An EP theorem for
dual linear complementarity problem

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We show that the dual LCP can be solved in polynomial time if the matrix is row sufficient, as for this case it will be shown that all feasible solutions are complementary. This result yields an improvement compared to earlier known polynomial time complexity results, namely an LCP is solvable in polynomial time for $P_*(\kappa)$-matrices with known $\kappa$. Due to the special structure of DLCP, the polynomial time complexity of interior point methods depends on the row sufficient property of the matrix. Furthermore we present an EP theorem for the dual LCP with arbitrary matrix $M$.

Throughout the paper the following notations are used. Scalars and indices are denoted by lowercase Latin letters, vectors by lowercase boldface Latin letters, matrices by capital Latin letters, and finally sets by capital calligraphic letters. Further, $(\oplus)$ denotes the diagonal matrix whose diagonal elements are the coordinates of vector $x$, i.e., $X = \text{diag}(x)$ and $I$ denotes the identity matrix of appropriate dimension. The vector $xs = Xs$ is the componentwise product (Hadamard product) of the vectors $x$ and $s$, and for $\alpha \in \mathbb{R}$ the vector $x^\alpha$ denotes the vector whose $i$th component is $x_i^\alpha$. We denote the vector of ones by $e$. Furthermore

$$F_P = \{ (x, s) \geq 0 : -Mx + s = q \}$$

is the set of the feasible solution of the LCP and

$$F_D = \{ (u, z) \geq 0 : u + M^T z = 0, \; q^T z = -1 \}$$

is the set of the feasible solutions of DLCP.

The rest of the paper is organized as follows. The following section reviews the necessary definitions and basic properties of the matrix classes used in this paper. In Section 3, we present our main result and compare it with the LCP duality theorem.

## 2 The matrix classes

The class of $P^*(\kappa)$-matrices, that can be considered as a generalization of the class of positive semidefinite matrices, were introduced by Kojima et al. [9].

**Definition 2.1.** Let $\kappa \geq 0$ be a nonnegative number. A matrix $M \in \mathbb{R}^{n \times n}$ is a $P^*(\kappa)$-matrix if

$$(1 + 4\kappa) \sum_{i \in I_+(x)} x_i(Mx)_i + \sum_{i \in I_-(x)} x_i(Mx)_i \geq 0, \text{ for all } x \in \mathbb{R}^n,$$
where \( \mathcal{I}_+(x) = \{1 \leq i \leq n : x_i(Mx)_i > 0\} \) and \( \mathcal{I}_-(x) = \{1 \leq i \leq n : x_i(Mx)_i < 0\} \).

The nonnegative real number \( \kappa \) denotes the weight that need to be used at the positive terms so that the weighted ‘scalar product’ is nonnegative for each vector \( x \in \mathbb{R}^n \). Therefore, naturally \( \mathcal{P}_*(0) \) is the class of positive semidefinite matrices (if we set aside the symmetry of the matrix \( M \)).

**Definition 2.2.** A matrix \( M \in \mathbb{R}^{n \times n} \) is called a \( \mathcal{P}_* \)-matrix if it is a \( \mathcal{P}_*(\kappa) \)-matrix for some \( \kappa \geq 0 \), i.e.

\[
\mathcal{P}_* = \bigcup_{\kappa \geq 0} \mathcal{P}_*(\kappa).
\]

The class of sufficient matrices was introduced by Cottle, Pang and Venkateswaran [2].

**Definition 2.3.** A matrix \( M \in \mathbb{R}^{n \times n} \) is a column sufficient matrix if for all \( x \in \mathbb{R}^n \)

\[
X(Mx) \leq 0 \text{ implies } X(Mx) = 0,
\]

and it is row sufficient if \( M^T \) is column sufficient. The matrix \( M \) is sufficient if it is both row and column sufficient.

Kojima et al. [9] proved that any \( \mathcal{P}_* \) matrix is column sufficient and Guu and Cottle [7] proved that it is row sufficient, too. Therefore, each \( \mathcal{P}_* \) matrix is sufficient. Váliaho proved the other direction of inclusion [11], thus the class of \( \mathcal{P}_*- \)matrices is same as the class of sufficient matrices.

Fukuda and Terlaky [6] proved a fundamental theorem for quadratic programming in oriented matroids. As they stated in their paper, the \( LCP \) duality theorem follows from that theorem for sufficient matrix \( LCP \)s.

**Theorem 2.4.** Let a sufficient matrix \( M \in \mathbb{Q}^{n \times n} \) and a vector \( q \in \mathbb{Q}^n \) be given. Then exactly one of the following statements hold:

1. \( (PLCP) \) has a solution \((x, s)\) whose encoding size is polynomially bounded.
2. \( (DLCP) \) has a solution \((u, v)\) whose encoding size is polynomially bounded.

A direct and constructive proof of the LCP duality theorem can be found in [4].

The concept of EP (existentially polynomial-time) theorems was introduced by Cameron and Edmonds [1]. It is a theorem of the form:

\[
[\forall x : F_1(x), F_2(x), \ldots, F_k(x)],
\]

where \( F_i(x) \) is a predicate formula which has the form

\[
F_i(x) = [\exists y_i \text{ such that } y_i \leq \|x\|^n \text{ and } f_i(x, y_i)].
\]

Here \( n_i \in \mathbb{Z}^+ \), \( \|z\| \) denotes the encoding length of \( z \) and \( f_i(x, y_i) \) is a predicate formula for which there is a polynomial size certificate.

