

OPERATOR EXTENSIONS ON HILBERT SPACE

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INTRODUCTION

This dissertation is based on the author's papers [16, 17, 18, 19, 20]. Some other results are also included to make it self-contained.

Throughout this paper \mathfrak{H} will denote a complex Hilbert space. Following Halmos' [8] terminology we use the word suboperator for continuous linear maps from some linear submanifold of \mathfrak{H} to the whole space. A suboperator is subpositive, subself-adjoint etc. if it admits a positive, self-adjoint etc. extension to the whole space, respectively. We are going to investigate the characterization problem of various classes of suboperators.

The answer to this problem is usually a construction which shows that, under certain obviously necessary condition imposed on the suboperator, how can someone recover one extension of the desired type.

The first result in this topic is due to M. G. Krein [13]. It states that every symmetric suboperator is subself-adjoint, but the main point of his theorem is that there is norm preserving self-adjoint extension. He was interested in this result since he could trace back the semi-bounded extension problem to this case by using the correspondence

$$A \longleftrightarrow (A - I)(A + I)^{-1}, \quad \text{where } \overline{\text{dom } A} = \mathfrak{H}.$$

Later on it turned out that seeking for positive extension of subpositive suboperators is a much more natural than seeking for a norm preserving self-adjoint extension of a symmetric suboperator. Nevertheless these two problems are equivalent.

The characterization problem of various classes of suboperators was also raised in [8]. Here the motivation came from the characterization problem of subnormal operators. An operator is subnormal if it admits a normal extension to a larger space. In this paper Halmos gave characterizations of subpositive, subprojection suboperators. He forced the operator matrix formalism which made his proofs a bit complicated comparing to the proofs of the present dissertation.

In the first chapter we develop the theory of positive and self-adjoint extensions following the method of Dr. Zoltán Sebestyén. We also introduce the notion of shortening and parallel sum of positive operators. Beside these we give a parametrisation of the set of positive extensions and different conditions on some sort of uniqueness.

In the second chapter we characterize subprojections, subreflections suboperators, the restrictions of partial and self-adjoint partial isometries as well as subcompact suboperators.

In the last chapter we extend the extension procedure described in Chapter I to the semibounded case. We also prove a sufficient condition for bounded operators to be intertwined with the smallest and with the largest selfadjoint (positive) extensions.

Many authors investigate the extensions of operators of finite index (especially of index one) using different methods see e.g. the series of papers of Hassi, Kältenback, de Snoo or a paper of A. Alonso and B. Simon [1]. As this dissertation contains mainly the author results in this topic these other approaches are not considered here.

In this paper positivity of a linear mapping means that the associated quadratic form is positive semi-definite. Therefore in the first chapter we suppress the adjective “self-adjoint” since a positive everywhere defined operator is necessarily self-adjoint. However in the third chapter, where we are dealing with the semi-bounded case, this is no more true. In the third chapter, therefore, we emphasize self-adjointness.

We use the name “Schwarz inequality” actually for two different relations. The first one sometimes is called the Cauchy-Schwarz inequality. It states that if a sesquilinear form ϕ is positive semi-definite then

$$|\phi(x, y)|^2 \leq \phi(x, x)\phi(y, y)$$

for each pair x, y from the domain of ϕ . The application of this inequality to the special case when $\phi(x, y) = \langle Ax, y \rangle$ with a positive bounded operator A on a Hilbert space gives that

$$\|Ax\|^2 \leq \|A\| \langle Ax, x \rangle,$$

which is also referred to as the Schwarz inequality.

Throughout the paper \mathbf{R} , \mathbf{C} denotes the real and complex field, respectively. $\overline{\mathfrak{M}}$ denotes the closure of the subspace \mathfrak{M} . $\mathcal{B}(\mathfrak{H})$, $\mathcal{B}_{sa}(\mathfrak{H})$ will stand for the set of all continuous and all self-adjoint operators on the Hilbert space \mathfrak{H} , respectively. We use lower case letters to denote suboperators. Extensions are denoted with the corresponding capital letters. There is one exception, the identity operator is always denoted with I regardless of its domain. However it is important to keep in mind that $\iota = I|_{\mathfrak{D}}$ is not the same as I . Indeed if we take adjoint then $I^* = I$ while ι^* is the projection onto $\overline{\mathfrak{D}}$. So very often formulas like $A|_{\mathfrak{D}}$ become PA after taking adjoint where P is the projection onto $\overline{\mathfrak{D}}$.

We will use the following two well known theorems on positive operators.

THEOREM 0.1. *An operator $A \in \mathcal{B}(\mathfrak{H})$ is an extremal point of the unit interval $[0, I]$ if and only if A is an orthogonal projection.*

THEOREM 0.2. *If $0 \leq A \leq B$ then $\text{ran } A^{1/2} \subset \text{ran } B^{1/2}$.*

Concerning adjoint operators in Chapter III. we will use the following well known facts

THEOREM 0.3. *Let A, B are densely defined operators on Hilbert space, assume that $A + B, AB$ are also densely defined. Then*

$$A^* + B^* \subset (A + B)^* \quad \text{and} \quad A^*B^* \subset (BA)^*.$$

If C is a bounded operator then

$$A^* + C^* = (A + C)^* \quad \text{and} \quad A^*C^* = (CA)^*$$

This last statement can be found for example in the “Functional Analysis” book of F. Riesz and B. Szőkefalvi-Nagy.

CHAPTER I

POSITIVE AND SELF-ADJOINT EXTENSIONS

Smallest positive extension

Concerning subpositive suboperators one of the fundamental constructions is due to Z. Sebestyén. It provides characterization of subpositive suboperators. It might be thought at first sight that the positivity of the quadratic form associated to the suboperator would have been enough. A very simple example shows that this is not the case. Let \mathfrak{H} be a two dimensional Hilbert space with orthonormal basis e_1, e_2 and \mathfrak{D} be the one dimensional subspace generated by e_1 . Define $b: \mathfrak{D} \rightarrow \mathfrak{H}$ the formula $be_1 = e_2$. Then the quadratic form on \mathfrak{D} associated to b is positive semidefinite but the determinant of any self-adjoint extension of b is -1 , therefore b is not subpositive.

However, the restriction of the Schwarz inequality is enough. This observation and the nice construction of the following theorem is due to Zoltán Sebestyén. It appeared e.g in [10, 11, 22, 23, 24, 25]. In this form it was also published in [16, 17, 19, 20]

THEOREM 1.1. *Let \mathfrak{D} be a linear subspace (not necessarily closed) of a complex Hilbert space \mathfrak{H} , and $b: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric linear map. Then b has a positive extension to \mathfrak{H} if and only if*

$$m := \sup \left\{ \|bx\|^2 : x \in \mathfrak{D}, \langle bx, x \rangle \leq 1 \right\} < \infty \quad (1)$$

Moreover, there is a smallest positive extension with smallest norm m .

PROOF. Denote \mathfrak{R} the range of b . Introduce an inner product $\langle \cdot, \cdot \rangle_0$ on \mathfrak{R} with the following definition

$$\langle bx, by \rangle_0 := \langle bx, y \rangle \quad \text{for } x, y \in \mathfrak{D}$$

It is well defined: if $x, x', y, y' \in \mathfrak{D}$ and $bx = bx', by = by'$ then

$$\langle bx, by \rangle_0 = \langle bx, y \rangle = \langle bx', y \rangle = \langle x', by' \rangle = \langle bx', by' \rangle_0$$

Finally $\langle \cdot, \cdot \rangle_0$ is positive definite since $\|bx\|^2 \leq m \langle bx, x \rangle$ for all $x \in \mathfrak{D}$.

Thus $(\mathfrak{A}, \langle \cdot, \cdot \rangle_0)$ is a pre-Hilbert space, \mathfrak{K}_0 denotes its completion. Define $J: \mathfrak{K}_0 \rightarrow \mathfrak{H}$ as the unique continuous extension of the natural embedding of $\mathfrak{A} (\subset \mathfrak{K}_0)$ into \mathfrak{H} . Then JJ^* is an extension of b . To prove this it is enough to show that for all $x \in \mathfrak{D}$ the element $bx - J^*x$ of \mathfrak{K}_0 is orthogonal to the dense subspace \mathfrak{A} of \mathfrak{K}_0 . But this is obvious, since

$$\langle by, J^*x \rangle_0 = \langle Jby, x \rangle = \langle by, x \rangle = \langle by, bx \rangle_0 \quad \text{for all } y \in \mathfrak{D}.$$

The norm of JJ^* is $\|J\|^2$ which is by definition m . The quadratic form associated to JJ^* is

$$\begin{aligned} \langle JJ^*x, x \rangle &= \langle J^*x, J^*x \rangle_0 = \\ \sup \left\{ |\langle J^*x, J^*y \rangle_0|^2 : y \in \mathfrak{D}, \langle J^*y, J^*y \rangle_0 \leq 1 \right\} &= \\ \sup \left\{ |\langle x, by \rangle|^2 : y \in \mathfrak{D}, \langle by, y \rangle \leq 1 \right\}. \end{aligned} \quad (2)$$

Let B be any positive extension of b . For each $x \in \mathfrak{H}$ we have that

$$\begin{aligned} \langle Bx, x \rangle &= \left\| B^{1/2}x \right\|^2 = \\ \sup \left\{ \left| \langle B^{1/2}x, B^{1/2}y \rangle \right|^2 : y \in \mathfrak{H}, \langle B^{1/2}y, B^{1/2}y \rangle \leq 1 \right\} &= \\ \sup \left\{ |\langle x, By \rangle|^2 : y \in \mathfrak{H}, \langle By, y \rangle \leq 1 \right\}. \end{aligned}$$

Taking into account that $b \subset B$ it follows that

$$\langle JJ^*x, x \rangle \leq \langle Bx, x \rangle \quad \text{for all } x \in \mathfrak{H}.$$

□

If we do not bother with the possible norm of the extension and assume that \mathfrak{D} is closed (this assumption does not really restrict the generality) then property (1) can be stated in terms of geometric behaviour of b . It can be found in [8].

Since $\langle bx, x \rangle = \langle Pbx, x \rangle$ where P is the projection to \mathfrak{D} , b fulfills (1) if and only if $(Pb)^{1/2}$ is a right divisor of $(I - P)b$, i.e. there is a continuous operator $T: \mathfrak{D} \rightarrow \mathfrak{D}^\perp$ such that $(I - P)b = T(Pb)^{1/2}$. Indeed using Douglas factorization theorem (1) implies the existence of T . Conversely applying the Schwarz inequality to Pb we get that

$$\langle bx, bx \rangle = \|Pbx\|^2 + \left\| T(Pb)^{1/2}x \right\|^2 \leq (\|Pb\| + \|T\|^2) \langle bx, x \rangle.$$

On the other hand we know (also from Douglas paper [6]) that $(I - P)b = T(Pb)^{1/2}$ if and only if $\text{ran}((I - P)b)^* \subset \text{ran}(Pb)^{1/2}$. Someone may think that perhaps

$\text{ran}(Pb)^{1/2}$ is replaceable with $\text{ran } Pb$. This is not true, however, since there are operators with trivial range intersection such that the range of their square root coincide and dense (see [7]).

Another way of expressing the quadratic form associated to $B_0 = JJ^*$ will be useful in the next statement. For any $x \in \mathfrak{H}$

$$\inf \{ \langle B_0(y-x), y-x \rangle : y \in \mathfrak{D} \} = 0$$

since $\langle B_0(y-x), y-x \rangle = \|J^*(y-x)\|^2$ and $\text{ran } b \subset \mathfrak{K}_0$ is dense, therefore

$$\langle B_0x, x \rangle = \sup \{ \langle by, x \rangle + \langle x, by \rangle - \langle by, y \rangle : y \in \mathfrak{D} \}.$$

This equation is just another way of expressing (2). Indeed taking first the supremum along the line $\mathbf{C}x$ we get back (2).

Now let $b = B|_{\mathfrak{D}}$ for some positive operator B , then

$$\langle (B - B_0)x, x \rangle = \inf \{ \langle B(x-y), x-y \rangle : y \in \mathfrak{D} \}. \quad (3)$$

It is surprising at first sight that this $B - B_0$ is maximal in some sense according to the following

COROLLARY 1.2. *Let B be a positive operator on \mathfrak{H} and \mathfrak{M} be a closed subspace of \mathfrak{H} . Then the set*

$$\{ C : 0 \leq C \leq B, \text{ran } C \subset \mathfrak{M} \} \quad (4)$$

has a largest element.

PROOF. Define $b: \mathfrak{M}^\perp \rightarrow \mathfrak{H}$ as $b = B|_{\mathfrak{M}^\perp}$. Then b is a subpositive suboperator by definition so it has a smallest positive extension B_0 .

We claim that $B - B_0$ is the largest operator in the given set (4). Indeed, $\ker(B - B_0) \supset \mathfrak{M}^\perp$ and $0 \leq B - B_0 \leq B$, so $B - B_0$ belongs (4). Let C any element of (4). Then $b \subset B - C$ since $\ker C \supset \mathfrak{M}^\perp$ therefore $B_0 \leq B - C$ thus $C \leq B - B_0$. \square

A slight generalization is possible here.

THEOREM 1.3. *Let \mathfrak{V} be a vector space, $\phi : \mathfrak{V} \rightarrow [0, \infty]$ be a quadratic form, i.e. the diagonal function of a sesquilinear function, and \mathfrak{W} be a linear subspace of \mathfrak{V} . Then among (not necessarily positive semidefinite but real valued) quadratic forms dominated by ϕ and vanishing on \mathfrak{W} there is a maximal one $\phi_{\mathfrak{W}}$ with respect to the pointwise order. This is defined by the formula*

$$\phi_{\mathfrak{W}}(v) = \inf \{ \phi(v-w) : w \in \mathfrak{W} \} \quad (5)$$

PROOF. The formula (5) defines a quadratic form since it satisfies the parallelogram law and upper-semicontinuous if restricted to finite dimensional subspaces.

It is maximal since for any positive definite quadratic form ψ vanishing on \mathfrak{M} and dominated by ϕ we get that for all $v \in \mathfrak{V}$ and $w \in \mathfrak{M}$

$$\psi(v) = \psi(v) + \psi(w) = \frac{\psi(v+w) + \psi(v-w)}{2} \leq \frac{\phi(v+w) + \phi(v-w)}{2}$$

Taking the infimum in w we get that

$$\psi(v) \leq \phi_{\mathfrak{M}}(v).$$

So $\phi_{\mathfrak{M}}$ is indeed the maximal element in a given set of quadratic with respect to the pointwise partial order. \square

Shortening of positive operators

$B - B_0$ is called the shortening of B to the subspace \mathfrak{M} and is denoted by $B_{\mathfrak{M}}$. The existence of shorted operator can be proved without referring to the extensions of subpositive suboperators. Moreover, we have to mention here that the latter argument was used by Krein to prove that the set of contractive self-adjoint extensions of symmetric linear map possesses a largest and a smallest element. Indeed, the way we have introduced shortening proves at ones that for any positive extension B of b , B_0 is just $B - B_{\mathfrak{M}^\perp}$.

