\|[\mathrm{D}R(\lambda,L)f](x)\big\| &\le \mathrm{e}^{-\lambda t_0}\|\Lambda(t_0)\|\Big(\sup_{x\in K''}|[R(\lambda,L)f)](x)| + \varepsilon\|R(\lambda,L)\|\cdot\|f\|\Big)+\\
&\quad+\int_0^{t_0}\|\Lambda(s)\|\,\mathrm{d}s\cdot\Big(\sup_{x\in K''}|f(x)|+\varepsilon\|f\|\Big)\le\\
&\le \mathrm{e}^{-\lambda t_0}\|\Lambda(t_0)\|\Big(C\sup_{x\in K'}|f(x)|+\varepsilon\|f\|+\varepsilon\|R(\lambda,L)\|\cdot\|f\|\Big)+\\
&\quad+\int_0^{t_0}\|\Lambda(s)\|\,\mathrm{d}s\cdot\Big(\sup_{x\in K''}|f(x)|+\varepsilon\|f\|\Big)\le M_{K,\varepsilon}\sup_{x\in K'}|f(x)|+M'\varepsilon\|f\|.
\end{aligned}$$

This justifies (4.9) and finishes the proof. ∎

We are now ready to state our main result. Let $F\in \mathrm{C_b}(H,H)$ and define the operator B by

$$D(B) := \mathrm{C}^1_\mathrm{b}(H),\quad (Bf)(x) := \langle F(x), \mathrm{D}f(x)\rangle.$$

PROPOSITION 4.3.4 — *Let B be as above. Then for all $f\in D(L)$ the orbits*

$$t\mapsto BP(t)f,\quad t\ge 0,$$

are locally norm-bounded.

PROOF — By Lemma 4.3.3 one sees that $B\in\mathscr{L}(D(L),\mathrm{C_b}(H))$. Since the set

$$\{P(t)f : t\in[0,t_0]\}$$

is $\|\cdot\|_L$ bounded for all $f\in D(L)$, the assertion follows immediately. ∎

PROPOSITION 4.3.5 — *Let B be as above. Then $B : (D(L),\tau_L)\to(\mathrm{C_b}(H),\tau_\mathrm{c})$ is continuous on $\|\cdot\|_L$ norm-bounded sets. In particular the orbits*

$$t\mapsto BP(t)f,\quad t\ge 0,$$

are τ_c-continuous for all $f\in D(L)$.

PROOF — The continuity of B is immediate from (4.9). Also the continuity of the orbits follows from this and Proposition 4.3.4. ∎

From (4.7) and ($**$) we conclude that (3.15) also holds for a sufficiently small $t_0 > 0$ and some $0 < q < 1$:

$$\int_0^{t_0} \|BP(s)f\| \,\mathrm{d}s \leq \|F\| \cdot \int_0^{t_0} \|\Lambda(s)\| \,\mathrm{d}s \cdot \|f\| \leq q\|f\| \quad \text{for all } f \in D(L). \tag{4.11}$$

Assumption (3.16) in Theorem 3.2.3 is justified by the integrated version of (4.6), i.e., for all $p_K \in \mathcal{P}$

$$\int_0^{t} p_K(BP(s)f) \,\mathrm{d}s \leq \|F\| \cdot \int_0^{t} \|\Lambda(s)\| \cdot p_K(P(s)f) \,\mathrm{d}s \quad \text{for all } f \in D(L). \tag{4.12}$$

Putting together these results and applying Theorem 3.2.3, we obtain the following.

THEOREM 4.3.6 — *Let P be the Ornstein – Uhlenbeck semigroup on $\mathrm{C}_{\mathrm{b}}(H)$ with generator L. For $F \in \mathrm{C}_{\mathrm{b}}(H, H)$ define the operator B as*

$$D(B) = \mathrm{C}_{\mathrm{b}}^1(H), \quad (Bf)(x) := \langle F(x), \mathrm{D}f(x)\rangle.$$

Then $(L + B, D(L))$ is the generator of a bi-continuous semigroup R on $\mathrm{C}_{\mathrm{b}}(H)$.

These semigroups are usually referred to as *transition semigroups*, and Section §2.2, in particular Theorem 2.2.5, immediately yields the following corollary.

COROLLARY 4.3.7 — *Let R be a transition semigroup on $\mathrm{C}_{\mathrm{b}}(H)$ as obtained previously and $\mathcal{K} \subseteq \mathscr{M}(H)$ be a norm-bounded, weak*-compact set. Then the family*

$$\{R'(t)\nu : t \in [0, t_0],\ \nu \in \mathcal{K}\}$$

is tight. In particular, for all $K \subseteq H$ compact set we have the tightness of the family

$$\{R'(t)\delta_x : t \in [0, t_0],\ x \in K\}.$$

As for open questions, we mention the case of F being unbounded. A particular result is established in [41] in case Q is boundedly invertible, and F is Gâteaux differentiable with bounded derivative. It seems that our perturbation method can not be directly applied for such drift operators, because of the nature of norm estimates on $\mathrm{C}_{\mathrm{b}}(H)$. Perturbation must be combined with approximation to obtain such results.

§ 4.4 TRANSITION SEMIGROUPS ON $\mathrm{C}_{\mathrm{b}}^{\theta}(H)$

We present a perturbation result for the Ornstein – Uhlenbeck semigroup on $\mathrm{C}_{\mathrm{b}}^{\theta}(H)$ similar to the result in Section §4.3. However, we consider a "bit more unbounded" perturbing operator B, which also involves the fractional powers of A. We would like to consider perturbations of the type

$$Bf(x) = \langle F(x), (-A)^{\gamma}\mathrm{D}f(x)\rangle, \quad \text{for } f \in D(L).$$

Such perturbation result was obtained by Da Prato in [19]. He used interpolation spaces and dissipativity, and proved the existence of a transition semigroup with generator $(L +$

$B, D(L))$ on $\overline{D(L)} \subseteq \mathrm{C}^{\theta}_{\mathrm{ub}}(H)$ when $\gamma = 1/2$ and the Hölder norm of $F \in \mathrm{C}^{\theta}_{\mathrm{b}}(H, H)$ is sufficiently small. We work on $\mathrm{C}^{\theta}_{\mathrm{b}}(H)$ and prove the perturbation result for $\gamma \in [0, 1/2)$ without any assumption on F. The case $\gamma = 1/2$ remains open, and because of the nature of the results below, its quite doubtful that without modifying the abstract perturbation result it can be settled. Da Prato asks whether the condition on $\|F\|_{\theta}$ can be dropped in general. The result presented here is a joint work with A. Es – Sarhir.

