Large-neighbourhood infeasible predictor-corrector algorithm for P-horizontal linear complementarity problems over Cartesian product of symmetric cones

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Abstract

We present an infeasible interior-point predictor-corrector algorithm, based on a large neighborhood of the central path, for the general P-horizontal linear complementarity problem over the Cartesian product of symmetric cones. The polynomial convergence is shown for the commutative class of search directions. We specialize our algorithm further by prescribing some scaling elements and also consider the case of feasible starting points. We believe this to be the first interior-point method based on large neighborhoods for the P-horizontal linear complementarity problems over the Cartesian product of symmetric cones.

Keywords: horizontal linear complementarity problem, infeasible interior-point method, large neighborhood, Euclidean Jordan algebra, Cartesian product of symmetric cones.

1 Introduction

In recent years, a lot of attention has been given to the optimization problems over symmetric cones [1–11] in the area of interior-point methods (IPMs). One may distinguish between IPMs, according to whether they are feasible or infeasible. A feasible IPM can be started only if a strictly feasible point is known. Usually such a starting point is not at hand. In that case, an infeasible IPM (IIPM) is recommended. Zhang [12, 13] established the convergence
of an infeasible-interior-point algorithm for horizontal linear complementarity problem (HLCP) and extended it to semidefinite programming (SDP). Potra and Sheng [14,15] presented IIPMs for solving P-linear complementarity problem (LCP). Rangarajan and Todd [16] proved convergence of an IIPM for self-scaled cones using the Nesterov-Todd (NT) directions for a large neighborhood of the central path. Later, Rangarajan [17] established polynomial-time convergence of IIPMs for symmetric cone programming (SCP) using the so-called negative infinity large neighborhood of the central path. The convergence has been shown for a commutative family of search directions used in Schmieta and Alizadeh [5].

Some of the above mentioned algorithms are based on the large neighborhoods, with the purpose to reduce the gap between the theory and implementation of IPMs. In fact, many primary IPMs are based on the small neighborhoods or the negative infinity large neighborhood. Recently, Ai and Zhang [18] applied an interesting idea and decomposed the classical Newton search directions to the nonnegative and nonpositive parts corresponding to the nonnegative and nonpositive parts of the right-hand side. Then they took along with these two directions two different and appropriate largest step sizes that maintain the iterates within a new neighborhood larger than the negative infinity large neighborhood. They showed that their method has the best iteration bound, the same as the small-neighborhood-based methods to solve monotone LCPs. Li and Terlaky [19] extended the Ai-Zhang directions to the class of SDPs. Their work was the first large neighborhood path-following IPM possessing the same complexity as small neighborhood one for SDPs. Liu et al. [20] also extended the Ai-Zhang’s large neighborhood to symmetric cones and obtained their new large neighborhood. Using the proposed large neighborhood, they designed a new IIPM over symmetric cones and showed the convergence for the commutative class of search directions used in [5]. The complexity of their large neighborhood IIPM has the same theoretical complexity bound as the best short step path-following IPMs. Based on the proposed large neighborhood of [20], Yang et al. [21] presented a variant of the Mizuno-Todd-Ye predictor-corrector IIPM [22] for SCP. In the corrector step, they considered a special parameterized form of the Newton system, which ensures that the duality gap and the infeasibility remain the same as in the predictor step. They proposed a step size in order to guarantee that the new iterate would be in the corresponding neighborhood. Finally, they achieved the iteration complexity for a commutative class of directions that match the best complexity results obtained for the solution of SCP.

P-HLCP over the Cartesian product of symmetric cones (the Cartesian P-
SCHLCP) is a more general problem than the ones mentioned above, and is also the one that has received less attention. Therefore, dealing with it, can obviate the need for addressing other issues. Recently, in [23], we have extended the general framework of this problem and proved the existence and uniqueness of its central path. Motivated by the above mentioned works, especially the ones presented in [20] and [21], we present here a large-neighborhood predictor-corrector IIPM for the Cartesian P-SCHLCP. Since SCHLCP is a generalization of conic linear programming (LP) problems, we lose the orthogonality of the search directions, which leads to a more complicated analysis. We analyze our algorithm for the commutative class of search directions used by Schmieta and Alizadeh [5] and then derive the iteration bounds for feasible and infeasible starting points. Notably, this is the first IPM based on the large neighborhoods for the general Cartesian P-SCHLCP problem.

The structure of the remainder of our work is as follows. In Section 2, we briefly review some concepts on the theory of Euclidean Jordan algebras (EJAs) and their associated symmetric cones. In Section 3, we introduce the Cartesian P-SCHLCP and the concept of its central path. Then, we give the systems defining the nonnegative and nonpositive search directions and define the concordant large neighborhood. In Section 4, we analyze the corrector step of the algorithm and find a suitable step length so that its requirements be fulfilled. Section 5 deals with the predictor step, analyzing it with a convenient selected step size. Then, in Section 6, based on the given analysis in sections 4 and 5, we compute the iteration complexity of the algorithm for the commutative class of search directions and specialize it with some well-known directions. In order to cover the feasible-start situations, we give a summarized analysis of the algorithm for the feasible starting points and gain the corresponding iteration bounds. Finally, conclusions are given in Section 7.

2 EJAs and their Associated Symmetric Cones

We assume that the reader has some familiarity with EJAs. For an exposition of the definitions and main results of this algebra, we refer the reader to the book by Faraut and Korányi [24].

Let $\nu \in \{1, \ldots, m\}$. Throughout our work, we assume that $(J_\nu, \circ, \langle \cdot, \cdot \rangle)$, shortly denoted by $J_\nu$, is a (real) EJA of rank $r_\nu$ and unit element $e^{(\nu)} \in J_\nu$. For $x, y \in J_\nu$, the symbol $\langle x, y \rangle$ denotes the inner product, and $x \circ y$ stands for the Jordan product. We use the notation $x^2 := x \circ x$, and we denote the corresponding symmetric cone as $K_\nu := \{x^2 : x \in J_\nu\}$. For simplicity, in the
rest of this section we drop the index $\nu$.

An element $c \in \mathcal{J}$ is an idempotent if $c^2 = c$. An idempotent is primitive if it is nonzero and cannot be written as a sum of two nonzero idempotents. A Jordan frame is a collection $\{c_1, \cdots, c_r\}$ of primitive idempotents satisfying $\sum_{i=1}^r c_i = e$ and $c_i \circ c_j = 0$ when $i \neq j$. Every element $x \in \mathcal{J}$ admits a spectral decomposition (see [24, Theorem III.1.2]); that is, there exists a Jordan frame $\{c_1, \cdots, c_r\}$ and real numbers $\lambda_1, \cdots, \lambda_r$ such that $x = \sum_{i=1}^r \lambda_i c_i$. The $\lambda_i$ values are uniquely determined by $x$ and are called the eigenvalues of $x$.

Let $x \in \mathcal{J}$ with spectral decomposition $x = \sum_{i=1}^r \lambda_i c_i$. We define the trace of $x$ by $\text{tr}(x) := \sum_{i=1}^r \lambda_i$ and the determinant of $x$ by $\det(x) := \prod_{i=1}^r \lambda_i$. The inner product $\langle x, y \rangle = \text{tr}(x \circ y)$ in $\mathcal{E} \mathcal{J} \mathcal{A} \mathcal{s}$ induces a norm as $\|x\|_F = \langle x, x \rangle^{1/2}$ which is well-known as the Frobenius norm. The element $x$ is invertible if no eigenvalue of $x$ is equal to zero, in which case one defines the inverse of $x$ as $x^{-1} := \sum_{i=1}^r \lambda_i^{-1} c_i$.

Two elements $a, b \in \mathcal{J}$ operator commute if $a \circ (b \circ z) = b \circ (a \circ z)$ for all $z \in \mathcal{J}$. From [5, Theorem 27], the operator commutation property is equivalent to the existence of a Jordan frame $\{c_1, \cdots, c_r\}$ and real numbers $\lambda_1, \cdots, \lambda_r$ and $\mu_1, \cdots, \mu_r$ such that $a = \sum_{i=1}^r \lambda_i c_i$ and $b = \sum_{i=1}^r \mu_i c_i$.

For every $y \in \mathcal{J}$, $L_x y := x \circ y$ is the Lyapunov transformation of the element $x \in \mathcal{J}$, and $Q_x := 2L_x^2 - L_{x^2}$ is called the quadratic representation of the element $x$. For $x \in \mathcal{K}$, $x^{1/2}$ is well-defined and $Q_{x^{1/2}} = Q_x^{1/2}$. Also, for an invertible $p \in \mathcal{J}$, it holds that $Q_{p^{-1}} = Q_p^{-1}$. From [24, Proposition III.2.2] we know that if $x$ is invertible, then $Q_x \mathcal{K}(\text{int} \mathcal{K}) = \mathcal{K}(\text{int} \mathcal{K})$, where int $\mathcal{K}$ represents the interior of the cone $\mathcal{K}$.

Throughout our work here, $\lambda_{\text{max}}(x)$ and $\lambda_{\text{min}}(x)$ show the largest and smallest eigenvalues of $x$, respectively. For any $\lambda \in \mathbb{R}$, $\lambda^+$ denotes its non-negative part, i.e., $\lambda^+ := \text{max}\{\lambda, 0\}$, and $\lambda^-$ the nonpositive part, i.e., $\lambda^- := \text{min}\{\lambda, 0\}$. Accordingly, if $x := \sum_{i=1}^r \lambda_i c_i$, then $x^+ = \sum_{i=1}^r \lambda_i^+ c_i$ and $x^- = \sum_{i=1}^r \lambda_i^- c_i$. The sign $\sim$ denotes similarity of two elements in $\mathcal{J}$.