Later shortening was rediscovered [2]. This time the motivation came from the theory of electrical networks. A basic notion in this theory is n -port. The behaviour of the n -port is described with the help of impedance operator B . In case the n -port contains only resistors the impedance operator is an $n \times n$ positive semidefinite matrix. If we short-circuit all but the first m ports ($m < n$) then the impedance operator of the obtained m -port is just the shortening of B to the subspace spanned by first m coordinate. It follows at once from (3) after we accepted Maxwell's principle.

Maxwell's principle states that among all possible current distributions the actual one that describes the behaviour of the circuit is the one that mini-mises energy dissipation. Here possible current distribution means any one of those that obey Kirchhoff's current law, whether it satisfies Kirchhoff's voltage drop law or not. It is an amusing fact that the above properties of shortening can be used to prove that Maxwell's principle actually implies Kirchhoff voltage drop law.

Shorted matrices also appear in connection with normal distributions. Let (Ω, μ) be a probability field, $X: \Omega \rightarrow \mathbf{R}^n$ a random vector with normal distribution \mathbf{Q} and P be an orthogonal projection of \mathbf{R}^n . Denote $Y = PX$ and Σ the co-variance matrix of X i.e.

$$\langle \Sigma \phi, \psi \rangle = \mathbb{E}(\phi(X)\psi(X)) - \mathbb{E}(\phi(X))\mathbb{E}(\psi(X))$$

for $\psi, \phi \in \mathbf{R}^n$ treated as linear functionals. Then the co-variance of the conditional distribution $\mathbf{Q}(\cdot|Y)$ is simply Σ shorted to $\text{ran } P^\perp$ denoted by Σ_P . Indeed it is easy to check that the covariance of conditional distribution is simply

$$\langle \Sigma_P \phi, \psi \rangle = \mathbb{E}((\phi(X) - \mathbb{E}(\phi(X)|Y))(\psi(X) - \mathbb{E}(\psi(X)|Y))|Y)$$

where ψ and ϕ are linear functionals on \mathbf{R}^n and $\mathbb{E}(\cdot|Y)$ stands for the conditional expectation. Since among random variables with normal distribution orthogonality implies independence, we have that the argument is independent from the σ -algebra the conditional expectation is taken to. So this conditional expectation is constant and its value is the simple expectation. In this way we have got that

$$\langle \Sigma_P \phi, \phi \rangle = \langle \Sigma(\phi - \phi'), \phi - \phi' \rangle$$

where ϕ' is the best approximation of ϕ from $\text{ran } P$ with respect to the form $\langle \Sigma, \cdot \rangle$, i.e. $\phi'(X) = \mathbb{E}(\phi(X)|Y)$. Comparing this with (3) we get the result.

Positive extensions of smallest possible norm

This section is taken from the publication [17, 16] In what follows denote $\mathfrak{D} \neq \{0\}$ a linear subspace $b : \mathfrak{D} \rightarrow \mathfrak{H}$ a symmetric linear map which fulfills the condition (1) of Theorem 1.1, and we set

$$\mathfrak{B} = \{B \in \mathcal{B}(\mathfrak{H}) : b \subset B, 0 \leq B \leq mI\}.$$

By Theorem 1.1 \mathfrak{B} is not empty, and the above constructed operator $B_0 = JJ^*$ is the smallest element in \mathfrak{B} . \mathfrak{B} also has a largest element, and this can be traced back to the existence of smallest element, observing that

$$mI - \mathfrak{B} = \{B \in \mathcal{B}(\mathfrak{H}) : mI - b \subset B, 0 \leq B \leq mI\}.$$

Thus since $mI - \mathfrak{B}$ is not empty it has a smallest element too, say $J_1J_1^*$, then $B_1 = mI - J_1J_1^*$ is the largest element in \mathfrak{B} . Therefore we have that

$$\mathfrak{B} \subset [B_0, B_1] = \{C \in \mathcal{B}(\mathfrak{H}) : B_0 \leq C \leq B_1\}.$$

Here we have equality instead of proper inclusion. Indeed, for any $C \in [B_0, B_1]$ and $x \in \mathfrak{D}$, $y \in \mathfrak{H}$ we have $0 \leq C - B_0$ and

$$\begin{aligned} |\langle (C - B_0)x, y \rangle|^2 &\leq \langle (C - B_0)x, x \rangle \langle (C - B_0)y, y \rangle \leq \\ &\leq \langle (B_1 - B_0)x, x \rangle \langle (C - B_0)y, y \rangle = 0. \end{aligned}$$

So $Cx = B_0x = bx$ i.e. $C \in \mathfrak{B}$.

Let us apply Theorem 1.1 to the operator $B_1 - B_0$ in place of b and \mathfrak{H} in place of \mathfrak{D} . Then we get a new Hilbert space \mathfrak{K} and $V : \mathfrak{K} \rightarrow \mathfrak{H}$ such that $B_1 - B_0 = VV^*$.

By standard arguments $\mathfrak{B} = [B_0, B_1] = \{B_0 + VCV^* : C \in \mathcal{B}(\mathfrak{K}), 0 \leq C \leq I_{\mathfrak{K}}\}$. Since V^* has dense range by definition the map

$$\varphi : \mathcal{B}(\mathfrak{K}) \rightarrow \mathcal{B}(\mathfrak{H}) \quad \varphi(C) = B_0 + VCV^*$$

is injective and therefore a bijection between $[0, I_{\mathfrak{K}}] \subset \mathcal{B}(\mathfrak{K})$ and $\mathfrak{B} = [B_0, B_1]$.

It is also clear that φ is affine, order preserving and continuous in the following sense: if $\mathcal{B}(\mathfrak{K})$ and $\mathcal{B}(\mathfrak{H})$ are both given either the weak, or the strong or the norm operator topologies then φ is continuous. It is well known that the closed unit ball is compact in the weak operator topology, therefore the unit interval which is a weak closed subset of it is also compact. Now the continuity of φ implies immediately that \mathfrak{B} is weak-compact (and therefore strong-closed) too. For a compact convex set in a topological vector space it is always interesting to determine the its extremal points. Since for the unit interval it is easy to prove that the set of extremal points coincides with the set of orthogonal projections we have just proved the following statement.

THEOREM 1.4. *Preserving the notation introduced above*

$$\mathfrak{B} = \{B_0 + VCV^* : C \in \mathcal{B}(\mathfrak{K}), 0 \leq C \leq I_{\mathfrak{K}}\}$$

and the set of extremal points of it is equal to

$$\{B_0 + VPV^* : P \in \mathcal{B}(\mathfrak{K}), P = P^2 = P^*\}.$$

The smallest extension B_0 has a distinguished property among all positive extensions of b .

THEOREM 1.5. *Let b and B_0 be as above and $b \subset B$ be a positive operator. Then $B = B_0$ if and only if $\text{ran } B^{1/2} \cap \mathfrak{D}^\perp = \{0\}$.*

PROOF. Let $x \in \text{ran } B^{1/2} \cap \mathfrak{D}^\perp$. Since $\text{ran } B^{1/2} = \text{ran } J$ we have $z \in \mathfrak{K}$ such that $Jz = x$, so

$$0 = \langle Jz, y \rangle = \langle z, J^*y \rangle \quad \text{for all } y \in \mathfrak{D}.$$

$\{J^*y : y \in \mathfrak{D}\} = \text{ran } b$ is a dense subspace of \mathfrak{K}_0 , hence $z = 0$ and $x = Jz = 0$.

Conversely, if $\text{ran } B^{1/2} \cap \mathfrak{D}^\perp = \{0\}$ then since $0 \leq B - B_0 \leq B$ we have $\text{ran } (B - B_0)^{1/2} \subset \text{ran } B^{1/2}$. We also have $\mathfrak{D} \subset \ker(B - B_0)$ therefore

$$\text{ran } (B - B_0)^{1/2} \subset \text{ran } B^{1/2} \cap \mathfrak{D}^\perp = \{0\},$$

so $B - B_0 = 0$. □

COROLLARY 1.6. *With the previous notation, if $M > m$ and \mathfrak{D} is non-dense in \mathfrak{H} then the operator interval*

$$\mathfrak{B}_M = \{B : b \subset B, 0 \leq B \leq MI\}$$

is non-degenerate.

PROOF. $MI - B_0 \in MI - \mathfrak{B}_M$ and $MI - B_0 \geq (M - m)I$, so it is invertible. On the other hand, the smallest extension of $MI - b$ also belongs to $MI - \mathfrak{B}_M$ the range of which does not meet \mathfrak{D}^\perp hence not invertible. The smallest and largest element of $MI - \mathfrak{B}_M$ is different so the same is true for \mathfrak{B}_M . \square

Now we turn to the question of uniqueness of extension. Corollary 1.6 states that except the trivial case when \mathfrak{D} is dense there are always many ways of extending b to a positive operator of norm not greater than M , provided that $M > m$. Therefore the right way of formulating the problem is what can be said about the uniqueness of norm m extension of b .

We know that b has unique positive extension with norm m if and only if the smallest and largest such extensions are equal or what is the same

$$mI = JJ^* + J_1J_1^*. \quad (6)$$

This condition was formulated by Kapos and Paez [10] in the following theorem.

THEOREM 1.7. *Let $b: \mathfrak{D} \rightarrow \mathfrak{H}$ be subpositive suboperator satisfying property (1) . b has unique positive operator extension of smallest norm m to \mathfrak{H} if and only if*

$$m\|x\|^2 = \sup\left\{|\langle x, by \rangle|^2 : y \in \mathfrak{D}, \langle by, y \rangle \leq 1\right\} + \sup\left\{|\langle x, my - by \rangle|^2 : y \in \mathfrak{D}, \langle my - by, y \rangle \leq 1\right\}$$

holds for each x from \mathfrak{H} .

PROOF. Apply (2) to (6). \square

Identity (6) can be formulated another way that involves x only from \mathfrak{D}^\perp , the orthocomplement of \mathfrak{D} and does contain only one expression on supremum.

THEOREM 1.8. *Let $b: \mathfrak{D} \rightarrow \mathfrak{H}$ be subpositive suboperator satisfying property (1) . b has unique positive operator extension of smallest norm m to \mathfrak{H} if and only if*

$$\|x\|^2 = \sup\left\{|\langle x, by \rangle|^2 : y \in \mathfrak{D}, \langle by, my - Pby \rangle \leq 1\right\} \quad (7)$$

holds for each x from \mathfrak{D}^\perp , where P denotes the orthogonal projection of \mathfrak{H} onto the norm closure of \mathfrak{D} .

PROOF. First we show that identity (6) is equivalent to the formally weaker

$$m \langle x, x \rangle = \langle JJ^*x, x \rangle + \langle J_1J_1^*x, x \rangle \quad \text{for all } x \in \mathfrak{D}^\perp \quad (8)$$

and after this that this weaker condition (8) is equivalent to (7).

To do this let us introduce some notation : let \mathfrak{A}_0 ($\mathfrak{H}_0, \langle \cdot, \cdot \rangle_0$) be the Hilbert space constructed from the range of b as in the proof of Theorem 1.1, and \mathfrak{A}_1 and ($\mathfrak{H}_1, \langle \cdot, \cdot \rangle_1$) be the corresponding entities when Theorem 1.1 is applied to $b_1 = mI|_{\mathfrak{D}} - b$. Denote ($\mathfrak{K}, \langle \cdot, \cdot \rangle_\oplus$) the direct sum of \mathfrak{H}_0 and \mathfrak{H}_1 , and define $V: \mathfrak{K} \rightarrow \mathfrak{H}$ by the formula

$$V(u, v) = Ju + J_1v \quad \text{for } u \in \mathfrak{H}_0 \text{ and } v \in \mathfrak{H}_1.$$

Since $V^*: \mathfrak{H} \rightarrow \mathfrak{K}$ is the 'row matrix' (J^*, J_1^*), we have

$$VV^* = JJ^* + J_1J_1^*.$$

For the first step we only have to prove that (8) implies (6). Notice first that it follows from the definition of \mathfrak{K} and V that the images of \mathfrak{D} and \mathfrak{D}^\perp under V^* are orthogonal linear subspaces of \mathfrak{K} . Indeed, for $x \in \mathfrak{D}$ and $y \in \mathfrak{D}^\perp$ we have that

$$\begin{aligned} \langle V^*x, V^*y \rangle_\oplus &= \langle VV^*x, y \rangle = \\ &= \langle (JJ^* + J_1J_1^*)x, y \rangle = \langle (b + mI - b)x, y \rangle = m \langle x, y \rangle = 0. \end{aligned}$$

On the other hand identity (6) restricted to the norm closure of \mathfrak{D} holds by definitions and continuity. Thus if (8) holds true then decomposing any $x \in \mathfrak{H}$ into $x = y + z$, where $y \in \overline{\mathfrak{D}}$ and $z \in \mathfrak{D}^\perp$ we get that

$$\langle (JJ^* + J_1J_1^*)x, x \rangle = \|V^*x\|^2 = \|V^*y\|^2 + \|V^*z\|^2.$$

Where the last term is equal to $m \|z\|^2$ by (8) and the previous one to $m \|y\|^2$ as we saw above, so assuming (8) one gets

$$\langle (JJ^* + J_1J_1^*)x, x \rangle = m \|x\|^2$$

which of course implies (6).