In order to $\mathrm{D}f(x) \in D((-A)^{\gamma})$ we need more assumptions on S. We restrict ourselves to the case of *symmetric Ornstein – Uhlenbeck semigroups:*

i) A is self-adjoint and there exists $\omega > 0$ such that

$$\langle Ax, x\rangle \le -\omega \|x\|^2, \quad x \in D(A).$$

ii) A^{-1} is of trace class.

Further we suppose that $Q = I$. Notice that in this case $S(t) = \mathrm{e}^{tA}$ by functional calculus[3], and also $(-A)^{\gamma}$ can be defined. Moreover, S is a holomorphic, compact C_0-semigroup on H. Then

$$Q(t) = \int_0^t S(s)S^*(s)\,\mathrm{d}s = \int_0^t S(2s)\,\mathrm{d}s = -A^{-1}2(S(2t) - I),$$

thus $\Lambda(t)$, defined in (4.4), takes the following form

$$\Lambda(t) = Q(t)^{-1/2}S(t) = \sqrt{2}(-A)^{1/2}S(t)(1 - S(2t))^{-1/2}.$$

Also from functional calculus we obtain for $\gamma \in (0, 1/2)$ that

$$\|(-A)^{\gamma}\Lambda(t)\| \le c_2 t^{-(\gamma+1/2)}, \quad t > 0,$$

where

$$c_2 = \sup_{\zeta > 0} \sqrt{2\zeta^{2\gamma+1}} \frac{\mathrm{e}^{-\zeta}}{\sqrt{1 - \mathrm{e}^{-2\zeta}}}.$$

We need a series of lemmas of analogous estimates to that of Section §4.3.

LEMMA 4.4.1 — *Let $f \in \mathrm{C}_{\mathrm{b}}(H)$ and $t > 0$. Then $\mathrm{D}P(t)f(x) \in D((-A)^{\gamma})$ for all $x \in H$. Moreover $(-A)^{\gamma}\mathrm{D}P(t)f \in \mathrm{C}_{\mathrm{b}}(H, H)$ and we have*

$$\|(-A)^{\gamma}[\mathrm{D}P(t)f](x)\| \le c_2 t^{-(\gamma+1/2)} |P(t)f^2(x)|^{1/2}, \quad t > 0. \tag{4.13}$$

In particular

$$\|(-A)^{\gamma}\mathrm{D}P(t)f\| \le c_2 t^{-(\gamma+1/2)} \|f\|, \quad t > 0, \tag{4.14}$$

and for all $\varepsilon > 0$ and $K \subseteq H$ compact there exists $K' \subseteq H$ compact such that

$$p_K((-A)^{\gamma}\mathrm{D}P(t)f) \le c_2 t^{-(\gamma+1/2)}(p_{K'}(f) + \varepsilon \cdot \|f\|), \quad t > 0. \tag{4.15}$$

[3] For self-adjoint operators we have e.g., the functional calculi via spectral measures, multiplication operators or Cauchy integrals , all of them yielding the same operators in this case.

Proof — Let $f \in \mathrm{C}_{\mathrm{b}}(H)$, $t > 0$ and $h \in D((-A)^\gamma)$. Then similarly to (4.5) we obtain

$$\langle [\mathrm{D}P(t)f](x), (-A)^\gamma h\rangle = \int_H \langle Q(t)^{-1/2}\cdot, (-A)^\gamma \Lambda(t)h\rangle f(S(t)x + \cdot)\,\mathrm{d}\mathcal{N}(0, Q(t))$$

for $x \in H$. By the Hölder inequality it follows that

$$|\langle [\mathrm{D}P(t)f](x), (-A)^\gamma h\rangle|^2 \leq |P(t)f^2(x)| \int_H |\langle Q(t)^{-1/2}\cdot, (-A)^\gamma \Lambda(t)h\rangle|^2\,\mathrm{d}\mathcal{N}(0, Q(t)) \leq$$
$$\leq \|f\|^2 \cdot \|(-A)^\gamma \Lambda(t)h\|^2 \leq |P(t)f^2(x)| \cdot c_2^2 t^{-2(\gamma+1/2)}\|h\|^2, \quad x \in H.$$

Thus, indeed $[\mathrm{D}P(t)f](x) \in D((-A)^\gamma)$ and inequality (4.13) follows from the arbitrariness of h. The other two inequalities are immediate consequences of the above. ∎

Lemma 4.4.2 — *Let $f \in \mathrm{C}_{\mathrm{b}}^{\theta}(H)$, $\theta \in (0,1)$ and $t > 0$. Then $(-A)^\gamma \mathrm{D}P(t)f \in \mathrm{C}_{\mathrm{b}}^{\theta}(H, H)$ and we have*

$$[(-A)^\gamma \mathrm{D}P(t)f]_\theta \leq c_2 t^{-(\gamma+1/2)}[f]_\theta, \quad t > 0, \tag{4.16}$$

and for all compact set $K \subset H$ and $\varepsilon > 0$ we have

$$p_{K,\theta}((-A)^\gamma \mathrm{D}P(t)f) \leq c_2 t^{-(\gamma+1/2)}\left(p_{K',\theta}(f) + \varepsilon \cdot [f]_\theta\right), \quad t > 0 \tag{4.17}$$

for an appropriate $K' \subseteq H$ compact set.

Proof — Let $f \in \mathrm{C}_{\mathrm{b}}^{\theta}(H)$, $h \in H$. Then for any $x, y \in H$ we find

$$\begin{aligned}&\langle (-A)^\gamma([\mathrm{D}P(t)f](x) - [\mathrm{D}P(t)f](y)), h\rangle = \\ &= \int_H \langle Q(t)^{-1/2}\cdot, (-A)^\gamma \Lambda(t)h\rangle (f(S(t)x + \cdot) - f(S(t)y + \cdot))\,\mathrm{d}\mathcal{N}(0, Q(t)).\end{aligned} \tag{4.18}$$