### 3 The Cartesian $P_*(\kappa)$-SCHLCP and the Algorithm

Let $\mathcal{J} := \mathcal{J}_1 \times \mathcal{J}_2 \times \cdots \times \mathcal{J}_m$ be the Cartesian product space with its cone of squares $\mathcal{K} := \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_m$. For any $x \in \mathcal{J}$, we use the notation $x = (x^{(1)}, x^{(2)}, \cdots, x^{(m)})^T$, where $x^{(\nu)} \in \mathcal{J}_\nu, \nu = 1 \cdots m$. Let $\mathcal{M}, \mathcal{N} : \mathcal{J} \rightarrow \mathcal{J}$ be linear operators and $b \in \mathcal{J}$. The Cartesian SCHLCP is to find $x, s \in \mathcal{J}$...
such that
\[ Mx + Ns = b, \quad x \circ s = 0, \quad x \in \mathcal{K}, \quad s \in \mathcal{K}. \]  

(1)

This general problem is fully handled in [23]. Note that the well-known linear complementarity problem over the Cartesian product of symmetric cones (Cartesian SCLCP) is a special case of (1), when \( \mathcal{N} \) is restricted to the negative identity operator, \(-\mathcal{I}\). Due to lack of such restriction, the Cartesian SCHLCP is a slightly more general problem than the Cartesian SCLCP. It should be noted that for an invertible \( \mathcal{N} \), (1) is simply rewritable to a Cartesian SCLCP. Otherwise, as studied in [23], the Cartesian SCHLCP and the Cartesian SCLCP are equivalent in a certain sense.

**Example 3.1.** Consider the case \( m = 3 \) and assume that \( \mathcal{J}_1 = S^2 \) is the Jordan algebra of real symmetric \( 2 \times 2 \) matrices with its cone of squares \( S^2_+ \). Moreover, let \( \mathcal{J}_2 = L^2 \) be the Jordan algebra associated with the second-order cone \( L^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 \geq x_2^2, x_1 \geq 0\} \). Finally, let \( \mathcal{J}_3 = \mathbb{R}^2 \) be the Jordan algebra with the non-negative orthant \( \mathbb{R}^2_+ \) as its cone of squares. Now, any \( x \in \mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2 \times \mathcal{J}_3 \) can be written as \( x = (x^{(1)}, x^{(2)}, x^{(3)})^T \). Moreover, for \( x, s \in \mathcal{J} \) we have \( x \circ s = (x^{(1)} \circ s^{(1)}, x^{(2)} \circ s^{(2)}, x^{(3)} \circ s^{(3)})^T \), where
\[
x^{(1)} \circ s^{(1)} = \frac{1}{2} \left( x^{(1)} s^{(1)} + s^{(1)} x^{(1)} \right), \quad x^{(3)} \circ s^{(3)} = \left[ x^{(3)}_1 s^{(3)}_1, x^{(3)}_2 s^{(3)}_2 \right]^T, \\
x^{(2)} \circ s^{(2)} = \left[ (x^{(2)})^T s^{(2)}, x^{(2)}_1 s^{(2)}_1 + s^{(2)}_1 x^{(2)}_2 \right]^T.
\]

We mention that \( x^{(1)} s^{(1)} \) means the usual matrix product. In the remaining part of the example we will use the vectorization operator \( \text{sv} : S^2 \to \mathbb{R}^3 \), which can be defined for all \( \xi \in S^2 \) as \( \text{sv}(\xi) = [\xi_{1,1}, \sqrt{2} \xi_{1,2}, \xi_{2,2}]^T \). Moreover, we introduce the notation \( \text{sv}(x) = \begin{bmatrix} \text{sv}(x^{(1)})^T, (x^{(2)})^T, (x^{(3)})^T \end{bmatrix}^T \) for each \( x \in \mathcal{J} \). Now, suppose that \( M \) and \( N \) are \( 7 \times 7 \) real matrices, \( \text{smat} \) is the inverse of the operator \( \text{sv} \) and
\[
Mx = \left( \text{smat} \left( (M\text{sv}(x))^{(1)} \right), (M\text{sv}(x))^{(2)}, (M\text{sv}(x))^{(3)} \right)^T, \\
Ns = \left( \text{smat} \left( (N\text{sv}(s))^{(1)} \right), (N\text{sv}(s))^{(2)}, (N\text{sv}(s))^{(3)} \right)^T.
\]

(4) 

(5)

We consider problem (1) with the “\( \circ \)” operation given by (2)-(3), \( Mx \) and \( Ns \) defined as in (4)-(5) and \( \mathcal{K} = S^2_+ \times L^2_+ \times \mathbb{R}^2_+ \). Observe that the problem cannot
be solved by an interior-point algorithm which considers only an HLCP or a second-order cone LCP or a semidefinite LCP. Therefore, the more general Cartesian SCHLCP must be analyzed.

For other examples related to the Cartesian SCHLCP, the reader is referred to [23].

We call the Cartesian SCHLCP as the Cartesian $P^*_\kappa$-SCHLCP, if the linear transformations $\mathcal{M}$ and $\mathcal{N}$ have the Cartesian $P^*_\kappa$ property, i.e.,

$$\mathcal{M}u + \mathcal{N}v = 0 \implies \langle u, v \rangle \geq -4\kappa \sum_{\nu \in I_+} \langle u^{(\nu)}, v^{(\nu)} \rangle,$$

for all $u, v \in \mathcal{J}$, where $\kappa \geq 0$ is a given constant and $I_+ = \{ \nu : \langle u^{(\nu)}, v^{(\nu)} \rangle > 0 \}$.

### 3.1 The central path for the Cartesian $P^*_\kappa$-SCHLCP

To solve the Cartesian $P^*_\kappa$-SCHLCP, IPMs perturb the right hand side of the second equality in (1) and obtain the following system with cone constraints

$$\mathcal{M}x + \mathcal{N}s = b, \quad x \circ s = \sigma \mu e, \quad x \in \mathcal{K}, \ s \in \mathcal{K},$$

where $\mu > 0$ is the gap measurement and $\sigma \in (0, 1)$ is a centralizing parameter. To ensure the existence and uniqueness of the solution for the above system, as described in [23], we scale it as follows. Let $\mathcal{C}(x, s)$ be the set of all invertible elements such that their scaled elements operator commute, i.e.,

$$\mathcal{C}(x, s) = \{ p \mid p \text{ is invertible, and } L_{Q_p}^{-1} p s = L_{Q_p}^{-1} L_{Q_p}^{-1} p s \}.$$

This is the so-called commutative class of search directions which were designed by Monteiro and Zhang [25] for SDP and later extended word-for-word to optimization problems over all symmetric cones by Schmieta and Alizadeh [5]. Choosing $p \in \mathcal{C}(x, s)$ and denoting $\tilde{x} = Q_p x$, $\tilde{s} = Q_p^{-1} s$, $\tilde{\mathcal{M}} = \mathcal{M} Q_p^{-1}$ and $\tilde{\mathcal{N}} = \mathcal{N} Q_p$, using the well-known Lemma 28 in [5], we can rewrite the system (6) equivalently as follows:

$$\tilde{\mathcal{M}} x + \tilde{\mathcal{N}} s = b, \quad \tilde{x} \circ \tilde{s} = \sigma \mu e, \quad \tilde{x} \in \mathcal{K}, \ \tilde{s} \in \mathcal{K}.$$

We assume that (1) satisfies the interior-point condition (IPC), i.e., it has feasible solutions in $\text{int} \mathcal{K}$; this assumption implies that the perturbed system (8) has a unique solution for each $\mu > 0$ [23]. This solution, denoted by $(\tilde{x}(\mu), \tilde{s}(\mu))$, is called the $\mu$-center of the Cartesian $P^*_\kappa$-SCHLCP.
3.2 Determining the search directions and the large neighborhood

To obtain the search directions for the Cartesian $P_*(\kappa)$-SCHLCP, the usual approach is to use Newton’s method and to linearize the system (8), with $\mu$ fixed. Let $(\tilde{x}, \tilde{s}) \in \text{int } K \times \text{int } K$. We want to find displacements $\tilde{\Delta}x$ and $\tilde{\Delta}s$ such that

$$\begin{align*}
\tilde{M} \tilde{\Delta}x + \tilde{N} \tilde{\Delta}s &= b - \tilde{M}\tilde{x} - \tilde{N}\tilde{s}, \\
\tilde{s} \circ \tilde{\Delta}x + \tilde{x} \circ \tilde{\Delta}s &= \sigma \mu e - \tilde{x} \circ \tilde{s}.
\end{align*}$$

(9)

Here, we decompose the usual Newton search directions $\tilde{\Delta}x$ and $\tilde{\Delta}s$ to non-negative and nonpositive parts and consider the following two systems:

$$\begin{align*}
\tilde{M} \tilde{\Delta}x_+ + \tilde{N} \tilde{\Delta}s_- &= b - \tilde{M}\tilde{x} - \tilde{N}\tilde{s}, \\
L\tilde{s} \circ \tilde{\Delta}x_+ + L\tilde{x} \circ \tilde{\Delta}s_- &= (\sigma \mu e - \tilde{x} \circ \tilde{s})^-,
\end{align*}$$

(10)

and

$$\begin{align*}
\tilde{M} \tilde{\Delta}x_- + \tilde{N} \tilde{\Delta}s_+ &= 0, \\
L\tilde{s} \circ \tilde{\Delta}x_- + L\tilde{x} \circ \tilde{\Delta}s_+ &= (\sigma \mu e - \tilde{x} \circ \tilde{s})^+.
\end{align*}$$

(11)

Like the Newton-systems investigated by Ai and Zhang [18] for LCPs, the nonpositive part $(\sigma \mu e - \tilde{x} \circ \tilde{s})^-$ is responsible for reducing the duality gap, and the nonnegative part $(\sigma \mu e - \tilde{x} \circ \tilde{s})^+$ is used to control the centrality.