For the second step we have to prove that (6) is equivalent to (7). To do this let us calculate $\|V^*x\|^2$ for $x \in \mathfrak{D}^\perp$, as the norm of the linear functional that V^*x represents:

$$\begin{aligned} \|V^*x\|^2 &= \sup \left\{ |\langle V^*x, \xi \rangle_\oplus|^2 : \xi \in \mathfrak{K}, \langle \xi, \xi \rangle_\oplus \leq 1 \right\} = \\ &= \sup \left\{ |\langle x, V\xi \rangle|^2 : \xi \in \mathfrak{A}_0 \oplus \mathfrak{A}_1, \langle \xi, \xi \rangle_\oplus \leq 1 \right\} = \end{aligned}$$

$$= \sup \left\{ |\langle x, bu - bv \rangle|^2 : u, v \in \mathfrak{D}, \langle bu, u \rangle + \langle mv - bv, v \rangle \leq 1 \right\}.$$

The last condition can then be rewritten with $\xi = u - v$, $\eta = u + v$ by observing that

$$\begin{aligned} & m\{\langle bu, u \rangle + \langle mv - bv, v \rangle\} = \\ &= \frac{m}{4} \{ \langle b(\xi + \eta), \xi + \eta \rangle - \langle b(\eta - \xi), \eta - \xi \rangle + m \langle \eta - \xi, \eta - \xi \rangle \} = \\ &= \left\langle b\xi, \frac{m}{2}\eta \right\rangle + \left\langle \frac{m}{2}\eta, b\xi \right\rangle + \left\| \frac{m}{2}\xi \right\|^2 + \left\| \frac{m}{2}\eta \right\|^2 - \left\langle \frac{m}{2}\xi, \frac{m}{2}\eta \right\rangle - \left\langle \frac{m}{2}\eta, \frac{m}{2}\xi \right\rangle = \\ &= \left\| \frac{m}{2}\xi \right\|^2 + \left\| \frac{m}{2}\eta \right\|^2 + \left\langle (b - \frac{m}{2}I)\xi, \frac{m}{2}\eta \right\rangle + \left\langle \frac{m}{2}\eta, (b - \frac{m}{2}I)\xi \right\rangle = \\ &= \left\| \frac{m}{2}\xi \right\|^2 + \left\| \frac{m}{2}\eta + (b - \frac{m}{2}I)\xi \right\|^2 - \left\| (b - \frac{m}{2}I)\xi \right\|^2. \end{aligned}$$

The infimum of $m\{\langle bu, u \rangle + \langle mv - bv, v \rangle\}$ where $u, v \in \mathfrak{D}$ and $\xi = u - v$ is fixed is therefore equal to

$$\begin{aligned} & \left\| \frac{m}{2}\xi \right\|^2 - \left\| P(b - \frac{m}{2}I)\xi \right\|^2 = \\ &= \left\| \frac{m}{2}\xi \right\|^2 - \left\| \frac{m}{2}\xi \right\|^2 - \|Pb\xi\|^2 + \left\langle Pb\xi, \frac{m}{2}\xi \right\rangle + \left\langle \frac{m}{2}\xi, Pb\xi \right\rangle = \\ &= m \langle \xi, b\xi \rangle - \|Pb\xi\|^2 = \langle b\xi, m\xi - Pb\xi \rangle, \end{aligned}$$

where P is the orthogonal projection of \mathfrak{H} to the norm closure of \mathfrak{D} . We get finally that

$$m^{-1} \|V^*x\|^2 = \sup \left\{ |\langle x, by \rangle|^2 : y \in \mathfrak{D}, \langle by, my - Pby \rangle \leq 1 \right\}.$$

This proves the equivalence of (8) and (7) and completes the proof of the theorem. \square

COROLLARY 1.9. *Let \mathfrak{D} be a closed subspace of the Hilbert space \mathfrak{H} , $b: \mathfrak{D} \rightarrow \mathfrak{H}$ a suboperator satisfying property (1). b has unique positive extension of smallest norm m if and only if there exists a co-isometry $T: \mathfrak{D} \rightarrow \mathfrak{D}^\perp$ in such a way that $C = T(mA - A^2)^{1/2}$, where $A = Pb$, $C = (I - P)b$ and P is the orthogonal projection onto \mathfrak{D} .*

PROOF. Since b satisfies (1) we have that

$$\|Cx\|^2 + \|Ax\|^2 \leq m \langle Ax, x \rangle \quad \text{for all } x \in \mathfrak{D}.$$

Therefore

$$\langle C^*Cx, x \rangle \leq \langle Ax, mx - Ax \rangle \quad \text{for all } x \in \mathfrak{D}.$$

In view of Douglas' factorization theorem this implies that there is a contraction T from \mathfrak{D} into \mathfrak{D}^\perp such that $C = T(mA - A^2)^{1/2}$. We may also assume that the

kernel of T contains the kernel of $mA - A^2$, (this additional requirement completely determines T). For each $x \in \mathfrak{D}^\perp$ we have

$$\begin{aligned} & \sup \left\{ |\langle x, by \rangle|^2 : y \in \mathfrak{D}, \langle by, my - Pby \rangle \leq 1 \right\} = \\ & \sup \left\{ |\langle x, Cy \rangle|^2 : y \in \mathfrak{D}, \langle Ay, my - Ay \rangle \leq 1 \right\} = \\ & \sup \left\{ \left| \langle T^*x, (mA - A^2)^{1/2}y \rangle \right|^2 : y \in \mathfrak{D}, \left\| (mA - A^2)^{1/2}y \right\|^2 \leq 1 \right\}. \end{aligned}$$

By assumption the range of T^* is contained in the norm closure of the range of $(mA - A^2)^{1/2}$, therefore the last supremum is simply $\|T^*x\|^2$.

Hence, applying Theorem 1.8, we get that b has unique such extension if and only if T^* is an isometry i.e. T is a co-isometry. \square

REMARK. According to the notation of Corollary 1.9

$$C^*C = (mA - A^2)^{1/2}T^*T(mA - A^2)^{1/2}.$$

If T^* is an isometry, then T^*T is an orthogonal projection and therefore C^*C is an extremal point in the operator interval $[0, mA - A^2]$, where 0 stands for the identically zero operator. Conversely if C^*C is an extremal point in $[0, mA - A^2]$ then T^*T is an orthogonal projection and therefore T^* has to be a partial isometry. So the corollary can be stated as follows. Under the above assumption b has unique positive extension with smallest norm m if and only if C^*C is an extremal point of $[0, mA - A^2]$ and C has dense range (equivalently C^* is injective).

We need a well known lemma about operator ranges.

LEMMA 1.10. *Let $D = EE^*$ be a bounded positive operator on a Hilbert space \mathfrak{H} . Then*

$$\text{ran } D^{1/2} = \text{ran } E = \left\{ \xi \in \mathfrak{H} : \sup \left\{ |\langle \xi, y \rangle|^2 : y \in \mathfrak{H}, \langle Dy, y \rangle \leq 1 \right\} < \infty \right\}.$$

PROOF. Let $\xi = Ex$, then

$$|\langle \xi, y \rangle|^2 = |\langle x, E^*y \rangle|^2 \leq \|x\|^2 \cdot \langle EE^*y, y \rangle.$$

On the other hand, if ξ is in the above given set then $E^*y \mapsto \langle y, \xi \rangle$ is a bounded linear functional, denote x one of its possible representing vector. Then $\langle E^*y, x \rangle = \langle y, \xi \rangle$, i.e. $Ex = \xi$. Since we can replace E and E^* by $D^{1/2}$ in this argument, the lemma is proved. \square

The next lemma characterizes extremal points of an operator interval.

LEMMA 1.11. *Let D, E be positive operators and suppose that $0 \leq E \leq D$. E is an extremal point in $[0, D]$ if and only if*

$$\operatorname{ran} E^{1/2} \cap \operatorname{ran} (D - E)^{1/2} = \{0\} \quad (9)$$

PROOF. First assume that E is an extremal point of the interval $[0, D]$. Then $E = D^{1/2}PD^{1/2}$ and $E - D = D^{1/2}(I - P)D^{1/2}$, where P is an orthogonal projection. Therefore the range of $E^{1/2}$ and $(D - E)^{1/2}$ is equal to the range of $D^{1/2}P$ and $D^{1/2}(I - P)$ respectively. So the only if part is almost obvious.

Conversely, since $0 \leq E \leq D$, there is a positive contraction P in such a way that $E = D^{1/2}PD^{1/2}$. We can also demand that the range of P is contained in the closure of the range of D (in this case P is uniquely determined).

Since $\operatorname{ran} E^{1/2}$ and $\operatorname{ran} (D - E)^{1/2}$ is equal to $\operatorname{ran} D^{1/2}P^{1/2}$ and $\operatorname{ran} D^{1/2}(I - P)^{1/2}$ respectively, and $D^{1/2}$ restricted to the closure of the range of P is injective we have that condition (9) implies that $\operatorname{ran} P^{1/2} \cap \operatorname{ran} (I - P)^{1/2} = \{0\}$. But then $P(I - P)$ is identically zero, $P = P^2$. So P is an orthogonal projection and D is an extremal point. \square

THEOREM 1.12 (KREIN'S CRITERION). *Let $b: \mathfrak{D} \rightarrow \mathfrak{H}$ be subpositive suboperator satisfying property (1). b has unique positive operator extension of smallest norm m to \mathfrak{H} if and only if*

$$\sup \left\{ |\langle x, by \rangle|^2 : y \in \mathfrak{D}, \langle by, my - by \rangle \leq 1 \right\} = \infty \quad (10)$$

holds for each $x \neq 0$ from \mathfrak{D}^\perp .

PROOF. Without loss of generality we may assume that \mathfrak{D} is closed. Write b as a column vector as in Theorem 1.8. According to Corollary 1.9, Remark and Lemma 1.11 we have only to prove that (10) is equivalent to the following: C^* is injective and the ranges of C^* and $(mA - A^2 - CC^*)^{1/2}$ have zero as the only common vector.

Since for $x \in \mathfrak{D}^\perp, y \in \mathfrak{D}$

$$\langle x, by \rangle = \langle x, Cy \rangle = \langle C^*x, y \rangle$$

and

$$\langle by, my - by \rangle = \langle mby, y \rangle - \|by\|^2 = \langle mAy, y \rangle - \langle A^2y, y \rangle - \langle C^*Cy, y \rangle$$

(10) can be rewritten as

$$\sup \left\{ |\langle C^*x, y \rangle|^2 : y \in \mathfrak{D}, \langle (mA - A^2 - C^*C)y, y \rangle \leq 1 \right\} = \infty \quad (11)$$

for all $x \neq 0$ from \mathfrak{D}^\perp .

Now (11) obviously implies that C^* is injective and $\operatorname{ran} C^* \cap \operatorname{ran} (mA - A^2 - C^*C)^{1/2} = \{0\}$ (use Lemma 1.10).

Conversely if C^* is injective and $x \in \mathfrak{D}^\perp$ is not zero then C^*x is also non zero and therefore it is not an element of the range of $(mA - A^2 - C^*C)^{1/2}$ which by virtue of Lemma 1.10 implies (11). \square

Self-adjoint norm preserving extension

The theory of subself-adjoint suboperators goes along the same line, except that we have to transform the problem first and then results have to be retransformed.

THEOREM 1.13. *Let $s: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric bounded linear map. Then s has a self-adjoint extension with the same bound. Moreover, the set of all self-adjoint extensions of smallest possible norm has a largest and a smallest element.*

PROOF. Since

$$\langle (\|s\| I - s)x, (s + \|s\| I)x \rangle = \|s\|^2 \|x\|^2 - \|sx\|^2 \geq 0$$

both $\|s\| I - s$ and $s + \|s\| I$ fulfills condition (1) of Theorem 1.1 with constant $m = 2\|s\|$. Denote A_- and A_+ the smallest extension of $\|s\| I - s$ and $\|s\| I + s$, respectively. Then $S_- = \|s\| I - A_-$ and $S_+ = A_+ - \|s\| I$ are extensions of s . We claim that the former is the largest and later is the smallest extension of s of smallest possible norm. Indeed, let S any self-adjoint extension of s with norm $\|s\|$, then $\|s\| I - S$ is a positive an extension of $\|s\| I - s$ therefore $\|s\| I - S \geq A_-$ and for similar reason $\|s\| I + S \geq A_+$. Hence

$$S_+ = A_+ - \|s\| I \leq S \leq \|s\| I - A_- = S_-.$$

□

The name ‘Krein’s criterion’ of Theorem 1.12 can be supported with the following. We have just seen that the extension problem of a symmetric suboperator s to a self-adjoint operator can be traced back to the positive case treated before. s has only one norm preserving self-adjoint extension if and only if $b = \|s\| I + s$ has unique positive extension with smallest possible norm $2\|s\|$. Applying Theorem 1.12 to this special case we arrive at Krein’s classical condition, which is formulated in [13] with $\|s\| = 1$.

THEOREM 1.14. *Let $s: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric continuous linear operator. s has unique self-adjoint norm preserving extension if and only if*

$$\sup \left\{ |\langle x, sy \rangle|^2 : y \in \mathfrak{D}, \|s\|^2 \|y\|^2 - \|sy\|^2 \leq 1 \right\} = \infty$$

holds true for every $x \neq 0$ element from \mathfrak{D}^\perp .

We have seen that in the self-adjoint case there is at least one norm preserving self-adjoint extension. One can naturally ask whether the constant m defined in (1) how relates the norm of b . It is easy to see that $\|b\| \leq m$ since $\|bx\|^2 \leq m \langle bx, x \rangle \leq m \|b\| \cdot \|x\|^2$. On the other hand it may happen that $\|b\| < m$ as the next very

simple example shows. Let \mathfrak{H} be the two dimensional Hilbert space with the basis e_1, e_2 and $\mathfrak{D} = \mathbf{C} \cdot e_1$, $be_1 = \alpha e_1 + \beta e_2$ with $\alpha, \beta > 0$. Then

$$m = \frac{\alpha^2 + \beta^2}{\alpha} > \sqrt{\alpha^2 + \beta^2} = \|b\|.$$

This example is, however, rather particular as the domain of definition of the above b is one-dimensional. The rest of this section is devoted to the investigation of the case $\|b\| = m$. Without loss of generality we will assume that \mathfrak{D} is closed.

Denote $A = Pb$ and $C = (I - P)b$ where P is the orthogonal projection of \mathfrak{H} to \mathfrak{D} . If $m = \|b\|$ then for all $x \in \mathfrak{D}$

$$\|bx\|^2 \leq m \langle bx, x \rangle = \|b\| \langle Pbx, x \rangle \leq \|b\| \|Pb\| \|x\|^2$$

and $\|b\| = m = \|Pb\|$ follows easily. Therefore

$$\|bx\|^2 = \|Ax\|^2 + \|Cx\|^2 \leq \|A\| \langle Ax, x \rangle$$

so

$$\|Cx\|^2 \leq \langle (\|A\| A - A^2)x, x \rangle$$

Except the trivial case when C is identically zero (this corresponds to the case when b is a positive selfadjoint operator on \mathfrak{D}) this can occur only if A is not a scalar multiple of an orthogonal projection of \mathfrak{D} . This fact explains the above example also. If \mathfrak{D} is one dimensional all operator on it are scalar multiple of a projection.

THEOREM 1.15. *Let $b : \mathfrak{D} \rightarrow \mathfrak{H}$ be a subpositive suboperator and denote P the projection of \mathfrak{H} onto \mathfrak{D} . If $Pb : \mathfrak{D} \rightarrow \mathfrak{D}$ is a scalar multiple of a projection and $(I - P)b$ not identically zero then the norm of any positive operator extension B of b is strictly larger than the norm of b , i.e. $\|B\| > \|b\|$.*

Parallel sum of positive operators

At the end of this chapter we develop a result, we need in the next chapter. These results are from [3].

LEMMA 1.16. *Let A, B positive operators on the Hilbert space \mathfrak{H} . Then there is a positive operator with the following quadratic form.*

$$x \mapsto \inf \{ \langle Ay, y \rangle + \langle Bz, z \rangle : y, z \in \mathfrak{H}, y + z = x \} \quad (12)$$

PROOF. Let $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}$, and define the operator $C : \mathfrak{K} \rightarrow \mathfrak{K}$ with the formula $C(x \oplus y) = A(x + y) \oplus A(x + y) + By$. Denote D the shortening of C to $\mathfrak{H} \oplus 0$. Then

$$\begin{aligned} \langle D(x \oplus 0), x \oplus 0 \rangle &= \inf \{ \langle C(x \oplus y), x \oplus y \rangle : y \in \mathfrak{H} \} \\ &= \inf \{ \langle A(x + y), x + y \rangle + \langle By, y \rangle : y \in \mathfrak{H} \}. \end{aligned}$$

Thus $\iota^* D \iota$ is the desired operator where $\iota: \mathfrak{H} \rightarrow \mathfrak{K}$ is the embedding $\iota x = x \oplus 0$. \square

The operator representing the quadratic form (12) is called the parallel sum of A and B and denoted by $A : B$. From the quadratic form of the parallel sum it is clear that parallel summation is a commutative and associative operation. Interesting properties of parallel summation are collected in the next lemma, where so-lim means limit in the strong operator topology.