By the Hölder inequality it follows that

$$\begin{aligned}&\sup_{x\neq y\in H} \frac{|\langle (-A)^\gamma [\mathrm{D}P(t)f](x) - (-A)^\gamma [\mathrm{D}P(t)f](y), h\rangle|^2}{|x-y|^{2\theta}} \leq \\ &\leq [f]_\theta^2 \int_H |\langle Q(t)^{-1/2}\cdot, (-A)^\gamma \Lambda(t)h\rangle|^2\,\mathrm{d}\mathcal{N}(0, Q(t)) = \\ &= [f]_\theta^2 \cdot \|(-A)^\gamma \Lambda(t)h\|^2 \leq [f]_\theta^2 \cdot c_2^2 t^{-2(\gamma+1/2)}\|h\|^2.\end{aligned}$$

From the arbitrariness of h, inequality (4.16) follows. For $K \subseteq H$ compact we obtain again by the Hölder inequality from (4.18), that

$$\begin{aligned}&|\langle (-A)^\gamma([\mathrm{D}P(t)f](x) - [\mathrm{D}P(t)f](y)), h\rangle|^2 \leq \\ &\leq c_2^2 t^{-2(\gamma+1/2)}\|h\|^2 \cdot \int_H |f(S(t)x + \cdot) - f(S(t)y + \cdot)|^2\,\mathrm{d}\mathcal{N}(0, Q(t)) \leq \\ &\leq c_2^2 t^{-2(\gamma+1/2)}\left(p_{K',\theta}^2(f) + \varepsilon \cdot [f]_\theta^2\right) \cdot \|x-y\|^{2\theta} \cdot \|h\|^2,\end{aligned}$$

when $x, y \in K$ using the regularity of $\mathcal{N}(0, Q(t))$. This establishes inequality (4.17), and the proof is complete. ∎

LEMMA 4.4.3 — *Let $f \in \mathrm{C}_b(H)$. Then $[\mathrm{D}R(\lambda, L)f](x) \in D((-A)^\gamma)$, $\lambda > 0$, $x \in H$, and we have*

$$\|(-A)^\gamma \mathrm{D}R(\lambda, L)f\| \leq c_2\lambda^{1/2-\gamma}\Gamma(1/2-\gamma)\|f\|,$$

and for all compact set $K \subset H$ and $\varepsilon > 0$, we have

$$p_K((-A)^\gamma \mathrm{D}R(\lambda, L)f) \leq c_2\lambda^{1/2-\gamma}\Gamma(1/2-\gamma)\,(p_{K'}(f) + \varepsilon\|f\|)$$

for an appropriate $K' \subseteq H$ compact set.

PROOF — Let $f \in \mathrm{C}_b(H)$. Taking the Laplace transform of $(-A)^\gamma \mathrm{D}P(t)f$ and using (4.14), we have for $h \in H$

$$|\langle(-A)^\gamma[\mathrm{D}R(\lambda, L)f](x), h\rangle| \leq c_2\|h\| \cdot \|f\| \int_0^{+\infty} t^{-(\gamma+1/2)}\mathrm{e}^{-\lambda t}dt =$$
$$= c_2\lambda^{1/2-\gamma}\Gamma(1/2-\gamma)\|f\| \cdot \|h\|.$$

To prove the second statement, we take the Laplace transform again, but make use of inequality (4.17) and thus obtain

$$\|(-A)^\gamma[\mathrm{D}R(\lambda, L)f](x)\| \leq c_2\lambda^{1/2-\gamma}\Gamma(1/2-\gamma) \cdot (p_{K'}(f) + \varepsilon \cdot \|f\|)\,, \quad x \in K,$$

which finishes the proof. ∎

LEMMA 4.4.4 — *Let $f \in \mathrm{C}_b^\theta(H)$ and $t > 0$. Then for $\lambda > 0$ and we have*

$$[(-A)^\gamma \mathrm{D}R(\lambda, L)f]_\theta \leq c_2\lambda^{1/2-\gamma}\Gamma(1/2-\gamma)[f]_\theta$$

and for all $\varepsilon > 0$ and compact set $K \subset H$ there exist $K' \subseteq H$ compact set such that

$$p_{K,\theta}((-A)^\gamma \mathrm{D}R(\lambda, L)f) \leq c_2\lambda^{1/2-\gamma}\Gamma(1/2-\gamma)\,(p_{K,\theta}(f) + \varepsilon \cdot [f]_\theta)\,.$$

PROOF — By using the Laplace transformation, the proof is similar to that of Lemma 4.4.2 ∎

At this point we are able to state the perturbation result. For $F \in \mathrm{C}_b^\theta(H, H)$ and $x \in H$ we define the operator

$$Bf(x) := \langle F(x), (-A)^\gamma \mathrm{D}f(x)\rangle, \quad \text{for } f \in D(L).$$

THEOREM 4.4.5 — *The sum $L+B$ with domain $D(L)$ generates a bi-continuous semigroup R on $\mathrm{C}_b^\theta(H)$*

PROOF — Since we proved in Theorem 2.3.3 that $P(t)$ is a contractive, τ_c^θ-bi-continuous semigroup, it remains to show that the conditions given in Theorem 3.2.3 are fulfilled. Note that by Lemmas 4.4.3 and 4.4.4 we obtain that for all $f \in D(L)$ the orbit $s \mapsto BP(t)f$ is norm-bounded on any compact interval $[0, t_0] \subseteq \mathbb{R}_+$ and it is τ_c^θ-continuous.

To check the condition (3.15), take $x \in H$ and $f \in D(L)$. By making use of Lemmas 4.4.1 and 4.4.2, we obtain for $s > 0$

$$[BP(s)f](x) = \langle F(x), (-A)^{\gamma}[\mathrm{D}P(s)f](x)\rangle,$$

hence

$$\begin{aligned}\|BP(s)f\|_{\theta} \leq \|F\|_{\theta} \cdot \|(-A)^{\gamma}\mathrm{D}P(s)f\|_{\theta} \leq \\ \leq \|F\|_{\theta} c_2 s^{-(\gamma+1/2)}\|f\|_{\theta}.\end{aligned}$$

Choose $t \in [0, t_0]$ sufficiently small, we obtain

$$\begin{aligned}\int_0^t \|BP(s)f\|_{\theta}\,\mathrm{d}s \leq c_2\|F\|_{\theta} \cdot \|f\|_{\theta} \int_0^t s^{-(\gamma+1/2)} ds \leq \\ \leq q\|f\|_{\theta},\end{aligned}$$

with an appropriate $0 < q < 1$. By using (4.15) in Lemma 4.4.1 and (4.17) in Lemma 4.4.2, we check the last condition of Theorem 3.2.3. Hence we conclude the proof. ∎

We remark that the case $\gamma = 0$ was treated in generality in Section §4.3 without extra assumptions on S. On $\mathrm{C}_{\mathrm{b}}^{\theta}(H)$ it is also possible to prove the result of Section §4.3, i.e., when $\gamma = 0$.