The large neighborhood of the central path introduced by Ai and Zhang was generalized for SCP in [20]. Now, we extend this concept to the Cartesian $P_*(\kappa)$-SCHLCPS. Let $N(\tau, \beta) := \{ (x, s) \in \text{int } K \times \text{int } K : \| (\tau \mu e - w)_+ \|_F \leq \beta \tau \mu \}$, where $\tau, \beta \in (0, 1)$ and $w = Q_{x^{1/2}}s$. This neighborhood is a large neighborhood and like the evidence exhibited in [20] one can demonstrate that it is larger than the classical negative infinity large neighborhood.

Let $(x, s)$ satisfy the system (6). We get the new iterate $(x(\alpha), s(\alpha))$ as follows:

$$(x(\alpha), s(\alpha)) = (x, s) + \alpha_1(\Delta x_-, \Delta s_-) + \alpha_2(\Delta x_+, \Delta s_+),$$

(12)

where $\alpha_1, \alpha_2 \in (0, 1]$ are suitable step sizes and

$$\begin{align*}
\Delta x_- &= Q_{p^{-1}}\tilde{\Delta}x_-, \\
\Delta s_- &= Q_p\tilde{\Delta}s_-, \\
\Delta x_+ &= Q_{p^{-1}}\tilde{\Delta}x_+, \\
\Delta s_+ &= Q_p\tilde{\Delta}s_+.
\end{align*}$$

(13)

(14)

The corresponding scaled new iterate is similarly defined as follows:

$$(\tilde{x}(\alpha), \tilde{s}(\alpha)) = (\tilde{x}, \tilde{s}) + \alpha_1(\tilde{\Delta}x_-, \tilde{\Delta}s_-) + \alpha_2(\tilde{\Delta}x_+, \tilde{\Delta}s_+).$$

(15)
We also define
\[
\tilde{\Delta}x(\alpha) := \alpha_1 \Delta x_+ + \alpha_2 \Delta x_-, \quad \tilde{\Delta}s(\alpha) := \alpha_1 \Delta s_+ + \alpha_2 \Delta s_-
\]
\[\alpha := (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R}.
\] (16)

Now, the general framework of our infeasible large-neighborhood predictor-corrector algorithm is given in Figure 1.

**Infeasible Large-Neighborhood Predictor-Corrector Algorithm**

**Input parameters:** required precision \(\varepsilon > 0\), neighborhood parameters \(0 < \beta \leq \frac{1}{2}\) and \(0 < \tau \leq \frac{1}{8}\), and \((x^0, s^0) \in N(\tau, \beta)\);

**Output:** a sequence of iterates \((x^k, s^k) : k = 0, 1, 2, \ldots\);

**Step 0.** Set \(k = 0\) and \(\phi^0 := 1\);

**Step 1.** If \(\langle x^k, s^k \rangle \leq \varepsilon\) then stop. Else choose \(p \in C(x^k, s^k)\) and go to Step 2;

**Step 2.** (Predictor step). Set \(\sigma = 0\). Obtain the Newton direction \((\tilde{\Delta}x_-, \tilde{\Delta}s_-)\) from (10). Calculate \((\Delta x_-^k, \Delta s_-^k)\) via (13). Find the largest step size \(0 < \alpha^k := \alpha^k_1 \leq 1\) such that
\[
\langle x(\alpha^k_1), s(\alpha^k_1) \rangle \in N(\tau, 2\beta),
\]
\[
\langle x(\alpha^k_1), s(\alpha^k_1) \rangle \leq \langle x^k, s^k \rangle \left(1 - \frac{2}{5} \alpha^k_1\right).
\]
Set \((x^k, s^k) := (x(\alpha^k_1), s(\alpha^k_1))\) and go to step 3;

**Step 3.** (Corrector step). Set \(\sigma = \tau\). Obtain the Newton directions \((\tilde{\Delta}x_-, \tilde{\Delta}s_-)\) and \((\tilde{\Delta}x_+, \tilde{\Delta}s_+)\) from (10) and (11), respectively. Calculate \((\Delta x_-, \Delta s_-)\) and \((\Delta x_+, \Delta s_+)\) via (13) and (14), respectively. Find the largest step size \(0 < \alpha^k := (\alpha^k_1, \alpha^k_2) \leq (1, 1)\) such that
\[
\langle x(\alpha^k), s(\alpha^k) \rangle \in N(\tau, \beta),
\]
\[
\langle x(\alpha^k), s(\alpha^k) \rangle \geq (1 - \alpha^k_1) \phi^k \langle x^0, s^0 \rangle,
\]
\[
\langle x(\alpha^k), s(\alpha^k) \rangle \leq \langle x^k, s^k \rangle \left(1 - \frac{1}{3} \alpha^k_1\right).
\]
Set \((x^{k+1}, s^{k+1}) := (x(\alpha^k), s(\alpha^k))\) and \(\phi^{k+1} := (1 - \alpha^k_1) \phi^k\);

**Step 4.** Set \(k := k + 1\) and go to step 1;

Figure 1: The Algorithm
Remark 3.1. (i) Let $\bar{w}$ be the $w$-point corresponding to $\bar{x}$ and $\bar{s}$, i.e., $\bar{w} = Q_{\bar{x}1/2}\bar{s}$. Similar to the Proposition 21 of [5] we can prove that $\bar{w} \sim w$. Therefore, $(x, s) \in N(\tau, \beta)$ if and only if $(\bar{x}, \bar{s}) \in N(\tau, \beta)$, i.e., the neighborhood $N(\tau, \beta)$ is scaling invariant. Also, note that since $L_{\bar{x}}L_{\bar{s}} = L_sL_{\bar{x}}$, $\bar{x}$ and $\bar{s}$ share a common Jordan frame. Since $\bar{x}1/2$ is a polynomial in $\bar{x}$, it has a common Jordan frame with $\bar{x}$ and then $L_{\bar{x}1/2}L_{\bar{s}} = L_sL_{\bar{x}1/2}$. Therefore, we have $\bar{w} = \bar{x} \circ \bar{s}$. But, in general, $\bar{w}(\alpha) \neq \bar{x}(\alpha) \circ \bar{s}(\alpha)$, where $\bar{w}(\alpha)$ is the $w$ point corresponding to $\bar{x}(\alpha)$ and $\bar{s}(\alpha)$.

(ii) Note that $w \in \text{int } K$; then, using (i), it follows that $\bar{x} \circ \bar{s} \in \text{int } K$. Due to the fact that in the predictor step $\sigma = 0$, we have

$$\Delta x_+ \circ \bar{s} + \bar{x} \circ \Delta s_+ = (-\bar{x} \circ \bar{s})^+ = 0.$$ 

Consequently, the Newton constituent direction with respect to the nonnegative part is simply zero.

(iii) Note that the last requirements for the step lengths of the corrector and predictor steps in the algorithm ensure the reduction of the gap measurement $\mu$ in the concordant step. The second requirement of the corrector step, provides the reduction of the infeasibility of the iterates during each main iteration of the algorithm.

Let $(x^k, s^k)$ be obtained by the algorithm. We can easily deduce that

$$b - Mx^k - Ns^k = \phi^k (b - Mx^0 - Ns^0),$$

where $\phi^k = \prod_{i=0}^{k-1} (1 - \alpha_i^k)$. From the above equation, we have $\phi^k = \frac{\|b - Mx^k - Ns^k\|_F}{\|b - Mx^0 - Ns^0\|_F}$. Hence $\phi^k$ shows the relative infeasibility of $(x^k, s^k)$. Therefore, we maintain the condition

$$\langle x^k, s^k \rangle \geq \phi^k \langle x^0, s^0 \rangle,$$

in each iteration of the algorithm, which establishes that infeasibility approaches zero as complementarity goes to zero.

In the following, we introduce a notation to facilitate the analysis of the algorithm. Consider a reference point $(u^0, v^0)$ which satisfies the equality constraints defining the Cartesian $P_*(\kappa)$-SCHLCP, but not necessarily contained in the cone $K$, such that for the starting point $(x^0, s^0) \in \text{int } K \times \text{int } K$ we have $x^0 - u^0, s^0 - v^0 \in \text{int } K$. Now, we use the method of defining a "slack sequence", as proposed by Rangarajan [17] and later on applied in [21]. Let

$$u^{k+1} := x^{k+1} + (1 - \alpha_1^k)(x^k - x^k), \quad v^{k+1} := s^{k+1} + (1 - \alpha_1^k)(v^k - s^k).$$
By the above definitions, we easily find that
\[ x^{k+1} - u^{k+1} = \phi^{k+1}(x^0 - u^0) \in \text{int } \mathcal{K}, \]
\[ s^{k+1} - v^{k+1} = \phi^{k+1}(s^0 - v^0) \in \text{int } \mathcal{K}, \]
\[ \mathcal{M}u^k + \mathcal{N}v^k = b, \text{ for all } k. \]

Let \((u^0, v^0)\) be the solution of the following minimization problem:
\[ \min \left\{ \|(u, v)\|_F : \mathcal{M}u + \mathcal{N}v = b \right\}. \]

As usual, we choose the following starting point \((x^0, s^0)\):
\[ x^0 = s^0 = \rho^0 e; \quad \rho^0 > \max \left\{ |\lambda_{\max}(u^0)|, |\lambda_{\max}(v^0)| \right\}. \]

This implies that \(x^0 - u^0 \in \text{int } \mathcal{K}\) and \(s^0 - v^0 \in \text{int } \mathcal{K}\). We also define
\[ \rho^* := \max \left\{ \lambda_{\max}(x^*), \lambda_{\max}(s^*) : (x^*, s^*) \text{ solves } (1) \right\}. \]

Noticing that \(\rho^0\) may grow arbitrarily, we assume additionally that for a constant \(\Psi > 0\), the following holds:
\[ \frac{\rho^*}{\rho^0} \leq \Psi. \]

For simplicity, we will often write \(x, s, \phi\) for \(x^k, s^k, \phi^k\), respectively. The indices should be clear from the context.