LEMMA 1.17. *Let \mathfrak{H} and \mathfrak{K} be Hilbert spaces, $T: \mathfrak{H} \rightarrow \mathfrak{K}$ be a continuous linear map, A, B, C and D be positive operators on \mathfrak{K} . Then*

- (1) $A : B \leq A$
- (2) $\text{ran}(A : B)^{1/2} = \text{ran} A^{1/2} \cap \text{ran} B^{1/2}$
- (3) if $A \leq C$ and $B \leq D$ then $A : B \leq C : D$
- (4) $T(A : B)T^* \leq TAT^* : TBT^*$ (transformer inequality)
- (5) if $\ker T^* = \{0\}$ then $TAT^* : TBT^* = T(A : B)T^*$ (transformer equality)
- (6) if A_n and B_n are decreasing sequences of positive operators with $\text{so-lim} A_n = A$, $\text{so-lim} B_n = B$ then $\text{so-lim}(A_n : B_n) = A : B$ (upper semi-continuity)

PROOF.

- (1) It follows from (12) with $y = x$, $z = 0$.
- (2) From the previous point it follows that

$$\text{ran}(A : B)^{1/2} \subset \text{ran} A^{1/2} \cap \text{ran} B^{1/2}.$$

If $x \in \text{ran} A^{1/2} \cap \text{ran} B^{1/2}$ then there are $m_A > 0$ and $m_B > 0$ such that for each y and z

$$|\langle x, y \rangle|^2 \leq m_A \langle Ay, y \rangle \quad \text{and} \quad |\langle x, z \rangle|^2 \leq m_B \langle Bz, z \rangle$$

Thus

$$|\langle x, y + z \rangle|^2 \leq 2|\langle x, y \rangle|^2 + 2|\langle x, z \rangle|^2 \leq 2(m_A + m_B)(\langle Ay, y \rangle + \langle Bz, z \rangle).$$

Taking infimum on the right hand side such that $y + z = u$ is fixed one gets that

$$|\langle x, u \rangle|^2 \leq 2(m_A + m_B) \langle (A : B)u, u \rangle$$

and $x \in \text{ran}(A : B)^{1/2}$ by Lemma 1.10.

- (3) Follows immediately from the definition of the quadratic form of parallel sum.
- (4) The quadratic form belonging to the right hand side is

$$\inf \{ \langle Ay, y \rangle + \langle Bz, z \rangle : y, z \in \text{ran} T^*, T^*x = y + z \}$$

while the same for the left hand side is

$$\inf \{ \langle Ay, y \rangle + \langle Bz, z \rangle : y, z \in \mathfrak{K}, T^*x = y + z \}$$

The only difference is the range over which y and z varies, so the given inequality is clear.

- (5) Follows from the proof of the previous point.
 (6) It follows from the third point of this Lemma that for each n

$$A : B \leq A_n : B_n.$$

Since strong operator limit preserves this relation we only have to prove the opposite inequality. Let x be fixed and ε be any positive number then there is y and z such that

$$\langle Ay, y \rangle + \langle Bz, z \rangle < \langle (A : B)x, x \rangle + \varepsilon.$$

From $A = \text{so-lim } A_n$ $\langle A_n y, y \rangle \rightarrow \langle Ay, y \rangle$, and similarly for B and z . Thus

$$\lim_{n \rightarrow \infty} \langle (A_n : B_n)x, x \rangle \leq \langle (A : B)x, x \rangle + \varepsilon$$

for any positive ε and x , i.e. $\text{so-lim}_{n \rightarrow \infty} (A_n : B_n) \leq A : B$. □

The existence of parallel sum has the interesting consequence that operator ranges constitutes a lattice with the usual subspace operations. Indeed $\text{ran } X + \text{ran } Y = \text{ran } (XX^* + YY^*)^{1/2}$ and $\text{ran } X \cap \text{ran } Y = \text{ran } ((XX^*) : (YY^*))^{1/2}$.

In case A and B are both invertible the defining relation of $A : B$ is explicitly solvable.

LEMMA 1.18. *Let A and B be invertible positive operators then $A : B = (A^{-1} + B^{-1})^{-1}$.*

PROOF. For $x \in \mathfrak{H}$ let $w = (A^{-1} + B^{-1})^{-1}x$, and $y = A^{-1}w$, $z = B^{-1}w$ Then $x = y + z$ and

$$\langle Ay, y \rangle + \langle Bz, z \rangle = \langle AA^{-1}w, y \rangle + \langle BB^{-1}w, z \rangle = \langle w, y \rangle + \langle w, z \rangle = \langle w, x \rangle$$

For any other choice $y' = y + u$, $z' = z - u$ we have that

$$\begin{aligned} \langle Ay', y' \rangle + \langle Bz', z' \rangle &= \\ \langle Ay, y \rangle + 2\Re \langle Ay, u \rangle + \langle Au, u \rangle + \langle Bz, z \rangle - 2\Re \langle Bz, u \rangle + \langle Bu, u \rangle. \end{aligned}$$

Since $Bz = Ay = w$ we have that

$$\langle Ay', y' \rangle + \langle Bz', z' \rangle = \langle w, x \rangle + \langle Au, u \rangle + \langle Bu, u \rangle$$

which is not smaller than $\langle w, x \rangle$, thus $\langle (A: B)x, x \rangle = \langle (A^{-1} + B^{-1})^{-1}x, x \rangle$ for all $x \in \mathfrak{H}$. In other words $A: B = (A^{-1} + B^{-1})^{-1}$. \square

The previous result suggests that $2(A: B)$ can be regarded as the harmonic mean of the operators A and B .

Using upper semi-continuity we can give a formula for $A: B$. Indeed,

$$A: B = \text{so-lim}_{\varepsilon \rightarrow 0} ((A + \varepsilon I)^{-1} + (B + \varepsilon I)^{-1})^{-1} = \text{so-lim}_{\varepsilon \rightarrow 0} (A + \varepsilon I)(A + B + 2\varepsilon I)^{-1}(B + \varepsilon I),$$

where we have used that $r^{-1} + s^{-1} = r^{-1}ss^{-1} + r^{-1}rs^{-1} = r^{-1}(s + r)s^{-1}$ in any unital ring. If A and B commute we can use functional calculus, so especially in case of $B = I - A$. Since

$$\lim_{\varepsilon \rightarrow 0+0} \frac{1}{\frac{1}{x + \varepsilon} + \frac{1}{1 - x + \varepsilon}} = x(1 - x)$$

we have that $A: (I - A) = A(I - A)$. This formula remains true if the identity operator is replaced by any projection P satisfying $PA = AP = A$ as in this case we can work in the Hilbert space $\text{ran } P$.

Formula (3) and (12) together give an extremely easy proof for the fact that the parallel summation and shortening operation commutes, i.e.

$$A_{\mathfrak{M}}: B = (A: B)_{\mathfrak{M}} = A_{\mathfrak{M}}: B_{\mathfrak{M}}$$

for any pair A, B of positive operators and subspace \mathfrak{M} . Indeed the quadratic form of any of these operators is

$$x \mapsto \inf \{ \langle Ay, y \rangle + \langle Bz, z \rangle : y + z \in x + \mathfrak{M}^{\perp} \}.$$

All of the above results on parallel summation may be put into a much more general framework, namely into the theory of operator connections and means elaborated by T. Ando F. Kubo and many others.

Connection is by definition a binary operation σ on positive operators with the following properties:

- (1) Monotonicity in both variable, i.e. $0 \leq A \leq C$ and $0 \leq B \leq D$ implies $A\sigma B \leq C\sigma D$.
- (2) Transformer inequality, i.e. for any bounded linear operator T and positive operators A, B we have $T(A\sigma B)T^* \leq TAT^*\sigma TBT^*$.
- (3) Upper-semicontinuity, i.e. if A_n and B_n are decreasing sequences of positive operators than $(\text{so-lim } A_n)\sigma(\text{so-lim } B_n) = \text{so-lim}(A_n\sigma B_n)$.

An operator mean is by definition a normalized connection, i.e. $I\sigma I = I$.

So far we have the following examples for connections:

- (i) $A\sigma B = A$
- (ii) $A\sigma B = A: B$
- (iii) $A\sigma B = B\sigma' A$ where σ' is any of the previous two.

and the linear combinations of these with positive coefficients.

Kubo and Ando prove in [3] that, indeed, any connection is strong limit of linear combination with positive coefficients of the first two type of connections ((i),(ii)). They use Loewner theory of operator monotone functions.

It is clear from the monotonicity property of connections that the set of positive extensions with smallest positive norm is closed under any connection operation. One can ask the following, is that true that any positive extension of smallest positive norm of a subpositive suboperator is some connection of the largest and the smallest such extension. The answer is definitely no. In case when the largest and smallest such extension commute then any connection of them is in the strong closure of the C^* subalgebra of $\mathcal{B}(\mathfrak{H})$ they generate. However, except in trivial one dimensional case or unicity of such extensions, this can not fill any non degenerate operator interval.

CHAPTER II

RESTRICTIONS OF VARIOUS TYPES OF OPERATORS

In this chapter we give necessary and sufficient conditions for a suboperator to have compact, projection, reflection, partial isometry or self-adjoint partial isometry extension.

Restrictions of orthogonal projections and reflections

The first part of the next statement is from Zoltán Sebestyén [23], the rest of this section is from [16]

THEOREM 2.1. *Let $p: \mathfrak{D} \rightarrow \mathfrak{D}$ be a symmetric linear map. p a subprojection suboperator if and only if*

$$\|px\|^2 = \langle px, x \rangle \quad \text{for all } x \in \mathfrak{D}. \quad (1)$$

The set of projection extensions of p coincides with the set of extremal points of the set of all positive extension of p of norm not greater than 1.

PROOF. The necessity of the condition is obvious. Conversely, note that the embedding J used in Theorem 1.1 is an isometry by the assumption on p . Therefore $J^*J = I_{\mathfrak{R}}$ and $JJ^*JJ^* = JJ^*$, i.e. the smallest positive extension $P_0 = JJ^*$ of p is projection.

Since $I - p$ is also a subprojection suboperator, its smallest extension $J_1J_1^*$ is also a projection. Thus $P_1 = I - J_1J_1^*$, the largest extension of p of norm one, is a projection too.

From Theorem 1.4 we know that the set of positive extension of p of norm not greater than one is

$$\{P_0 + (P_1 - P_0)C(P_1 - P_0) : 0 \leq C \leq P_1 - P_0\},$$

where we have identified the Hilbert space constructed from $P_1 - P_0$ with the range of $P_1 - P_0$.

It is clear that if $0 \leq C \leq P_1 - P_0$ is a projection then $P_0 + C = P_0 + (P_1 - P_0)C(P_1 - P_0)$ is a projection extension of p .

Conversely let Q be an orthogonal projection which extends p . Then $P_0 \leq Q \leq P_1$. Since Q and P_0 are comparable the difference of them is also a projection and $Q - P_0 \leq P_1 - P_0$. So $C = Q - P_0$ shows that Q is an extremal point. \square

COROLLARY 2.2. *Let p be a subprojection suboperator. p admits unique projection extension if and only if the domain and range of p span a dense linear subspace of \mathfrak{H} .*

FIRST PROOF. $\text{ran } P_1 - P_0 \subset \mathfrak{D}^\perp \cap (\text{ran } P_0)^\perp$, so the condition is sufficient. Conversely, $\text{ran } (I - P_1) = \text{ran } (I - p)$, thus

$$\text{ran } P_1 = (\text{ran } (I - p))^\perp \supset \mathfrak{D}^\perp \cap \text{ran } p^\perp.$$

Thus in case $\text{ran } p^\perp \cap \mathfrak{D}^\perp \neq \{0\}$ then $P_1 \neq P_0$. □

SECOND PROOF. Let us use Theorem 1.8. We have to check that

$$\|x\|^2 = \sup \left\{ |\langle x, py \rangle|^2 : y \in \mathfrak{D}, \langle py, y - Ppy \rangle \leq 1 \right\}$$

for any $x \neq 0$ from \mathfrak{D}^\perp , where P denotes the projection to the closure of \mathfrak{D} . But now

$$\langle py, y - Ppy \rangle = \|py\|^2 - \|Ppy\|^2 = \|(I - P)px\|^2,$$

hence this condition is fulfilled if and only if the range of $(I - P)p$ is dense in \mathfrak{D}^\perp , i.e. $\text{ran } p$ and \mathfrak{D} generate a dense linear subspace of \mathfrak{H} . □

The formula $U = 2P - I$ defines a bijection between the projection P and the reflection (i.e. self-adjoint unitary operator) U . Therefore the counterpart of Theorem 2.1 for self-adjoint extensions is

THEOREM 2.3. *Let $u : \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric isometric suboperator. Then the set of the reflection extensions of u coincides with the set of extremal self-adjoint extensions of norm one.*

Moreover there is only one reflection extension of u if and only if the range and the domain of u span a dense subset of \mathfrak{H} .

PROOF. The first of the statement is clear. For the second part it has to be observed, that $\text{ran } (I + u)/2$ and \mathfrak{D} generate the same linear space as $\text{ran } u$ and \mathfrak{D} . □

Other characterizations of uniqueness was formulated by L. Kapos and J. Paez [10].

THEOREM 2.4. *Let $p : \mathfrak{D} \rightarrow \mathfrak{H}$ be a subprojection suboperator. p has unique orthogonal projection extension if and only if*

$$\|x\|^2 = \sup \left\{ |\langle x, py \rangle|^2 : y \in \mathfrak{D}, \langle py, y \rangle \leq 1 \right\} + \sup \left\{ |\langle x, y - py \rangle|^2 : y \in \mathfrak{D}, \langle y - py, y \rangle \leq 1 \right\}$$

holds for each x in \mathfrak{H} .

THEOREM 2.5. *Let $u : \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric linear isometry. Then u has unique self-adjoint isometry extension if and only if*

$$2\|x\|^2 = \sup \left\{ |\langle x, y + uy \rangle|^2 : y \in \mathfrak{D}, \langle y + uy, y \rangle \leq 1 \right\} + \\ \sup \left\{ |\langle x, y - uy \rangle|^2 : y \in \mathfrak{D}, \langle y - uy, y \rangle \leq 1 \right\}$$

holds for each x in \mathfrak{H} .

PROOF. Both statement expresses that there is unique extension of the appropriate type if and only if the range and the domain of the suboperator together span a dense linear subspace. \square

Characterization of subprojections can be extended easily to the characterization of restriction of a spectral resolution. This was done by Z. Sebestyén, J. Stochel, J. Paez [26]. Here we just mention the sufficient and obviously necessary condition of this statement, see Lemma 1 of [26].