§ 4.5 POSITIVITY OF TRANSITION SEMIGROUPS

Our aim is to show that the transition semigroups obtained as a Miyadera – Voigt perturbations of the Ornstein – Uhlenbeck semigroup are positive on $\mathrm{C}_{\mathrm{b}}(H)$. The notation applied here is what is introduced in Sections §4.3 and §4.4. We start with a reformulation of Theorem 3.3.1 for the perturbation of the Ornstein – Uhlenbeck semigroup with non-linear drifts. The results presented here are joint work with A. Es-Sarhir.

LEMMA 4.5.1 — *Let $F_n \in \mathrm{C}_{\mathrm{b}}(H, H)$ be a norm-bounded sequence and suppose that*

$$F = \tau_{\mathrm{c}} - \lim_{n\to\infty} F_n \quad \textit{with } F \in \mathrm{C}_{\mathrm{b}}(H, H).$$

Then we may consider the corresponding drift operators

$$(B_n)f(x) := \langle F_n(x), \mathrm{D}f(x)\rangle \quad \textit{for } f \in \mathrm{C}_{\mathrm{b}}^1(H),$$

and similarly B. For the transition semigroups R_n and R generated by $(L + C_n, D(L))$ and $(L + B, D(L))$ respectively, we have

$$R(t)f = \tau_{\mathrm{c}} - \lim_{n\to\infty} R_n(t)f \quad \textit{uniformly on compact intervals } [0, t_0] \subseteq \mathbb{R}_+.$$

PROOF — We check the conditions of Theorem 3.3.1. Let $M := \sup_{n\in\mathbb{N}} \|F_n\|$. If $f \in D(L)$ then $f \in \mathrm{C}_{\mathrm{b}}^1(H)$ by Lemma 4.3.2, and

$$|B_n f(x) - Bf(x)| \leq \|F_n(x) - F(x)\| \cdot \|\mathrm{D}f(x)\| \leq \|F_n(x) - F(x)\| \cdot \|\mathrm{D}f\|.$$

This shows that $B_n f \xrightarrow{\tau_c} Bf$ and hence $(L+B_n)f \xrightarrow{\tau_c} (L+B)f$ for all $f \in D(L)$. It remains only to show the two estimates (3.20) and (3.21). By (4.11) we can write

$$\int_0^t \|B_n P(s) f\| \,\mathrm{d}s \leq \|F_n\| \cdot \int_0^t \|\Lambda(s)\| \,\mathrm{d}s \cdot \|f\| \leq M \int_0^t \|\Lambda(s)\| \,\mathrm{d}s \cdot \|f\|,$$

thus there exists $t_0 > 0$ and $q < 1$ such that

$$\int_0^{t_0} \|B_n P(s) f\| \,\mathrm{d}s \leq q\|f\|$$

holds for all $f \in D(L)$ and $n \in \mathbb{N}$. This establishes (3.20). On the other hand, inequality (4.12) yields

$$\int_0^t p_K(B_n P(s) f) \,\mathrm{d}s \leq M \cdot p_K(P(t)f) \cdot \int_0^t \|\Lambda(t)\| \,\mathrm{d}t,$$

which shows by the locality of P that (3.21) is also fulfilled. The proof is thus complete by applying Theorem 3.3.1. ■

We turn to observing the positivity of the perturbed transition semigroups. To this end, we recall some results from [74]. Theorem 4.2 of [74] is valid under weaker assumptions than presented there. In fact, in [74] the positivity of transition semigroups is proved on $\mathrm{L}^2(H,\mu)$ with invariant measure μ, we do it on $\mathrm{C_b}(H)$. Basically, we repeat the proof of Theorem 4.2 of [74] with small modifications, and we split the reasoning into a series of lemmas. The first lemma is proved in [74].

Lemma 4.5.2 — *Let $F \in \mathrm{Lip_b}(H,H)$, then there exist a positive, contractive C_0-semigroup on $\mathrm{C_{ub}}(H)$ with generator $(C, D(C))$, such that $\mathrm{C^1_{ub}}(H) \subseteq D(C)$ and*

$$Cf(x) = \langle F(x), \mathrm{D}f(x)\rangle = Bf(x)$$

for all $f \in \mathrm{C_{ub}}(H)$.

Lemma 4.5.3 — *Let $F \in \mathrm{Lip_b}(H,H)$. Then the bi-continuous semigroup R generated by $(L+B, D(B))$ is positive on $\mathrm{C_b}(H)$.*

Proof — Consider the operator C from Lemma 4.5.2 and take its Hille – Yosida approximation on $\mathrm{C_{ub}}(H)$, i.e., $C_n = nCR(n,C)$; this converges to C pointwise on $D(C)$ (see [31, Thm. II.3.5]). Since by Lemma 4.3.3 we have that $\operatorname{rg}(R(\lambda,L)) \subseteq \mathrm{C^1_{ub}}(H)$, we also have that $CR(\lambda,L) = BR(\lambda,L)$. On the other hand, from (3.14) we see that $\operatorname{rg}(R(\lambda,L+B)) \subseteq \mathrm{C^1_{ub}}(H)$. By (4.10) we see that $\|BR(\lambda,L)\| \leq q_0 < 1$ for some q_0 and all $\lambda > \lambda_0 \in \mathbb{R}_+$; suppose also that $\lambda_0 \geq 1$. Since $(C, D(C))$ generates a contraction C_0-semigroup, it is dissipative and $\|nR(n,C)\| \leq 1$ for $n \in \mathbb{N}$ (see the Lumer – Phillips theorem [5, Thm. 3.4.5]). This implies

$$\|C_n R(\lambda,L)\| = \|nR(n,C)BR(\lambda,L)\| \leq \|nR(n,C)\| \cdot \|BR(\lambda,L)\| \leq q_0$$

for $\lambda > \lambda_0$ (recall that $\operatorname{rg}(BR(\lambda, L)) \subseteq \mathrm{C}_{\mathrm{ub}}(H)$ thus the factorisation above holds). Since $D(L) \subseteq \mathrm{C}_{\mathrm{ub}}(H)$, we have that $C_n + L$ makes sense, and by the Neumann series for $\lambda > \lambda_0$

$$R(\lambda, C_n + L)f = R(\lambda, L)\sum_{k=0}^{\infty}[C_n R(\lambda, L)]^k f$$

