Multiple-point formulas and their applications

PhD Thesis

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2008

Contents

1 Introduction

The structure of the thesis is the following. First we introduce the reader to those objects in differential topology which are investigated in the thesis (section 2). Then we state and prove a generalization of Herbert's multiple-point formula (section 3.1). In the rest of section 3 we give various applications of our main formula.

In section 4 we generalize and a give a new geometric proof of a theorem of Szűcs that describes the relationship between the multiple-points of an immersion and the singularities of its 1-codimensional projection (see [10]).

The last part of the thesis is devoted to various product constructions. Using the results from section 3 we investigate Cartesian products of immersions (section 5.1). Then we define multiplication on prim and Morin maps (sections 5.2, 5.3). As an application of the results on the products of immersions we can compute the ring structure defined on the Morin maps (section 5.4). Finally section 5.5 investigates singular strata of products of generic maps.

The results of section 3 are joint work with Gábor Braun (see [2]), while the results of section 5 are joint work with András Szűcs (see [12]).

Acknowledgements. I would like to express my deep gratitude to my supervisor András Szűcs. He did not only teach me almost everything I know about topology, but without his constant help and encouragement I never could have written this thesis.

2 Basic notions

In this section we briefly review the main definitions and recall the main constructions used in the thesis. These are all classical notions in topology so we shall only mention their most relevant properties and give references for the reader who is interested in the details. We shall also fix the notation in this section.

2.1 Manifolds and maps

Throughout the whole thesis "manifold" will always mean "smooth (C^∞-differentiable), Hausdorff manifold". We shall also understand manifolds to be closed (i.e. compact and without boundary) unless otherwise explicitly stated or clear from the context. Manifolds will usually be denoted by a capital letter with a superscript which stands for the dimension of the manifold (e.g. M^n). The superscript will sometimes be omitted for easier reading.

2.1.1 Maps

The main objects of investigation will be maps between manifolds. We will always assume that all maps are smooth and proper. (Properness means that the preimage of any compact set is compact. This is automatic when the source manifold is compact.) Given a map $f : M^n \to N^{n+k}$ the integer k is called the codimension of the map. Recall that such a map induces a map $df : TM \to TN$ of tangent bundles, called the differential of f. The classes of maps we are interested in are identified using the differential as follows.

Definition 1. A map $f : M^n \to N^{n+k}$ of nonnegative codimension is called

1. an *immersion* if for any $x \in M$ the restricted map $df_x : T_xM \to T_xN$ has rank n (i.e. its kernel is trivial). A map will be denoted by $f : M^n \looparrowright N^{n+k}$ when we want to emphasize that it is an immersion.

2. a *Morin* map if $\dim\ker df_x \leq 1$ for every $x \in M$.

3. a *prim* (*pr*ojected *im*mersion) map if it is Morin and the line bundle $\ker df \subset TM$ is trivialized. (As we will later see this is equivalent to saying that there is an immersion $\tilde{f} : M^n \looparrowright N^{n+k} \times \mathbb{R}$ whose projection to N is f, and this $\tilde{f}$ is given up to regular homotopy. This explains the name of these maps.)

2.1.2 Genericity

In most of the cases we are only interested in a special subset of these maps which behave nicer than the rest. We shall refer to this subset as the "generic" maps. What we exactly mean by this varies with the type of maps under consideration.

Definition 2. We fix our notion of genericity for the maps defined above.

1. An immersion $f : M \to N$ is generic if it is self-transverse, which means that if $f(x_1) = f(x_2) = \cdots = f(x_k) = y \in N$ but $x_i \neq x_j$ $(i \neq j)$ then the subspaces Im $df_{x_1}, \ldots,$ Im df_{x_k} are in general position in T_yN.

2. A Morin map $f : M \to N$ is generic if it is self-transverse and its 1-jet extension is transverse to the set of all corank 1 germs.

3. A prim map $f : M \to N$ is generic if it is generic as a Morin map and its lift $\tilde{f} : M \to N \times \mathbb{R}$ is a generic immersion.

It is well known that in all the above cases the generic maps form an open dense subset among all the maps of the given type. This implies that by a small perturbation we can always make our maps generic, and that a sufficiently small perturbation of a generic map is still generic.

2.1.3 Powers and diagonals

We shall often use Cartesian products of manifolds. For easier reference we introduce the following notation.

Definition 3. For a manifold M^n let $M^{(r)} = M \times M \times \cdots \times M$ denote its r-fold Cartesian power. For a map $f : M \to N$ let $f^{(r)} : M^{(r)} \to N^{(r)} = f \times f \times \cdots \times f$ denote its r-fold Cartesian power.

Definition 4. Let

$$\Delta_r(V) = \{(x_1, \ldots, x_r) \in V^{(r)} : (\exists\, i \neq j)(x_i = x_j)\}$$
$$\delta_r(V) = \{(x, \ldots, x) \in V^{(r)}\}$$

denote the fat and the narrow diagonals of $V^{(r)}$ for any manifold V.

Remark 1. Some properties of a map $f : M \to N$ can be nicely expressed using these constructions. It is easy to see that f is self-transverse if and only if $f^{(r)}$ is transverse to $\delta_r(N)$ outside $\Delta_r(M)$ for every r. On the other hand f is an immersion if and only if for every r the closure of ${f^{(r)}}^{-1}(\delta_r(N)) \setminus \Delta_r(M)$ is disjoint from $\Delta_r(M)$. (For details see [17].)

2.2 Cobordism

Manifolds and their maps could be considered up to various equivalence relations leading to completely different areas of topology. Our interest is in the relation called cobordism.

Definition 5. Two manifolds M^n and N^n of the same dimension are said to be *cobordant* if there exists a compact manifold W^{n+1} with boundary $\partial W = M \amalg N$. It is well know that this is indeed an equivalence relation. The equivalence classes are called cobordism classes.

It is easy to see that disjoint union gives a well defined addition on the cobordism classes of n-dimensional manifolds. The empty manifold acts as the unit element, and every class has an inverse. This makes the set of cobordism classes into a group denoted by $\mathcal{N}_n$ and called the n-dimensional cobordism group.

The cobordism class of the direct product of two manifolds depends only on the cobordism classes of the manifolds. This observation allows one to define a multiplication on the cobordism classes. This makes the group $\oplus_{n=1}^{\infty}\mathcal{N}_n$ into a graded ring denoted by $\mathcal{N}_*$ and called the cobordism ring.

Sometimes we want to consider oriented manifolds only. The cobordism relation has its direct analogue for oriented manifolds as well. The resulting ring is denoted by Ω_*.

The cobordism relation can be easily extended from manifolds to maps. Let $\mathcal{F}$ denote a family of maps to a fixed manifold N. For example $\mathcal{F}$ could be all the immersions mapping to N, or all the prim maps mapping to N, etc. The maps in $\mathcal{F}$ will be referred to as $\mathcal{F}$-maps.

Definition 6. Two $\mathcal{F}$-maps $f_i : M_i^n \to N^{n+k}$ are said to be $\mathcal{F}$-*cobordant* if there is a compact manifold W^{n+1} with boundary $\partial W = M_1 \amalg M_2$ and an $\mathcal{F}$-map $F : W \to N \times [1, 2]$ such that $F^{-1}(N \times \{i\}) = M_i$ and $F|_{M_i} = f_i \times \{i\}$.

Similarly to the manifold case this is an equivalence relation. The equivalence classes (again called cobordism classes) form a group with the disjoint union of source manifolds being the addition. These groups are denoted $\mathcal{N}_n(N), \mathrm{Imm}_n(N), \mathrm{Prim}_n(N), \mathrm{Morin}_n(N)$ when respectively the family $\mathcal{F}$ consist of all maps, immersions, prim maps or Morin maps.

In the case of $N = \mathbb{R}^{n+k}$ we shall write $\mathrm{Imm}(n, k)$ instead of $\mathrm{Imm}_n(\mathbb{R}^{n+k})$ and similarly for the other type of maps.

Definition 7. Just as in the case of manifolds, we get oriented versions of these cobordism groups if we restrict ourselves to maps between oriented manifolds.

The oriented cobordism groups are denoted by $\Omega_n(N)$, $\mathrm{Imm}_n^{SO}(N)$, $\mathrm{Prim}_n^{SO}(N)$, $\mathrm{Morin}_n^{SO}(N)$ respectively. In the case of $N = \mathbb{R}^{n+k}$ we will use the notation $\mathrm{Imm}^{SO}(n, k)$ etc.

2.3 Multiple-point manifolds

Consider a generic immersion $f : M^n \to N^{n+k}$. The *r-fold points* of f are those points in N whose preimage consists of exactly r different points. We shall denote this set N_r. This is not always a closed set in N. Its closure $\bar{N}_r$ consists of those points that have at least r distinct preimages. Set $M_r = f^{-1}(N_r)$ to obtain the r-fold points of f in the source manifold, and let $\bar{M}_r$ denote its closure.

The sets M_r and N_r are generally not submanifolds of M and N but they are images of (non-generic) immersions of manifolds. Here we recall the well-known construction to fix the notation: Let

$$\hat{M}_r(f) = \{(x_1, \dots, x_r) \in M^{(r)} : f(x_1) = \dots = f(x_r), (i \neq j) \Rightarrow (x_i \neq x_j)\}.$$

The symmetric group S_r acts on this set freely in the obvious way. Let $[x_1, \dots, x_r]$ denote the equivalence class of $(x_1, \dots, x_r)$. On the other hand S_{r-1} also acts freely on the last $r-1$ coordinates. Here the equivalence class of $(x_1, \dots, x_r)$ is denoted by $(x_1, [x_2, \dots, x_r])$. The sets of equivalence classes are denoted by

$$\widetilde{\Delta}_r(f) = \hat{M}_r(f)/S_r$$
$$\Delta_r(f) = \hat{M}_r(f)/S_{r-1}.$$

There are natural, well defined mappings

$$\begin{aligned} \widetilde{f}_r &: \widetilde{\Delta}_r(f) \to N & \widetilde{f}_r([x_1, \dots, x_r]) &:= f(x_1) \\ f_r &: \Delta_r(f) \to M & f_r(x_1, [x_2, \dots, x_r]) &:= x_1 \\ s_r &: \Delta_r(f) \to \widetilde{\Delta}_r(f) & s_r(x_1, [x_2, \dots, x_r]) &:= [x_1, \dots, x_r]. \end{aligned}$$

The images of $\widetilde{f}_r$ and f_r are clearly $\bar{N}_r$ and $\bar{M}_r$ and they are bijective to the points that have multiplicity exactly r. On the other hand s_r is clearly an r-sheeted covering.

The sets $\widetilde{\Delta_r}(f)$ and $\Delta_r(f)$ are called the r-fold multiple-point manifolds of f in the target and source respectively. They are indeed manifolds. Consider the r-fold product $f^{(r)} : M^{(r)} \to N^{(r)}$. Clearly

$$\hat{M}_r(f) = (f^{(r)})^{-1}(\delta_r(N)) \setminus \Delta_r(M).$$

Since f is a generic immersion (see Remark 1), $f^{(r)}$ is transversal to $\delta_r(N)$ and thus $\hat{M}_r(f)$ is a manifold of dimension $n-(r-1)k$. So after factoring out with the free group actions we still get manifolds.

Remark 2. Let us note how the multiple-point manifolds depend on the map f. If f is changed by a regular homotopy, then the cobordism classes of $f_r, \widetilde{f}_r, \Delta_r(f)$ and $\widetilde{\Delta_r}(f)$ remain the same. This is easily seen by a generic-position argument. This remains valid for $\widetilde{f}_r$ and $\widetilde{\Delta_r}(f)$ even if f is changed by a cobordism. This explains why it is natural to consider cobordism when dealing with multiple-points.

2.4 Singular points

Definition 8. Given a smooth map $f : M \to N$ where $\dim M \leq \dim N$, a point $x \in M$ is said to be a Σ^i point if the corank (i.e. the dimension of the kernel) of $df_x : T_xM \to T_{f(x)}N$ is at least i. The set of such points is denoted by $\Sigma^i(f)$. If $i_1 \geq i_2$ then we can define $\Sigma^{i_1,i_2}(f) = \Sigma^{i_2}(f|_{\Sigma^{i_1}(f)})$. This method can be continued recursively to give the definition of $\Sigma^{(i_1,i_2,\dots,i_r)}$ points, where $i_1 \geq i_2 \geq \cdots \geq i_r$. This classification of singular points is called the Thom-Boardman type. For details see e.g. [1].

Remark 3. If $f : M \to N$ is a *Morin* map then it has no Σ^2 points. The singularities of such maps are classified by their Thom-Boardman type, which can only be $\Sigma^{\overbrace{(1,1,\dots,1)}^{r}} = \Sigma^{1_r}$ for some $r \geq 0$. (In the notation of [1] this is A_r.)

3 Multiple-point formulas

Multiple-point manifolds have been long studied from different viewpoints. From the point of topology the question arises as follows. Given a generic immersion $f : M^n \to N^{n+k}$, is it possible to express topological invariants of its multiple-point manifolds using invariants of M, N and f?

The first such questions considered were the homology classes of the multiple-point manifolds. It turned out that indeed these homology classes are related to each other in a simple way that includes only little information from f (namely the Euler class of its normal bundle and f^*, the induced map in cohomology). The formula was first stated by Lashof and Smale [9] but it turned out to be partially false. It was corrected by Herbert [5] and later Ronga [17] gave a simple and very geometric proof.

Theorem 1 (Herbert's formula). *Let $f : M^n \to N^{n+k}$ be a generic immersion. Then the closures of the r-tuple point sets, $\bar{N}_r(f)$ and $\bar{M}_r(f)$ carry fundamental classes with $\mathbb{Z}_2$ coefficients. Denoting by n_r and m_r their Poincaré duals in N and M respectively and setting $e = e(\nu_f)$ we have:*

$$m_r = f^*(n_{r-1}) - e \cdot m_{r-1}$$

If both M and N are oriented and k is even, then the multiple-point manifolds can be given a natural orientation. Thus one can interpret m_r and n_r as cohomology classes with integer coefficients and the formula remains valid in its original form.

Remark 4. For oriented maps of odd codimension the multiple-points in the target might be non-orientable and in the source there is no canonical choice of orientation.

Here we present a generalization of this formula that allows us to move from homology classes to cobordism classes and is based on ideas of Ronga, Kamata and Szűcs.