The \( LCP \) duality theorem in EP form was given by Fukuda, Namiki and Tamura [5]:

**Theorem 2.5.** Let a matrix \( M \in \mathbb{Q}^{n \times n} \) and a vector \( q \in \mathbb{Q}^n \) be given. Then at least one of the following statements hold:

1. **problem LCP** has a complementary feasible solution \((x, s)\), whose encoding size is polynomially bounded.
2. **problem DLCP** has a complementary feasible solution \((u, z)\), whose encoding size is polynomially bounded.
3. matrix \( M \) is not sufficient and there is a certificate whose encoding size is polynomially bounded.

## 3 The main result

In this section we show that if the matrix is row sufficient then all feasible solutions of \( DLCP \) are complementary as well. Based on this result we get an EP theorem for problem \( DLCP \).

**Lemma 3.1.** Let the matrix \( M \) be row sufficient. If \((u, z) \in \mathcal{F}_D\), then \((u, z)\) is a solution of \( DLCP \).

**Proof:** The vector \((u, z)\) is a feasible solution of \( DLCP \), therefore \( u, z \geq 0 \) and \( u = -M^T z \), so the complementarity gap is nonnegative

\[
0 \leq u z = -z M^T z = -Z M^T z.
\]

\( Z \) being a vector of the form \( Z = \sum_{i=1}^n y_i e_i \) where \( e_i \) denotes the i-th standard vector. Then

\[
Z M^T z = 0,
\]

thus \( u z = 0 \).

**Corollary 3.2.** Let the matrix \( M \) be row sufficient. Then problem \( DLCP \) can be solved in polynomial time.
**Proof:** By Lemma 3.1, if \( M \) be row sufficient one need only to solve the feasibility problem of \( DLCP \), that is a linear feasibility problem what can be solved in polynomial time e.g., by interior point methods [10].

We must to note that there is no known polynomial time algorithm for checking whether a matrix is row sufficient or not. The following theorem presents what can be proved about an LCP problem with arbitrary matrix using a polynomial time algorithm.

**Theorem 3.3.** Let matrix \( M \in \mathbb{Q}^{n \times n} \) and vector \( q \in \mathbb{Q}^n \) be given. Then it can be shown in polynomial time that at least one of the following statements hold:

1. Problem \( DLCP \) has a complementary feasible solution \((u, z)\), whose encoding size is polynomially bounded.

2. Problem \( LCP \) has a feasible solution, whose encoding size is polynomially bounded.

3. Matrix \( M \) is not row sufficient and there is a certificate whose encoding size is polynomially bounded.

**Proof:** Apply a polynomial time algorithm to solve the feasibility problem of \( DLCP \), i.e., to find a point in the set \( F_D \). This is a linear feasibility problem, thus it can be solved in polynomial time with e.g., interior point methods using the self-dual embedding technique (see [10]). If \( F_D = \emptyset \), then by the Farkas Lemma \( F_P \neq \emptyset \), and a primal feasible point can be read out from the solution of the embedding problem, thus we get the second case. Otherwise, we get a point \((u, z) \in F_D\). If the complementarity condition, \( uz = 0 \) holds too, then the point \((u, z)\) is a solution of \( DLCP \), so get the first case. Finally if the feasible solution \((u, z)\) is not complementary, then according to Lemma 3.1 vector \( z \) provides a certificate that matrix \( M \) is not a row sufficient matrix. As the encoding size of the solution of the self-dual embedding problem, after a proper rounding procedure, is polynomially bounded, the third option holds in this case.

Observe that Theorem 3.3 is in EP form. Both Theorems 2.5 and 3.3 deal with problem \( LCP \), but Theorem 2.5 approaches the problem from the primal, while Theorem 3.3 from the dual side. The advantages of Theorem 3.3 is to determine certificates in polynomial time. The proof of Theorem 2.5 is constructive also, it is based on criss-cross algorithm (for details see [4, 5]). The \( LCP \) duality theorem gives in first two case not only feasible, but complementer solution. We deal with the second case of Theorem 3.3 in paper [8], where we present modified interior point methods, which either solve the \( LCP \) with arbitrary matrix, or provide a certificate in polynomial time, that the matrix of the problem is not sufficient.

At the end let us note that using this technique the following result can be obtained as well.

**Theorem 3.4.** Let the matrix \( M \) be column sufficient and \( q = 0 \), then every feasible solution of \( (LCP) \) is complementarity, too. Furthermore, if there is a nontrivial solution, it can be presented in polynomial time.

**Proof:** Similarly to the proof of Lemma 3.1, we can show the first statement, namely that the set of feasible solutions is the same as the set of solutions. If there is a nontrivial solution, then there is a nontrivial feasible solution, namely \( x \neq 0 \). Arbitrary nontrivial feasible solution satisfies the inequality \( e^T x > 0 \). In this case, the set of feasible solutions form a cone, so there is such a feasible solution, which satisfies the equation \( e^T x = 1 \). Therefore solving (in polynomial time) the linear programming problem

\[
(x, s) \in F_P, \quad e^T x = 1,
\]

we get a nontrivial feasible solution, which is a nontrivial solution of the \( (LCP) \) too.

**References**


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