In the rest of this section, we provide two lemmas, which will be used later.

**Lemma 3.1.** Given the Newton equations, the following identity holds:
\[ \ddot{x}(\alpha) \circ \ddot{s}(\alpha) = v(\alpha) + \Delta x(\alpha) \circ \Delta s(\alpha), \]
where \(v(\alpha) := \ddot{x} \circ \ddot{s} + \alpha_1(\sigma e - \ddot{x} \circ \ddot{s})^- + \alpha_2(\sigma e - \ddot{x} \circ \ddot{s})^+\).

**Proof.** By definition of \(\ddot{x}(\alpha)\) and \(\ddot{s}(\alpha)\), using (10) and (11), the result is at hand. \(\square\)

**Lemma 3.2.** One has \(v(\alpha) \succeq_K 0\).

**Proof.** Let \(\sigma = \tau\) and \(\lambda_i := \lambda_i(\ddot{x} \circ \ddot{s})\). Assume additionally that for some \(0 \leq k \leq r\), we have
\[ \lambda_i \geq \tau \mu \text{ for } i = 1, \ldots, k, \quad \text{and} \quad \lambda_i < \tau \mu \text{ for } i = k + 1, \ldots, r. \]
One can derive that
\[
v(\alpha) = \sum_{i=1}^{r} \lambda_i c_i + \alpha_1 \sum_{i=1}^{k} (\tau \mu - \lambda_i) c_i + \alpha_2 \sum_{i=k+1}^{r} (\tau \mu - \lambda_i) c_i
\]
\[
= \sum_{i=1}^{k} (\alpha_1 \tau \mu + (1 - \alpha_1) \lambda_i) c_i + \sum_{i=k+1}^{r} (\alpha_2 \tau \mu + (1 - \alpha_2) \lambda_i) c_i,
\]
which proves the lemma for the case \(\sigma = \tau\). For \(\sigma = 0\), we have
\[
v(\alpha) = \tilde{x} \circ \tilde{s} + \alpha_1 (-\tilde{x} \circ \tilde{s})^- = (1 - \alpha_1) \tilde{x} \circ \tilde{s} \succeq_{\mathcal{K}} 0.
\]
and the proof is completed. \(\square\)

4 Analysis of the Corrector Step

This section deals with the corrector step and gives an analysis so that its three requirements are ensured with some convenient selected step sizes. In this section we assume that \(\sigma = \tau\).

In the sequel, we let \(G := L^{-1}_s L_s\). It is easy to see that
\[
(L_s L_s)^{-1/2} L_s = G^{1/2} \quad \text{and} \quad (L_s L_s)^{-1/2} L_s = G^{-1/2}.
\]

**Lemma 4.1.** (cf. Lemma 3.2 in [26]) Let the Cartesian SCHLCP be \(P_*(\kappa)\), \(a \in \mathcal{J}\) and \((\tilde{f}, \tilde{g}) \in \mathcal{J} \times \mathcal{J}\) be the unique solution of the following system:
\[
\mathcal{M} \tilde{f} + \mathcal{N} \tilde{g} = 0, \quad L_s \tilde{f} + L_s \tilde{g} = a. \tag{25}
\]
Then, \((\tilde{f}, \tilde{g})\) satisfies the following inequalities:
\[
\langle \tilde{f}, \tilde{g} \rangle \geq -\kappa \|\bar{a}\|_F^2, \quad \| (\tilde{f}, \tilde{g}) \|_G^2 \leq (1 + 2\kappa) \|\bar{a}\|_F^2,
\]
where \(\bar{a} := (L_s L_s)^{-1/2} a\) and \(\| (\tilde{f}, \tilde{g}) \|_G^2 := \| G^{-1/2} \tilde{f} \|_F^2 + \| G^{1/2} \tilde{g} \|_F^2\).

**Proof.** Left-multiplying the second equation in the system (25) by \((L_s L_s)^{-1/2}\), we derive \(G^{-1/2} \tilde{f} + G^{1/2} \tilde{g} = \bar{a}\). Thus, for each \(\nu = 1, \ldots, m\), we have
\[
4 \langle \tilde{f}^{(\nu)}, \tilde{g}^{(\nu)} \rangle \leq \| (G^{-1/2} \tilde{f} + G^{1/2} \tilde{g})^{(\nu)} \|_F^2 = \|\bar{a}^{(\nu)}\|_F^2.
\]

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Since $\mathcal{M}\tilde{f} + \mathcal{N}\tilde{g} = 0$, if $I_+ = \{\nu : \langle \tilde{f}^{(\nu)}, \tilde{g}^{(\nu)} \rangle > 0 \}$, by the $P_\kappa(\kappa)$ property of $(\mathcal{M}, \mathcal{N})$, we obtain

$$\langle \tilde{f}, \tilde{g} \rangle \geq -4\kappa \sum_{\nu \in I_+} \langle \tilde{f}^{(\nu)}, \tilde{g}^{(\nu)} \rangle \geq -\kappa \|\tilde{a}\|_F^2.$$  

This implies the first assertion. Using this inequality, we can write

$$\left\| (\tilde{f}, \tilde{g}) \right\|_G^2 = \left\| G^{-1/2} f \right\|_F^2 + \left\| G^{1/2} g \right\|_F^2 = \left\| G^{-1/2} \tilde{f} + G^{1/2} \tilde{g} \right\|_F^2 - 2 \langle \tilde{f}, \tilde{g} \rangle \leq \|\tilde{a}\|_F^2 + 2\kappa \|\tilde{a}\|_F^2 = (1 + 2\kappa) \|\tilde{a}\|_F^2,$$

which proves the second assertion of the lemma to complete the proof. □

**Lemma 4.2.** Suppose that $\tilde{(x, s)} \in N(\tau, \beta)$ and $\beta < \frac{1}{2}$. Then,

$$\left\| \alpha_1 (L\tilde{x}L\tilde{s})^{-1/2} (\tau \mu e - \tilde{x} \circ \tilde{s}) + \alpha_2 (L\tilde{x}L\tilde{s})^{-1/2} (\tau \mu e - \tilde{x} \circ \tilde{s})^+ \right\|_F^2 \leq \alpha_1^2 \tau \mu + \alpha_2^2 \beta \tau \mu.$$

**Proof.** Using the fact that $\tilde{(x, s)} \in N(\tau, \beta)$, we derive that $\lambda_i(\tilde{x} \circ \tilde{s}) \geq (1 - \beta)\tau \mu$, for $i = 1, \cdots, r$. Therefore, the lemma can be proved in the same way as Lemma 5.3 of [20]. □

**Lemma 4.3.** (cf. Lemma 3.3 in [26]) If $\tilde{(x, s)} \in N(\tau, \beta)$, $\beta < \frac{1}{2}$ and $\tau < \frac{1}{8}$, then

$$\left\| (\tilde{\Delta}x(\alpha), \tilde{\Delta}s(\alpha)) \right\|_G \leq \sqrt{1 + 2\kappa} \left( \alpha_1 \sqrt{\beta \tau \mu} + \alpha_2 \sqrt{\tau \mu} \right) + \left( 1 + \sqrt{2(1 + 2\kappa)} \right) \zeta,$$

where $\zeta := \min \left\{ \|\tilde{(u, v)}\|_G : \tilde{\mathcal{M}}\tilde{u} + \tilde{\mathcal{N}}\tilde{v} = \alpha_1 (b - \tilde{\mathcal{M}}\tilde{x} - \tilde{\mathcal{N}}\tilde{s}) \right\}$.