3.1 The generalized multiple-point formula

Given a generic immersion $f : M^n \to N^{n+k}$ we can try to determine the cobordism class of its multiple-point manifolds instead of the homology classes they represent. It is well known that the cobordism class of a manifold depends only on its characteristic numbers, thus it suffices to calculate these numbers. Instead of evaluating a characteristic class of the multiple-point manifold on its fundamental class we can take its push-forward into M and evaluate it on $[M]$. (From now on $[M]$ will denote the fundamental class of the manifold M, unless otherwise mentioned.) So our goal now is to express these push-forwards in terms of M, N and f. To do so we set up formulas involving the push-forwards.

As it turns out these formulas are direct generalizations of Herbert's original formula. This is simply because the represented cohomology class is exactly the push-forward of the unit element in the cohomology ring of the multiple-point manifold.

Our result and the idea of the proof has its roots in the works of Szűcs and Kamata. Szűcs [23] used the Herbert-Ronga formula in the oriented case for double-point manifolds in K-theory and translated it to ordinary cohomology via the Chern character to obtain a sequence of formulas involving push-forwards of Pontrjagin classes of the double-point manifold.

Kamata [6] used the Herbert-Ronga formula in the unoriented cobordism cohomology and translated it via the Boardman homomorphism to obtain a sequence of formulas involving the push-forwards of the Stiefel-Whitney classes of the multiple-point manifolds.

Later in [24] Szűcs investigated the case of oriented manifolds immersed in Euclidean space. Using a filtration on the multiple-point manifold he could calculate its Pontrjagin numbers without pushing them forward to M.

The method presented here gives a general result containing all three above results at the same time and which avoids complicated homological calculations or the use of natural transformations between extraordinary cohomology theories and ordinary cohomology.

3.1.1 Ronga's lemma

We need the notion of sub-cartesian diagram and two lemmas from [17]. In the sequel ν always denotes the normal bundle of an immersion:

Lemma 1 (Ronga). *f_r and $\widetilde{f}_r$ are proper immersions with normal bundles $\nu_{\widetilde{f}_r} = (\nu_f^{(r)}|_{\hat{M}_r(f)})/S_r$ and $\nu_{f_r} = (0 \times \nu_f^{(r-1)}|_{\hat{M}_r(f)})/S_{r-1}$.*

Definition 9. A commutative diagram of proper immersions:

$$\begin{array}{ccc} Z & \xrightarrow{f_B} & B \\ \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

is said to be *sub-cartesian* if

(i) $f_A \times f_B : Z \to A \times B$ is an embedding onto $\{(a,b) \in A \times B : \alpha(a) = \beta(b)\}$.

(ii) the following sequence is exact:

$$0 \to TZ \xrightarrow{d(f_A \times f_B)} f_A^*TA \times f_B^*TB \xrightarrow{(d\alpha - d\beta)} f_A^*\alpha^*TX$$

The first condition of the definition says that Z is the intersection of A and B where multiple intersections (that is: points that are multiple for α or β) are counted with appropriate multiplicity. The second condition says that the intersection is clean in the terminology of Quillen (cf [14]), that is, the tangent space of the intersection manifold is locally the same as the intersection of the two tangent spaces.

Definition 10. In a sub-cartesian diagram $E = \operatorname{coker}(d\alpha - d\beta)$ is called the *excess vector* bundle.

Let h^* denote any generalized cohomology theory with products and let e_h denote the Euler class of a vector bundle in h^*.

Lemma 2 (Ronga). *If A, B, X and Z are h-orientable then $\alpha^*\beta_!(c) = f_{A!}(e_h(E) \cdot f_B^*(c))$ holds for any $c \in h^*(B)$.*

We would like to apply Lemma 2 with $X = N, A = M, B = \widetilde{\Delta_{r-1}}(f), \alpha = f, \beta = \widetilde{f_{r-1}}$ since the r-tuple points are in the intersection of the $r-1$-tuple points with the image of f. It is easily seen that in this case we have to set $Z = \Delta_r(f) \cup \Delta_{r-1}(f)$, that is the disjoint union of the r-tuple point manifold and the $r-1$-tuple point manifold. The maps f_A and f_B have to be defined as below to make the diagram a pull-back diagram:

$$\begin{aligned} f_A|_{\Delta_r(f)} &:= f_r & \qquad f_B|_{\Delta_r(f)} &:= p_r \\ f_A|_{\Delta_{r-1}(f)} &:= f_{r-1} & \qquad f_B|_{\Delta_{r-1}(f)} &:= s_{r-1} \end{aligned}$$

Here $p_r : \Delta_r(f) \to \widetilde{\Delta_{r-1}}(f)$ is the projection map defined by the formula

$$p_r(x_1, [x_2, \ldots, x_r]) = [x_2, \ldots, x_r].$$

This way we get a sub-cartesian diagram. The genericity of f implies that the excess vector bundle over $\Delta_r(f)$ is the zero bundle and Lemma 1 implies that it is $f_{r-1}^* \nu_f$ over $\Delta_{r-1}(f)$.

If all manifolds involved are h-orientable we can apply Lemma 2 to an element $c \in h^*(\widetilde{\Delta_{r-1}}(f))$ to get

$$\begin{aligned} f^*(\widetilde{f_{r-1!}}(c)) &= f_{r!}(p_r^*(c)) + f_{r-1!}(e_h(f_{r-1}^* \nu_f) \cdot s_{r-1}^*(c)) \\ &= f_{r!}(p_r^*(c)) + e_h(\nu_f) \cdot f_{r-1!}(s_{r-1}^*(c)). \qquad (1) \end{aligned}$$

3.1.2 The main formula

Let γ denote a multiplicative characteristic class in h^*. That is, for any bundle $\xi : E \to B$ there is a class $\gamma(\xi) \in h^*(B)$ such that this class is natural with respect to induced bundles and $\gamma(\xi_1 \oplus \xi_2) = \gamma(\xi_1) \cdot \gamma(\xi_2)$. This definition is valid for bundles with any given structure group. Well-known examples of such a γ are the total Stiefel-Whitney class when $h^* = H(\cdot, \mathbb{Z}_2)$ or the Euler class. We shall allow $\gamma(\xi)$ to be an infinite sum but we will assume that there is an other multiplicative characteristic class β such that $\gamma \cdot \beta \equiv 1$ (so the Euler class is excluded now).

Let us choose $c = \gamma(\nu_{\widetilde{f_{r-1}}})$. Then $f_B^*(c) = f_B^*(\gamma(\nu_{\widetilde{f_{r-1}}})) = \gamma(f_B^*(\nu_{\widetilde{f_{r-1}}}))$. We compute the two parts $f_B^*|_{\Delta_r(f)}(c)$ and $f_B^*|_{\Delta_{r-1}(f)}(c)$ separately.

First notice that since $\Delta_r(f)$ is the transversal intersection of M and $\widetilde{\Delta_{r-1}}(f)$ it follows that $f_B^*(\nu_{\widetilde{f_{r-1}}})|_{\Delta_r(f)} = \nu_{f_A|_{\Delta_r(f)}} = \nu_{f_r}$. Thus

$$f_{r!}(f_B^*(c)) = f_{r!}(\gamma(f_B^*(\nu_{\widetilde{f_{r-1}}}))) = f_{r!}(\gamma(\nu_{f_r})) \tag{2}$$

To calculate the other part first we make a trivial remark that we will use later.

Remark 5. For any immersion $g : V \to W$ we have $\nu_g \oplus TV = g^*TW$ so $\gamma(\nu_g) = \frac{g^*\gamma(TW)}{\gamma(TV)}$. Using the standard formula $f_!(f^*x \cdot y) = x \cdot f_!(y)$ this implies that

$$g_!(\gamma(\nu_g)) = g_!(g^*\gamma(TW) \cdot \beta(TV)) = \gamma(TW) \cdot g_!(\beta(TV)) \tag{3}$$

It is easy to see that $f_B|_{\Delta_{r-1}(f)}$ is an $r-1$-sheeted covering of $\widetilde{\Delta_{r-1}}(f)$. This implies that the bundle $f_B^*\nu_{\widetilde{f_{r-1}}}|_{\Delta_{r-1}(f)}$ is equal to the normal bundle of the composite map $\widetilde{f_{r-1}} \circ f_B|_{\Delta_{r-1}(f)}$ which is in turn equal to $f \circ f_{r-1}$ since our sub-cartesian diagram is by definition commutative. Thus we have the following sequence of equations:

$$\begin{aligned} f_{r-1!}(f_B^*(c)) &= f_{r-1!}(\gamma(f_B^*(\nu_{\widetilde{f_{r-1}}}))) = f_{r-1!}(\gamma(\nu_{f\circ f_{r-1}})) \\ &= f_{r-1!}(f_{r-1}^*(\gamma(\nu_f)) \cdot \gamma(\nu_{f_{r-1}})) = \gamma(\nu_f) \cdot f_{r-1!}(\gamma(\nu_{f_{r-1}})) \end{aligned} \tag{4}$$

(We used the fact that $\nu_{f\circ f_{r-1}} = \nu_{f_{r-1}} \oplus f_{r-1}^*(\nu_f)$ and the standard formula which we also used in the previous remark.)

Lemma 3. *Let h^* be a generalized cohomology theory with products. Then for any invertible multiplicative characteristic class γ taking values in h^* and any generic immersion $f : M \to N$ for which all the arising manifolds are h-orientable we have*

$$f^*\widetilde{f_{r-1}}_!(\gamma(\nu_{\widetilde{f_{r-1}}})) = f_{r!}(\gamma(\nu_{f_r})) + e_h(\nu_f) \cdot \gamma(\nu_f) \cdot f_{r-1!}(\gamma(\nu_{f_{r-1}}))$$

Proof. Plug (2) and (4) into (1). □

The class γ of the normal bundles of f_r, f_{r-1} and $\widetilde{f_{r-1}}$ are hard to evaluate directly and so we write them in the terms of classes of the tangent bundles of the multiple-point manifolds. To this end we use (3). We get

$$\begin{aligned} f^*\widetilde{f_{r-1}}_!(\gamma(\nu_{\widetilde{f_{r-1}}})) =& f^*(\gamma(TN)) \cdot f^*\widetilde{f_{r-1}}_!(\beta(T\widetilde{\Delta_{r-1}}(f))) \\ f_{r!}(\gamma(\nu_{f_r})) =& \gamma(TM) \cdot f_{r!}(\beta(T\Delta_r(f))) \\ e_h(\nu_f) \cdot \gamma(\nu_f) \cdot f_{r-1!}(\gamma(\nu_{f_{r-1}})) =& e_h(\nu_f) \cdot f^*(\gamma(TN)) \cdot \beta(TM) \cdot \gamma(TM) \cdot \\ & \cdot f_{r-1!}(\beta(T\Delta_{r-1}(f))) \end{aligned}$$

Combining these formulas with Lemma 3 and dividing by $\gamma(TM)$ we get

$$\begin{aligned} f_{r!}(\beta(T\Delta_r(f))) = \frac{f^*\gamma(TN)}{\gamma(TM)} \cdot \Big(& f^*\widetilde{f_{r-1}}_!(\beta(T\widetilde{\Delta_{r-1}}(f))) \\ & - e_h(\nu_f) \cdot f_{r-1!}(\beta(T\Delta_{r-1}(f)))\Big) \end{aligned}$$

We can think of this formula as a recursion which expresses an invariant of the r-tuple point manifold in terms of invariants of the $r-1$-tuple point manifolds. Let us denote by $m_r = f_{r!}(\beta(T\Delta_r(f)))$ and by $n_r = \widetilde{f}_{r!}(\beta(T\widetilde{\Delta}_r(f)))$ the quantities we are interested in.

Main formula.

$$m_r = \gamma(\nu_f) \cdot (f^* n_{r-1} - e_h(\nu_f) \cdot m_{r-1}) \tag{5}$$

The difficulty in applying this formula is that a priori we know nothing about n_{r-1}. But in favourable cases we can relate it to m_{r-1} thereby obtaining a real recursion-formula on the m_r.

Lemma 4. *$f_!(m_r) = p \cdot n_r$ where $p \in h^0(\widetilde{\Delta_r}(f))$ is a cohomology class such that $s_{r*}[\Delta_r(f)] = p \cdot [\widetilde{\Delta_r}(f)]$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \Delta_r(f) & \xrightarrow{s_r} & \widetilde{\Delta_r}(f) \\ \Big\downarrow{\scriptstyle f_r} & & \Big\downarrow{\scriptstyle \widetilde{f_r}} \\ M & \xrightarrow{f} & N \end{array}$$

As s_r is an r-sheeted covering, the tangent bundle of $\Delta_r(f)$ is induced from the tangent bundle of $\widetilde{\Delta_r}(f)$ by s_r. By Poincaré duality $p = s_{r!}(1) = s_{r!}s_r^*(1) \in h^0(\widetilde{\Delta_r}(f))$. Thus:

$$\begin{aligned} f_!(m_r) &= f_! f_{r!}(\beta(T\Delta_r(f))) = \widetilde{f}_{r!} s_{r!}(\beta(s_r^*(T\widetilde{\Delta_r}(f)))) \\ &= \widetilde{f}_{r!}(s_{r!}s_r^*(\beta(T\widetilde{\Delta_r}(f)))) \\ &= \widetilde{f}_{r!}(\beta(T\widetilde{\Delta_r}(f)) \cdot s_{r!}s_r^*(1)) = \widetilde{f}_{r!}(p \cdot \beta(T\widetilde{\Delta_r}(f))) = p \cdot n_r. \end{aligned}$$

□

Now if $p \in h^0(\widetilde{\Delta_r}(f))$ is an invertible element then it follows that $f_!(m_r)$ is divisible by p and we can rewrite Lemma 4 in the form $n_r = \frac{f_!(m_r)}{p}$. This is the case for example if we take cohomology $\mathbb{Q}$ coefficients and restrict ourselves to oriented manifolds:

Theorem 2. *Let $f : M^m \to N^{m+k}$ be a generic immersion of even codimension between oriented manifolds and choose a cohomology theory with coefficient ring $h^0(pt) \cong \mathbb{Q}$. Then we have*

$$m_r = \gamma(\nu_f) \cdot \left(\frac{f^* f_!(m_{r-1})}{r-1} - e_h(\nu_f) m_{r-1} \right)$$

Proof. Since M and N are oriented, so is $\hat{M}_r(f)$. The action of the symmetric groups S_r and S_{r-1} are orientation preserving since k is even. So both $\Delta_r(f)$ and $\widetilde{\Delta_r}(f)$ are oriented and $s_r : \Delta_r(f) \to \widetilde{\Delta_r}(f)$ is also orientation preserving. This means that $p = r$ in Lemma 4. Thus we are done if we combine the Main Formula and Lemma 4. □

Remark 6. If k is odd then the S_r action contains orientation reversing involutions on $\hat{M}_r(f)$ and so either $\widetilde{\Delta_r}(f)$ is unorientable or its components have no preferred orientation.