for all $f \in \mathrm{C}_{\mathrm{b}}(H)$, therefore $\|R(\lambda, C_n + L)\| \leq ((1 - q_0)\lambda)^{-1}$. Thus we may write

$$\begin{aligned}\|R(\lambda, L + C_n) - R(\lambda, L + B)f\| &= \|R(\lambda, L + C_n)(B - C_n)R(\lambda, L + B)f\| \leq \\ &\leq \tfrac{1}{(1-q_0)\lambda}\|(C - C_n)R(\lambda, L + B)f\| \to 0\end{aligned}$$

as $n \to \infty$ for all $f \in \mathrm{C}_{\mathrm{b}}(H)$. If we show that $R(\lambda, L + C_n)f \geq 0$, the positivity of $R(\lambda, L + B)f$ will also follow. Fix $n \in \mathbb{N}$ and compute the resolvent $R(\lambda, L + C_n)$. Note that $C_n = n(-I + nR(n, C))$, thus $C_n + n \geq 0$, as $R(n, C_n) \geq 0$. Further

$$\|(C_n + n)R(\lambda + n, L)\| \leq q_0 + \tfrac{n}{\lambda + n} < 1$$

for all $\lambda > \lambda_1 \in \mathbb{R}_+$ by the dissipativity of L. This implies that the resolvent $R(\lambda, C_n + L) = R(\lambda + n, C_n + n + L)$ is given by the Neumann series

$$R(\lambda + n, C_n + n + L) = R(\lambda + n, L)\sum_{k=0}^{\infty}[(C_n + n)R(\lambda + n, L)]^k,$$

from which we infer that $R(\lambda + n, C_n + n + L) \geq 0$, since the operators appearing in the series are all positive. On the other hand, let

$$\lambda_2 = \inf\{\lambda : \ \lambda > \lambda_0, \ R(\mu, L + C_n) \geq 0 \text{ for all } \mu > \lambda\}.$$

By the above, we have $\lambda_2 \leq \lambda_1$. If $\lambda_2 > \lambda_0$, we take $0 < \varepsilon < \min\{\lambda_2 - \lambda_0, 1\}$ and $\lambda_2 < \lambda < \lambda_2 + \varepsilon/2$, then

$$R(\lambda - \delta, L + C_n) = \sum_{k=0}^{\infty}\delta^k R(\lambda, L + C_n)^{k+1} \geq 0$$

for all $\delta \in [0, \varepsilon)$. This shows that for all $\mu > \lambda - \varepsilon$ one has $R(\mu, L + C_n) \geq 0$. Since $\lambda_0 < \lambda - \varepsilon < \lambda_2$, we arrived to a contradiction, i.e., $\lambda_2 = \lambda_0$ must hold. Thus $R(\lambda, L + B_n) \geq 0$ for all $\lambda > \lambda_0$. Hence $R(\lambda, L + B) \geq 0$, and referring to Theorem 1.4.1 finishes the proof. ■

Lemma 4.5.4 — *If $F \in \mathrm{C}_{\mathrm{ub}}(H, H)$ then we also have the positivity of R on $\mathrm{C}_{\mathrm{b}}(H)$.*

Proof — For the proof, we recall from [78] or [80] (cf. Lemma A.0.7) that for all $F \in \mathrm{C}_{\mathrm{ub}}(H, H)$ there exists $F_n \in \mathrm{Lip}_{\mathrm{b}}(H, H)$ with $\|F_n - F\| \to 0$ and refer to Lemma 4.5.1. ■

At this point, we may further strengthen the result in [74] as follows.

Theorem 4.5.5 — *For $F \in \mathrm{C}_{\mathrm{b}}(H, H)$ the transition semigroup $R(t)$ is positive on $\mathrm{C}_{\mathrm{b}}(H)$.*

PROOF — Using Lemma A.0.7 we approximate $F \in \mathrm{C_b}(H,H)$ by a norm-bounded sequence $F_n \in \mathrm{C_{ub}}(H,H)$ in τ_c. Denote by B_n the corresponding drift operators. By Lemma 4.5.4 the semigroup R_n generated by $(L+B_n, D(L))$ is positive and by Lemma 4.5.1 we have that $R_n(t)f \xrightarrow{\tau_c} R(t)f$, thus $R(t)$ must be positive. ∎

Now we turn to the positivity of transition semigroups obtained in Theorem 4.4.5.

THEOREM 4.5.6 — *Let $\gamma \in [0,1/2)$. The transition semigroup R generated by $(L + \langle F, (-A)^{\gamma}\mathrm{D}f\rangle, D(L))$ is positive on $\mathrm{C_b}(H)$.*

PROOF — Again we apply Theorem 3.3.1. For $\gamma \in [0,1/2)$ let $\varepsilon > 0$ be sufficiently small such that $\gamma+\varepsilon < 1/2$. For $n \in \mathbb{N}$, we define the sequence of operators B_n, given by

$$B_n f(x) := \langle (-A_n)^{\gamma}(-A)^{-(\gamma+\varepsilon)}F(x), (-A)^{\gamma+\varepsilon}\mathrm{D}f(x)\rangle \quad \text{for } f \in D(L),$$

where $(-A_n)^{\gamma} \in \mathscr{L}(H)$ are the Yosida approximations of $(-A)^{\gamma}$. From the definition of B_n we have

$$|B_n f(x) - Bf(x)| \le \|((-A_n)^{\gamma}(-A)^{-\gamma} - I)(-A)^{-\varepsilon}F(x)\| \cdot \|(-A)^{\gamma+\varepsilon}\mathrm{D}f\|$$

for all $f \in D(L)$. Thus $B_n f \to Bf$ in the supremum norm for all $f \in D(L)$. This shows that the convergence assumption of the generators in Theorem 3.3.1 is satisfied. For $F_n(x) := (-A_n)^{\gamma}(-A)^{-(\gamma+\varepsilon)}F(x)$ we have that $F_n \in \mathrm{C_b}(H,H)$, hence Theorem 4.3.6 applies and we obtain the bi-continuous semigroup R_n with generator $(L+B_n, D(L))$. On the other hand, we have that $\|F_n\| \le K < +\infty$, thus using (4.13) we have

$$\int_0^t \|B_n P(s) f\| \,\mathrm{d}s \le K \cdot c_2 t^{1/2-(\gamma+\varepsilon)}\|f\| \quad \text{for all } f \in D(L).$$