**Proof.** Let $(\tilde{u}, \tilde{v})$ be the solution of the minimization problem defining $\zeta$. We multiply both sides of the systems (10) and (11) with $\alpha_1$ and $\alpha_2$, respectively. By summing up the obtained results, we have

$$\tilde{\mathcal{M}}(\tilde{\Delta}x(\alpha) - \tilde{u}) + \tilde{\mathcal{N}}(\tilde{\Delta}s(\alpha) - \tilde{v}) = 0,$$

$$L\tilde{x}(\tilde{\Delta}x(\alpha) - \tilde{u}) + L\tilde{s}(\tilde{\Delta}s(\alpha) - \tilde{v}) = \alpha_1 (\tau \mu e - \tilde{x} \circ \tilde{s})^- + \alpha_2 (\tau \mu e - \tilde{x} \circ \tilde{s})^+$$

$$- (L\tilde{x}\tilde{v} + L\tilde{s}\tilde{u}) := h.$$  

Using Lemma 4.1, we obtain

$$\left\| (\tilde{\Delta}x(\alpha) - \tilde{u}, \tilde{\Delta}s(\alpha) - \tilde{v}) \right\|_G \leq \sqrt{1 + 2\kappa} \left\| (L\tilde{x}L\tilde{s})^{-1/2} h \right\|_F.$$

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We may write
\[
\begin{align*}
\| (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} h \|_F &= \alpha_1 (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} (\tau \mu e - \tilde{x} \circ \tilde{s})^- + \alpha_2 (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} (\tau \mu e - \tilde{x} \circ \tilde{s})^+ \\
&- (G^{-1/2} \tilde{u} + G^{1/2} \tilde{v}) \|_F \\
&\leq \alpha_1 (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} (\tau \mu e - \tilde{x} \circ \tilde{s})^- + \alpha_2 (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} (\tau \mu e - \tilde{x} \circ \tilde{s})^+ \|_F \\
&+ \| G^{-1/2} \tilde{u} \|_F + \| G^{1/2} \tilde{v} \|_F \\
&\leq \alpha_1 \sqrt{r \mu} + \alpha_2 \sqrt{\beta \tau \mu} + \sqrt{2} \| (\tilde{u}, \tilde{v}) \|_G ,
\end{align*}
\]

where the second inequality is due to Lemma 4.2 and the third one can be implied by some elementary calculations. From the above argument, we have
\[
\| (\tilde{x}(\alpha), \tilde{s}(\alpha)) \|_G = \| (\tilde{x}(\alpha) - \tilde{u} + \tilde{u}, \tilde{s}(\alpha) - \tilde{v} + \tilde{v}) \|_G \\
\leq \| (\tilde{x}(\alpha) - \tilde{u}, \tilde{s}(\alpha) - \tilde{v}) \|_G + \| (\tilde{u}, \tilde{v}) \|_G \\
\leq \sqrt{1 + 2 \kappa} \left( \alpha_1 \sqrt{r \mu} + \alpha_2 \sqrt{\beta \tau \mu} + \sqrt{2} \| (\tilde{u}, \tilde{v}) \|_G \right) + \| (\tilde{u}, \tilde{v}) \|_G \\
= \sqrt{1 + 2 \kappa} \left( \alpha_1 \sqrt{r \mu} + \alpha_2 \sqrt{\beta \tau \mu} \right) + (1 + \sqrt{2(1 + 2 \kappa)}) \| (\tilde{u}, \tilde{v}) \|_G ,
\]

which combined with the definition of \( \zeta \), implies the lemma. \( \Box \)

Here, we proceed by deriving a bound for \( \zeta \).

Lemma 4.4 (cf. Lemma 4.1 in [17]). If \( G = L_{\tilde{s}}^{-1} L_{\tilde{s}} \), then \( \lambda_{\text{max}} \left( Q_{\tilde{x}}^{-1} G \right) = \frac{1}{\lambda_{\text{min}} \left( Q_{\tilde{x}}^{1/2} \tilde{s} \right)} \). Assume, additionally, that \( q \in K \). Then, \( \left\| Q_{\tilde{x}}^{1/2} q \right\|_F \leq \langle q, \tilde{x} \rangle \).

Proof. The proof is similar to the one given as Lemma 4.1 in [17]. \( \Box \)

Lemma 4.5. (cf. Lemma 11 in [21]) Let \((\tilde{x}, \tilde{s}) \in N(\tau, \beta)\), \( \zeta \) be as defined in Lemma 4.3 and \((u, v)\) correspond to the slack sequence as defined before. Then,
\[
\zeta \leq \alpha_1 \frac{\langle s, x - u \rangle + \langle x, s - v \rangle}{\sqrt{(1 - \beta) \tau \mu}} .
\]
Proof. Defining \( \tilde{u} = Q_p u \) and \( \tilde{v} = Q_{p-1} v \) for \( p \in \mathcal{C} \), from (19), (20) and (21), we have \( \tilde{s} - \tilde{v} \in \mathcal{K}, \tilde{x} - \tilde{u} \in \mathcal{K} \) and \( \tilde{M} \tilde{u} + \tilde{N} \tilde{v} = b \). Now, let \( \hat{u} = \alpha_1 (\tilde{x} - \tilde{u}) \) and \( \hat{v} = \alpha_1 (\tilde{s} - \tilde{v}) \). Then, \( \hat{u} \in \mathcal{K} \) and \( \hat{v} \in \mathcal{K} \) and

\[
\tilde{M}(-\hat{u}) + \tilde{N}(-\hat{v}) = \alpha_1 \tilde{M}(\tilde{u} - \tilde{x}) + \alpha_1 \tilde{N}(\tilde{v} - \tilde{s})
= \alpha_1 (\tilde{M} \tilde{u} + \tilde{N} \tilde{v} - (\tilde{M} \tilde{x} + \tilde{N} \tilde{s}))
= \alpha_1 (b - \tilde{M} \tilde{x} - \tilde{N} \tilde{s}). \tag{26}
\]

Since \( \tilde{x} \) and \( \tilde{s} \) operator commute, \( G \) and \( Q_{\tilde{x}} \) operator commute and we may write

\[
\left\| G^{1/2} \hat{v} \right\|_F^2 = \langle \hat{v}, G \hat{v} \rangle = \left\langle Q_{\tilde{x}}^{1/2} \hat{v}, Q_{\tilde{x}}^{-1} G Q_{\tilde{x}}^{1/2} \hat{v} \right\rangle \leq \lambda_{\max} \left( Q_{\tilde{x}}^{-1} G \right) \left\| Q_{\tilde{x}}^{1/2} \hat{v} \right\|_F^2
\leq \frac{\langle \tilde{x}, \hat{v} \rangle^2}{\lambda_{\min}(\tilde{w})} = \alpha_1^2 \frac{\langle \tilde{x}, \tilde{s} - \tilde{v} \rangle^2}{\lambda_{\min}(\tilde{w})} = \alpha_1^2 \frac{\langle x, s - v \rangle^2}{\lambda_{\min}(w)} \leq \alpha_1^2 \frac{(1 - \beta) \tau_\mu}{(1 - \beta) \tau_\mu},
\]

where the second inequality is derived from Lemma 4.4 and the last one is obtained by the fact that \((\tilde{x}, \tilde{s}) \in N(\tau, \beta)\). In a similar way, we have

\[
\left\| G^{-1/2} \hat{u} \right\|_F^2 \leq \alpha_1^2 \frac{(s, x - u)^2}{(1 - \beta) \tau_\mu}. \tag{26}
\]

Therefore, by (26) and the definition of \( \zeta \), we conclude that

\[
\zeta \leq \left\| (-\hat{u}, -\hat{v}) \right\|_G = \left\| (\hat{u}, \hat{v}) \right\|_G \leq \sqrt{\alpha_1^2 \frac{(x, s - v)^2 + (s, x - u)^2}{(1 - \beta) \tau_\mu}}
\leq \sqrt{\alpha_1^2 \frac{(x, s - v)^2 + (s, x - u)^2 + 2 \langle x, s - v \rangle \langle s, x - u \rangle}{(1 - \beta) \tau_\mu}}
= \alpha_1 \frac{(x, s - v) + (s, x - u)}{\sqrt{(1 - \beta) \tau_\mu}}.
\]

The proof is completed. \( \Box \)

In the next lemma, we obtain an upper bound for the expression \( \langle x, s - v \rangle + \langle s, x - u \rangle \).

**Lemma 4.6.** (cf. Lemma 11 in [21]) Let \( (u, v) \) correspond to the slack sequence and \( (x^*, s^*) \) be an optimal solution of the Cartesian \( P_*(\kappa) \)-SCHLCP. Then, \( \langle x, s - v \rangle + \langle s, x - u \rangle \leq (1 + 4\kappa)(5 + 4\Psi)\tau_\mu \).

**Proof.** From (21), we have \( M u + N v = b \). On the other hand, the optimality of \( (x^*, s^*) \) implies that \( M x^* + N s^* = b \). Hence, \( M (x^* - u) + N (s^* - v) = 0 \). By the \( P_*(\kappa) \) property, we have

\[
\langle x^* - u, s^* - v \rangle \geq -4\kappa \sum_{\nu \in I_+} \langle (x^* - u)^{(\nu)}, (s^* - v)^{(\nu)} \rangle.
\]
Since \( x, s, x^*, s^*, x - u \) and \( s - v \) belong to \( \mathcal{K} \), we may write
\[
\langle x - u, s - v \rangle + \langle x^*, s - v \rangle + \langle x - u, s^* \rangle + \langle x, s \rangle - \langle x - u, s \rangle - \langle x, s - v \rangle \\
\geq -4\kappa \sum_{\nu \in I_+} \left( \langle x^* - x + (x - u) \rangle^{(\nu)}, \langle s^* - s + (s - v) \rangle^{(\nu)} \right) \\
\geq -4\kappa \sum_{\nu \in I_+} \left( \langle x - u \rangle^{(\nu)}, \langle s - v \rangle^{(\nu)} \right) \\
+ \left( \langle x^* \rangle^{(\nu)}, \langle s \rangle^{(\nu)} \right) + \left( \langle x - u \rangle^{(\nu)}, \langle s^* \rangle^{(\nu)} \right) + \left( x^{(\nu)}, s^{(\nu)} \right) \\
\geq -4\kappa \left( \langle x - u, s - v \rangle + \langle x^*, s - v \rangle + \langle x - u, s^* \rangle + \langle x, s \rangle \right).
\]