To make explicit calculations with the formula of Theorem 2 one has to deal with the term $f^*f_!$. One way to do this is to suppose that $f^* = 0$ in positive dimensions. We are going to use this approach in section 3.2. However there is an other case when we can resolve it, and that is when there is a bundle ξ over N such that $TM = f^*(\xi)$.

Theorem 3. *Let f and h^* be as in Theorem 2. Further assume that there is a bundle $\xi : E \to N$ such that $TM = f^*\xi$, and that $e(\nu_f) = f^*(y)$ for an $y \in h^*(N)$. Then for every $r \geq 1$ there is a cohomology class $k_r \in h^*(N)$ such that $m_r = f^*(k_r)$ and the following simple recursion formula holds:*

$$k_r = \gamma(TN)\beta(\xi)\left(\frac{c}{r-1} - y\right) \cdot k_{r-1}$$

where $c = f_!f^*(1) \in h^*(N)$.

Proof. The statement easily follows by induction on r. For $r = 1$ we have $m_1 = \beta(TM) = \beta(f^*(\xi)) = f^*(\beta(\xi))$ so we can choose $k_1 = \beta(\xi)$.

For the inductional step notice that $\gamma(\nu_f) = \frac{f^*\gamma(TN)}{\gamma(TM)} = f^*(\gamma(TN)\beta(\xi))$. So by the inductional hypothesis and Theorem 2

$$m_r = \gamma(\nu_f) \cdot \left(\frac{f^*f_!(m_{r-1})}{r-1} - e_h(\nu_f)m_{r-1}\right)$$
$$= f^*\left(\gamma(TN)\beta(\xi)\left(\frac{f_!f^*(k_{r-1})}{r-1} - y \cdot k_{r-1}\right)\right).$$

Thus we may choose

$$\begin{aligned} k_r &:= \gamma(TN)\beta(\xi)\left(\frac{f_! f^*(k_{r-1})}{r-1} - y\cdot k_{r-1}\right) \\ &= \gamma(TN)\beta(\xi)\left(\frac{k_{r-1}\cdot c}{r-1} - y\cdot k_{r-1}\right) = \gamma(TN)\beta(\xi)\left(\frac{c}{r-1} - y\right)\cdot k_{r-1} \end{aligned}$$

which finishes the proof. □

3.2 Special cases

In this section we shall show how the results of Kamata [6] and Szűcs [24] follow from our method. We apply the general machinery of section 3.1.2 with the right choice of the generalized cohomology theory h^* and the multiplicative characteristic class β to obtain the various special cases.

3.2.1 The unoriented case

Let our cohomology theory $h^*(X) = H^*(X,\mathbb{Z}_2)[[t_1,t_2,\dots]]$ be the ring of formal power-series of infinite variables over $H^*(X,\mathbb{Z}_2)$. For an n-dimensional bundle $\xi : E \to B$ let us define

$$w_t(\xi) = \prod_{i=1}^{n}\left(1+\alpha_i t_1 + \alpha_i^2 t_2 + \alpha_i^3 t_3 + \cdots\right)$$

where the total Stiefel-Whitney class of ξ is expanded by the splitting principle as

$$w(\xi) = (1+\alpha_1)\cdots(1+\alpha_n).$$

Since $w_t(\xi)$ is symmetric in the variables α_i it is really a characteristic class. Its multiplicativity and naturality easily follow from that of $w(\xi)$. It is also invertible since $w_t(\xi)$ always starts with $1+\dots$. Thus we may choose $\gamma = w_t$. It is clear that $e_h(\xi) = e(\xi) \in H^*(B,\mathbb{Z}_2)$ where $e(\xi)$ is just the ordinary Euler class of ξ.

Theorem 4 (Kamata). *Let $f : M^m \to N^{m+k}$ be a self-transverse immersion for which f^* is the constant map in positive dimension. (This is satisfied if for example f is null-homotopic.) Then*

$$f_{r!}\left(\frac{1}{w_t(T\Delta_r(f))}\right) = e(\nu_f)^{r-1} \cdot \left(\frac{1}{w_t(TM)}\right)^r$$

in $H^*(M, \mathbb{Z}_2)[[t_1, t_2, \dots]]$.

Proof. Let us look at the Main Formula with the choice of $\gamma = w_t$ and $\beta = \frac{1}{w_t}$. Since $f^* = 0$ and we are working with $\mathbb{Z}_2$ coefficients, the formula simplifies to $m_r = w_t(\nu_f)e(\nu_f)m_{r-1}$. Thus by induction we have $m_r = (w_t(\nu_f)e(\nu_f))^{r-1} m_1$. Here $w_t(\nu_f) = f^*(w_t(TN)) \cdot \beta(TM) = \beta(TM)$. On the other hand f_1 is just the identity map of M so $m_1 = \beta(TM)$. Thus $m_r = \beta(TM)^r \cdot e(\nu_f)^{r-1}$ and this is exactly what we wanted to prove. □

Evaluating both sides on $[M]$ and noticing that

$$\langle f_{r!}(x), [M]\rangle = \langle x, [\Delta_r(f)]\rangle$$

we get

Corollary 1 (Kamata).

$$\left\langle \frac{1}{w_t(T\Delta_r(f))}, [\Delta_r(f)] \right\rangle = \left\langle e(\nu_f)^{r-1} \cdot \left(\frac{1}{w_t(TM)}\right)^r, [M] \right\rangle.$$

Remark 7. It would have been simpler to use $\beta(\xi) = w_t(\xi)$ since this way we get a formula for the push-forward of $w_t(T\Delta_r(f))$ instead of its reciprocal. Then instead of the above corollary we would get

$$\langle w_t(T\Delta_r(f)), [\Delta_r(f)]\rangle = \left\langle e(\nu_f)^{r-1} \cdot (w_t(TM))^r, [M] \right\rangle.$$

Though this form is better for any application, we wanted to state the theorem exactly as Kamata stated it in [6].

This formula then implies:

Corollary 2 (Kamata). *If a self-transverse immersion $f : M^m \to N^{m+k}$ is null-homotopic and M is null-cobordant then so are all the multiple-point manifolds $\Delta_r(f)$.*

Proof. The idea of the proof is the following. The single equation of the above corollary actually implies equation of every coefficient in the formal power series. The coefficients on the left hand side are all the characteristic (Stiefel-Whitney) numbers of the multiple-point manifold. Similarly on the right hand side the coefficients are Stiefel-Whitney numbers of M. If M is null-cobordant, then all its Stielfel-Whitney numbers are zero. Thus the same holds for the multiple-point manifold, hence it is also null-cobordant. For more details see [6]. □

Remark 8. We cannot see any easy way to avoid the use of formal power series. If, for instance, we choose $\beta(\xi) = w(\xi)$ the total Stiefel-Whitney class, then the corollary will express only one Stielfel-Whitney polynomial in each dimension, instead of expressing all of them at the same time.

3.2.2 The oriented case

In this section our cohomology theory $h^*(X)$ will be $H^*(X, \mathbb{Q})[[t_1, t_2, \dots]]$. Let $f : M^m \to N^{m+k}$ be a generic immersion where M and N are oriented and k is even. In the case when f^* is the zero homomorphism in positive dimensions, we will be able to express the push-forward of any Pontrjagin polynomial of the multiple-point manifolds in terms of the Pontrjagin classes of M and $e(\nu_f)$.

We present the same result in two different forms. There seems to be no simple direct proof of the fact that the two forms are actually equivalent.

Symmetric polynomials As in the previous section, we have a multiplicative characteristic class which is defined for a bundle $\xi : E \to B$ with the formula

$$\beta(\xi) := \prod_{i=1}^{n} \left(1 + y_i t_1 + y_i^2 t_2 + \cdots\right)$$

where the total Pontrjagin class of ξ is written as

$$p(\xi) = (1 + y_1)\cdots(1 + y_n).$$

Here the y_i are 4-dimensional cohomology classes and the j^{th} Pontrjagin class of ξ is $p_j(\xi) = \sigma_j(y_1, \ldots, y_n)$, the j^{th} symmetric polynomial in the variables y_i. Since β is symmetric in the variables y_i it is well-defined. It is also obviously natural and invertible. It is also multiplicative since the total Pontrjagin class is multiplicative modulo 2-torsion but with $\mathbb{Q}$ coefficients there is no 2-torsion.

Let us apply the Main Formula with $\gamma = 1/\beta$. As in the previous section the $f^* = 0$ assumption simplifies the formula and we get $m_r = -\beta(TM)e(\nu_f)m_{r-1}$. As $m_1 = \beta(TM)$, we have by induction that

$$f_{r!}(\beta(T\Delta_r(f))) = (-e(\nu_f))^{r-1} \cdot \beta(TM)^r. \tag{6}$$

This formula is actually not a single equation, since both sides are formal power series with infinite variables. Thus they can only be equal if the coefficients of all the corresponding monomials are the same. So we have an equation for every monomial of the form $t_1^{b_1} t_2^{b_2} \cdots t_s^{b_s}$, $s \geq 0, b_i \geq 0$. These equations contain all the information needed to calculate the push-forwards of the Pontrjagin polynomials of the multiple-point manifold. To extract this information we use the Hirzebruch base of symmetric polynomials.

For a partition $I = (a_1, \ldots, a_r)$ of $|I| = a_1 + \cdots + a_r$ let $x_I \in H^{4|I|}(B, \mathbb{Q})$ denote the smallest symmetric polynomial containing the monomial $y_1^{a_1} y_2^{a_2} \cdots y_r^{a_r}$. For example

$$\begin{aligned} x_{(1)} &= y_1 + \cdots + y_n = p_1(\xi) \\ x_{(1,1)} &= y_1 y_2 + y_1 y_3 + \cdots = p_2(\xi) \\ x_{(2)} &= y_1^2 + \cdots + y_n^2 = p_1(\xi)^2 - 2p_2(\xi) \end{aligned}$$

If (by abuse of notation) we introduce the following operation on partitions:

$$I(\underline{b}) = I(b_1, \dots, b_s) := (\underbrace{1, \dots, 1}_{b_1}, \underbrace{2, \dots, 2}_{b_2}, \dots, \underbrace{s, \dots, s}_{b_s})$$

then it is easy to see that

$$\begin{aligned} \beta(\xi) &= (1 + y_1 t_1 + y_1^2 t_2 + \dots)(1 + y_2 t_1 + y_2^2 t_2 + \dots) \cdots (1 + y_n t_1 + y_n^2 t_2 + \dots) \\ &= \sum_{(b_1, \dots, b_s), s \geq 0, b_i \geq 0} x_{I(\underline{b})} t_1^{b_1} t_2^{b_2} \cdots t_s^{b_s} \quad (7) \end{aligned}$$

Now we can think of (6) as a formula that tells us the push-forward of any characteristic polynomial $x_I, I = (a_1, \dots, a_r)$ of the multiple-point manifold. It is exactly $(-e(\nu_f))^{r-1}$ times the coefficient of $t_1^{b_1} t_2^{b_2} \cdots t_s^{b_s}$ in $\beta(TM)$, where $b_i = |\{j : a_j = i\}|$.

As any x_I is a polynomial of Pontrjagin classes and every such polynomial is a linear combination of the x_I, we get the push-forward of all the Pontrjagin polynomials. And finally the formula

$$\langle f_{r!}(x), [M] \rangle = \langle x, [\Delta_r(f)] \rangle$$

gives us all the Pontrjagin numbers of the multiple-point manifolds.

Pontrjagin polynomials With the use of a different multiplicative characteristic class we can express push-forwards of Pontrjagin polynomials directly with Pontrjagin classes of M and $e(\nu_f)$. The formula we are going to prove this way was first proved by Szűcs in [24] by a completely different method.

For an orientable bundle $\xi : E \to B$ let

$$\beta(\xi) = \prod_{i=1}^{N} (1 + p_1(\xi) t_i + p_2(\xi) t_i^2 + \cdots) \in H^*(B, \mathbb{Q})[[t_1, \dots, t_N]]$$

where N is a large number. As $p(\xi \oplus \eta) = p(\xi) \cdot p(\eta)$, an easy calculation shows that each factor of our class β is indeed multiplicative. Naturality and invertibility of β is obvious.

Just as in the previous section, formula (6) holds for our class β. On the left-hand side the coefficient of $t_1^{b_1}\cdots t_N^{b_N}$ is the push-forward of the Pontrjagin polynomial $p_1^{b_1}\cdots p_N^{b_N}$ of the multiple-point manifold. It is also easy to see the coefficient of the same monomial in the right hand side of (6). As

$$\beta(TM)^r = \prod_{i=1}^{N}(1 + p_1(TM)t_i + p_2(TM)t_i^2 + \cdots)^r,$$

the $t_1^{b_1}$ part comes from the first factor, the $t_2^{b_2}$ from the second, and so on. It is also easy to see that the coefficient of $t_1^{b_1}$ in $(1+p_1(TM)t_1+p_2(TM)t_1^2+\cdots)^r$ is exactly the $4b_i$ dimensional part of $p(TM)^r = (1+p_1(TM)+p_2(TM)+\cdots)^r$.

Let us denote by q_j the $4j$ dimensional part of $p(TM)^r$. For a partition $I = (b_1, b_2, \ldots, b_N)$ let us denote $p^r(TM)_I = q_{b_1}\cdots q_{b_N}$, and let the usual Pontrjagin polynomial $p_I(T\Delta_r(f)) = p_{b_1}p_{b_2}\cdots p_{b_N}(T\Delta_r(f)$.

Theorem 5.

$$f_{r!}(p_I(T\Delta_r(f))) = (-e(\nu_f))^{r-1}p^r(TM)_I$$

Corollary 3 (Szűcs).

$$\langle p_I(T\Delta_r(f)), [\Delta_r(f)]\rangle = \big\langle(-e(\nu_f))^{r-1}p^r(TM)_I, [M]\big\rangle \tag{8}$$

3.3 Numerical calculations

Szűcs used his original formulas to show that there are cobordism classes of manifolds that do not contain double-point manifolds. Here we carry out similar calculations for multiple-point manifolds of arbitrary multiplicity.

We are going to use the machinery of the previous section to obtain numerical results on the cobordism classes of multiple-point manifolds. We will show that many cobordism classes do not contain manifolds that arise as the multiple-point manifold of an immersion $f: M^m \to N^{m+k}$ with $f^* = 0$.