THEOREM 2.6. *A suboperator valued measure $F : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{D}, \mathfrak{H})$ is a restriction to \mathfrak{D} of a spectral measure if and only if*

- (a) $\|F(\sigma)x\| = \langle F(\sigma)x, x \rangle$ for $\sigma \in \mathcal{A}$, $x \in \mathfrak{D}$,
- (b) $\langle F(\sigma)x, F(\tau)y \rangle = 0$ for $\sigma, \tau \in \mathcal{A}$, $\tau \cap \sigma = \emptyset$; $x, y \in \mathfrak{D}$

Restrictions of partial isometries

Characterization of subprojection suboperators can be used to prove the following

LEMMA 2.7. *Let \mathfrak{D} be a closed subspace of \mathfrak{H} and $A : \mathfrak{D} \rightarrow \mathfrak{D}$ be a positive contraction. A is a compression of a projection on \mathfrak{H} to \mathfrak{D} (i.e. there is projection Q such that $PQ|_{\mathfrak{D}} = A$, where P denotes the projection to \mathfrak{D}) if and only if*

$$\dim \mathfrak{D}^{\perp} \geq \dim \overline{\text{ran}(A - A^2)}. \quad (2)$$

PROOF. Assume that A is the compression of Q . Then for $x \in \mathfrak{D}$ we have that

$$\langle Qx, x \rangle = \langle Ax, x \rangle = \|(I - P)Qx\|^2 + \|PQx\|^2.$$

Rearranging this equation

$$\|(I - P)Qx\|^2 = \langle Ax, x \rangle - \|Ax\|^2 = \left\| (A - A^2)^{1/2}x \right\|^2$$

we see that there is partial isometry $U : \mathfrak{D} \rightarrow \overline{\text{ran}(A - A^2)^{1/2}}$ with which $(I - P)Q = U(A - A^2)^{1/2}$. Since the initial space of U is larger than $\overline{\text{ran}(A - A^2)^{1/2}}$ and its final space is contained in \mathfrak{D}^{\perp} , we have shown that (2) is necessary.

Conversely, let $U: \mathfrak{D} \rightarrow \mathfrak{D}^\perp$ be a partial isometry proving (2). Then the suboperator $q: \mathfrak{D} \rightarrow \mathfrak{H}$, $qx = Ax + U(A - A^2)^{1/2}x$ is a subprojection since it fulfills (1). Let Q be a projection extension of q then $A = PQ|_{\mathfrak{D}}$ so (2) is sufficient. \square

U is a partial isometry if and only if U^*U is a projection. We can use this property to describe restrictions of partial isometries.

THEOREM 2.8. *Let \mathfrak{D} be a closed subspace, $u: \mathfrak{D} \rightarrow \mathfrak{H}$ be a contractive linear map and denote $A = u^*u$. Then u has a partial isometry extension if and only if*

$$\dim \mathfrak{D}^\perp \geq \dim \overline{\text{ran}(A - A^2)}.$$

PROOF. If $u = U|_{\mathfrak{D}}$ where U is a partial isometry then $u^*u = PU^*U|_{\mathfrak{D}}$, i.e. $A = u^*u$ is a compression of a projection on \mathfrak{H} . So necessity is obvious from Lemma 2.7.

Conversely, using Lemma 2.7 we see that there is a projection Q on \mathfrak{H} such that $u^*u = PQ|_{\mathfrak{D}}$, we may also assume that $\{Qx : x \in \mathfrak{D}\}$ is dense in $\text{ran } Q$. Let $V: \text{ran } Q \rightarrow \mathfrak{H}$ be the unique continuous extension of $V(Qx) = ux$, $x \in \mathfrak{D}$ to $\text{ran } Q$. V is an isometry and the partial isometry $U = VQ$ extends u by its definition. \square

Of course there is no uniqueness in case partial isometry extension unless \mathfrak{D} is dense.

Theorem 2.8 and Lemma 2.7 are due to Ákos Magyar and Zoltán Sebestyén [15]. The rest of this section is taken from [18].

The idea exploited in Theorem 2.8 can be used to describe the restrictions of self-adjoint partial isometries also. We need some lemmas.

LEMMA 2.9. *Let \mathfrak{D} be a closed subspace, $v: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric contractive suboperator and denote P_v the projection of \mathfrak{H} onto the norm closure of $\text{ran } v$. v admits a self-adjoint partial isometry extension if and only if there is projection Q on \mathfrak{H} such that*

$$P_v \leq Q \tag{3}$$

and

$$\|Qx\| = \|vx\| \quad \text{for all } x \in \mathfrak{D}. \tag{4}$$

PROOF. Assume that V is a self-adjoint partial isometry extending v . Then the range projection of V will provide a good choice for Q . Indeed $P_v \leq Q$ is clear and $\|vx\| = \|Vx\| = \|Qx\|$ by properties of self-adjoint partial isometries.

Conversely, assume the existence of Q with property (3) and (4). Then the map $u(Qx) = vx$, $x \in \mathfrak{D}$ is symmetric since using (3) it follows that

$$\langle u(Qx), Qy \rangle = \langle vx, Qy \rangle = \langle vx, y \rangle = \langle x, vy \rangle = \langle Qx, u(Qy) \rangle$$

and isometric by (4). So by Theorem 2.3 u possesses reflection extension to any subspace containing the range and the domain of u , especially to $\text{ran } Q$, say U . It is clear that UQ is one of the self-adjoint partial isometry extensions of v . \square

So what we are interested in is the projection $Q - P_v$. The next lemma will help. It is a slight generalization of Lemma 2.7.

LEMMA 2.10. *Let \mathfrak{H} and \mathfrak{K} be two Hilbert spaces and $T: \mathfrak{K} \rightarrow \mathfrak{H}$ be a continuous linear map. A positive operator $A \leq TT^*$ on \mathfrak{H} is of the form TQT^* with a projection Q on \mathfrak{K} if and only if*

$$\dim \overline{\text{ran } A^{1/2} \cap \text{ran } (TT^* - A)^{1/2}} \leq \dim \ker T. \quad (5)$$

PROOF. For any positive operator $0 \leq C \leq TT^*$ there is a unique positive operator $\tilde{C} \in \mathcal{B}(\mathfrak{K})$ such that $\ker T \subset \ker \tilde{C}$. It also true that if D is another positive operator satisfying $C = TDT^*$ then $P(D - \tilde{C})P = 0$ where P is the projection of \mathfrak{K} onto the closure of $\text{ran } T^*$. So $A = TQT^*$ if and only if $\tilde{A} = PQP$. By Lemma 2.7 this is equivalent to $\dim \text{ran } \tilde{A}(I - \tilde{A}) \leq \dim \ker P$. $\tilde{A}(P - \tilde{A})$ is the parallel sum \tilde{A} and $P - \tilde{A}$ so by the Lemma 1.17 transformer equality $TA(P - A)T^* = T\tilde{A}T^*: T(P - \tilde{A})T^* = A: (TT^* - A)$ and thus there is an isometry between

$$\text{ran } A^{1/2} \cap \text{ran } (TT^* - A)^{1/2} = \text{ran } \left(T\tilde{A}T^*: T(P - \tilde{A})T^* \right)^{1/2} \text{ and } \text{ran } (\tilde{A}(I - \tilde{A}))^{1/2}$$

This completes the proof. \square

THEOREM 2.11. *Let \mathfrak{D} be a closed subspace, $v: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric contractive suboperator and denote P_v the projection of \mathfrak{H} onto the norm closure of $\text{ran } v$ and \mathfrak{H}_0 the closed subspace generated by $\text{ran } v$ and \mathfrak{D} . Then v admits a self-adjoint partial isometry extension V if and only if*

$$\|P_v x\| \leq \|vx\| \quad \text{for all } x \in \mathfrak{D}, \quad (6)$$

and

$$\dim \left(\text{ran } (v^*v - PP_v|_{\mathfrak{D}})^{1/2} \cap \text{ran } (I|_{\mathfrak{D}} - v^*v)^{1/2} \right) \leq \dim \mathfrak{H}_0^\perp \quad (7)$$

where P denotes the projection onto \mathfrak{D} .

PROOF. Assume that $v \subset V$ and V is a self-adjoint partial isometry. Denote E the range projection of V . For $x \in \mathfrak{D}$

$$\|vx\| = \|Vx\| = \|Ex\| \geq \|P_v x\|$$

since $V^*V = V^2 = E \geq P_v$. So (6) is necessary. We have also that $P(E - P_v)|_{\mathfrak{D}} = v^*v - PP_v|_{\mathfrak{D}}$ i.e. $v^*v - PP_v|_{\mathfrak{D}}$ is of the form TQT^* where $\mathfrak{K} = \text{ran } I - P_v$,

$T^* = (I - P_v)|_{\mathfrak{D}}$ and $Q = E - P_v$. Since $(\text{ran}(I - P_v)|_{\mathfrak{D}})^\perp \mathfrak{K} = \mathfrak{H}_0^\perp$ Lemma 2.10 shows that (7) is also necessary.

Conversely, from (6) it follows that $v^*v - PP_v|_{\mathfrak{D}}$ is a positive contraction, and (7) implies the existence of Q such that $Q \leq I - P_v$ and $\|(P_v + Q)x\| = \|vx\|$ for all $x \in \mathfrak{D}$. By virtue of Lemma 2.9 v has a self-adjoint partial isometry extension. \square

There are almost trivial examples ($\dim \mathfrak{H} = 2$, $\dim \mathfrak{D} = 1$) showing that it is possible that a symmetric suboperator has contractive self-adjoint and partial isometry extension, but does not admit self-adjoint partial isometry extension. e.g. denote e_1, e_2 an orthonormal basis for \mathfrak{H} , $e_1 \in \mathfrak{D}$ and let $ve_1 = \lambda e_1 + \mu e_2$, where $\lambda, \mu \in (0, 1)$ such that $\lambda < (\lambda^2 + \mu^2) < 1$.

Next we deal with uniqueness.

THEOREM 2.12. *With previous notation, assume that v satisfies (6) then v has a unique self-adjoint partial isometry extension if and only if*

$$\overline{\mathfrak{D} + \text{ran } v} = \mathfrak{H}. \quad (8)$$

and

$$\text{ran}(v^*v - PP_v|_{\mathfrak{D}})^{1/2} \cap \text{ran}(I|_{\mathfrak{D}} - v^*v)^{1/2} = \{0\} \quad (9)$$

PROOF. Since the conditions of Theorem 2.11 is fulfilled v actually has a self-adjoint partial isometry extension. Let us use the notation of the proof of Theorem 2.11. It is clear that if $\text{ran } T^*$ is not dense in \mathfrak{K} then there are many ways of finding the range projection of the prospective extensions. Therefore (8) is necessary. Thus (9) is also necessary for the extendibility to a self-adjoint partial isometry.

Conversely, there is a unique way of choosing the projection Q with the properties $Q \leq I - P_v$ and $\|(P_v + Q)x\| = \|vx\|$, therefore if V_1 and V_2 both are self-adjoint partial isometries extending v then their range projections and $P_v + Q$ coincide. Hence both V_1 and V_2 extend the map $(P_v + Q)x \mapsto vx$, $x \in \mathfrak{D}$ to a reflection of the space $\text{ran}(P_v + Q)$. The subspace

$$\{P_v x + Qx : x \in \mathfrak{D}\} + \text{ran } v$$

is dense in $\text{ran}(P_v + Q)$. Thus Theorem 2.3 implies $V_1 = V_2$. \square

COROLLARY 2.13. *With the previous notation assume that v satisfies (6) and (8). Then v has unique self-adjoint partial isometry extension if and only if for each $x \in \mathfrak{D}$*

$$\inf \left\{ \|vy\|^2 - \langle P_v y, y \rangle + \|z\|^2 - \|z\|^2 : y + z = x \right\} = 0 \quad (10)$$

PROOF. By Lemma 1.17 we have that for x (10) holds if and only if

$$(v^*v - PP_v|_{\mathfrak{D}}): (I|_{\mathfrak{D}} - v^*v)x = 0. \quad (11)$$

This is true for each $x \in \mathfrak{D}$ if and only if this parallel sum is zero, so, again by Lemma 1.17, this implies (9).

Conversely, (9) implies (11) by Lemma 1.17 and also (10). \square

Restrictions of self-adjoint compact operators

Our next goal is to describe all compact extensions of b in case there are any. In this section we borrow some idea of Ákos Magyar and Zoltán Sebestyén [15]. A not too deep generalization is also carried out, since we consider the \mathcal{C}_p classes also.

Let K be a compact operator and (s_n) be the decreasing sequence of the eigenvalues of $(K^*K)^{1/2}$ counted by multiplicity. The operator K is said to belong to \mathcal{C}_p , $1 \leq p < \infty$ if $\sum_{n=1}^{\infty} s_n^p < \infty$. It is known that if K belongs to \mathcal{C}_p then TK and K^* belong to \mathcal{C}_p also for any bounded linear operator T . We say that a suboperator belongs to \mathcal{C}_p if its continuous extension to $\overline{\mathfrak{D}}$ belongs to \mathcal{C}_p (for details see [5]).

LEMMA 2.14. *Let A and B be linear bounded operators such that $\|Ax\| \leq \|Bx\|$ for all x . If B is compact then so is A , if B belongs to \mathcal{C}_p then so does A .*

PROOF. All these statements follow from the previous remark as $A^*A \leq B^*B$ implies that B is a right divisor of A , i.e. $A = TB$ where T is a bounded operator. \square

Since for positive compact operator A the interval $[0, A]$ contains only compact operators a promising candidate for a compact extension is the smallest positive one.

THEOREM 2.15. *Let $b : \mathfrak{D} \rightarrow \mathfrak{H}$ a symmetric suboperator. It has a positive compact operator extension to \mathfrak{H} if and only if the set*

$$\{bx : x \in \mathfrak{D}, \langle bx, x \rangle \leq 1\} \quad (12)$$

is completely bounded.

PROOF. Let B be a compact operator such that $b \subset B$ then

$$\{bx : x \in \mathfrak{D}, \langle bx, x \rangle \leq 1\} \subset \left\{ Bx : x \in \mathfrak{H}, \|x\|^2 \leq \|B\| \right\}.$$

Since this latter set is completely bounded, the necessity is obvious.

Conversely, if the set (12) is completely bounded then it is bounded, and b fulfills property (1.1) of Theorem 1.1. It follows from the construction of Theorem 1.1 that J is compact provided that the set (12) is completely bounded. Hence JJ^* the smallest positive extension of b is compact. \square

Now here comes the promised description of all compact positive extension.

THEOREM 2.16. *Let $b : \mathfrak{D} \rightarrow \mathfrak{H}$ a symmetric suboperator such that (12) is completely bounded. The set of compact positive extensions of b of smallest possible norm is*

$$\{B_0 + VCV^* : C \in \mathcal{B}(\mathfrak{K}), 0 \leq C \leq I, C \text{ is compact}\}$$

where V and \mathfrak{K} are as defined in Theorem 1.4.

Obviously the given set contains compact operator extensions of b . To prove the contrary we need a lemma which is essentially due to Ákos Magyar. He used it to prove that a compact symmetric transformation of \mathfrak{D} into \mathfrak{H} have a compact self-adjoint extension.

LEMMA 2.17. *Let $s : \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric compact suboperator and $\lambda \in \mathbf{R} \setminus \{0\}$ such that $b = \lambda I + s$ has a positive extension.*

Then $JJ^|_{\mathfrak{D}^\perp}$ is a compact operator where JJ^* stands for the smallest positive extension of b .*

Moreover if s belongs to \mathcal{C}_p then $J^|_{\mathfrak{D}^\perp}$ also does so.*

PROOF. Without loss of generality we may assume that \mathfrak{D} is a closed subspace. Let us decompose \mathfrak{D} into the sum of two subspaces

$$\mathfrak{D} = \mathfrak{V} \oplus (\mathfrak{D} \ominus \mathfrak{V})$$

where \mathfrak{V} is the eigenspace belonging to the eigenvalue $-\lambda$ (\mathfrak{V} may be zero dimensional).