Therefore, we obtain
\[
\langle x - u, s \rangle + \langle x, s - v \rangle \leq (1 + 4\kappa) \left( \langle x - u, s - v \rangle + \langle x^*, s - v \rangle + \langle x - u, s^* \rangle + \langle x, s \rangle \right) \\
= (1 + 4\kappa) \left( \frac{\langle x - u, s - v \rangle + \langle x^*, s - v \rangle + \langle x - u, s^* \rangle}{\langle x, s \rangle} + 1 \right) \langle x, s \rangle.
\]

From (19), we have \( x - u = \phi(x^0 - u^0) \) and \( s - v = \phi(s^0 - v^0) \). Also, as stated before, in all iterations we maintain the condition \( \langle x^k, s^k \rangle \geq \phi^k \langle x^0, s^0 \rangle \). So, we may write
\[
\langle x - u, s \rangle + \langle x, s - v \rangle \leq \\
(1 + 4\kappa) \left( \frac{\phi^2 \langle x^0 - u^0, s^0 - v^0 \rangle + \phi \left( \langle x^*, s^0 - v^0 \rangle + \langle x^0 - u^0, s^* \rangle \right)}{\phi \langle x^0, s^0 \rangle} + 1 \right) r\mu.
\]

Note that by the Cauchy-Schwarz inequality, we have \( |\langle p, q \rangle| \leq \|p\|_F \|q\|_F \leq r |\lambda_{\max}(p)| |\lambda_{\max}(q)| \), for \( p, q \in \mathcal{J} \). Hence, using (22) and (23), we get
\[
\langle x - u, s \rangle + \langle x, s - v \rangle \leq (1 + 4\kappa) \left( \frac{4r\phi(\rho^0)^2 + 4r\rho^0 \rho^*}{r(\rho^0)^2} + 1 \right) r\mu \\
\leq (1 + 4\kappa) \left( \frac{4r(\rho^0)^2 + 4r\rho^0 \rho^*}{r(\rho^0)^2} + 1 \right) r\mu \\
= (1 + 4\kappa) \left( 5 + \frac{4\rho^*}{\rho^0} \right) r\mu \leq (1 + 4\kappa) (5 + 4\Psi) r\mu,
\]
which completes the proof. \( \square \)

Due to Lemmas 4.5 and 4.6, we obtain
\[
\zeta \leq \alpha_1 \frac{(1 + 4\kappa)(5 + 4\Psi) r \sqrt{\mu}}{\sqrt{(1 - \beta)\tau}}.
\]
Substituting this inequality in Lemma 4.3, we have

\[
\left\| \begin{pmatrix} \Delta x(\alpha) \\ \Delta s(\alpha) \end{pmatrix} \right\|_G \\
\leq \sqrt{1 + 2\kappa} \left( \alpha_1 \sqrt{r\mu} + \alpha_2 \sqrt{\beta\tau\mu} \right) + \\
\alpha_1 \frac{(1 + \sqrt{2(1 + 2\kappa)})(1 + 4\kappa)(5 + 4\Psi)}{\sqrt{(1 - \beta)\tau}} r\sqrt{\mu}
\]

\[
\leq (1 + 4\kappa)^{3/2} \left( \alpha_1 \sqrt{r\mu} + \alpha_2 \sqrt{\beta\tau\mu} + \alpha_1 \frac{(1 + \sqrt{2})(5 + 4\Psi)}{\sqrt{(1 - \beta)\tau}} r\sqrt{\mu} \right)
\]

\[
= (1 + 4\kappa)^{3/2} \left( \frac{\alpha_1}{\sqrt{r}} + \alpha_2 \frac{\sqrt{\beta\tau}}{r} + \alpha_1 \omega \right) r\sqrt{\mu},
\]

where \( \omega := \frac{(1 + \sqrt{2})(5 + 4\Psi)}{\sqrt{(1 - \beta)\tau}} \geq 34 \). Here, the inequality is obtained due to \( \tau < \frac{1}{8} \), \( 1 - \beta < 1 \) and \( \Psi \geq 0 \).

The following lemma can be proved in the same way as Lemma 33 of [5].

**Lemma 4.7** (cf. Lemma 33 in [5]). Let \( u, v \in J \) and \( G \) be a positive definite matrix and \( \text{cond}(G) \) denotes the condition number of \( G \), i.e., \( \frac{\lambda_{\text{max}}(G)}{\lambda_{\text{min}}(G)} \); then,

\[
\|u\|_F \|v\|_F \leq \sqrt{\text{cond}(G)} \left( \|G^{-1/2}u\|_F^2 + \|G^{1/2}v\|_F^2 \right).
\]

Consider \( G = L^{-1}_s L_{\tilde{x}} \) and let \( \text{cond}(G) \leq l \). From (27), we have

\[
\left\| \begin{pmatrix} \Delta x(\alpha) \\ \Delta s(\alpha) \end{pmatrix} \right\|_F \leq \frac{\sqrt{l}}{2} (1 + 4\kappa)^{3/2} \left( \frac{\alpha_1}{\sqrt{r}} + \alpha_2 \frac{\sqrt{\beta\tau}}{r} + \alpha_1 \omega \right)^2 r^2 \mu. \tag{28}
\]

Subsequently, using the obtained bound in (28), we show how to ensure the requirements of the corrector step being satisfied by a suitable step length.

**Lemma 4.8.** Let \( \alpha_1 = \alpha_2 \sqrt{\frac{\beta\tau}{r}} \), \( \alpha_2 \leq \frac{\sqrt{\tau}}{(1 + 4\kappa)^{3/2}\sqrt{\omega^2 r}} \), \( \beta < \frac{1}{2} \), \( \tau < \frac{1}{8} \) and \( (\tilde{x}, \tilde{s}) \in N(\tau, \beta) \). Then, \( \langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle \leq \langle \tilde{x}, \tilde{s} \rangle \left( 1 - \frac{1}{3} \alpha_1 \right) \).
Proof. From Lemma 3.1 and the Cauchy-Schwarz inequality, we derive
\[ \langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle \]
\[ = \langle \tilde{x}, \tilde{s} \rangle + \alpha_1 \langle e, (\tau \mu e - x \circ s)^- \rangle + \alpha_2 \langle e, (\tau \mu e - x \circ s)^+ \rangle + \langle \tilde{\Delta} x(\alpha), \tilde{\Delta} s(\alpha) \rangle \]
\[ \leq r \mu + \alpha_1 \langle e, (\tau \mu e - x \circ s) \rangle + \alpha_2 \|e\|_F \| (\tau \mu e - x \circ s)^+ \|_F + \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F \]
\[ \leq r \mu - \alpha_1 (1 - \tau) r \mu + \alpha_2 \sqrt{r} \beta \tau \mu + \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F \]
\[ \leq r \mu - \alpha_2 \sqrt{\frac{3}{\tau}} (1 - \tau) r \mu + \alpha_2 \sqrt{r} \beta \tau \mu + \alpha_2^2 (1 + 4 \kappa) \beta \tau \sqrt{l} \left( \frac{2}{\sqrt{r}} + \omega \right)^2 r \mu \]
\[ = r \mu \left( 1 - \alpha_2 \sqrt{\frac{3}{\tau}} \left( 1 - \tau - \sqrt{\beta \tau} - \alpha_2 (1 + 4 \kappa) \frac{\beta \tau \sqrt{l}}{2} \left( \frac{2}{\sqrt{r}} + \omega \right)^2 \right) \right) \]
\[ \leq \langle \tilde{x}, \tilde{s} \rangle \left( 1 - \alpha_2 \sqrt{\frac{3}{\tau}} \left( 1 - \tau - \sqrt{\beta \tau} - \sqrt{\beta \tau} \right) \right) \leq \langle \tilde{x}, \tilde{s} \rangle \left( 1 - \frac{1}{3} \alpha_1 \right), \]
where the third inequality is obtained by substituting the upper bound for \( \alpha_2 \). The last two inequalities are due to \( \omega \geq 34, r \geq 1, \beta < \frac{1}{2} \) and \( \tau < \frac{1}{8} \). \( \square \)

Lemma 4.9. Let \( (\tilde{x}, \tilde{s}) \in N(\tau, \beta) \). Then,
\[ \| (\tau \mu(\alpha) e - v(\alpha))^+ \|_F \leq (1 - \alpha_2) \beta \tau \mu(\alpha). \]

Proof. From Lemmas 3.2 and 4.8, we have \( v(\alpha) \succeq_{K} 0 \) and \( \mu(\alpha) \leq \mu \), respectively. Therefore, we may write
\[ \| (\tau \mu(\alpha) e - v(\alpha))^+ \|_F \leq \left\| \left( \tau \mu(\alpha) e - \frac{\mu(\alpha)}{\mu} v(\alpha) \right)^+ \right\|_F \]
\[ = \frac{\mu(\alpha)}{\mu} \| (\tau \mu e - v(\alpha))^+ \|_F \]
\[ = \frac{\mu(\alpha)}{\mu} \left\| (\tau \mu e - \tilde{x} \circ \tilde{s} - \alpha_1 (\tau \mu e - \tilde{x} \circ \tilde{s})^- - \alpha_2 (\tau \mu e - \tilde{x} \circ \tilde{s})^+) \right\|_F \]
\[ = \frac{\mu(\alpha)}{\mu} \left\| ((1 - \alpha_1)(\tau \mu e - \tilde{x} \circ \tilde{s})^- + (1 - \alpha_2)(\tau \mu e - \tilde{x} \circ \tilde{s})^+) \right\|_F \]
\[ = \frac{\mu(\alpha)}{\mu} \| (1 - \alpha_2)(\tau \mu e - \tilde{x} \circ \tilde{s})^+ \|_F \leq \frac{\mu(\alpha)}{\mu} (1 - \alpha_2) \beta \tau \mu = (1 - \alpha_2) \beta \tau \mu(\alpha), \]
to complete the proof. \( \square \)
Lemma 4.10. Let $\alpha_1 = \alpha_2 \sqrt{\frac{\beta \tau}{r}}$, $\alpha_2 \leq \frac{\sqrt{\tau}}{(1+4\kappa)^3 \sqrt{r} \omega^2 r}$, $\beta < \frac{1}{2}$ and $\tau < \frac{1}{8}$. Then,

$$\left\| (\tilde{\Delta} x(\alpha) \circ \tilde{\Delta} s(\alpha)) \right\|_F \leq \alpha_2 \beta \tau \mu(\alpha).$$