Lemma 5. *Let $V = \mathbb{CP}^{2k_1} \times \mathbb{CP}^{2k_2} \times \cdots \times \mathbb{CP}^{2k_s}$. Then*

$$\langle p_1^{k_1+\cdots+k_s}(V), [V] \rangle = \frac{(k_1 + \cdots + k_s)!}{k_1! \cdots k_s!} \prod_{i=1}^{s} (2k_i + 1)^{k_i}.$$

Proof. It is well-known that $p(\mathbb{CP}^n) = (1+y)^{n+1}$ where $y \in H^4(\mathbb{CP}^n)$ is the square of the Euler class of the canonical line bundle over $\mathbb{CP}^n$. So

$$p(V) = p(\mathbb{CP}^{k_1}) \times p(\mathbb{CP}^{k_2}) \times \cdots \times p(\mathbb{CP}^{k_s}) = (1+y)^{2k_1+1} \times \cdots \times (1+y)^{2k_s+1}.$$

The 4 dimensional part of this is

$$p_1(V) = \sum_{i=1}^{s} (2k_i + 1) \, (\underbrace{1 \times \cdots \times 1}_{i-1} \times y \times \underbrace{1 \times \cdots \times 1}_{s-i})$$

and so

$$p_1^{k_1+\cdots+k_s} = (y^{k_1} \times \cdots \times y^{k_s}) \cdot \prod_{i=1}^{s} (2k_i + 1)^{k_i} \cdot \prod_{i=1}^{s} \binom{k_i + \cdots + k_s}{k_i}.$$

Since $y^{k_1} \times \cdots \times y^{k_s}$ is the generator of $H^{4(k_1+\cdots+k_s)}(V)$ this finishes the proof. ☐

Lemma 6. *The greatest common divisor of all the numbers $\langle p_1^n(V), [V] \rangle$ where V runs over the $4n$ dimensional oriented closed manifolds is 1 if $n \not\equiv 1(3)$ and is 1 or 3 if $n \equiv 1(3)$.*

Proof. The function $f(V) = \langle p_1^n(V), [V] \rangle$ is an $f : \Omega_{4n} \to \mathbb{Z}$ homomorphism. By the previous lemma we have

$$f(\mathbb{CP}^{2n}) = (2n+1)^n = A \text{ and } f(\mathbb{CP}^2 \times \mathbb{CP}^{2n-2}) = \binom{n}{1} \cdot 3^1 \cdot (2n-1)^{n-1} = B.$$

It is clear that $n, 2n-1$ and $2n+1$ are pairwise coprime, so the greatest common divisor (A, B) of A and B equals to $(3, 2n+1)$. Thus $(A, B) = 1$ if $n \not\equiv 1(3)$ else $(A, B) = 3$. Since $f : \Omega_{4n} \to \mathbb{Z}$ is a homomorphism, there is a manifold V^{4n} such that $f(V) = (A, B)$ and this proves the lemma. ☐

Theorem 6. *If a* $4t$ *dimensional oriented manifold* V^{4t} *is the* r*-tuple-point manifold of an immersion* $f : M^m \to N^{m+k}$ *(ie.* $V = \Delta_r(f)$*) with* $f^* = 0$ *in positive dimension, then* $\langle p_1^t(V), [V]\rangle$ *is divisible by* r^t*.*

Proof. Let us calculate $\langle p_1^t(TV), [V]\rangle$. By (8) we have

$$\begin{aligned}\langle p_1^t(TV), [V]\rangle &= \left\langle (-e(\nu_f))^{r-1} p^r(TM)_{(1,\dots,1)}, [M]\right\rangle \\ &= \left\langle (-e(\nu_f))^{r-1} \left(r \cdot p_1(TM)\right)^t, [M]\right\rangle = r^t \left\langle (-e(\nu_f))^{r-1} p_1^t(TM), [M]\right\rangle\end{aligned}$$

□

Define the homomorphism $\Delta_r : Imm^{SO}(m,k) \to \Omega_{m-k(r-1)}$ from the cobordism group of immersions of oriented m-manifolds into $\mathbb{R}^{m+k}$ to the oriented cobordism group by

$$\Delta_r(f) := [\Delta_r(f)],$$

where $[\cdot]$ now denotes cobordism class. This homomorphism is well-defined when m and k are even. Combining the last theorem with the last lemma we get:

Corollary 4. *If* $m - k(r-1)$ *is divisible by four then* $|coker(\Delta_r)| \geq r^{\frac{m-k(r-1)}{4}}/3^{\varepsilon}$ *where* $\varepsilon = 1$ *if* $3|r$ *and* $m - k(r-1) \equiv 1(3)$*, else* $\varepsilon = 0$*.*

This is a generalization of Szűcs's result obtained in [23].

4 Multiple-points and projections

There is a surprising relation between the multiple-points of an immersion $g : M \looparrowright N \times \mathbb{R}$ and the singularities of its projection $f : M \to N$ that was found by Szűcs in [25] (see also [20]). Namely he showed that if N is a Euclidean space then the $r+1$-tuple-points of g are cobordant to the Σ^{1_r} points of f. The proof of this result involved computing the characteristic numbers of the two manifolds and observing that they coincide.

It is very natural to ask whether this cobordism can be "seen" in an explicit way hidden in the geometry of f, not just as mere luck that all the characteristic numbers coincide.

We shall answer this question in the affirmative by constructing a cobordism that connects the two manifolds. This allows us to slightly extend the original theorem: instead of cobordism of manifolds we obtain singular bordism of maps, and we prove the theorem for any smooth target manifold N.

Theorem 7. *Let $f : M^n \to N^{n+k}$ be a prim map, and let $g : M \looparrowright N \times \mathbb{R}$ be its lift to an immersion. Then for any $r \geq 1$ we have $g_r \sim \Sigma^{1_{r-1}}(f)$, that is they represent the same element in the singular bordism group $\mathcal{N}(M)$.*

If M and N are oriented and the codimension k is odd, then $g_r \sim_{SO} \Sigma^{1_{r-1}}(f)$, that is they represent the same element in the singular oriented bordism group $\Omega(M)$.

4.1 Preparations

Let us fix a prim map $f : M^n \to N^{n+k}$, its lift $g : M \looparrowright N \times \mathbb{R}$ and $r \geq 2$ (for $r = 1$ the statement is obvious). We shall introduce intermediate manifolds and their maps to M which we shall call 'mixed'-point manifolds. For any $1 \leq i \leq r$ let us consider those points in M that are i-tuple points of g and at the same time $\Sigma^{1_{r-i}}$ points of f. These points do not necessarily form a submanifold of M, but we can construct their resolution just like we did for the multiple-points: Let us consider the map

$$G_i := g|_{\Sigma^{1_{r-i}}(f)} \times g \times \cdots \times g : \Sigma^{1_{r-i}}(f) \times M \times \cdots \times M \to (N \times \mathbb{R}) \times \cdots \times (N \times \mathbb{R}),$$

where we take $i-1$ factors of M on the left, and thus i factors of $(N \times \mathbb{R})$ on the right.

Definition 11. The fat diagonal of $\Sigma^{1_r}(f) \times M^{(i-1)}$ can be defined analogously to $\Delta_i(M)$, since $\Sigma^{1_r}(f) \subset M$ is a submanifold. Let us denote

$$\Delta_i^r(M) = \{(x_1, x_2, \ldots, x_i) \in \Sigma^{1_r}(f) \times M^{(i-1)} : \exists j \neq l, x_j = x_l\}.$$

Since f is a generic prim map and g its generic lift we have that G_i is transverse to the narrow diagonal $\delta_i(N \times \mathbb{R})$ outside of the fat diagonal $\Delta_i^{r-i}(M)$. Since g is an immersion the set $\hat{M}_{i,r-i}(f) := G_i^{-1}(\delta_i(N \times \mathbb{R})) \setminus \Delta_i^{r-i}(M)$ is a closed submanifold in $\Sigma^{1_{r-i}} \times M^{(i-1)}$. The symmetric group S_{i-1} acts on $\Sigma^{1_{r-i}} \times M^{(i-1)}$ by permuting the last $i-1$ coordinates. This action restricted to $\hat{M}_{i,r-i}(f)$ is free, so we can factorize and get the manifold

$$\Lambda_r^i = \hat{M}_{i,r-i}(f)/S_{i-1}.$$

A point of Λ_r^i can be referred to as $(x_1, [x_2, \ldots, x_i])$ where the x_j's are all different, $g(x_1) = g(x_2) = \cdots = g(x_i)$ and $x_1 \in \Sigma^{1_{r-i}}(f)$. In this notation the desired resolution

$$\lambda_r^i : \Lambda_r^i \to M$$

is given by

$$(x_1, [x_2, \ldots, x_i]) \mapsto x_1.$$

(The maps f, g are omitted from the notation.) It is easy to see that the manifold Λ_r^i has dimension $n-(r-1)(k+1)$ and in particular $\Lambda_r^r = \Delta_r(g), \lambda_r^r = g_r$ and $\lambda_r^1 : \Lambda_r^1 \to M$ is the natural inclusion $\Sigma^{1_{r-1}}(f) \hookrightarrow M$. Thus the theorem follows from the following lemma.

Lemma 7. $\lambda_r^1 \sim \lambda_r^2 \sim \cdots \sim \lambda_r^r$.

The proof consists of two very different ingredients. The first ingredient is the global construction of the desired cobordisms using the map f. The constructed spaces are easy to describe but they are not obviously manifolds. The precise proof that they are indeed manifolds requires detailed study of the map f near its singular points. Thus the second ingredient is a local computation using normal forms. This computation is only a technical point so first we give the proofs omitting the computational details. Then in section 4.3 we finally show how to carry out the computations used earlier.

4.2 Construction of the cobordisms

Let us again consider the map

$$G_i := g|_{\Sigma^{1_{r-i}}(f)} \times g \times \cdots \times g : \Sigma^{1_{r-i}}(f) \times M \times \cdots \times M \to (N \times \mathbb{R}) \times \cdots \times (N \times \mathbb{R}).$$

Let us define

$$\Delta_i^+ = \{((x,s),(x,t),\ldots,(x,t)) \in (N \times \mathbb{R})^{(i)} : s \geq t\}.$$

Outside of $\Delta_i^{r-i}(M)$ the map G_i is transverse to Δ_i^+ and $\partial\Delta_i^+ = \delta_i(N \times \mathbb{R})$, since both f and g are generic and thus self-transverse.

Let us now define $H' = G_i^{-1}(\Delta_i^+) \setminus \Delta_i^{r-i}(M)$. Transversality implies that H' is a (not necessarily compact) manifold with boundary $G_i^{-1}(\delta_i(N \times \mathbb{R})) \setminus \Delta_i^{r-i}(M) = \hat{M}_{i,r-i}(f)$. Let us denote the closure of H' in $\Sigma^{1_{r-i}}(f) \times M^{(i-1)}$ by H. Obviously $H \setminus H' \subset \Delta_i^{r-i}(M)$. We have seen in section 4.1 that $\partial H'$ is a closed manifold disjoint from the fat diagonal. Thus $\partial H'$ is disjoint from $H \setminus H' \subset \Delta_i^{r-i}(M)$.

Let us take a point $(x_1,\ldots,x_i) \in H \setminus H'$. Then by definition of H' there exist points y_j^k $(k \geq 1, i \geq j \leq 1)$ that fulfill all the following requirements:

1. For every j we have $\lim_{k\to\infty} y_j^k = x_j$.

2. $y_1^k \in \Sigma^{1_{r-i}}(f)$.

3. For any fixed k the y_j^k's are all different.

4. $g(y_{j_1}^k) = g(y_{j_2}^k)$ for any $j_1, j_2 \geq 2$.

5. $f(y_{j_1}^k) = f(y_{j_2}^k)$ for any $j_1, j_2 \geq 1$.

Since g is a generic immersion, 3. and 4. imply that $\forall j > l \geq 2 (x_j \neq x_l)$. Then since $H \setminus H' \subset \Delta_i^{r-i}(M)$ there must be a $j > 1$ such that $x_1 = x_j$. Thus $y_1^k \to x_1$ and $y_j^k \to x_1$ as well. Furthermore $y_1^k \in \Sigma^{1_{r-i}}(f)$. Theorem 8 in section 4.3 can be applied and hence $x_1 \in \Sigma^{1_{r-i+1}}(f)$.

Conversely let us suppose that $x_1 \in \Sigma^{1_{r-i+1}}(f)$ and $x_2, \ldots, x_{i-1}$ are all different from each other and x_1 and $g(x_j)$ is the same for every $1 \leq j \leq i-1$. We want to show that in the neighborhood of $(x_1, x_1, x_2, \ldots, x_{i-1})$ the set H is a compact manifold with boundary and $(x_1, x_1, x_2, \ldots, x_{i-1})$ is on ∂H. First consider the first two factors separately from the others.

$$G_2 = g|_{\Sigma^{1_{r-i}}(f)} \times g : \Sigma^{1_{r-i}}(f) \times M \to (N \times \mathbb{R})^{(2)}.$$

Let us denote $H_2' = G_2^{-1}(\Delta_2^+)$. By Theorem 9 in section 4.3 we know that locally around (x_1, x_1) its closure $H_2 = \mathrm{cl}(H_2')$ is a compact manifold with boundary $\partial H_2 = \{(u, u) : u \in \Sigma^{1_{r-i+1}}(f)\}$. Clearly H is locally the complete intersection of $H_2 \subset \Sigma^{1_{r-i}}(f) \times M$ around (x_1, x_1) and $\hat{M}_{i-2}(g) \subset M^{(i-2)}$ around $(x_2, \ldots, x_{i-1})$. Thus the genericity of f and g implies that H is also locally a compact manifold with boundary ∂H the complete intersection of ∂H_2 and $\hat{M}_{i-2}(g)$.

Thus H is a compact manifold. Its boundary consist of two disjoint components $H \setminus H'$ and $\partial H' = \hat{M}_{i,r-i}(f)$. The symmetric group S_{i-1} acts on $\Sigma^{1_{r-i}}(f) \times M^{(i-1)}$ by permuting the last $i-1$ coordinates. By definition H' is invariant under this action. The above considerations show that $\partial H'$ and $H \setminus H'$ are also invariant, and the action is free on each. Thus we can factorize by this action on H and get that the quotient is again a compact manifold

$\hat{H}$ with boundary $\partial H'/S_{i-1}$ and $(H \setminus H')/S_{i-1}$. By definition $\partial H'/S_{i-1} = \hat{M}_{i,r-i}(f)/S_{i-1} = \Lambda_r^i$. On the other hand we have seen that

$$H \setminus H' = \{(x_1, x_2, \ldots, x_i) \in \Delta_i^{r-i+1}(M) \setminus \Sigma^{1_{r-i+1}}(f) \times \Delta_{i-1}(M) : \\ : g(x_j) = g(x_l) \ (1 \le j < l \le i)\}.$$

Thus there is a natural map $\phi : (H \setminus H')/S_{i-1} \to \Lambda_r^{i+1}$ that is given by $\phi(x_1, [x_2, \ldots, x_i]) = (x_1, [x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i])$ when $x_1 = x_j$. This map is clearly a diffeomorphism. Thus $(H \setminus H')/S_{i-1} = \Lambda_r^{i+1}$.