As a consequence q, $t_0 > 0$ in Theorem 3.3.1 can be chosen to be the same for all B_n. Further, it is also clear that for all $K \subseteq H$ compact set and $\varepsilon > 0$ there exist $M_{K,\varepsilon} > 0$ and $K' \subseteq H$ compact set such that

$$\int_0^t p_K(B_n P(s) f)\,\mathrm{d}s \le M_{K,\varepsilon} p_{K'}(f) + \varepsilon\|f\|, \quad \text{for all } t \in [0,t_0],\ f \in D(L) \text{ and } n \in \mathbb{N}.$$

These imply that the perturbed semigroups R_n satisfy the assumptions (3.20) and (3.21). Now, using Theorem 3.3.1 we obtain that

$$R_n(t)f \xrightarrow{\tau_c} R(t)f \quad \text{for all } f \in \mathrm{C_b}(H).$$

For $\bar{F}_n(x) := (-A_n)^{\gamma}F(x)$ we have that $\bar{F}_n \in \mathrm{C_b}(H,H)$, thus B_n take the alternative form

$$B_n f(x) = \langle \bar{F}_n(x), \mathrm{D}f(x)\rangle.$$

Hence by Theorem 4.5.5 R_n is positive, and this finishes the proof. ∎

Appendix A

Some results on topological vector spaces

In this closing chapter, we concentrate basically on the space $\mathrm{C_b}(\Omega)$. We present several topological and functional analytic properties, which are used throughout this work. First we give an approximation result, then we turn our attention to mixed topologies in general. The last section is devoted to compactness concepts related to $\mathrm{C_b}(\Omega)$, namely the Prokhorov theorem is presented there.

Let H be separable Hilbert space. We show the sequential denseness of $\mathrm{C_{ub}}(H,H)$ in $\mathrm{C_b}(H,H)$ with respect to τ_c. Since H generally fails to be locally compact, the cut-off technique using $\mathrm{C}_0(H,H)$ is not applicable. We instead use a smoothing to obtain an approximation of $F \in \mathrm{C_b}(H,H)$ with uniformly continuous functions.

LEMMA A.0.7 — *For all $F \in \mathrm{C_b}(H,H)$ there exists a sequence $F_n \in \mathrm{C_{ub}}(H,H)$ with*

$$\|F_n\| \leq \|F\| \quad \textit{and} \quad \tau_c - \lim_{n\to\infty} F_n = F.$$

PROOF — Let A be strictly positive, self-adjoint operator in H with A^{-1} trace class, it generates a contractive, compact C_0-semigroup S. Let $Q = I$, then the Ornstein–Uhlenbeck semigroup P exist on $\mathrm{C_b}(H)$ as considered in Section §2.3. For a function $F \in \mathrm{C_b}(H,H)$ and $h \in H$, $\|h\| \leq 1$ define

$$\langle F_n(x), h\rangle := P(1/n)(\langle F(x), h\rangle), \quad n = 1, 2, \ldots.$$

Then

$$|\langle F_n(x) - F(x), h\rangle| \leq \int_H |\langle F(\cdot + S(1/n)x) - F(x), h\rangle| \,\mathrm{d}\mathcal{N}(0, Q(1/n)) \leq$$
$$\leq \int_H \|F(\cdot + S(1/n)x) - F(x)\| \,\mathrm{d}\mathcal{N}(0, Q(1/n))$$

for all $x \in H$. One shows similarly to Theorem 2.3.1 that this converges to 0 uniformly for $x \in K \subseteq H$ compact as $n \to \infty$. It remains to show that $F_n \in \mathrm{C_{ub}}(H,H)$. Let

$x, y \in H$, then

$$|\langle F_n(x) - F_n(y), h\rangle| \leq \int\limits_H |\langle F(\cdot + S(1/n)x) - F(\cdot + S(1/n)y), h\rangle| \,\mathrm{d}\mathcal{N}(0, Q(1/n)) \leq$$

$$\leq \int\limits_K \|F(\cdot + S(1/n)x) - F(\cdot + S(1/n)y)\| \,\mathrm{d}\mathcal{N}(0, Q(1/n)) + 2\varepsilon\|F\|,$$

where the compact set $K \subseteq H$ is chosen to ε in accordance with the tightness of the measures $\mathcal{N}(0, Q(1/n))$. If $\|x - y\| \leq \delta$, then $\|S(1/n)x - S(1/n)y\| \leq \delta$ and the integral on K is small by the uniform continuity of F on the compact set $K' = K + S(1/n)B(x, \delta)$. ■

§ A.1 MIXED TOPOLOGIES

In this section, we are in the framework established by Assumption 1.2.1. We define a new topology "interpolating" between the norm and τ. The construction given here is basically due to Wiweger. In [87] he constructs mixed topologies in a more general setting even for not necessarily locally convex topologies. This approach to mixed topologies via seminorms was obtained by the author in this generality independently from Wiweger.

For a Banach space X endowed with a locally convex topology τ satisfying Assumption 1.2.1 and determined by the family $\mathcal{P}$ of seminorms we associate a new locally convex topology τ_m.

DEFINITION A.1.1 — For $p_n \in \mathcal{P}$ and $(a_n) \in c_0$, $a_n \geq 0$ we define the seminorm

$$\tilde{p}_{(p_n, a_n)}(x) := \sup_{n \in \mathbb{N}} a_n p_n(x).$$

The *mixed-topology* τ_m is determined by the family of seminorms

$$\tilde{\mathcal{P}} := \left\{ \tilde{p}_{(p_n, a_n)} : p_n \in \mathcal{P}, (a_n) \in c_0 \right\}.$$

It is clear, that

$$\tau \leq \tau_m \leq \tau_n,$$

where τ_n denotes the norm-topology.

LEMMA A.1.2 — *A sequence $x_n \in X$ is convergent in the topology τ_m, if and only if it is norm-bounded and τ convergent.*

PROOF — Note that it suffices to consider the case when the limit is 0. First let $x_n \xrightarrow{\tau_m} 0$ then clearly $x_n \xrightarrow{\tau} 0$. Suppose by contradiction that $\|x_n\|$ is unbounded. We may assume that $\|x_n\| \geq n$. For each x_n take $p_n \in \mathcal{P}$ such that $p_n(x_n) \geq \frac{n}{2}$. Thus for $\tilde{p}_{(p_k, 1/k)}$ we have

$$\tilde{p}_{(p_k, 1/k)}(x_n) = \sup_{\substack{k \in \mathbb{N} \\ k > 0}} \tfrac{1}{k} p_k(x_n) \geq \tfrac{1}{n} p_n(x_n) \geq \tfrac{1}{2},$$

which is a contradiction.