Proof. We have

$$\langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle = \mu r + \alpha_1 tr(\tau \mu e - x \circ s) - + \alpha_2 tr(\tau \mu e - x \circ s)^+ + \langle \tilde{\Delta} x(\alpha), \tilde{\Delta} s(\alpha) \rangle$$

$$= \mu r + \alpha_1 tr(\tau \mu e - x \circ s) + (\alpha_2 - \alpha_1) tr(\tau \mu e - x \circ s)^+ + \langle \tilde{\Delta} x(\alpha), \tilde{\Delta} s(\alpha) \rangle$$

$$\geq \mu r - \alpha_1 (1 - \tau) \mu r + \langle \tilde{\Delta} x(\alpha), \tilde{\Delta} s(\alpha) \rangle$$

$$\geq \mu r - \alpha_1 \mu r - \left\| \tilde{\Delta} x(\alpha) \right\|_F \left\| \tilde{\Delta} s(\alpha) \right\|_F,$$

where the first inequality arises from the fact that $\alpha_1 \leq \alpha_2$, by the assumption of the lemma. The second inequality is obtained due to the Cauchy-Schwarz inequality and $\tau \mu r > 0$. Therefore, we may write

$$\mu(\alpha) \geq \mu - \alpha_1 \mu - \frac{\left\| \tilde{\Delta} x(\alpha) \right\|_F \left\| \tilde{\Delta} s(\alpha) \right\|_F}{r}.$$

Thus, to prove the statement of the lemma, it suffices to show that

$$\left\| \tilde{\Delta} x(\alpha) \right\|_F \left\| \tilde{\Delta} s(\alpha) \right\|_F \leq \alpha_2 \beta \tau \left( \mu - \alpha_1 \mu - \frac{\left\| \tilde{\Delta} x(\alpha) \right\|_F \left\| \tilde{\Delta} s(\alpha) \right\|_F}{r} \right),$$

which is equivalent to

$$\left(1 + \alpha_2 \frac{\beta \tau}{r}\right) \left\| \tilde{\Delta} x(\alpha) \right\|_F \left\| \tilde{\Delta} s(\alpha) \right\|_F \leq \alpha_2 \beta \tau \mu \left(1 - \alpha_2 \sqrt{\frac{\beta \tau}{r}}\right).$$

Using (28), the last inequality is satisfied, whenever

$$\left(1 + \alpha_2 \frac{\beta \tau}{r}\right) (1 + 4\kappa)^3 \alpha_2 \frac{\sqrt{l} \beta \tau}{2} \left(\frac{2}{\sqrt{r}} + \omega\right)^2 r \mu \leq \alpha_2 \beta \tau \mu \left(1 - \alpha_2 \sqrt{\frac{\beta \tau}{r}}\right).$$

This is equivalent to

$$\left(1 + \alpha_2 \frac{\beta \tau}{r}\right) (1 + 4\kappa)^3 \alpha_2 \frac{\sqrt{l}}{2} \left(\frac{2}{\sqrt{r}} + \omega\right)^2 r \leq 1 - \alpha_2 \sqrt{\frac{\beta \tau}{r}}.$$
Since $\alpha_2 \leq 1$ and $r \geq 1$, we have $1 - \alpha_2 \sqrt{\frac{\beta \tau}{r}} \geq 1 - \sqrt{\beta \tau}$. Therefore, it suffices to have

$$
\left(1 + \alpha_2 \frac{\beta \tau}{r}\right)^{(1 + 4\kappa)^3} \frac{\sqrt{l}}{2} \left(\frac{2}{\sqrt{r}} + \omega\right)^2 r \leq 1 - \sqrt{\beta \tau}.
$$

Substituting the upper bound for $\alpha_2$, we derive the following upper bound for the left hand side of the above inequality:

$$
\alpha_2 (1 + 4\kappa)^3 \frac{\sqrt{l}}{2} \left(\frac{2}{\sqrt{r}} + \omega\right)^2 r + \alpha_2^2 (1 + 4\kappa)^3 \frac{\beta \tau \sqrt{l}}{2} \left(\frac{2}{\sqrt{r}} + \omega\right)^2 
\leq \frac{\sqrt{\tau}}{2} \left(\left(\frac{2}{\sqrt{r} \omega} + 1\right)^2 + \frac{\beta \tau^{3/2}}{(1 + 4\kappa)^3 \sqrt{l} \omega^2} \left(\frac{2}{\sqrt{r} \omega^2} + \frac{1}{\omega}\right)^2\right).
$$

Considering $\beta < \frac{1}{2}$, $\tau < \frac{1}{8}$, $l \geq 1$, $r \geq 1$, $\kappa \geq 0$, and $\omega \geq 34$, an upper bound for the last expression is 0.20, while $1 - \sqrt{\beta \tau} \geq 0.75$. This proves the lemma.

The subsequent two lemmas are originally shown in [27]. The proof of the second one is converted into English in [21].

**Lemma 4.11** (cf. Lemma 6 in [21]). If $x, s \in \text{int } K$ and $w = Q_{x_1/s} s$, then $\|(\tau \mu e - w)^+\|_F \leq \|(\tau \mu e - x \circ s)^+\|_F$, for $\tau \in (0, 1)$ and $\mu > 0$.

**Lemma 4.12** (cf. Lemma 5 in [21]). Let $a, b \in J$. Then,

$$
\|(a + b)^+\|_F \leq \|a^+\|_F + \|b^+\|_F.
$$

Based on Lemmas 4.9-4.12, the next lemma establishes that after performing the corrector step, the iterates fall in the desired neighborhood.

**Lemma 4.13.** Let $\alpha_1 = \alpha_2 \sqrt{\frac{\beta \tau}{r}}$, $\alpha_2 \leq \frac{\sqrt{\tau}}{(1 + 4\kappa)^3 \sqrt{l} \omega^2}$, $\beta < \frac{1}{2}$, $\tau < \frac{1}{8}$ and $(x, s) \in N(\tau, \beta)$. Then, $(\tilde{x}(\alpha), \tilde{s}(\alpha)) \in N(\tau, \beta)$.

**Proof.** Assuming $\tilde{w}(\alpha) = Q_{\tilde{x}(\alpha)^1/2} \tilde{s}(\alpha)$ and using Lemmas 4.9-4.12, we derive

$$
\|(\tau \mu(\alpha) e - \tilde{w}(\alpha))^+\|_F \leq \|(\tau \mu(\alpha) e - \tilde{x}(\alpha) \circ \tilde{s}(\alpha))^+\|_F 
= \left\|\left(\tau \mu(\alpha) e - v(\alpha) - \tilde{\Delta} x(\alpha) \circ \tilde{\Delta} s(\alpha)\right)^+\right\|_F 
\leq \|(\tau \mu(\alpha) e - v(\alpha))^+\|_F + \left\|\tilde{\Delta} x(\alpha) \circ \tilde{\Delta} s(\alpha)\right\|_F 
\leq (1 - \alpha_2)\beta \tau \mu(\alpha) + \alpha_2 \beta \tau \mu(\alpha) = \beta \tau \mu(\alpha).
$$
On the other hand, due to the inequalities above, we may write
\[
\| (\tau \mu(\alpha) e - \tilde{x}(\alpha) \circ \tilde{s}(\alpha))^+ \|_F \leq \beta \tau \mu(\alpha),
\]
which leads to
\[
\tau \mu(\alpha) - \lambda_{\min}(\tilde{x}(\alpha) \circ \tilde{s}(\alpha)) \leq \beta \tau \mu(\alpha).
\]
This is equivalent to
\[
\lambda_{\min}(\tilde{x}(\alpha) \circ \tilde{s}(\alpha)) \geq (1 - \beta) \tau \mu(\alpha) > 0.
\]
Consequently, \( \tilde{x}(\alpha) \circ \tilde{s}(\alpha) \in \text{int} \ K. \) Therefore, similar to the proof of Lemma 2.15 in [6], we obtain \( \det(\tilde{x}(\alpha)) \neq 0 \) and \( \det(\tilde{s}(\alpha)) \neq 0. \) Furthermore, since \( \tilde{x} \in \text{int} \ K \) and \( \tilde{s} \in \text{int} \ K, \) by the continuity property, we have \( \tilde{x}(\alpha) \in \text{int} \ K \) and \( \tilde{s}(\alpha) \in \text{int} \ K. \) The proof is completed.

Note that in the proof of Lemma 4.13 we used the fact that \( \mu(\alpha) > 0. \) This fact can be understood implicitly by the following lemma.