Finally projecting everything to the first coordinate we get a map $\hat{H} \to M$ that on the boundary coincides with λ_r^i and λ_r^{i+1}. Thus $\lambda_r^i \sim \lambda_r^{i+1}$. □

Remark 9. If the codimension k is odd, then the codimension of g is even. So if M and N are oriented, then H' can be given a natural orientation. This is preserved by the action of S_{i-1} and so the manifold $\hat{H}$ that creates the cobordism between λ_r^i and λ_r^{i+1} is oriented. Thus $\lambda_r^i \sim_{SO} \lambda_r^{i+1}$ and the oriented part of theorem follows as well.

4.3 Local computations

Let us consider a prim map $f : M^n \to N^{n+k}$. Let us write $n = r(k+1)+z$. Then the Σ^{1_r}-points of f form a z-dimensional submanifold in M. Let $x \in \Sigma^{1_r}(f) \setminus \Sigma^{1_{r+1}}(f)$. Then (according to e. g. [1]) it is possible to take small Euclidean neighborhoods of x and $f(x)$ and introduce local coordinates such that f takes the following local normal form (we take both x and $f(x)$ to be in the origin):

$$\begin{aligned} F : (\mathbb{R}^{r(k+1)+z}, 0) &\to (\mathbb{R}^{1+k+(r(k+1)-1)+z}, 0) \\ (t, \underline{y}_r, \underline{y}_{r-1}, \ldots, \underline{y}_1, \underline{s}) &\mapsto (p_0(t), p_1(t), \ldots, p_k(t), \underline{\underline{y}}, \underline{s}), \end{aligned}$$

where $\underline{y}_j = (y_{j,0}, y_{j,1}, \ldots, y_{j,k}) \in \mathbb{R}^{k+1}$ for every $1 \le j \le r-1$ and $\underline{y}_r = (y_{r,1}, y_{r,2}, \ldots, y_{r,k}) \in \mathbb{R}^k$. By $\underline{\underline{y}}$ we denote the collection of all y_i^j, so $\underline{\underline{y}} \in$

$\mathbb{R}^{r(k+1)-1}$. Finally $\underline{s} = (s_1, \ldots, s_z) \in \mathbb{R}^z$. The polynomials p_i are defined by $p_0(t) = t^{r+1} + y_{r-1,0}t^{r-1} + \cdots + y_{1,0}t$ which is of degree $r+1$ and for any $i > 0$ we have $p_i(t) = y_{r,i}t^r + \cdots + y_{1,i}t$ which is of degree r. We will think of the p_i mostly as polynomials of the single variable t.

Lemma 8.

1. *The point $(t, \underline{\underline{y}}, \underline{s})$ is a Σ^{1_j}-point of F if and only if $p_i'(t) = p_i''(t) = \cdots = p_i^{(j)}(t) = 0$ for every $0 \le i \le k$.*

2. *The set of such points form a submanifold in $\mathbb{R}^{r(k+1)+z}$ which can be smoothly parametrized by $\underline{s}, \underline{y}_r, \ldots, \underline{y}_{j+1}$.*

Proof. Part 2 easily follows from part 1, since if $j < r$ and $\underline{s}, \underline{y}_r, \ldots, \underline{y}_{j+1}$ are fixed, then $p_i^{(j)}(t) = 0$ is a non-degenerate linear equation for $\underline{y}_j$. This can be uniquely solved. Then $p_i^{(j-1)}(t) = 0$ is a non-degenerate linear equation for $\underline{y}_{j-1}$, and so on. Finally if $j = r$ then obviously the only solution is $y_{l,i} = 0$ for every i, l independently of $\underline{s}$. Thus it suffices to show part 1.

We will proceed by induction on j. The initial step $j = 1$ is easy to see: dF is singular if and only if $p_i'(t) = 0$ for every i and in this case $\ker dF$ is the t-axis. Now let us suppose we know the statement for $j - 1$ and take a point $x \in \Sigma^{1_j}(F)$. Then $x \in \Sigma^{1_{j-1}}(F)$ and $\ker d_xF \subset T_x\Sigma^{1_{j-1}}(F)$. Then there is a sequence of points $x(i) = (t(i), \underline{\underline{y}}(i), \underline{s}(i) \in \Sigma^{1_{j-1}}(F)$ such that $x(i) \to x = (t, \underline{\underline{y}}, \underline{s})$, $\frac{t(i)-t}{|x(i)-x|} \to 1$ and $\frac{|\underline{\underline{y}}(i)-\underline{\underline{y}}|}{|x(i)-x|} \to 0$. Let us focus on p_l where l is arbitrary but fixed, and temporarily denote it by p. We will also temporarily include in the notation of p all its hidden variables. Then

$$p^{(j)}(t, \underline{\underline{y}}) = \lim_{i\to\infty} \frac{p^{(j-1)}(t(i), \underline{\underline{y}}) - p^{(j-1)}(t, \underline{\underline{y}})}{t(i) - t} \overset{(1)}{=}$$

$$\overset{(1)}{=} \lim_{i\to\infty} \frac{p^{(j-1)}(t(i), \underline{\underline{y}}) - p^{(j-1)}(t(i), \underline{\underline{y}}(i))}{t(i) - t} \overset{(2)}{=} 0.$$

Here (1) holds since $p^{(j-1)}(t, \underline{\underline{y}}) = p^{(j-1)}(t(i), \underline{\underline{y}}(i)) = 0$ by the inductive hypothesis. (2) holds since $p^{(j-1)}$ is a fixed finite sum of expressions linear in $\underline{\underline{y}}$

and $\frac{|\underline{\underline{y}}(i)-\underline{\underline{y}}(i)|}{t(i)-t} \to 0$. This argument can be easily reversed and so the statement is true for j. This completes the induction and thus proves part 1. $\square$

Theorem 8. *Let $f : M^n \to N^{n+k}$ a generic Morin map. If there exist points $x_i \neq x'_i \in M; (i \geq 1)$ such that $x_i \to x$, $x'_i \to x$, $x_i \in \Sigma^{1_r}(f)$ and $f(x_i) = f(x'_i)$ for every i, then $x \in \Sigma^{1_{r+1}}(f)$.*

Proof. It is obvious that $x \in \Sigma^{1_r}(f)$. Let us suppose that $x \in \Sigma^{1_r}(f) \setminus \Sigma^{1_{r+1}}(f)$. We can consider f locally around x and introduce Euclidean neighborhoods as before, denoting the function in the new coordinate-system by F. As $x_i \to x$ and $x'_i \to x$, these points will fall into the chosen neighborhood with at most finite exceptions. From Lemma 8 it is obvious that the only Σ^{1_r}-points of F are those for which $t = 0$ and $\underline{\underline{y}} = \underline{\underline{0}}$, and $\underline{s}$ is arbitrary. On the other hand if $F(t, \underline{\underline{y}}, \underline{s}) = (0, 0, \ldots, 0, \underline{\underline{0}}, \underline{s})$ then obviously $t = 0$ and $\underline{\underline{y}} = \underline{\underline{0}}$. So none of the Σ^{1_r}-points of F are double points of F which is contradiction. $\square$

If $f : M^n \to N^{n+k}$ is actually a prim map with lifting $g : M^n \looparrowright N^{n+k} \times \mathbb{R}$ and $x \in \Sigma^{1_r}(f) \setminus \Sigma^{1_{r+1}}(f)$, then we can take the Euclidean coordinates around x and $f(x)$ introduced at the beginning of this section, and choose a last extra coordinate around $g(x)$ such that g takes the local form $G(x) = (F(x), t)$. Let $j < r$ and let us consider the set

$$A' = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : u \in \Sigma^{1_j}(F), F(u) = F(v), t(u) \geq t(v)\}$$

and its closure $A = \mathrm{cl}(A')$.

Theorem 9. *The set A is a manifold with boundary $\partial A = \{(u, u) : u \in \Sigma^{1_{j+1}}(F)\}$.*

Proof. Theorem 8 implies that a boundary point of A' must be in $\Sigma^{1_{j+1}}(F)$. We shall give an explicit smooth parametrization of A' on an open halfspace, and show that this extends smoothly and bijectively to a parametrization of $\Sigma^{1_{j+1}}(F)$ on the boundary of the halfspace. It is obvious that the variables

$\underline{s}$ play no role whatsoever, so without loss of generality we may assume that $z = 0$ and omit $\underline{s}$ from the further calculations.

The condition $F(u) = F(v)$ obviously implies $\underline{\underline{y}}(u) = \underline{\underline{y}}(v)$, so $(u, v) \in A'$ if and only if $t(u) > t(v)$, and $p_i'(t(u)) = p_i''(t(u)) = \cdots = p_i^{(j)}(t(u)) = p_i(t(u)) - p_i(t(v)) = 0$ for every i. (Here we think of p_i as a polynomial of one variable. Its coefficients depend on $\underline{\underline{y}}$, but since $\underline{\underline{y}}$ is independent of u and v, this notation makes sense.)

We claim that for any choice of parameters $t(v) > t(u), \underline{y}_r, \underline{y}_{r-1}, \ldots, \underline{y}_{j+2}$ there is a unique choice of $\underline{y}_{j+1}, \ldots, \underline{y}_1$ depending smoothly on the parameters such that the resulting pair of points $(u, v) \in A'$. (In case of $j = r - 1$ there is only a single parameter $t(v) > 0$.)

Let us first deal with the case $j < r - 1$. Then for each i the problem of finding $y_{j+1,i}, y_{j,i}, \ldots, y_{1,i}$ such that $p_i'(t(u)) = p_i''(t(u)) = \cdots = p_i^{(j)}(t(u)) = p_i(t(u)) - p_i(t(v)) = 0$ holds can be solved independently of each other. In fact the problem is the same for every i, so we fix an arbitrary i and denote $p_i(t) = p(t) = \lambda_r t^r + \cdots + \lambda_1 t$ temporarily. Let us write $p(t) = q(t) + \lambda_{j+2} t^{j+2} + \cdots + \lambda_r t^r = q(t) + r(t)$. Since $\lambda_r, \ldots, \lambda_{j+2}, t(u)$ and $t(v)$ are fixed parameters, we know the value of $r(t(u)), r(t(v)), r'(t(u)), r''(t(u)), \ldots, r^{(j)}(t(u))$. We have to find the coefficients of q. Let us write q as a Taylor polynomial around $t(u)$. Then

$$q(t) = q(t(u)) + \sum_{i=1}^{j} q^{(i)}(t(u)) \cdot \frac{(t - t(u))^i}{i!} + \lambda_{j+1} \cdot (t - t(u))^{j+1}. \tag{9}$$

Since $0 = p^{(i)}(t(u)) = q^{(i)}(t(u)) + r^{(i)}(t(u))$, in (9) the only unknown value is λ_{j+1}. By definition

$$q(t(v)) - q(t(u)) = p(t(v)) - r(t(v)) + r(t(u)) - p(t(u)) = r(t(u)) - r(t(v)),$$

and hence by substituting $t = t(v)$ in (9) we get that

$$\lambda_{j+1} = \left(\frac{1}{t(v) - t(u)}\right)^{j+1} \cdot \left(r(t(u)) - r(t(v)) - \sum_{i=1}^{j} q^{(i)}(t(u)) \cdot \frac{(t - t(u))^i}{i!}\right)$$

As every quantity on the right hand side is fixed and $t(u) > t(v)$ we find that the parameters uniquely and smoothly determine λ_{j+1}. Then all the remaining λ's are uniquely and smoothly determined by the Taylor expansion (9). Finally to see what happens on the boundary of the halfspace $t(u) > t(v)$ just observe, that the vanishing of the derivatives of p at $t(u)$ imply that $p(t) = p(t(u)) + (t - t(u))^{j+1} \cdot w(t)$ for some polynomial $w(t)$. Then the equation $p(t(v)) = p(t(u))$ is equivalent to $w(t(v)) = 0$. Then if $t(v) - t(u)$ converges to 0 the solution will converge to a $w(t)$ for which $w(t(u)) = 0$, which is equivalent to saying that $p^{(j+1)}(t(u)) = 0$. So the boundary of the halfspace $t(u) > t(v)$ parametrizes those points (u, u) for which $p'(t(u)) = p''(t(u)) = \cdots = p^{(j+1)}(t(u)) = 0$ which is equivalent to $u \in \Sigma^{1_{j+1}}(F)$.

Now consider the case $j = r - 1$. The only parameter is $t(v)$. Let us suppose that we have a solution u that satisfies all the equations. Let $i \geq 1$. Then $p_i(t)$ is a degree r polynomial for which the first r derivatives vanish at $t(u)$. Thus $p_i = c_i \cdot (t - t(u))^r$. Further we know that $p_i(t(v)) = p_i(t(u)) = 0$ while $t(v) > t(u)$. This is only possible if $c_i = 0$. So all the p_i's must be identically 0, except for p_0. Let us temporarily denote $p_0(t) = p(t) = t^{r+1} + \lambda_{r-1}t^{r-1} + \cdots + \lambda_1 t$. The constraints on the derivatives imply that

$$p(t) = p(t(u)) + p^{(r)}(t(u)) \cdot \frac{(t - t(u))^r}{r!} + (t - t(u))^{r+1}.$$

The polynomial p has no x^r term by definition, so $p^{(r)}(t(u)) = r!(r + 1)t(u)$, and so

$$p(t) = p(t(u)) + (t + r \cdot t(u))(t - t(u))^r.$$

Finally

$$p(t(u)) = p(t(v)) = p(t(u)) + (t(v) + r \cdot t(u))(t(v) - t(u))^r,$$

so $t(u) = -t(v)/r$, and $p(0) = 0$ determines $p(t(u))$. Thus indeed for any $t(v) > 0$ there is a unique solution u, this solution is smoothly parametrized by $t(v)$, and the boundary $t(v) = 0$ goes to the only Σ^{1_r}-point, the origin. □

5 Applications to product maps

The results of this section are the first steps in understanding how the direct product operation affects the singularities of maps. They show that indeed there is some well controllable effect, at least in the simplest cases. There are two main difficulties. The first one is that the direct product of generic maps will not be generic, so one has to take a small perturbation. This makes it hard to understand the singular strata geometrically. The second one is that generally the product of two singular maps even after a generic perturbation will have more complicated singularities then the original maps had.

In Section 5.1 we study products of immersions. Here only the first type of problem arises, namely that the self intersections will not be transverse. This can be overcome by employing the general multiple-point formula from section 3.1.2 that helps to compute the characteristic numbers of multiple-point manifolds.