Note first that $b|_{\mathfrak{D} \ominus \mathfrak{V}}$ is bounded from below. To see that, assume for the contrary that there exists (x_n) a sequence of elements of norm one in $\mathfrak{D} \ominus \mathfrak{V}$ such that bx_n converges to the null element. Since s is compact we may also assume that sx_n is convergent. But then $x_n = \frac{1}{\lambda}(bx_n - sx_n)$ convergent too, and for its limit y we have that $\lambda y = -sy$ and $\|y\| = 1$ which is impossible since y is orthogonal to \mathfrak{V} the eigenspace of $-\lambda$.

Let c be a lower bound for $b|_{\mathfrak{D} \ominus \mathfrak{V}}$ i.e. $c\|x\| \leq \|bx\|$ for all $x \in (\mathfrak{D} \ominus \mathfrak{V})$. The norm of J^*x ($x \in \mathfrak{D}^\perp$) fulfills

$$\begin{aligned} \|J^*x\|^2 &= \sup \left\{ |\langle J^*x, u \rangle_0|^2 : \|u\|_0 \leq 1 \right\} \\ &= \sup \left\{ |\langle x, Jbv \rangle|^2 : v \in \mathfrak{D}, \langle bv, v \rangle \leq 1 \right\} \\ &\leq \sup \left\{ |\langle x, sv \rangle|^2 : v \in (\mathfrak{D} \ominus \mathfrak{V}), \langle v, v \rangle \leq \frac{m}{c^2} \right\} = \frac{m \|s^*x\|^2}{c^2} \end{aligned}$$

where we used the following inequality

$$\frac{c^2}{m} \|v\|^2 \leq \frac{\|bv\|^2}{m} \leq \langle bv, v \rangle$$

and $s^* : \mathfrak{H} \rightarrow \mathfrak{D}$ denoted the adjoint map of s . Since the compactness properties of s imply the same properties of s^* Lemma 2.14 completes the proof. \square

COROLLARY 2.18. *Let b be a subpositive suboperator belonging to \mathcal{C}_p . The smallest positive extension of b also belongs to \mathcal{C}_p .*

COROLLARY 2.19. *Let b be a subpositive suboperator. b admits a positive Hilbert-Schmidt (\mathcal{C}_2) operator extension if and only if*

$$\sum_{\lambda} \|be_{\lambda}\|^2 < \infty$$

where $\{e_{\lambda}\} \subset \mathfrak{D}$ is orthonormal basis of $\overline{\mathfrak{D}}$.

PROOF OF THEOREM 2.16. First we show that b is a compact suboperator. This follows from the following

$$\{bx : x \in \mathfrak{D}, \|x\| \leq 1\} \subset \{bx : x \in \mathfrak{D}, \langle bx, x \rangle \leq \|b\|\}$$

Now put $-b$ in place of s the constant m defined in (1.1) in place of λ in the previous lemma to get that $J_1 J_1^*|_{\mathfrak{D}^{\perp}}$ is compact. As $VV^* = B_1 - B_0 = mI - J_1 J_1^* - JJ^*$ and $\mathfrak{D} \subset \ker B_1 - B_0$ we have that $VV^*|_{(\ker V^*)^{\perp}}$ is injective and is a difference of $mI|_{(\ker V^*)^{\perp}}$, the multiple of the restriction of identity, and a compact, hence it is bounded from below so the compactness of C and VCV^* are equivalent. This completes the proof. \square

We also learned from the proof that $(mI - (B_1 - B_0))|_{\mathfrak{D}^{\perp}}$ is compact provided that B_0 is such. Therefore, in case the set (12) is completely bounded and \mathfrak{D}^{\perp} is infinite dimensional B_1 never equals B_0 , i.e. we have no uniqueness. If the set (12) is completely bounded and \mathfrak{D}^{\perp} is finite dimensional all extensions are compact, but we cannot state more about uniqueness than it is stated in Theorem 1.8.

THEOREM 2.20. *Let $s: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric compact suboperator such that \mathfrak{D} is not dense in \mathfrak{H} . Then s admits compact extensions of norm M for any $M \geq \|s\|$.*

The arithmetic mean of the largest and the smallest extension of s of norm M is compact. Moreover this extension belongs to \mathcal{C}_p provided that s does so.

PROOF. We know from the previous chapter that the smallest positive extension A of $\|s\|I + s$ and largest positive extension A_M of the same suboperator of norm not greater than $M + \|s\| > 2\|s\|$ do not coincide so there is a rank one projection P such that $P \leq A_M + A$. So if there is at least one compact self-adjoint extension S of s then there are compact extensions of any possible norm ($\geq \|s\|$) in the form e.g. $S + tP$, where $t \in \mathbf{R}$ is a suitable number. So it is enough to find at least one compact extension. The second part of the theorem shows that it is possible.

Without loss of generality we may assume that \mathfrak{D} is closed. Denote A_+ , A_- the smallest positive extension of $MI + s$, $MI - s$, respectively, Then $A_+ - MI$ is the smallest self-adjoint extension of s of norm not greater than M (its norm is actually M) and $MI - A_-$ is the largest such extension.

By Lemma 2.17 $(A_+ - A_-)|_{\mathfrak{D}^{\perp}}$ is compact. Then

$$(A_+ - MI) - (A_- - MI) = sP - (A_+ - A_-)(I - P)$$

is compact since both summands are such. The last statement follows from Lemma 2.17 also. \square

Description of all compact selfadjoint extensions is similar to that of positive extensions.

THEOREM 2.21. *Let $s: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric compact suboperator. Denote S_+ , S_- the smallest and the largest extension of s of norm $M \geq \|s\|$, respectively. Write the positive operator $(S_- - S_+)/2$ in the form VV^* . Then the set of all self-adjoint compact extensions of s of norm not greater than M is*

$$\left\{ \frac{S_- + S_+}{2} + VCV^* : -I \leq C \leq I, C \text{ is compact} \right\}.$$

PROOF. It is obvious that the given set contains only compact extensions.

Conversely, let S be an extension of s of norm not greater than M . Then $S_+ \leq S \leq S_-$ thus $S - S_+ = VC'V^*$ for some $0 \leq C' \leq 2I$ and

$$S = S_+ + VC'V^* = \frac{S_+ + S_-}{2} - \frac{S_- - S_+}{2} + VC'V^* = \frac{S_+ + S_-}{2} + VCV^*$$

where $C = C' - I$. Applying Lemma 2.17 as in Theorem 2.16 we can see that $V^*|_{(\ker V^*)^\perp}$ is bounded from below, therefore either both $S - (S_+ + S_-)$ and C are compact or none of them. This completes the proof. \square

CHAPTER III

SEMIBOUNDED EXTENSIONS

Extension procedure similar to that of Chapter I can be carried out in the semi-bounded case also. Instead of stating the theorem in advance, first we establish the result. We use the symbol $A: \mathfrak{H} \supset \mathfrak{D} \rightarrow \mathfrak{H}$ to indicate that the given map A is defined on a (possibly proper) subset of \mathfrak{H} .

Krein–von Neumann extension

Let A be a positive self-adjoint (not necessarily bounded) operator on \mathfrak{H} . The quadratic form $x \mapsto \langle Ax, x \rangle$ is positive semi-definite on the domain of A thus the application of the Schwarz inequality gives that

$$\text{dom } A \subset \left\{ y \in \mathfrak{H} : \exists m_y > 0, \forall x \in \mathfrak{D}, |\langle Ax, y \rangle|^2 \leq m_y \langle Ax, x \rangle \right\}$$

is dense in \mathfrak{H} . Denote $a: \mathfrak{D} \rightarrow \mathfrak{H}$ a symmetric positive linear map. In what follows we always assume the denseness of the next subspace

$$\mathfrak{D}_*(a) = \left\{ y \in \mathfrak{H} : \exists m_y > 0, \forall x \in \mathfrak{D}, |\langle Ax, y \rangle|^2 \leq m_y \langle Ax, x \rangle \right\}.$$

An associated new Hilbert space \mathfrak{K} will be introduced based on the range space $\text{ran } a$ of the map a with a new inner product $\langle \cdot, \cdot \rangle_0$ defined by

$$\langle ax, ay \rangle_0 = \langle ax, y \rangle \quad \text{for all } x, y \in \mathfrak{D}.$$

It is well defined since for $x, y, x', y' \in \mathfrak{D}$ $ax = ax'$ and $ay = ay'$ we have

$$\langle ax, y \rangle = \langle ax', y \rangle = \langle x', ay \rangle = \langle x', ay' \rangle = \langle ax', y' \rangle.$$

Positive definiteness of $\langle \cdot, \cdot \rangle_0$ follows from the fact that $\mathfrak{D}_*(a)$ is dense in \mathfrak{H} . Indeed if $\langle ax, ax \rangle_0 = 0$ then for each $y \in \mathfrak{D}_*(a)$

$$|\langle ax, y \rangle|^2 \leq m_y \langle ax, x \rangle = m_y \langle ax, x \rangle_0 = 0$$

thus $ax = 0$.

Denote \mathfrak{K} the completion of the pre-Hilbert space $(\text{ran } a, \langle \cdot, \cdot \rangle_0)$. The natural embedding J of $\text{ran } a$ into \mathfrak{H} , $J : \text{ran } a \rightarrow \mathfrak{H}$, $J(ax) = ax$, ($x \in \mathfrak{D}$) is densely defined, therefore it has adjoint J^* acting between \mathfrak{H} and \mathfrak{K} . The domain of definition of J^* is just the set \mathfrak{D}_* . The positive self-adjoint operator $J^{**}J^*$ is an extension of a , since

$$\langle ax, ay \rangle_0 = \langle Jax, y \rangle$$

for all $x \in \mathfrak{D}$ by definition, i.e. $\mathfrak{D} \subset \text{dom } J^*$ and $J^*y = ay \in \mathfrak{K}$, for $y \in \mathfrak{D}$.

So we have just got the following

THEOREM 3.1. *Let \mathfrak{D} be a linear submanifold of \mathfrak{H} and $a : \mathfrak{D} \rightarrow \mathfrak{H}$ a positive, hence symmetric, linear map. a admits a positive self-adjoint (possibly unbounded) extension on \mathfrak{H} if and only if the set*

$$\mathfrak{D}_*(a) := \left\{ y \in \mathfrak{H} : \exists m_y, \forall x \in \mathfrak{D}, |\langle ax, y \rangle|^2 \leq m_y \langle ax, x \rangle \right\}$$

is dense in \mathfrak{H} .

The above statement appeared in [19,20,25]

It is possible to introduce a partial order of positive self-adjoint operators similar to that of bounded self-adjoint operators. However, we must be cautious with the domains. Let A and B be positive self-adjoint operators we say that $A \leq B$ if and only if $\text{dom } B^{1/2} \leq \text{dom } A^{1/2}$ and $\|A^{1/2}x\|^2 \leq \|B^{1/2}x\|^2$ for each $x \in \text{dom } B^{1/2}$.

The reasons for this game with the square roots are very simple, first the quadratic form associated to A $x \mapsto \langle Ax, x \rangle$, $x \in \text{dom } A$ can be extended by closure to the domain of $A^{1/2}$. Indeed, as a side result, we will see later that the domain of $A^{1/2}$ is just the subspace

$$\text{dom } A^{1/2} = \{y \in \mathfrak{H} : \exists (x_n) \subset \text{dom } A, y = \lim x_n, \langle A(x_n - x_m), x_n - x_m \rangle \rightarrow 0\}.$$

We have to mention here that this is the domain of any closed operator T factoring A in the form $A = T^*T$. Indeed, in this case we have $A^{1/2} = \overline{UT}$ where U is a partial isometry with initial space $\overline{\text{ran } T}$ and final space $\text{ran } A^{1/2}$.

Secondly, it may happen that $\text{dom } A \cap \text{dom } B = \{0\}$ while $\text{dom } A^{1/2} = \text{dom } B^{1/2}$.

One advantage of this definition is that taking inverse is order reversing regardless of the boundedness or unboundedness of the given operators. Another advantage is that with this notion of ordering $A \leq B$ implies $\text{ran } A^{1/2} \subset \text{ran } B^{1/2}$ as in the case of bounded positive operators. Indeed the map $B^{1/2}x \mapsto A^{1/2}x$, $x \in \text{dom } B^{1/2}$ is contractive therefore can be extended to the whole space as a contractive linear operator, let us denote it by T . Then $TB^{1/2} \subset A^{1/2}$ thus by taking adjoint we have that $A^{1/2} \subset (TB^{1/2})^* = B^{1/2}T^*$, where we have exploited the continuity of T .

Next we are going to prove that the above extension $J^{**}J^*$ is the smallest possible positive self-adjoint extension. Indeed let A be an arbitrary positive self-adjoint extension of a . We can also apply the above construction to A : in this way we have

a new Hilbert space, say $(\mathfrak{K}_A, \langle \cdot, \cdot \rangle_A)$, and a densely defined operator $J_A : \mathfrak{H} \supset \rightarrow \mathfrak{K}_A$ such that $J_A^{**} J_A^*$ is a self-adjoint extension of A , since A is the only self-adjoint extension of itself $A = J^{**} J^*$. Since A is an extension of a there is a natural isometric embedding V of the pre-Hilbert space $(\text{ran } a, \langle \cdot, \cdot \rangle_0)$ into \mathfrak{K}_A given by the formula $V(ax) = J_A^* x$, $x \in \mathfrak{D}$. We use the same symbol for the unique isometric extension to \mathfrak{K} . From the inclusion $J \subset J_A V$ it follows that

$$V^* J_A^* \subset (J_A V)^* \subset J^*$$

and using the continuity of V^* we infer that

$$J^{**} \subset (V^* J_A^*)^* \subset J_A^{**} V$$

Therefore

$$\text{dom } A^{1/2} = \text{dom } J_A^* = \text{dom } V^* J_A^* \subset \text{dom } J^* = \text{dom } (J^{**} J^*)^{1/2}$$

and for any $x \in \text{dom } A^{1/2}$ we have that

$$\left\| (J^{**} J^*)^{1/2} x \right\| = \|J^* x\|_0 = \|V^* J_A^* x\|_0 \leq \|J_A^* x\|_A = \|A^{1/2} x\|.$$

This proves that $J^{**} J^*$ is the smallest element in the set of all positive self-adjoint extensions of a .

The following statement is an immediate corollary of the above construction and was published in [19].

THEOREM 3.2. *Let $a : \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric linear map such that $\mathfrak{D}_*(a)$ is dense in \mathfrak{H} . Then the domain of definition of the square root of its smallest positive extension, the Krein–von Neumann extension, A_s is*

$$\text{dom } (A_s^{1/2}) = \mathfrak{D}_*(a) = \left\{ y \in \mathfrak{H} : \exists m_y, \forall x \in \mathfrak{D}, |\langle ax, y \rangle|^2 \leq m_y \langle ax, x \rangle \right\}.$$

The range of the square root of A_s satisfies

$$\text{ran } (A_s^{1/2}) = \{y \in \mathfrak{H} : \exists (x_n) \subset \mathfrak{D}, \|ax_n - y\| \rightarrow 0, \langle a(x_n - x_m), x_n - x_m \rangle \rightarrow 0\}.$$

For any positive self-adjoint operator extension A of the map a we have that

$$\text{dom } A_s^{1/2} \supset \text{dom } A^{1/2} \quad \text{and} \quad \text{ran } A_s^{1/2} \subset \text{ran } A^{1/2}.$$

PROOF. $\text{dom } A_s^{1/2} = \text{dom } J^*$ and $\mathfrak{D}_*(a)$ is just the set of those y -s for which $\mathfrak{K} \ni ax \mapsto \langle Jax, y \rangle$ is a continuous linear functional.