Conversely, assume that $x_n \xrightarrow{\tau} 0$ and x_n is norm-bounded. Take $a_n \in c_0$ and $p_n \in \mathcal{P}$. Then for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| \leq \varepsilon$ whenever $n > N$. Thus

$$\tilde{p}_{(p_k,a_k)}(x_n) = \sup_{k\in\mathbb{N}} a_k p_k(x_n) \leq \sup_{0\leq k\leq N} a_k p_k(x_n) + \sup_{k\geq N} a_k p_k(x_n) \leq C \cdot p'(x_n) + \varepsilon \|(a_k)\|,$$

which shows $x_n \xrightarrow{\tau_m} 0$. ∎

The following result translates the notion of bi-continuous semigroups to the language of mixed topologies.

PROPOSITION A.1.3 — *The class of bi-continuous semigroups and the class of τ_m-strongly continuous and locally sequentially τ_m-equicontinuous semigroups coincide.*

PROOF — Let T be a bi-continuous semigroup. Since $[0,1] \ni h \to T(h)x$ is norm-bounded and τ-continuous for all x, we obtain by Lemma A.1.2 that these orbits are also τ_m-continuous. The sequential τ_m-equicontinuity of the family

$$\{T(t) : t \in [0, t_0]\}$$

is simply a reformulation of Definition 1.2.3 in view of Lemma A.1.2.

For the converse, let T be a τ_m-strongly continuous and locally sequentially-equicontinuous semigroup. Then the orbits $[0,1] \ni h \to T(h)x$ are norm-bounded since they are τ_m-continuous. The τ-strong continuity is immediate, while the local bi-equicontinuity is again trivial by Lemma A.1.2. ∎

According to Wiweger, the topology we have introduced coincides with his mixed topology under some abstract conditions, in particular for $(\mathrm{C_b}(\Omega), \tau_c)$ and $(X', \sigma(X', X))$. Further he proves that in these cases τ_m is the finest locally convex topology which agrees with τ on norm-bounded sets.

§ A.2 THE SPACE $\mathrm{C_b}(\Omega)$

In this section, we observe the space $\mathrm{C_b}(\Omega)$ in detail, we equip it with various topologies, in general different from τ_c, and determine its dual. We also recall some results from measure theory.

a) Measures and $\mathrm{C_b}(\Omega)$. Let Ω be completely regular and topologically complete. The dual of $\mathrm{C_b}(\Omega)$ (as a Banach space) is isomorphic to the space $\mathscr{M}(\beta\Omega)$ of all bounded, complex, Borel measures on the Stone-Čech compactification $\beta\Omega$ of Ω.

Let $\mathcal{J}$ denote the set of norm-continuous linear functionals on $\mathrm{C_b}(\Omega)$ which are also τ_c-continuous on norm-bounded sets. Similarly to Section 1.3.d, one shows that it is a closed subspace of $\mathscr{M}(\beta\Omega)$. It is an interesting problem to find a characterisation of those measures in $\mathscr{M}(\beta\Omega)$ belonging to $\mathcal{J}$. To this end, we recall from [86, Ch. 1] that for a topologically complete space Ω, one always has that Ω is a G_δ set in $\beta\Omega$. Thus we can identify $\mathscr{M}(\Omega)$ with a subspace of $\mathscr{M}(\beta\Omega)$ by

$$\iota : \mathscr{M}(\Omega) \to \mathscr{M}(\beta\Omega),$$
$$[\iota(\nu)](B) := \nu(\Omega \cap B) \quad \text{for all } \nu \in \mathscr{M}(\Omega) \text{ and } B \subseteq \beta\Omega \text{ Borel set.}$$

Then ι is an injection with

$$\operatorname{rg}\iota = \{\mu : \mu \in \mathscr{M}(\beta\Omega),\ \mu(\beta\Omega \setminus \Omega) = 0\}$$

and $\iota(\mathscr{M}(\Omega)) \subseteq \mathscr{J}$. For the reverse inclusion take $\mu \in \mathscr{J}$ and suppose without loss of generality that μ is positive. Assume that $\mu \notin \iota(\mathscr{M}(\Omega))$, i.e., $\mu(\beta\Omega \setminus \Omega) > 0$. Since Ω is a G_δ, hence $\beta\Omega \setminus \Omega$ is an F_σ set, there exists a closed set $F \subseteq \beta\Omega$ with $F \cap \Omega = \emptyset$ and $\mu(F) > 0$. Consider further a compact set $K \subseteq \Omega$. This set is compact, thus closed in $\beta\Omega$. Therefore, when F is a closed subset as considered previously, we have a continuous function $f_{K,F}$ on $\beta\Omega$ with values in $[0,1]$ satisfying $f_{K,F}(x) = 0$ for all $x \in K$ and $f_{K,F}(x) = 1$ when $x \in F$. So we obtain

$$\langle f_{F,K}, \mu\rangle \geq \mu(F) > 0.$$

But this leads to contradiction by the arbitrariness of the compact set K and the τ_c-continuity of μ.

In Section 1.3.d, we introduced the dual space X° for a certain Banach space X endowed also with a locally convex topology τ. Clearly $\mathscr{J} \subseteq \mathrm{C_b}(\Omega)^\circ$. We would like to identify the Banach space $\mathrm{C_b}(\Omega)^\circ$ when Ω is a Polish space. To do this, we need the results presented in the following section.

b) Strict topologies on $\mathrm{C_b}(\Omega)$. When Ω fails to be locally compact, it may lack sufficiently many compact sets, thus one may find the locally convex topology τ_c unsatisfactory. Many other topologies were introduced on $\mathrm{C_b}(\Omega)$ by several authors, the literature on this topic seems to be utterly extensive. We shall therefore only stick to [76], where a nice account is given on the space of bounded, continuous functions and topological measure theory. We introduce three locally convex topologies on $\mathrm{C_b}(\Omega)$, and without going into the details we cite a couple of important results. Though it will turn out that these topologies coincide when Ω is a Polish space, we find it nevertheless worthwhile to mention all the three constructions.