In Section 5.3 we study Morin maps. In this case one has to deal with the second kind of problem. We get around this by increasing the dimension of the target space by one.

In Section 5.4 we define and set out to compute the ring $\mathrm{Mor}_{\mathbb{Q}}$ (the ring of rational cobordism classes of Morin maps). First, in Section 5.4.1, we identify the components of $\mathrm{Mor}_{\mathbb{Q}}$ as subgroups of the rational oriented cobordism ring $\Omega_* \otimes \mathbb{Q}$. Then combining the results of the previous sections we show that the singular strata behave nicely under the multiplication defined in Section 5.3. It turns out that this information is actually enough to compute $\mathrm{Mor}_{\mathbb{Q}}$.

Finally Section 5.5 deals with general singular maps. We show that a Cartan-type formula relates the homology class of Σ^1 points of two maps with that of their direct product. We compute the oriented Thom polynomial of the Σ^2 singularity with $\mathbb{Q}$ coefficients. Finally we derive a Cartan-type formula for the Σ^2 points as well.

5.1 Products of immersions

First we shall introduce a characteristic class β that assigns to any oriented vector bundle ξ over B an element

$$\beta(\xi)=\prod_{i=1}^{\infty}(1+p_1(\xi)t_i^1+p_2(\xi)t_i^2+\dots)\in H^*(B;\mathbb{Q})[[t_1,t_2,\dots]]$$

in the ring of formal power series of the variables t_i over the ring $H^*(B;\mathbb{Q})$. (Here $p_i(\xi)\in H^{4i}(B;\mathbb{Q})$ is the $4i$-dimensional Pontrjagin class of ξ). Since the Cartan formula holds for Pontrjagin classes modulo 2-torsion it follows that $\beta(\xi\oplus\eta)=\beta(\xi)\cdot\beta(\eta)$. (We have got rid of all torsions by taking $\mathbb{Q}$ coefficients.) It is also easily seen that β is natural, and always has an inverse element. When B is a manifold we shall abbreviate $\beta(TB)$ by $\beta(B)$.

Now let $f:M^n\to N^{n+k}$ be a generic immersion between oriented manifolds. The manifolds and the maps representing the *r-fold points* of f in the source and the target respectively are denoted as before by

$$f_r:\Delta_r(f)\to M,\ \text{and}$$
$$\widetilde{f_r}:\widetilde{\Delta_r}(f)\to N.$$

When the codimension of the map k is even, these manifolds are equipped with a natural orientation. Our goal is to obtain information about the cobordism classes of certain multiple-point manifolds. To this end we will try to compute their characteristic numbers. Let us denote

$$m_r=m_r(f)=f_{r!}(\beta(\Delta_r(f))),$$
$$n_r=n_r(f)=\widetilde{f}_{r!}(\beta(\widetilde{\Delta_r}(f))).$$

The reason for considering these elements is the following simple observation. Evaluating each coefficient of m_r on the fundamental class of M we get an element in $\mathbb{Q}[[t_1,t_2,\dots]]$. The coefficients of this power series are exactly the Pontrjagin numbers of $\Delta_r(f)$.

The classes m_r and n_r are related by the Main formula of section 3.1.2:

$$m_r \cdot \beta(\nu_f) = f^* n_{r-1} - e(\nu_f) m_{r-1} \tag{10}$$

We are going to apply this in the case when the target is a Euclidean space. This implies $f^* = 0$ so (10) is simplified to $m_r \cdot \beta(\nu_f) = -e(\nu_f) \cdot m_{r-1}$. Then applying this recursively one gets that $m_r \cdot \beta(\nu_f)^{r-1} = (-e(\nu_f))^{r-1} \cdot m_1$. But $m_1 = \beta(M)$ and $\beta(M) \cdot \beta(\nu_f) = \beta(\mathbb{R}^n) = 1$, so we end up with

$$m_r = (-e(\nu_f))^{r-1} \cdot \beta(M)^r.$$

Now we can state and prove the main result of this section.

Theorem 10. *Let $g_i : M_i^{n_i} \to \mathbb{R}^{n_i+k_i}; (i = 1, 2)$ be generic immersions. Then the r-tuple point manifold $\Delta_r(g_1 \times g_2) \sim (-1)^{r-1} \Delta_r(g_1) \times \Delta_r(g_2)$ where $\sim$ stands for "unoriented-cobordant".*

If the M_i are oriented and the k_i are even, then their r-tuple point manifolds are oriented cobordant.

Proof. We will only consider the oriented case. The unoriented version is proved exactly the same way, except that there is no need to study Pontrjagin classes.

Let $f = g_1 \times g_2$. Then

$$\begin{aligned} m_r(f) = \quad & (-e(\nu_f))^{r-1} \cdot \beta((M_1 \times M_2))^r = \\ = \quad & (-e(\nu_{g_1} \times \nu_{g_2}))^{r-1} \cdot \beta(TM_1 \times TM_2)^r = \\ = \quad & (-1)^{r-1} \left((-e(\nu_{g_1})^{r-1} \cdots \beta(M_1)^r\right) \times \left((-e(\nu_{g_2})^{r-1} \cdots \beta(M_2)^r\right) = \\ = \quad & (-1)^{r-1} m_r(g_1) \times m_r(g_2). \end{aligned}$$

The following equations are easily checked.

$$\begin{aligned} \langle \beta(\Delta_r(f)), [\Delta_r(f)] \rangle = \langle m_r(f), [M_1 \times M_2] \rangle = \langle m_r(g_1 \times g_2), [M_1 \times M_2] \rangle = \\ = (-1)^{r-1} \langle \beta(\Delta_r(g_1), [\Delta_r(g_1)] \rangle \cdot \langle \beta(\Delta_r(g_2), [\Delta_r(g_2)] \rangle = \\ = (-1)^{r-1} \langle \beta((\Delta_r(g_1) \times \Delta_r(g_2))), [\Delta_r(g_1) \times \Delta_r(g_2)] \rangle \end{aligned}$$

We have obtained equality of two formal power series, so the corresponding coefficients must be equal on the two sides. As the coefficients are the Pontrjagin numbers of the manifolds involved, we get that the Pontrjagin numbers of the two manifolds are all equal.

To finish the proof we have to repeat the whole argument using an analogous class instead of β, namely

$$\beta'(\xi) = \prod_{i=1}^{\infty}(1 + w_1(\xi)t_i^1 + w_2(\xi)t_i^2 + \dots) \in H^*(B, \mathbb{Z}_2)[[t_1, t_2, \dots]].$$

It is obvious that all the above hold for β' as well. Thus not only the Pontrjagin numbers, but also the Stiefel-Whitney numbers of the two manifolds are equal. Since the oriented cobordism class is determined by these numbers, the claim of the theorem follows.

□

This result will no longer hold if we consider a general target space N. However the Pontrjagin and Stiefel-Whitney numbers of the multiple-point manifolds of $g_1 \times g_2$ are still expressible in terms of g_1, g_2 and their multiple-point manifolds. This expression is particularly simple for the double-point set.

First we need a small result about the embedded manifold representing a vector bundle's Euler class. Let $\xi \to B$ be a vector bundle over a manifold B. Let $s : B \to \xi$ be a section transverse to the 0-section. Let us denote by Δ_ξ the submanifold in B that is the inverse image of the 0-section by s, and let $\delta_\xi : \Delta_\xi \to B$ denote the inclusion.

Lemma 9. $\langle \beta(\Delta_\xi), [\Delta_\xi] \rangle = \langle \beta(B) \cdot \frac{e(\xi)}{\beta(\xi)}, [B] \rangle$.

Proof. It suffices to show that

$$\delta_{\xi !}(\beta(\Delta_\xi)) = \beta(B) \cdot \frac{e(\xi)}{\beta(\xi)}.$$

By the construction of Δ_ξ we have the following pull-back diagram:

$$\begin{array}{ccc} \Delta_\xi & \xrightarrow{\delta_\xi} & B \\ \Big\downarrow{\scriptstyle \delta_\xi} & & \Big\downarrow{\scriptstyle \text{0-section}} \\ B & \xrightarrow{s} & \xi \end{array}$$

Hence the normal bundle of δ_ξ is just the pull-back of the normal-bundle of the 0-section. This latter is just ξ. Thus we have

$$T\Delta_\xi \oplus {\delta_\xi}^*\xi = {\delta_\xi}^* TB,$$

which in turn implies that

$$\beta(\Delta_\xi) = {\delta_\xi}^* \left(\frac{\beta(B)}{\beta(\xi)} \right).$$

Applying the push-forward to this equation gives the proof of the lemma, since $f_!(f^*x) = f_!(1) \cdot x$ is well known and obviously $\delta_{\xi!}(1) = e(\xi)$. ☐

Theorem 11. *Let $g_i : M_i^{n_i} \to N_i^{n_i+k_i}; (i = 1, 2)$ be generic immersions. Then*

$$\Delta_2(g_1 \times g_2) \sim \Delta_2(g_1) \times \Delta_2(g_2) + \Delta_2(g_1) \times \Delta_{\nu_{g_2}} + \Delta_{\nu_{g_1}} \times \Delta_2(g_2)$$

where $\sim$ stands for "unoriented-cobordant". (Recall that ν_{g_i} is the normal bundle of g_i and Δ_{g_i} is the zero set of a generic section of ν_{g_i}.) If the M_i are oriented and the k_i are even, then the same is true up to oriented cobordism.

Proof. We proceed in a similar manner as in the previous theorem. Let us put $f = g_1 \times g_2$ and $M = M_1 \times M_2$ again. Then using (10) we get

$$\begin{aligned} \beta(\nu_f) \cdot m_2(f) + e(\nu_f) \cdot \beta(M) = f^* f_!(\beta(M)) = \\ = g_1^* g_{1!}(\beta(M_1)) \times g_2^* g_{2!}(\beta(M_2)) = \\ = \Big(\beta(\nu_{g_1}) m_2(g_1) + e(\nu_{g_1}) \cdot \beta(M_1) \Big) \times \Big(\beta(\nu_{g_2}) m_2(g_2) + e(\nu_{g_2}) \cdot \beta(M_2) \Big), \end{aligned}$$

and thus

$$\beta(\nu_f) \cdot m_2(f) = \beta(\nu_f) \cdot \Big(m_2(g_1) \times m_2(g_2) + \\ + m_2(g_1) \times \beta(M_2) \frac{e_{\nu_{g_2}}}{\beta(\nu_{g_2})} + \beta(M_1) \frac{e_{\nu_{g_1}}}{\beta(\nu_{g_1})} \times m_2(g_2) \Big)$$

Now we can divide by $\beta(\nu_f)$ as it is an invertible element. We evaluate both sides on $[M] = [M_1] \times [M_2]$. Finally we have to apply the previous lemma to get that all the corresponding characteristic numbers are equal for the two manifolds in question. As before, we can repeat the argument for Stiefel-Whitney numbers in $\mathbb{Z}_2$ coefficients and Pontrjagin numbers in $\mathbb{Q}$ coefficients, so we get both parts of the theorem at the same time. □

Remark 10.

1. It is possible to carry out similar calculations for triple points or points of higher (say r) multiplicity. But the number of terms involved in these formulas grow exponentially with r and the authors did not manage to find a nice way to write them down, not even recursively.

2. It would be possible to obtain similar formulas not only for the cobordism classes of the underlying multiple-point manifolds, but for the cobordism classes of the maps f_r themselves. To do this one would need to consider the characteristic numbers of these maps instead of the characteristic numbers of the manifolds. These calculations are more or less the same as the ones described here, but they are harder to keep track of.

3. It seems that the same results could be obtained using techniques of Eccles and Grant from [4].

4. We would like to point out that Theorem 11 is a non-trivial generalization of the oriented case of Theorem A in [3], which considers the case of $n = k$.

5.2 Products of prim maps

Let us recall that a generic map $f : M \to N$ is called prim (*pr*ojected *im*mersion) if it can be lifted to a generic immersion, $\tilde{f} : M \to N \times \mathbb{R}$. We will always denote the lifting by a tilde.

The cobordism class of a prim map f (as defined in section 2.2) will be denoted by $[f]$. (For details see e.g. [20].)

By definition a prim map is necessarily a Morin map. Prim maps provide a good link between immersions and Morin maps in the sense that they can be handled using regular immersion techniques and on the other hand Morin maps are "almost prim". We shall exploit this idea by first defining multiplication of prim maps (using their liftings to immersions) and then show how this gives a multiplication on Morin maps (using results from [26]). We will only work with prim maps whose target space is Euclidean.

Let us denote $l_0 : pt \hookrightarrow \mathbb{R}$ the inclusion of a point into the line.

Lemma 10.

a) Any two generic hyperplane projections of an immersion represent the same prim cobordism class.

b) Hyperplane projections of cobordant immersions represent the same prim cobordism class.

Proof. a) Instead of taking two projections of the same immersion we can take projection to the same hyperplane of two immersions which differ only by a rotation. This rotation can be realized by a regular homotopy. Regular homotopy is a special case of cobordism, hence a) will follow from b).

b) A generic hyperplane projection of the cobordism connecting the two immersions gives a prim cobordism of the prim maps obtained on the boundaries. □

Definition 12. Given two prim maps $f_i : M_i \to \mathbb{R}^{n_i}$ $(i = 1, 2)$ consider the product map

$$g = f_1 \times f_2 \times l_0 : M_1 \times M_2 \to \mathbb{R}^{n_1+n_2} \times \mathbb{R}.$$

The map g might not yet be prim, but we can turn it into such by a small perturbation. Take liftings $\tilde{f}_1$ and $\tilde{f}_2$ that are sufficiently close to $f_1 \times l_0$ and $f_2 \times l_0$. Now $\tilde{f}_1 \times \tilde{f}_2 : M_1 \times M_2 \to \mathbb{R}^{n_1+n_2} \times \mathbb{R}^2$ is a non-generic immersion. Let us take a sufficiently small perturbation of this product so that it becomes a generic immersion. Finally take a generic projection of this immersion to a hyperplane "close" to $\mathbb{R}^{n_1+n_2} \times \mathbb{R}$, where the last $\mathbb{R}$ factor is the diagonal in $\mathbb{R}^2$. We obviously get a prim map g' that can be arbitrarily close to g. Let us denote $g' = f_1 * f_2$ and let us define the multiplication on prim cobordism classes as follows: $[f_1] * [f_2] = [f_1 * f_2]$.