To see that $\text{ran } A_s^{1/2}$ is just the given set notice that $\text{ran } A_s^{1/2} = \text{ran } J^{**} = \text{ran } \bar{J}$ and \bar{J} is the closure of the graph of J .

The last statement follows from the fact that A_s is the smallest positive self-adjoint extension. \square

Friedrichs extension

One cannot expect the existence of largest positive extension of a unless \mathfrak{D} is dense in \mathfrak{H} . Indeed for any positive self-adjoint extension A of a , $A + P$, where P is the projection to \mathfrak{D}^\perp , is a positive self-adjoint extension of a and $A \leq A + P$. However, if \mathfrak{D} is dense a similar construction gives the largest positive self-adjoint extension of a .

So, assume that \mathfrak{D} is dense. In this case $a: \mathfrak{D} \rightarrow \mathfrak{H}$ can be treated as a densely defined operator $Q: \mathfrak{H} \supset \mathfrak{D} \rightarrow \mathfrak{K}$, where \mathfrak{K} is the constructed Hilbert space as above. Q has adjoint $Q^*: \mathfrak{K} \supset \mathfrak{H} \rightarrow \mathfrak{H}$. The positive self-adjoint operator Q^*Q^{**} is just the Friedrichs extension (i.e. the largest positive self-adjoint extension) of a . To prove this, one has to notice that with the above notation $Q \subset J^*$ therefore $J \subset J^{**} \subset Q^*$ thus for $x \in \mathfrak{D}$

$$Q^*Q^{**}x = Q^*Qx = JQx = ax.$$

We have to prove also the maximality of Q^*Q^{**} . Let A be an arbitrary positive self-adjoint extension of a . We can also apply the above construction to A : in this way we have a new Hilbert space, say $(\mathfrak{K}_A, \langle \cdot, \cdot \rangle_A)$, and a densely defined operator $Q_A: \mathfrak{H} \supset \mathfrak{D} \rightarrow \mathfrak{K}_A$ such that $A = Q_A^*Q_A^{**}$. Since A is an extension of a there is a natural isometric embedding V of the pre-Hilbert space $(\text{ran } a, \langle \cdot, \cdot \rangle_0)$ into \mathfrak{K}_A given by the formula $V(Qx) = Q_Ax$. We use the same symbol for the unique isometric extension to \mathfrak{K} . From the inclusion $VQ \subset Q_A$ and the continuity of V we infer that

$$Q^*V^* = (VQ)^* \supset Q_A^* \quad \text{and} \quad V^{**}Q^{**} \subset (VQ)^{**} \subset Q_A^{**}.$$

Therefore

$$\text{dom } (Q^*Q^{**})^{1/2} = \text{dom } Q^{**} = \text{dom } V^{**}Q^{**} \subset \text{dom } Q_A^{**} = \text{dom } A^{1/2}$$

and for any $x \in \text{dom } Q^{**} = \text{dom } (Q^*Q^{**})^{1/2}$ we have that

$$\left\| (Q^*Q^{**})^{1/2}x \right\| = \|Q^{**}x\|_0 = \|(VQ)^{**}x\|_1 = \|Q_1^{**}x\|_1 = \|(B)^{1/2}x\|.$$

This proves that Q^*Q^{**} is the largest element in the set of all positive self-adjoint extensions of a .

In this way we have got the following theorem appeared in [19].

THEOREM 3.3. *Let $a: \mathfrak{D} \rightarrow \mathfrak{H}$ be a densely defined operator bounded from below by zero. Then the domain of definition of the square root of its largest positive self-adjoint extension, the Friedrichs extension, A_l is*

$$\text{dom } A_l^{1/2} = \{y \in \mathfrak{H} : \exists (x_n) \subset \mathfrak{D}, \|x_n - y\| \rightarrow 0, \langle a(x_n - x_m), x_n - x_m \rangle \rightarrow 0\}.$$

The range of the square root of A_l is the set

$$\text{ran } A_l^{1/2} = \left\{ y \in \mathfrak{H} : \exists m_y, \forall x \in \mathfrak{D}, |\langle x, y \rangle|^2 \leq m_y \langle ax, x \rangle \right\}.$$

For any positive self-adjoint operator extension A of the map a we have moreover that

$$\operatorname{dom} A^{1/2} \supset \operatorname{dom} A_l^{1/2} \quad \text{and} \quad \operatorname{ran} A^{1/2} \subset \operatorname{ran} A_l^{1/2}.$$

PROOF. Similar to the proof of Theorem 3.2. Indeed $\operatorname{ran} A_l^{1/2} = \operatorname{ran} Q^*$ and we can use Lemma 1.10. Although it is stated in terms of continuous operator, the proof works equally well for unbounded ones.

$\operatorname{dom} A_l^{1/2} = \operatorname{dom} Q^{**} = \operatorname{dom} \overline{Q}$ gives the domain characterization.

The last statement is consequence of the maximality of A_l . \square

The range and domain characterization Theorem 3.2 and Theorem 3.3 are also valid for the square root of any positive self-adjoint operator, since they are maximal symmetric operators.

The set of positive extensions, uniqueness

Next we examine uniqueness. We have seen that this question is interesting only if \mathfrak{D} is dense.

THEOREM 3.4. *Let \mathfrak{D} be a dense subset of \mathfrak{H} and $a: \mathfrak{D} \rightarrow \mathfrak{H}$ a positive linear map. a admits unique positive self-adjoint extension if and only if for each $0 \neq y \in \operatorname{ran}(I + a)^\perp$*

$$\sup \left\{ |\langle x, y \rangle|^2 : x \in \mathfrak{D}, \langle ax, x \rangle \leq 1 \right\} = \infty \quad (1)$$

FIRST PROOF. There is only one positive self-adjoint extension of a if and only if $A_l = A_s$ and this happens exactly when $Q^{**} = J^*$. Since $Q^{**} \subset J^*$ we can calculate the orthocomplement of the graph of Q^{**} in the graph of J^* . $x \oplus Qx \perp y \oplus J^*y$ if and only if

$$0 = \langle x, y \rangle + \langle Qx, J^*y \rangle_0 = \langle x, y \rangle + \langle JQx, y \rangle = \langle x + ax, y \rangle.$$

So $y \oplus J^*y \in J^* \ominus Q^{**}$ if and only if $y \in \operatorname{ran}(I + a)^\perp$ and $y \in \mathfrak{D}_*(a)$. Since (1) simply formulates that $\operatorname{ran}(I + A)^\perp \cap \mathfrak{D}_*(a) = \{0\}$ both the necessity and sufficiency are clear. \square

SECOND PROOF. a has only one self-adjoint positive extension if and only if the map $b: \operatorname{ran}(I + a) \rightarrow \mathfrak{D}$, $b(I + a)x = x$ has only one positive extension of norm one. Indeed, b fulfills (1.1) since

$$\|b(I + a)x\|^2 = \|x\|^2 \leq \langle ax, x \rangle + \langle x, x \rangle = \langle b(I + a)x, x \rangle,$$

and the range of any extension of b contains $\operatorname{ran} b = \operatorname{dom} a$ which is dense thus for any positive extension B of b , $B^{-1} - I$ is a positive self-adjoint extension of a . We can apply the uniqueness theorems to b . Theorem 1.12 gives that a has unique positive self-adjoint extension if and only if (1) holds. \square

The second proof provide the ground for the next

THEOREM 3.5. *Let a be as in the previous theorem. The followings are equivalent*

- (1) *a admits only one positive self-adjoint extension*
- (2) *for each $x \in \mathfrak{H}$*

$$\|x\|^2 = \sup \left\{ |\langle x, y \rangle|^2 : y \in \mathfrak{D}, \langle (I+a)y, y \rangle \leq 1 \right\} + \sup \left\{ |\langle x, ay \rangle|^2 : y \in \mathfrak{D}, \langle (I+a)y, ay \rangle \leq 1 \right\},$$

- (3) *for each $x \in \text{ran}(I+a)^\perp$*

$$\|x\|^2 = \sup \left\{ |\langle x, y \rangle|^2 : y \in \mathfrak{D}, \langle y, ay + Py \rangle \leq 1 \right\}$$

where P denotes the projection to $\text{ran}(I+a)^\perp$,

- (4) *for each $0 \neq y \in \text{ran}(I+a)^\perp$*

$$\sup \left\{ |\langle x, y \rangle|^2 : x \in \mathfrak{D}, \langle ax, x \rangle \leq 1 \right\} = \infty.$$

PROOF. Apply Theorem 1.7, Theorem 1.8, Theorem 1.12 respectively as in the second proof of Theorem 3.4. \square

The above statements Theorem 3.4, Theorem 3.5 are not all original. The first one appeared in Krein's classical paper [13], in the other statement (2) is a translation of a result of Kapos to the unbounded case while (3) is a result of the author.

Commutation with the extensions

First we prove a general theorem. The ideas used here are similar the ones exploited in [20, 25].

THEOREM 3.6. *Let $a, b: \mathfrak{D} \rightarrow \mathfrak{H}$ be a positive linear map such that $\mathfrak{D}_*(a)$ and $\mathfrak{D}_*(b)$ are dense.*

Let C and D be continuous linear operators on \mathfrak{H} leaving \mathfrak{D} invariant and such that

$$aCx = D^*bx, \quad bDx = C^*ax \quad \text{for all } x \in \mathfrak{D}.$$

Then either $A = A_s, B = B_s$, the smallest, or in case \mathfrak{D} is dense $A = A_l, B = B_l$ the largest positive self-adjoint extension of a, b , respectively, we have

$$D^*B \subset AC, \quad C^*A \subset BD.$$

PROOF. Let \mathfrak{K}_a and \mathfrak{K}_b denote the Hilbert spaces constructed from the range of a and b , respectively. Also let us index with a and b the operators in the above extension procedure.

We define two continuous linear operators between $\text{ran } a \subset \mathfrak{K}_a$ and $\text{ran } b \subset \mathfrak{K}_b$:

$$\hat{C}Q_b x := Q_a C x \quad \text{and} \quad \hat{D}Q_a x = Q_b D x \quad \text{for all } x \in \mathfrak{D}.$$

To show that \hat{C} and \hat{D} are well-defined and continuous we find estimates for the norm of $\hat{C}Q_b x$ and $\hat{D}Q_a x$ step by step. First we have for any $x \in \mathfrak{D}$ that

$$\begin{aligned} \left\langle \hat{C}Q_b x, \hat{C}Q_b x \right\rangle_a &= \langle a C x, C x \rangle = \langle D^* b x, C x \rangle = \langle b x, D C x \rangle = \langle Q_b x, Q_b(D C x) \rangle_b \\ &\leq \langle Q_b x, Q_b x \rangle_b^{1/2} \langle Q_b(D C x), Q_b(D C x) \rangle_b^{1/2} \\ &= \langle Q_b x, Q_b x \rangle_b^{1/2} \left\langle \hat{D} \hat{C} Q_b x, \hat{D} \hat{C} Q_b x \right\rangle_b^{1/2}. \end{aligned}$$

Exchanging the role of C and DC and repeating this argument we obtain

$$\begin{aligned} \left\langle \hat{C}Q_b x, \hat{C}Q_b x \right\rangle_a &\leq \langle Q_b x, Q_b x \rangle_b^{1/2 + \dots + 1/2^n} \left\langle Q_b(DC)^{2^{n-1}} x, Q_b(DC)^{2^{n-1}} x \right\rangle_b^{1/2^n} \\ &= \langle Q_b x, Q_b x \rangle_b^{1-1/2^n} \left\langle b x, (DC)^{2^n} x \right\rangle_b^{1/2^n} \\ &\leq \langle Q_b x, Q_b x \rangle_b^{1-1/2^n} \|b x\|^{1/2^n} \left\| (DC)^{2^n} \right\|^{1/2^n} \|x\|^{1/2^n}. \end{aligned}$$

Passing with n to infinity we get

$$\left\langle \hat{C}Q_b x, \hat{C}Q_b x \right\rangle_a \leq r(DC) \langle Q_b x, Q_b x \rangle_b \quad \text{for each } x \in \mathfrak{D},$$

where $r(DC) \leq \|DC\|$ stands for the spectral radius of DC . This tells us that \hat{C} is continuous and well defined. Exchanging the role C and D we see that \hat{D} is also continuous and well-defined. Thus both \hat{C} and \hat{D} have unique continuous extensions on \mathfrak{K}_b and \mathfrak{K}_a , respectively, which we also denote by the same symbols, as this causes no confusion.

Now we show that $\hat{C}^* = \hat{D}$ since they coincide on $\text{ran } Q_a \subset \mathfrak{K}_a$:

$$\left\langle Q_b x, \hat{C}^* Q_a y \right\rangle_b = \left\langle \hat{C}Q_b x, Q_a y \right\rangle_a = \langle a C x, y \rangle = \langle b x, D y \rangle = \left\langle Q_b x, \hat{D}Q_a y \right\rangle_b$$

holds true for each x and y in \mathfrak{D} .

(1) If $A = A_l$, $B = B_l$ then from $\hat{C}Q_b \subset Q_a C$ and $\hat{D}Q_a \subset Q_b D$ we get

$$Q_b^* \hat{C}^* = (\hat{C}Q_b)^* \supset (Q_a C)^* \supset C^* Q_a^*$$

and similarly $Q_a^* \hat{D}^* \supset D^* Q_b^*$. Taking adjoint again we have $\hat{C}Q_b^{**} \subset Q_a^{**} C$ and $\hat{D}Q_a^{**} \subset Q_b^{**} D$.

Now we can put the pieces together:

$$D^*B_l = D^*Q_b^*Q_b^{**} \subset Q_a^*\hat{D}^*Q_b^{**} = Q_a^*\hat{C}Q_b^{**} \subset Q_a^*Q_a^{**}C = A_lC,$$

and similarly

$$C^*A_l \subset B_lD.$$

(2) If $A = A_s$, $B = B_s$ then from

$$J_b\hat{D}Q_ax = J_bQ_bDx = bDx = C^*ax = C^*J_aQ_ax \quad \text{for each } x \in \mathfrak{D}$$

we get that $C^*J_a \subset J_b\hat{D}$ because the domain of the former is $\text{ran } Q_a$ and on $\text{ran } Q_a$ they coincide. Taking adjoint we get that

$$\hat{C}J_b^* = \hat{D}^*J_b^* \subset (J_b\hat{D})^* \subset (C^*J_a)^* = J_a^*C$$

and changing the role of C and D , $\hat{D}J_a^* \subset J_b^*D$. Taking adjoint again

$$C^*J_a^{**} \subset (J_a^*C)^* \subset (\hat{C}J_b^*)^* = J_b^{**}\hat{C}^* = J_b^{**}\hat{D}.$$

Finally as in the previous case

$$C^*A_s = C^*J_a^{**}J_a^* \subset J_b^{**}\hat{D}J_a^* \subset J_b^{**}J_b^*D = B_sD,$$

and exchanging the role of C and D

$$D^*B_s \subset A_sC.$$

□

COROLLARY 3.7. *Let $a: \mathfrak{D} \rightarrow \mathfrak{H}$ be a positive linear map such that $\mathfrak{D}_*(a)$ is dense and B is a continuous self-adjoint linear operator on \mathfrak{H} leaving \mathfrak{D} invariant and commuting with a .*

Then B commutes with the Krein–von Neumann extension of a . B also commutes with the Friedrichs extension of a provided that \mathfrak{D} is dense in \mathfrak{H} .