Let Ω be completely regular. Consider the family $\mathcal{G}$ of all absolutely convex and absorbent sets $U \subseteq \mathrm{C_b}(\Omega)$ which satisfy the following: for all $r > 0$ there exist a τ_c-neighbourhood V_r of 0, such that $U \cap B(0,r) \supseteq V_r \cap B(0,r)$. Then $\mathcal{G}$ is a base for a locally convex topology denoted by β_0. Clearly, it is the finest locally convex topology which agrees with τ_c on all balls $B(0,r)$, thus it is indeed the mixed topology τ_m.

A bounded continuous function extends continuously from Ω to $\beta\Omega$. For each compact set $K \subseteq \beta\Omega \setminus \Omega$ define

$$\mathrm{C_b^K}(\Omega) := \{g :\ g \in \mathrm{C_b}(\Omega),\ g|_K = 0\}.$$

Then consider the locally convex topologies β_K determined by the family of seminorms

$$\mathcal{P}^K := \left\{ \|fg\| : \ g \in \mathrm{C}_\mathrm{b}^K(\Omega) \right\}.$$

Then β and β_1 are defined by inductive limits as follows[1]

$$\beta := \varinjlim_{\substack{K \subseteq \beta\Omega \setminus \Omega \\ \text{compact}}} \beta_K \quad \text{and} \quad \beta_1 := \varinjlim_{\substack{K \subseteq \beta\Omega \setminus \Omega \\ \text{zero set}}} \beta_K.$$

For these topologies the following relations hold

$$\tau_\mathrm{c} \le \tau_m = \beta_0 \le \beta \le \beta_1 \le \tau_n.$$

It is known that in all places $<$ is possible. However, when Ω is nice, the topologies between τ_c and τ_n coincide.

THEOREM A.2.1 — *Let Ω either be a σ-compact, locally compact space, or a Polish space[2]. Then*

a) $\beta_0 = \beta = \beta_1$.

b) $\beta = \mu(\mathrm{C_b}(\Omega), \mathscr{M}(\Omega))$ *the Mackey topology (cf. Section §4.2).*

c) A linear operator $T : \mathrm{C_b}(\Omega) \to \mathrm{C_b}(\Omega)$ *for a locally convex space E is β-continuous, if and only if it is β-sequentially continuous. The same holds for linear functionals.*

d) $(\mathrm{C_b}(\Omega), \beta)$ *is sequentially barelled[3], i.e., every* $\varphi_n \in (\mathrm{C_b}(\Omega), \beta)'$ *weak*-null sequence is equicontinuous.*

THEOREM A.2.2 — *If Ω is a Polish space then* $(\mathrm{C_b}(\Omega), \beta)' = \mathrm{C_b}(\Omega)^\circ = \mathscr{M}(\Omega)$*, and* $\beta = \mu(\mathrm{C_b}(\Omega), (\mathrm{C_b}(\Omega), \beta)')$*, i.e.,* $(\mathrm{C_b}(\Omega), \beta)$ *is a Mackey-space.*

PROOF — By Theorem A.2.1 $\varphi \in \mathscr{M}(\beta\Omega)$ is β-continuous, if and only if it is sequentially τ_m-continuous. Thus Proposition A.1.3 yields that φ is β-continuous when it is sequentially τ_c-continuous on norm-bounded sets. Section A.2.a finishes the proof. ∎

We turn our attention to measures, and observe the relationship between their certain measure theoretic and functional analytic properties.

DEFINITION A.2.3 — A family of measures $\mathcal{K} \subseteq \mathscr{M}(\Omega)$ is called *tight*, if for all $\varepsilon > 0$ there exist $K \subseteq \Omega$ compact such that $|\varphi|(\Omega \setminus K) \le \varepsilon$ holds for all $\varphi \in \mathcal{K}$.

A similar result to the following theorem is known strict topologies and Ω completely regular (see [45, Sec. 12.6]).

THEOREM A.2.4 — *Let (Ω, d) be a metric space. Then $\mathcal{K} \subseteq \mathscr{M}(\Omega)$ is tight and norm-bounded, if and only if it is τ_c-equicontinuous on norm bounded sets.*

[1] Under a zero set we understand the zero level set of a continuous function.

[2] The metric space itself must not, but the induced topology should be complete.

[3] c_0-sequentially barelled [45, Sec. 12]

PROOF — Assume first the tightness and the norm-boundedness. Then the estimate

$$|\langle f, \varphi \rangle| \leq \int_K |f| \,\mathrm{d}|\varphi| + \varepsilon \|f\|$$

for all $f \in \mathrm{C_b}(\Omega)$ and $\varphi \in \mathcal{K}$ yields the equicontinuity of $\mathcal{K}$.

For the contrary, let the family $\mathcal{K} \subseteq \mathscr{M}(\Omega)$ be τ_c-equicontinuous on norm-bounded sets. Assume by contradiction that there exists $\varepsilon > 0$ such that for all $K \subseteq \Omega$ compact there exist a functional $\varphi_K \in \mathcal{K}$ with $|\varphi_K|(\Omega \setminus K) \geq \varepsilon$. For $K \subseteq \Omega$ compact, the sets $K_n := \{x : d(x, K) < 1/n\}$ are open, and one has $\cap_{n=1}^{\infty} K_n = K$. Take $n \in \mathbb{N}$ such that $|\varphi_K|(\Omega \setminus K_n) \geq \varepsilon/2$. Define the function $f|_K := 0$ and $f|_{\Omega \setminus K_n} := 1$ and extend it to the whole of Ω with values in $[0, 1]$ (Tietze's theorem). Then

$$\int_\Omega f \,\mathrm{d}|\varphi_K| \geq \int_{\Omega \setminus K_n} f \,\mathrm{d}|\varphi_K| \geq \varepsilon.$$

This contradicts the equicontinuity assumption since in all τ_c-neigbourhood of 0 one finds such f. ∎

Combining the previous results we conclude the following. Note that since $(\mathrm{C_b}(\Omega), \beta)$ is a Mackey-space the β-equicontinuous sets of $\mathscr{M}(\Omega) = (\mathrm{C_b}(\Omega), \beta)'$ are exactly the relative weak*-compact sets.

THEOREM A.2.5 [Prokhorov] — *Let $\mathcal{K} \subseteq \mathscr{M}(\Omega)$ be norm-bounded set, then the following are equivalent*

i) $\mathcal{K}$ is tight, i.e., for all $\varepsilon > 0$ there exist $K \subseteq \Omega$ compact such that $|\mu|(\Omega \setminus K) \leq \varepsilon$.

ii) $\mathcal{K}$ is relatively $\sigma(\mathscr{M}(\Omega), \mathrm{C_b}(\Omega))$-compact.

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