Theorem 12. *The above definition is correct, that is the cobordism class $[f_1 * f_2]$ is independent of the choice of f_1 and f_2 within their cobordism class and of any other choices made in the definition. The multiplication defined in this way together with the addition being the disjoint union gives rise to a ring structure.*

Proof. The liftings are given up to regular homotopy. Also the perturbation of $\tilde{f}_1 \times \tilde{f}_2$ is unique up to regular homotopy. Thus Lemma 10 implies that the resulting prim map is independent of these choices.

Now suppose $[f_1] = [g_1]$. Then there is a prim cobordism H joining f_1 and g_1. We can take its lifting $\tilde{H}$ which is an immersed cobordism between $\tilde{f}_1$ and $\tilde{g}_1$, and so $\tilde{f}_1 \times \tilde{f}_2$ and $\tilde{g}_1 \times \tilde{f}_2$ are regularly homotopic via $\tilde{H} \times \tilde{f}_2$. So their projections are prim cobordant, and this is what we wanted to prove. (The definition is symmetric so the other factor can be handled the same way.)

The last claim only requires checking distributivity, which is obvious. ☐

Remark 11. It is clear that we can carry out the same construction for oriented prim maps.

5.3 Products of Morin maps

Let us consider the set of rational cobordism classes of all Morin maps to Euclidean spaces. This set is a commutative group with addition induced by the disjoint union of maps. In this section we endow this group with a ring structure. The main tool in constructing the multiplication will be prim maps

We shall only consider maps between oriented manifolds. Let us denote the group of cobordism classes of oriented Morin maps $f : M^n \to \mathbb{R}^{n+k}$ by $\mathrm{Morin}^{SO}(n,k)$ and the cobordism classes of prim maps $f : M^n \to \mathbb{R}^{n+k}$ by $\mathrm{Prim}^{SO}(n,k)$. As a prim map is automatically Morin and prim cobordant maps are Morin cobordant as well, we have a natural forgetting map $F : \mathrm{Prim}^{SO}(n,k) \to \mathrm{Morin}^{SO}(n,k)$, that induces a map $F_{\mathbb{Q}} : \mathrm{Prim}^{SO}(n,k) \otimes \mathbb{Q} \to \mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q}$. The following key result, which roughly says that every Morin map is almost prim, is proved in [26]:

Lemma 11. *The map $F_{\mathbb{Q}}$ is epimorphic.*

This lemma says that every Morin map has a non-zero multiple that is Morin-cobordant to a prim map. Using this result and the construction in the previous section we can now define a multiplication on $\left(\bigoplus_{n,k} \mathrm{Morin}^{SO}(n,k)\right) \otimes \mathbb{Q}$.

Definition 13. Let us take two Morin maps $g_i : M_i^{n_i} \to \mathbb{R}^{n_i+k_i}$. By Lemma 11 we can find prim maps f_1 and f_2 that are rationally Morin cobordant to g_1 and g_2. Let us define $[g_1] * [g_2] \stackrel{\mathrm{def}}{=} F_{\mathbb{Q}}([f_1 * f_2])$, where $[f]$ denotes the rational Morin cobordism class of the Morin map f.

Theorem 13. *The above definition is correct, that is $[g_1]*[g_2]$ is independent of the choices made. The multiplication defined this way gives rise to a ring structure on $\left(\bigoplus_{n,k} \mathrm{Morin}^{SO}(n,k)\right) \otimes \mathbb{Q}$.*

Proof. There is only one thing left that needs to be checked: if f_1 and f_1' are Morin cobordant prim-representatives of g_1, then $f_1 * f_2$ is Morin cobordant to $f_1' * f_2$. Let us take the Morin cobordism H connecting f_1 and f_1'. Then $H \times (f_2 \times l_0)$ is still a Morin cobordism after a sufficiently small perturbation, since the factor $f_2 \times l_0$ can be perturbed to an immersion. This Morin cobordism connects exactly the two desired maps. $\square$

5.4 Ring structure of Morin maps

Definition 14. Let $\mathrm{Mor}_{\mathbb{Q}}$ denote the group $\bigoplus_{n,k} \mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q}$ with this ring structure. $\mathrm{Mor}_{\mathbb{Q}}$ is a bigraded ring, the two grades being n and $k+1$. Note that this implies that the direct sum $\bigoplus_{k \text{ odd}} \mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q}$ is a subring or $\mathrm{Mor}_{\mathbb{Q}}$.

5.4.1 The structure of $\mathrm{Morin}^{SO}(n,k)$

Let $f : M^n \to \mathbb{R}^{n+k}$ be a generic oriented Morin map of codimension k. To such a map we can associate the subset $\Sigma^{1_r}(f) \subset M^n$ defined in Remark 3 in section 2.4. This subset is actually a submanifold. The cobordism class of this submanifold is invariant under a Morin cobordism of f, since the Σ^{1_r} points of the cobordism of f give a cobordism between the Σ^{1_r} points of f. If we tensor with $\mathbb{Q}$ then this submanifold becomes an element in $\Omega_* \otimes \mathbb{Q}$. Thus by abuse of notation we get maps

$$\Sigma^{1_r} : \mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q} \to \Omega_* \otimes \mathbb{Q}, \quad [f] \mapsto [\Sigma^{1_r}(f)],$$

whose image we denote by $\mathrm{Im}(\Sigma^{1_r})$. Let $\mathrm{Imm}^{SO}(n,k)$ denote the oriented cobordism group of immersions of n-dimensional oriented manifolds to $\mathbb{R}^{n+k}$. Our goal in this section is to prove the following theorem:

Theorem 14.

1. *If* $k = 2l$ *then* $\mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q} = Imm^{SO}(n,k+1)$ *and* $Im(\Sigma^{1_r}) = 0$ *for any* $r \geq 1$.

2. *If* $k = 2l+1$ *then*

$$\mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q} = \bigoplus_{r \geq 0} Im(\Sigma^{1_r}) = \bigoplus_{i \geq 0} Im(\Sigma^{1_{2i}})$$

where $Im(\Sigma^{1_{2i+1}}) = 0$ *and* $Im(\Sigma^{1_{2i}}) = \{[L] \in \Omega_{n-2i(k+1)} \otimes \mathbb{Q} : p_I[L] = 0$ *for any Pontrjagin monomial* p_I *which has a factor* p_j *with index* $j \geq l\}$

Proof. Part (i) is stated explicitly in [26] in Section 14/B.

To show part (ii) we need some preparations. For any stable singularity type η there is a bundle $\tilde{\xi}_\eta$ that plays the role of the universal normal bundle for this singularity type. This means the following: Whenever for a map $f : M \to N$ one of its most complicated singularities is η then the η-points of f form a submanifold of M. The restriction of f to this submanifold is an immersion to N. The normal bundle of this immersion is induced from $\tilde{\xi}_\eta$. (See [16] for details.)

Let us write $\tilde{\xi}_r = \tilde{\xi}_{\Sigma^{1_r}}$ for short. Let $\mathrm{Imm}^{\tilde{\xi}_r}(n - r(k+1), r(k+1)+k)$ denote the cobordism group of oriented immersions $f : M^{n-r(k+1)} \to \mathbb{R}^{n+k}$ whose normal bundles are induced from $\tilde{\xi}_r$.

We need two results from [26] which we state here in a lemma.

Lemma 12. *Let* $k \geq 1$ *be odd. Then*

1.

$$\mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q} = \bigoplus_{i=0}^{\infty} Imm^{\tilde{\xi}_{2i}}(n - 2i(k+1), 2i(k+1)+k) \otimes \mathbb{Q}. \quad (11)$$

2.

$$H_{n+k}(T\tilde{\xi}_{2i}; \mathbb{Q}) = H_{n-2i(k+1)}(BSO(k); \mathbb{Q}).$$

Proof. Part (i) is an immediate consequence of Example 118 in [26].

For part (ii) we cite from [26] that the bundle $\tilde{\xi}_\eta$ has a counterpart denoted by ξ_η which is the universal normal bundle of the η-points of a map in the source manifold. The two bundles ξ_η and $\tilde{\xi}_\eta$ have the same base space BG_η where G_η is the maximal compact subgroup of the symmetry group of the singularity η. This implies that the homologies of $T\tilde{\xi}_\eta$ and $T\xi_\eta$ are the same up to a dimension shift equal to $\operatorname{rank}\tilde{\xi}_\eta - \operatorname{rank}\xi_\eta = k$, i.e. $H_{n+k}(T\tilde{\xi}_r;\mathbb{Q}) = H_n(T\xi_r;\mathbb{Q})$. Lemma 102/b in [26] implies that for even r we have $H_n(T\xi_r;\mathbb{Q}) = H_{n-r(k+1)}(BSO(k);\mathbb{Q})$. The statement follows. □

Now we can return to the proof of Theorem 14 which will follow fairly easily from part (i) of Lemma 12. Let us introduce the notation $a = n - r(k+1)$ and $b = r(k+1) + k = n + k - a$ for convenience. Let us consider the sequence of forgetting maps

$$\operatorname{Imm}^{\tilde{\xi}_r}(a,b) \xrightarrow{\tilde{\alpha}} \operatorname{Imm}^{SO}(a,b) \xrightarrow{\tilde{\beta}} \Omega_a, \tag{12}$$

where we first forget about the extra structure on the normal bundle, and then forget about the immersion and just take the underlying source manifold. On the level of classifying spaces this corresponds to the standard maps:

$$T\tilde{\xi}_r \to T\gamma_b^{SO} \to T\gamma^{SO},$$

and on the level of base spaces to

$$BSO(k) \xrightarrow{\alpha} BSO(b) \xrightarrow{\beta} BSO.$$

α induces $\tilde{\xi}_r$ from γ_b^{SO} and β is just the standard inclusion map.

Using the well-known Pontrjagin-Thom construction and part (ii) of Lemma 12 we get that

$$\operatorname{Imm}^{\tilde{\xi}_r}(a,b) \otimes \mathbb{Q} \cong \pi^S_{n+k}(T\tilde{\xi}_r) \otimes \mathbb{Q} \cong H_{n+k}(T\tilde{\xi}_r;\mathbb{Q}) = H_a(BSO(k);\mathbb{Q}).$$

Similarly we have

$$\operatorname{Imm}^{SO}(a,b) \otimes \mathbb{Q} \cong \pi^S_{n+k}(T\gamma_b^{SO}) \otimes \mathbb{Q} \cong H_{n+k}(T\gamma_b^{SO};\mathbb{Q}) = H_a(BSO(b);\mathbb{Q}),$$

and $\Omega_a \otimes \mathbb{Q} \cong H_a(BSO;\mathbb{Q})$. In this context the forgetting map in (12) becomes

$$H_a(BSO(k);\mathbb{Q}) \xrightarrow{\alpha_*} H_a(BSO(b);\mathbb{Q}) \xrightarrow{\beta_*} H_a(BSO;\mathbb{Q}).$$

We want to prove that this map is injective, and compute its image. Since we work with rational coefficients it is enough to show that the dual map

$$H^a(BSO(k);Q) \xleftarrow{\alpha^*} H^a(BSO(b);\mathbb{Q}) \xleftarrow{\beta^*} H^a(BSO;\mathbb{Q})$$

is surjective, and compute its kernel. $H^*(BSO;\mathbb{Q}) = \mathbb{Q}[p_1,p_2,\dots]$ and since k is odd $H^*(BSO(k);\mathbb{Q}) = \mathbb{Q}[p_1,p_2,\dots,p_{\frac{k-1}{2}}]$. The induced homomorphism $\alpha^*\beta^*$ takes the total Pontrjagin class $p = 1+p_1+p_2+\dots$ to the total Pontrjagin class of $\tilde{\xi}_r$ preserving the grading. It is known that $p(\tilde{\xi}_r) = p(\gamma_k^{SO})^r = (1+p_1+p_2+\dots p_{\frac{k-1}{2}})^r \in H^*(BSO(k);\mathbb{Q})$. Now easy computation shows that indeed every $p_i \in H^*(BSO(k);\mathbb{Q})$ is in the image of $\alpha^*\beta^*$ so the map is surjective. On the other hand the kernel is generated exactly by those Pontrjagin monomials in which there is at least one factor with index larger than $\frac{k-1}{2}$. Thus the image of $\tilde{\beta}\tilde{\alpha} : \mathrm{Imm}^{\tilde{\xi}_r}(a,b) \to \Omega_a$ is generated by the cobordism classes of exactly those manifolds L for which $p_I[L]=0$ for any Pontrjagin monomial p_I which has a factor p_j with index $j \geq \frac{k-1}{2}$.

To finish the proof of the theorem we just have to observe that Σ^{1_r} is the composition of the projection in the splitting (11) and the forgetting map in (12): $\mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q} \to \mathrm{Imm}^{\tilde{\xi}_r}(n-r(k+1), r(k+1)+k) \otimes \mathbb{Q} \to \Omega_{n-r(k+1)} \otimes \mathbb{Q}$.

□

5.4.2 Ring homomorphisms

We can consider Σ^{1_r} as a map from $\mathrm{Mor}_{\mathbb{Q}}$ to the rational oriented cobordism ring:

$$\Sigma^{1_r} : \bigoplus_{k,n} \mathrm{Morin}^{SO}(n,k) \otimes \mathbb{Q} \to \Omega_* \otimes \mathbb{Q}.$$

Theorem 15. *The map Σ^{1_r} is a ring homomorphism. In other words*

$$\Sigma^{1_r}(f * g) \sim \Sigma^{1_r}(f) \times \Sigma^{1_r}(g)$$

holds for any two Morin maps f, g to Euclidean spaces where $\sim$ now stands for rationally cobordant (in the oriented sense).

Proof. The only case that requires proof is when k is odd and $r > 0$ is even, since otherwise $\Sigma^{1_r}(f)$ is always 0 and for $r = 0$ the statement is obvious. (Note that $\Sigma^0(f) = \Sigma^{1_0}(f)$ is the rational cobordism class of the source manifold of f.) We will proceed along the lines explained earlier, that is we will use prim maps as a link between Morin maps and immersions. Then the multiplicative properties of multiple-points of immersions will provide the result.

Let us first consider prim maps. The same argument as above gives a map

$$\Sigma^{1_r}_{Prim} : \left(\bigoplus_{k \text{ odd}, n} \mathrm{Prim}^{SO}(n,k) \right) \otimes \mathbb{Q} \to \Omega_* \otimes \mathbb{Q}$$

where $\Sigma^{1_r}_{Prim} = \Sigma^{1_r} \circ F_{\mathbb{Q}}$.