Application to the bounded case

The results of this section concerning commutation can be found in a paper of Z. Sebestyén [24], the only exception is Lemma 3.9. The results stated here is somewhat more general as we do not restrict ourself to extensions of smallest possible norms.

Although Theorem 3.6 is stated for semi-bounded extension it applies to the bounded case also. Indeed if $b: \mathfrak{D} \rightarrow \mathfrak{H}$ fulfills (1.1) then

$$\mathfrak{D}_*(b) = \left\{ y \in \mathfrak{H} : \exists m_y, \forall x \in \mathfrak{D}, |\langle bx, y \rangle|^2 \leq m_y \langle bx, x \rangle \right\} = \mathfrak{H}$$

since $m_y = m \cdot \|y\|^2$ is good choice for any y . Therefore we have the next

COROLLARY 3.8. Let $b: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric linear map fulfilling (1.1), and C, D be bounded operators leaving \mathfrak{D} invariant. If for all $x \in \mathfrak{D}$

$$C^*bx = bDx \quad \text{and} \quad D^*bx = dCx$$

then $B_0D = C^*B_0$ where B_0 stands for the smallest positive extension of b .

Moreover if $C = C^* = D$ then $B_M C = C B_M$, where $M \geq m$ and B_M is the largest positive extension of norm M .

PROOF. The first part of the statement is just the application of Theorem 3.6 with b in place of a and b and B_0 in place of A_s and B_s .

To prove the second part notice that in this case $(MI - b)C = CMI + bC = C(MI + b)$ so by the first part $MI - B_M$ the smallest positive extension of $MI - b$ commutes with C . Therefore B_M does the same. \square

We say that C and D anticommute if $CD = -DC$. A positive operator cannot anticommute with any operator unless their product is zero.

LEMMA 3.9. Let D be a bounded positive operator and C be a bounded operator such that $CD = -DC$ then $CD = 0$.

FIRST PROOF. Let $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}$, $A(x \oplus y) = Dx \oplus -Dy$, $B(x \oplus y) = Cy \oplus Cx$. If E denotes the spectral resolution of D then it is easy to see that F the spectral resolution of A can be expressed as $F(\delta) = E(\delta) \oplus E(-\delta)$ for any Borel set $\delta \subset \mathbf{R}$. Since C anticommutes with D we have that B commutes with A and therefore with any of the latter's spectral projections. Let $\delta \subset [0, \infty)$, $BF(\delta) = F(\delta)B$ means that $CE(\delta) = E(-\delta)C = 0$ by the positivity of D , i.e. $CD = C \int \lambda E(d\lambda) = 0$. \square

SECOND PROOF. Using the relation $CD = -DC$ it follows that $C^*CD = -C^*DC$ is negative operator, so

$$\text{sp}(C^*CD) \subset (-\infty, 0].$$

On the other hand

$$\text{sp}(C^*CD) \cup \{0\} = \text{sp}(D^{1/2}C^*CD^{1/2}) \cup \{0\} \subset [0, \infty)$$

also true. Therefore $\text{sp}(C^*CD) = \{0\}$ so $CD^{1/2} = 0$ and $CD = 0$. \square

The second proof, which does not use the spectral resolution, is due to Zoltán Sebestyén.

From the Lemma 3.9 we have learnt that for positive bounded C , $CD = -DC$ implies $CD = 0 = DC$ that is C and D actually commute moreover from the proof it is also clear that they are “singular” in the sense that $\ker C$ is an invariant subspace for D and $D|_{\ker C} = 0$.

Nevertheless self-adjoint operator can anticommute and the counterpart of Corollary 3.8 for the self-adjoint case is the following

COROLLARY 3.10. *Let $s: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric bounded linear map and T a continuous self-adjoint operator leaving \mathfrak{D} invariant.*

If $Ts \subset sT$ then the smallest and the largest extensions of s of norm $M \geq \|s\|$ are commuting with T .

If $Ts \subset -sT$ then the arithmetic mean of the largest and smallest extensions of norm $M \geq \|s\|$ anticommutes with T .

PROOF. The first statement follows immediately from Corollary 3.8.

To prove the second assertion we apply Theorem 3.6. Put T in place of C and D , $MI + s$ in place of a and $MI - s$ in place of b . From our condition it follows that $T(MI - s) \subset (MI + s)T$ and $T(MI + s) \subset (MI - s)T$. So we can apply Theorem 3.6 and we get that for the smallest extension A_+ and A_- of $MI + s$ and $MI - s$, respectively, we have that $A_+T = TA_-$. But then for $(A_+ - A_-)/2$ the arithmetic mean of the largest and smallest self-adjoint extensions of s of norm M we have that

$$\frac{(A_+ - A_-)}{2}T = T\frac{A_- - A_+}{2} = -T\frac{A_+ - A_-}{2}.$$

□

At the very end we give the range characterization of the square root of the largest positive extension of smallest possible norm. This is a result from [19].

THEOREM 3.11. *Let $b: \mathfrak{D} \rightarrow \mathfrak{H}$ be a symmetric linear map fulfilling (1.1). Then the range of the square root of the largest positive extension B_1 of b of smallest possible norm m is equal to \mathfrak{M} given by*

$$\mathfrak{M} = \left\{ y \in \mathfrak{H} : \exists m_y > 0, \forall x \in \mathfrak{D}, |\langle y, mx - bx \rangle|^2 \leq m_y \langle bx, mx - bx \rangle \right\}.$$

PROOF. Since $\|B_1\| = m$ the operator $B_1(mI - B_1)$ and $mI - B_1$ are positive operators so the application of Schwarz inequality gives that if $y = B_1^{1/2}z$ then

$$\begin{aligned} |\langle y, mx - bx \rangle|^2 &= \left| \langle B_1^{1/2}z, mx - B_1x \rangle \right|^2 \\ &= \left| \langle (mI - B_1)^{1/2}z, (B_1(mI - B_1))^{1/2}x \rangle \right|^2 \\ &\leq \langle z, mz - B_1z \rangle \langle B_1x, mx - B_1x \rangle. \end{aligned}$$

So $m_y = \langle z, mz - B_1z \rangle$ is a right choice to see that $\text{ran } B_1^{1/2} \subset \mathfrak{M}$.

To prove the opposite inclusion it is enough to show that there is an extension of b of norm m the range of which is equal to the given set \mathfrak{M} . So let \mathfrak{H}_\circ be the closure of \mathfrak{M} and $a: \text{ran } b \rightarrow \mathfrak{H}_\circ$ be the map $a(bx) = mPx - bx$ for $x \in \mathfrak{D}$, where P denotes the orthogonal projection of \mathfrak{H} to \mathfrak{H}_\circ . Note first that a is well defined since from the definition of \mathfrak{M} it easily follows that it is orthogonal to the kernel

of b , second that $\text{ran } b \subset \mathfrak{H}_0$ so a maps indeed a subset of \mathfrak{H}_0 into \mathfrak{H}_0 , and finally that a is positive since $\langle a(bx), bx \rangle = \langle mx - bx, bx \rangle \geq 0$.

We can now apply the construction mentioned above to a and \mathfrak{H}_0 since the corresponding set is dense in \mathfrak{H}_0 :

$$\begin{aligned} \mathfrak{D}_*(a) &= \left\{ y \in \mathfrak{H}_0 : \exists m_y, \forall x \in \text{ran } b, |\langle ax, y \rangle|^2 \leq m_y \langle ax, x \rangle \right\} \\ &= \left\{ y \in \mathfrak{H}_0 : \exists m_y, \forall x \in \mathfrak{D}, |\langle mPx - bx, y \rangle|^2 \leq m_y \langle mPx - bx, bx \rangle \right\} \\ &= \left\{ y \in \mathfrak{H}_0 : \exists m_y, \forall x \in \mathfrak{D}, |\langle mx - bx, y \rangle|^2 \leq m_y \langle mx - bx, bx \rangle \right\} = \mathfrak{M}. \end{aligned}$$

So a has a positive self-adjoint extension A to \mathfrak{H}_0 . We claim that $C = m(A+I)^{-1}P$ is an extension of b with norm m and the range of the square root of it is equal to \mathfrak{M} . The norm of C is at most m and for $x \in \mathfrak{D}$ we have $(a+I)bx = mPx - bx + bx = mPx$ from which we get that $Cx = bx$ for all $x \in \mathfrak{D}$. Hence C extends b and therefore the norm of C is at least m too. Moreover we have that

$$\text{ran } C^{1/2} = \text{dom } (A+I)^{1/2} = \text{dom } A^{1/2} = \mathfrak{M}.$$

This completes the proof. □

One can easily verify that the operator C constructed in the proof is just B_1 the largest positive extension of b with norm m .

Using Theorem 3.11 we can give another proof of Theorem 1.12. Indeed, b has unique positive extension of norm m if and only if the range of the square root of the largest such extension and \mathfrak{D}^\perp have 0 as the only common vector. In other words for any non-zero vector v from \mathfrak{D}^\perp , v does not belong the set

$$\left\{ y \in \mathfrak{H} : \exists m_y > 0, \forall x \in \mathfrak{D}, |\langle y, mx - ax \rangle|^2 \leq m_y \langle ax, mx - ax \rangle \right\}.$$

That is

$$\sup \left\{ |\langle v, mx - bx \rangle|^2 : x \in \mathfrak{D}, \langle bx, mx - bx \rangle \leq 1 \right\} = \infty$$

which is just the result of Theorem 1.12, since $\langle v, mx \rangle = 0$.

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SUMMARY

This dissertation deals with the operator extension problem in Hilbert space. To be more precise we concern only bounded self-adjoint and semibounded self-adjoint extensions. The fundamental construction used throughout the paper is due to Z. Sebestyén and described in Chapter I. Using this, we give a characterization of subpositive suboperators and prove the existence of smallest positive extension. This result is due to Z. Sebestyén [22, 24].

As an application we introduce the notion of shortening. Two examples are given where shorted operators arises naturally. As far as the author is aware the relation of shorted matrices and normal distribution remained hidden so far. The example of electrical networks is taken from [2].

The set of all positive extension of smallest possible norm and the extremal points of this convex set are described. This part is from the author paper [17] as well as the description of all positive (self-adjoint) compact extensions.

We also give some necessary and sufficient condition on uniqueness of extension. These results are from [16] except Theorem 1.7 which is from [10].

Next we apply the result of positive extensions to the self-adjoint case, and examine the relation of the constant m defined at the beginning of the chapter to the norm of the subpositive suboperator.

At the end of the chapter shortening is used to prove the existence of parallel sum of positive operators. The properties of parallel summation are also collected and some useful formula are derived.

In the second chapter restrictions of projections, reflections and (self-adjoint) partial isometries are characterized. These results are due to L. Kapos, Á. Magyar and Z. Sebestyén [9, 11, 15, 23] except the result on uniqueness. The results concerning self-adjoint partial isometries are completely new and are from [18]. At the end of the chapter we concern restrictions of compact self-adjoint operators. This section is based on ideas of Á. Magyar and Z. Sebestyén [14]. A not too deep generalization is carried out as we consider the \mathcal{C}_p classes also. The description of the set of all compact extensions is also given.

In the third chapter we give a construction for the Krein-von Neumann extension as well as the Friedrichs extension. The results are from [25] and [19, 20]. We investigate the range and the domain of the square root of the extremal extensions as well as uniqueness of the extension. A quite general theorem is proved about the

commutation with the extremal extensions. Then from this theorem we derive some result about the bounded case concerning commutation and anticommutation. At the end of the chapter description of the range of the square root of the largest positive positive extension with smallest positive norm is given.

ÖSSZEFOGLALÓ

Ez a dolgozat Hilbert operátorok kiterjesztési problémájával foglalkozik. Kicsit pontosabban korlátos önadjungált (pozitív) és félig korlátos önadjungált kiterjesztésekkel. Az alapkonstrukció, amit az egész dolgozatban használok Dr. Sebestyén Zoltántól származik, és az első fejezetben ismertetem. Ezt felhasználva jellemzem a szubpozitív szuboperátorokat, és bizonyítom a legkisebb pozitív kiterjesztés létezését. Ezek az eredmények Dr. Sebestyén Zoltántól származnak [22, 24].

Ezek alkalmazásaként bevezetem az „operátorrövidítés” fogalmát és két természetesen adódó példát mutatok be. Amennyire tudomásom van róla a „rövidített mátrixok” és a normális eloszlás kapcsolata mindeddig nem vetődött fel. A elektromos hálózatokkal kapcsolatos példa [2]-ből való.

Leírom a pozitív kiterjesztések halmazát és ennek a konvex halmaznak az extrémális pontjait. Ez a rész megtalálható [17] dolgozatomban, csakúgy mint a pozitív kompakt kiterjesztéseké.

Megadok továbbá számos szükséges és elégséges feltételt arra vonatkozóan, hogy mikor létezik egyetlen pozitív kiterjesztés a lehető legkisebb normával. Ezek az Annales-ben megjelent dolgozatomból [16] valók, kivéve az 1.7 telt.

Ezután a pozitív kiterjesztésekről szóló eredményeket az önadjungált esetre alkalmazom, és megvizsgálom a fejezet elején definiált konstans m és a szubpozitív szuboperátor normájának a viszonyát.

A fejezet végén operátorrövidítés segítségével bizonyítom a parallel összeg létezését. A parallel összegzés tulajdonságait és néhány hasznos formulát is levezetek.

A második fejezetben jellemzem a pojekciók, reflexiók és parciális isometriák megszorításaként előálló szuboperátorokat. Ezek az eredmények Kapos Lászlótól, Magyar Ákostól ill. Dr. Sebestyén Zoltántól származnak [9, 11, 15, 23], kivéve az egyértelműséggel foglalkozó állításokat. Az önadjungált parciális izometriákról szóló rész teljesen új eredményket ír le [18]. A fejezet végén kompakt önadjungált operátorok megszorításait vizsgálom. Ez a rész Magyar Ákos és Dr. Sebestyén Zoltán gondolatait használja [14], bár annál valamelyest általánosabb hiszen a \mathcal{C}_p osztályokkal is foglalkozom. Megadom továbbá a kompakt kiterjesztések halmazának egy paraméterezését.

A harmadik fejezetben megadok egy konstrukciót a Krein–Neumann és a Friedrichs kiterjesztésre. Ezek az eredmények [25] és [19, 20] dolgozatokból vannak.

Megvizsgálom az extrémális kiterjesztések négyzetgyökének értékészletét és értelmezési tartományát. Egy meglehetősen általános tételt bizonyítok az extrémális kiterjesztések kommutálási tulajdonságairól. Ebből a tételből levezetem a korlátos önadjungált esetekről szóló kommutálási és antikommutálási tételt. A dolgozat végén a legkisebb normájú legnagyobb pozitív kiterjesztés négyzetgyökének értékészletét jellemzem.