Given an immersion $f : M^n \to \mathbb{R}^{n+k+1}$, let us denote by $\pi(f)$ its generic projection to a hyperplane. This map is a prim map whose prim cobordism class is well defined and depends only on the cobordism class of the immersion f according to Lemma 10. The direct sum $\bigoplus_{k \text{ odd}, n} \mathrm{Imm}^{SO}(n, k+1)$ has a natural ring structure with multiplication being the direct product. It is clear from the definitions that the map

$$\pi : \bigoplus_{k \text{ odd}, n} \mathrm{Imm}^{SO}(n, k+1) \to \bigoplus_{k \text{ odd}, n} \mathrm{Prim}^{SO}(n,k); \quad [f] \mapsto [\pi(f)]$$

is a ring homomorphism with respect to the direct product on the left, and $*$-product on the right. The same remains true after forming the tensor product with $\mathbb{Q}$.

In Theorem 10 we have shown that

$$\tilde{M}_{r+1} : \bigoplus_{k \text{ odd},n} \mathrm{Imm}^{SO}(n, k+1) \to \Omega_*$$

is a ring homomorphism, and obviously the same is true after forming the tensor product with $\mathbb{Q}$.

To finish the proof we have to use Theorem 7 from section 4 in the following form:

Theorem 16. $\tilde{M}_{r+1} \otimes \mathrm{id}_{\mathbb{Q}} = (\pi \otimes \mathrm{id}_{\mathbb{Q}}) \circ \Sigma^{1_r}_{Prim}$ *i.e. the rational cobordism class of the manifold of* $r+1$*-tuple points of an immersion* $f : M^n \to R^{n+k+1}$ *coincides with that of the manifold of* Σ^{1_r} *points of its hyperplane projection.*

(The same result can be found in [25].) Thus the following diagram is commutative.

$$\begin{array}{ccc}
\left(\bigoplus_{k \text{ odd},n} \mathrm{Imm}^{SO}(n,k+1)\right) \otimes \mathbb{Q} & & \\
\downarrow {\scriptstyle \pi \otimes \mathrm{id}_{\mathbb{Q}}} & \searrow {\scriptstyle \tilde{M}_{r+1} \otimes \mathrm{id}_{\mathbb{Q}}} & \\
\left(\bigoplus_{k \text{ odd},n} \mathrm{Prim}^{SO}(n,k)\right) \otimes \mathbb{Q} & \xrightarrow{\Sigma^{1_r}_{Prim}} & \Omega_* \otimes \mathbb{Q} \\
\downarrow {\scriptstyle F_{\mathbb{Q}}} & \nearrow {\scriptstyle \Sigma^{1_r}} & \\
\left(\bigoplus_{k \text{ odd},n} \mathrm{Morin}^{SO}(n,k)\right) \otimes \mathbb{Q} & &
\end{array}$$

The vertical maps are ring epimorphisms and $\tilde{M}_{r+1}$ is a ring homomorphism. This implies that $\Sigma^{1_r}_{Prim}$ and Σ^{1_r} are ring homomorphisms too. □

We can summarize our results as follows. The ring $\mathrm{Mor}_{\mathbb{Q}}$ is the direct sum of clearly identified subgroups of $\Omega_* \oplus \mathbb{Q}$ as stated in Theorem 14. An element in $\mathrm{Mor}_{\mathbb{Q}}$ is completely determined by its grading and its collection of Σ^{1_r} strata. Then Theorem 15 shows that the multiplication in $\mathrm{Mor}_{\mathbb{Q}}$ corresponds to the direct product of cobordism classes representing the singular strata. The manifolds corresponding to the various Σ^{1_r} strata multiply independently of each other. Thus the ring $\mathrm{Mor}_{\mathbb{Q}}$ is completely computed.

5.5 Singular strata of direct products

Our goal in this final section is to show that the cohomology class represented by the submanifold formed by the closure of the set of certain singular points of a direct product $f \times g$ depends only on those of f, g and some maps closely related to them. Before formulating the theorems, we have to introduce some notation.

Definition 15. For $j \geq 0$ let $q_j : * \to S^j$ denote the inclusion of a point into S^j and let $q_{-j} : S^j \to *$ be the map that takes the sphere to a point. Now for any integer j we define $f'_j = f \times q_j$ and take f_j to be a generic perturbation of f'_j.

Finally let $\mathrm{id}_M^j = \mathrm{id}_M \times q_j$.

5.5.1 The Σ^1 stratum

In this section we work with Z_2 coefficients. Let $\Sigma^1 f = \Sigma^1(f)$ denote the closure of the set of all singular points in the source manifold of f. (The parenthesis are omitted for easier reading of the formulas below.) The Thom polynomial of this singularity type is w_{k+1}. That is, given a map $f : M^n \to N^{n+k}$, the cohomology class Poincaré dual to the homology class represented by $\Sigma^1 f$ is equal to $w_{k+1}(\nu_f)$ where ν_f stands for the virtual normal bundle of f. This dual cohomology class will be denoted by $[\Sigma^1 f]$ for simplicity.

Theorem 17. *Let* $f : M_1^{n_1} \to N^{n_1+k_1}, g : M_2^{n_2} \to N_2^{n_2+k_2}$ *be two generic* *maps. Then for a generic perturbation of their product we have*

$$[\Sigma^1 f \times g] = \sum_{j \geq 1} \Big([\Sigma^1 f_{j-1}] \times (\mathrm{id}^j_{M_2})^*[\Sigma^1 g_{(-j)}] + (\mathrm{id}^j_{M_1})^*[\Sigma^1 f_{(-j)}] \times [\Sigma^1 g_{j-1}] \Big)$$

Proof. As a first step let us notice that since $\nu_{f\times g} = \nu_f \times \nu_g$ we can write

$$\begin{aligned} w_{k_1+k_2+1}(\nu_{f\times g}) = & \sum_{r=0}^{k_1+k_2+1} w_r(\nu_f) \times w_{k_1+k_2+1-r}(\nu_g) = \\ & = \sum_{j\geq 1} \Big(w_{k_1+j}(\nu_f) \times w_{k_2-j+1}(\nu_g) + w_{k_1-j+1}(\nu_f) \times w_{k_2+j}(\nu_g) \Big) \end{aligned}$$

Now we have to take a closer look at $w_{k_1+j}(\nu_f)$. If $k_1 + j - 1$ were equal to the codimension of f then this characteristic class would just represent the singular locus of f. When this is not the case, we have to find an appropriate replacement of f that has the right codimension, whose normal bundle however is stably equivalent to that of f. This replacement map is exactly f_{j-1}. Indeed, $\nu_{f_{j-1}} = \nu_f \oplus \varepsilon^{j-1}$ so $w_{k_1+j}(\nu_f) = w_{k_1+j}(\nu_{f_{j-1}})$ which in turn is equal to $[\Sigma^1 f_{j-1}]$ since this map has the right codimension.

The argument is just slightly more complicated in the case of $w_{k_2-j+1}(\nu_g)$. Here first we take the map $g_{(-j)} : M_2^{n_2} \times S^j \to N_2^{n_2+k_2}$. This has codimension $k_2 - j$ so $[\Sigma^1 g_{(-j)}] = w_{k_2-j+1}(\nu_{g_{(-j)}})$. The only problem is that this class lives in the cohomology of $M_2 \times S^j$. This is why we have to pull it back to M_2 by $\mathrm{id}^j_{M_2}$. Since the composition of $\mathrm{id}^j_{M_2}$ and $g_{(-j)}$ is just a perturbation of g and $w(\nu_{q_j}) = 1$ it follows that $(\mathrm{id}^j_{M_2})^* w_{k_2-j+1}(\nu_{g_{(-j)}}) = w_{k_2-j+1}(\nu_g)$.

Putting all these together gives the result of the theorem. □

5.5.2 The Σ^2 stratum

A very similar result can be proved about the Σ^2 stratum of oriented maps. First we need to compute the Thom polynomial of the Σ^2 stratum in the oriented case. We will work with rational coefficients, but the same argument also works for Z_p coefficients where p is an odd prime.

Theorem 18. *Let $f : M^n \to N^{n+k}$ be a generic map where $k = 2t-2$. Then the rational cohomology class dual to the closure of the set of Σ^2-points of f (for short $[\Sigma^2 f]$) equals $p_t(\nu_f)$, where $p_t \in H^{4t}(M;\mathbb{Q})$ is the t^{th} Pontrjagin class.*

Proof. By definition the Thom polynomial tp_{Σ^2} of the Σ^2-stratum is a cohomology class in $H^{4t}(BSO;\mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \ldots]$. We want to show that $tp_{\Sigma^2} = p_t$. It is enough to show that these two cohomology classes evaluated on any homology class in $H_{4t}(BSO;\mathbb{Q})$ are equal.

Lemma 13. *All homology classes in $H_{4t}(BSO;\mathbb{Q})$ can be represented by a normal map, i.e. by a map $h : L^{4t} \to BSO$ of an oriented $4t$-manifold L^{4t} corresponding to the stable normal bundle of L^{4t}.*

Proof. It is enough to consider $BSO(N)$, $(N \gg 1)$. By the Pontrjagin-Thom construction an embedding $L^{4t} \hookrightarrow S^K$ gives a map $h' : S^K \to MSO(K-4t)$ such that $L^{4t} = h'^{(-1)}(BSO(K-4t))$ and the restriction map $h'|_{L^{4t}} : L^{4t} \to BSO(K-4t)$ corresponds to the normal bundle of the embedding $L^{4t} \hookrightarrow S^K$. The homotopy class $[h'] \in \pi_K(MSO(K-4t))$ is mapped by the composition of the Hurewicz homomorphism and the Thom isomorphism into a homology class $x = h'_*([L^{4t}]) \in H_{4t}(BSO(K-4t))$. Hence this class x is represented by a normal map. Since the Hurewicz homomorphism in stable dimensions $(K \geq 8t+2)$ is a rational isomorphism, we obtain the statement of the lemma. □

To evaluate a $4t$ dimensional cohomology class on a $4t$ dimensional homology class represented by a manifold, one just pulls back the cohomology class to the manifold and evaluates it on the fundamental class.

Now it is enough to prove, that for every oriented M^{4t} the map $\nu^* : H^{4t}(BSO;\mathbb{Q}) \to H^{4t}(M;\mathbb{Q})$ induced by the normal mapping $\nu : M^{4t} \to BSO$ takes p_t and tp_{Σ^2} to the same cohomology class in $H^{4t}(M;\mathbb{Q})$. As $\nu^*(p_t) = p_t(\nu_M)$ and $\nu^*(tp_{\Sigma^2})$ is the dual of the Σ^2 stratum of a generic map

$M^{4t} \to \mathbb{R}^{6t-2}$ we reduced the problem of finding the Thom polynomial tp_{Σ^2} to the special case of maps $M^{4t} \to \mathbb{R}^{6t-2}$.

If we take an immersion $f : M^{4t} \to \mathbb{R}^{6t}$, and project it to two non-parallel hyperplanes, then we get a map $f' : M^{4t} \to \mathbb{R}^{6t-2}$. Let us denote the two hyperplanes H_1, H_2. The projection of f to H_i shall be called f_i. It is obvious that those and only those points belong to $\Sigma^2 f'$ which belong to $\Sigma^1 f_1$ and $\Sigma^1 f_2$ at the same time. This means that for this f' we have $[\Sigma^2 f'] = [\Sigma^1 f_1] \cup [\Sigma^1 f_2]$. The two cohomology classes on the right are both equal to the Thom polynomial of the Σ^1 singularity, which is the Euler class of the normal bundle of f (see e.g. [25]). As this normal bundle has rank $2t$, the square of its Euler class is equal to $p_t(\nu_f)$, which is the same as $p_t(\nu_M)$. Thus we have proved our claim for those maps $M^{4t} \to \mathbb{R}^{6t-2}$ where the source manifold can be immersed into $\mathbb{R}^{6t}$.

Let us recall that by $\mathrm{Imm}^{SO}(4t, 2t)$ we denoted the cobordism group of oriented immersions from $4t$ dimensional manifolds to $\mathbb{R}^{6t}$. There is the natural forgetting map $\psi : \mathrm{Imm}^{SO}(4t, 2t) \to \Omega_{4t}$ taking the cobordism class of an immersion to that of its underlying manifold. To finish the proof of the theorem it is sufficient to show, that this map has finite cokernel. According to the Pontrjagin-Thom construction and Serre's theorem on the stable Hurewicz homomorphism

$$\mathrm{Imm}^{SO}(4t, 2t) \cong \pi^S_{6t} MSO(2t) \overset{\mathbb{Q}}{\cong} H_{6t}(MSO(2t); \mathbb{Q}))$$

and

$$\Omega_{4t} \cong \pi^S_{4t}(MSO) \overset{\mathbb{Q}}{\cong} H_{4t}(MSO; \mathbb{Q}),$$

where $\overset{\mathbb{Q}}{\cong}$ means: "isomorphic if tensored with $\mathbb{Q}$". Thus ψ has finite cokernel if and only if

$$\psi_H : H_{6t}(MSO(2t); \mathbb{Q})) \to H_{4t}(MSO; \mathbb{Q})$$

is epimorphic. The latter is equivalent to (by taking the dual morphism in cohomology)

$$\psi^* : H^{4t}(MSO; \mathbb{Q}) \to H^{6t}(MSO(2t); \mathbb{Q}))$$

being monomorphic. We can apply the Thom-isomorphism to further reduce the problem to showing that

$$\psi_B^* : H^{4t}(BSO;\mathbb{Q}) \to H^{4t}(BSO(2t);\mathbb{Q})$$

is monomorphic. It is easy to see that ψ_B^* is induced by the natural inclusion map $BSO(2t) \hookrightarrow BSO$. The cohomology ring of $BSO(2t)$ is the polynomial ring $\mathbb{Q}[p_1, p_2, \ldots, p_{t-1}, \chi_{2t}]$ generated by the Pontrjagin classes and the Euler class, whose square is p_t. On the other hand $H^*(BSO;\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots]$. As ψ_B^* takes each Pontrjagin class to the same Pontrjagin class, we get that ψ_B^* is indeed injective in dimension $4t$. This completes the proof of $tp_{\Sigma^2} = p_t$. □

The proof of the next theorem copies the proof of the previous section. The equality below in the Theorem is meant in the cohomology groups with rational coefficients or with Z_p for any odd prime p.

Theorem 19. *Let $f : M_1^{n_1} \to N^{n_1+k_1}, g : M_2^{n_2} \to N_2^{n_2+k_2}$ be two generic maps of even codimension. Then for a generic perturbation of their product we have*

$$[\Sigma^2 f \times g] = \sum_{j\geq 1}([\Sigma^2 f_{2j-2}] \times (\mathrm{id}_{M_2}^{2j})^*[\Sigma^2 g_{(-2j)}] + (\mathrm{id}_{M_1}^{2j})^*[\Sigma^2 f_{(-2j)}] \times [\Sigma^2 g_{2j-2}])$$

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