**Lemma 4.14.** Let \( \alpha_1 = \alpha_2 \sqrt{\frac{\beta \tau}{r}}, \alpha_2 \leq \frac{\sqrt{\tau}}{(1+4\kappa)\sqrt{\omega^2 r}} \), \( \beta < \frac{1}{2}, \tau < \frac{1}{8} \) and \( (\tilde{x}, \tilde{s}) \in N(\tau, \beta) \). Then, \( \langle x(\alpha), s(\alpha) \rangle \geq (1 - \alpha_1) \phi \langle x^0, s^0 \rangle \).

**Proof.** As derived in the proof of Lemma 4.10, we have
\[
\langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle \geq r \mu - \alpha_1 (1 - \tau) \mu r - \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F
\]
\[
= (1 - \alpha_1) \mu r + \alpha_1 \tau r \mu - \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F
\]
\[
= (1 - \alpha_1) \langle \tilde{x}, \tilde{s} \rangle + \alpha_1 \tau r \mu - \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F
\]
\[
\geq (1 - \alpha_1) \phi \langle x^0, s^0 \rangle + \alpha_1 \tau r \mu - \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F,
\]
where, the last inequality is due to (18). Hence, to obtain the result, it suffices to prove that \( \alpha_1 \tau r \mu - \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F \geq 0. \) We may write
\[
\alpha_1 \tau r \mu - \| \tilde{\Delta} x(\alpha) \|_F \| \tilde{\Delta} s(\alpha) \|_F
\]
\[
\geq \alpha_2 \sqrt{\frac{\beta \tau}{r}} \tau r \mu - (1 + 4\kappa)^3 \alpha_2 \sqrt{\frac{\beta \tau}{r}} \left( \frac{2}{\sqrt{r}} + \omega \right)^2 \mu r
\]
\[
= \alpha_2 \sqrt{\beta \tau r} \tau \mu \left( 1 - \alpha_2 (1 + 4\kappa)^3 \frac{\sqrt{\beta}}{2 \sqrt{r}} \left( \frac{2}{\sqrt{r}} + \omega \right)^2 \sqrt{r} \right)
\]
\[
\geq \alpha_2 \sqrt{\beta \tau r} \tau \mu \left( 1 - \frac{\sqrt{\beta}}{2 \sqrt{r}} \left( \frac{2}{\sqrt{r} \omega} + 1 \right)^2 \right) \geq 0,
\]
where the first inequality is obtained from (28), the second one is derived by substituting the upper bound of $\alpha_2$, and the last one is due to $\beta < \frac{1}{2}$, $\tau < \frac{1}{8}$, $r \geq 1$ and $\omega \geq 34$.

5 Analysis of the Predictor Step

This section considers the predictor step in more details, in order to fulfill its requirements with a suitable step length. Throughout this section, we assume that $\sigma = 0$.

Lemma 5.1. If $(\tilde{x}, \tilde{s}) \in N(\tau, \beta)$, then $\left\| (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} \tilde{x} \circ \tilde{s} \right\|_F^2 = r \mu$.

Proof. One has

$$\left\| (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} \tilde{x} \circ \tilde{s} \right\|_F^2 = \left\| L_{\tilde{s}}^{-1/2} L_{\tilde{x}}^{1/2} \tilde{s} \right\|_F^2 = \left\| G^{1/2} \tilde{s} \right\|_F^2 = \langle G^{1/2} \tilde{s}, G^{1/2} \tilde{s} \rangle = \langle G \tilde{s}, \tilde{s} \rangle = \langle L_{\tilde{s}}^{-1} L_{\tilde{x}} \tilde{s}, \tilde{s} \rangle = \langle L_{\tilde{x}} \tilde{s}, e \rangle = \langle \tilde{x}, \tilde{s} \rangle = r \mu,$$

which proves the lemma. \qed

Lemma 5.2. Let $(\tilde{x}, \tilde{s}) \in N(\tau, \beta)$, $\beta < \frac{1}{2}$ and $\tau < \frac{1}{8}$. Then,

$$\left\| (\tilde{\Delta} x_-, \tilde{\Delta} s_-) \right\|_G \leq \sqrt{(1 + 2\kappa)} r \mu + (1 + \sqrt{2(1 + 2\kappa)}) \bar{\zeta},$$

where $\bar{\zeta} := \min \{ \| (u, v)\|_G : \tilde{M} \tilde{u} + \tilde{N} \tilde{v} = b - \tilde{M} \tilde{x} - \tilde{N} \tilde{s} \}$.

Proof. Using system (10), we have

$$\tilde{M} \tilde{\Delta} x_- + \tilde{N} \tilde{\Delta} s_- = b - \tilde{M} \tilde{x} - \tilde{N} \tilde{s},$$
$$L_{\tilde{x}} \tilde{\Delta} s_- + L_{\tilde{s}} \tilde{\Delta} x_- = (-\tilde{x} \circ \tilde{s})^- = -\tilde{x} \circ \tilde{s}.$$

Letting $(\tilde{u}, \tilde{v})$ to be the solution of the minimization problem defining $\bar{\zeta}$, we obtain

$$\tilde{M}(\tilde{\Delta} x_- - \tilde{u}) + \tilde{N}(\tilde{\Delta} s_- - \tilde{v}) = 0,$$
$$L_{\tilde{x}}(\tilde{\Delta} x_- - \tilde{u}) + L_{\tilde{s}}(\tilde{\Delta} s_- - \tilde{v}) = -\tilde{x} \circ \tilde{s} - (L_{\tilde{x}} \tilde{v} + L_{\tilde{s}} \tilde{u}).$$
From the above system and Lemmas 4.1 and 5.1, we conclude that

\[
\left\| (\tilde{\Delta} x_\tau - \bar{u}, \tilde{\Delta} s_\tau - \bar{v}) \right\|_G \\
\leq \sqrt{1 + 2\kappa} \left( \sqrt{r\mu} + \left\| G^{-1/2}\bar{u} \right\|_F + \left\| G^{1/2}\bar{v} \right\|_F \right) \\
\leq \sqrt{1 + 2\kappa} \left( \sqrt{r\mu} + \sqrt{2} \left\| (\bar{u}, \bar{v}) \right\|_G \right).
\]

Now, similar to the proof of Lemma 4.3, we have

\[
\left\| (\tilde{\Delta} x_\tau, \tilde{\Delta} s_\tau) \right\|_G \\
\leq \left\| (\tilde{\Delta} x_\tau - \bar{u}, \tilde{\Delta} s_\tau - \bar{v}) \right\|_G + \left\| (\bar{u}, \bar{v}) \right\|_G \\
\leq \sqrt{1 + 2\kappa} \left( \sqrt{r\mu} + \sqrt{2} \left\| (\bar{u}, \bar{v}) \right\|_G \right) + \left\| (\bar{u}, \bar{v}) \right\|_G \\
= \sqrt{(1 + 2\kappa)r\mu} + (1 + \sqrt{2(1 + 2\kappa)})\bar{\zeta}.
\]

The proof is completed.

Lemma 5.3. Let \((\bar{x}, \bar{s}) \in N(\tau, \beta)\). Then, \(\bar{\zeta} \leq \frac{\langle x, s-v \rangle + \langle s, x-u \rangle}{\sqrt{(1-\beta)\tau\mu}}\).

Proof. The proof is similar to the proof of Lemma 4.5, and therefore is omitted.

Similar to the proof of Lemma 4.6, we obtain

\[
\tilde{\zeta} \leq \frac{(1 + 4\kappa)(5 + 4\Psi)r\sqrt{\mu}}{\sqrt{(1 - \beta)\tau}}.
\]

Substituting this inequality in Lemma 5.2, we get

\[
\left\| (\tilde{\Delta} x_\tau, \tilde{\Delta} s_\tau) \right\|_G \leq \sqrt{(1 + 2\kappa)r\mu} + \frac{(1 + \sqrt{2(1 + 2\kappa)})(1 + 4\kappa)(5 + 4\Psi)r\sqrt{\mu}}{\sqrt{(1 - \beta)\tau}} \\
\leq (1 + 4\kappa)^{3/2} \left( \frac{1}{\sqrt{r}} + \omega \right) r\sqrt{\mu},
\]

Therefore, using Lemma 4.7, we have

\[
\left\| \tilde{\Delta} x_\tau \right\|_F \left\| \tilde{\Delta} s_\tau \right\|_F \leq (1 + 4\kappa)^3 \frac{\sqrt{\ell}}{2} \left( \frac{1}{\sqrt{r}} + \omega \right)^2 r^2\mu,
\]

where, \(l \geq \text{cond}(G)\).
Lemma 5.4. If $\alpha_1 \leq \frac{1}{(1+4\kappa)^3\sqrt{\ell\omega^2r}}$, then after performing the predictor step
with step length $\alpha_1$, we have $\langle \tilde{x}(\alpha_1), \tilde{s}(\alpha_1) \rangle \leq (1 - \frac{2}{5} \alpha_1) \langle \tilde{x}, \tilde{s} \rangle$.

Proof. For the predictor step, we have

$$\tilde{x}(\alpha_1) \circ \tilde{s}(\alpha_1) = (1 - \alpha_1) \tilde{x} \circ \tilde{s} + \alpha_1^2 \Delta x_\circ \Delta s_\circ.$$  \hspace{1cm} (30)

Using (29), we may write

$$\langle \tilde{x}(\alpha_1), \tilde{s}(\alpha_1) \rangle = \langle \tilde{x}, \tilde{s} \rangle \left( 1 - \alpha_1 + \frac{\alpha_1}{2} \left( \frac{1}{\sqrt{r\omega}} + 1 \right)^2 \right)$$

$$\leq \langle \tilde{x}, \tilde{s} \rangle \left( 1 - \alpha_1 + \frac{3}{5} \alpha_1 \right)$$

$$= \left( 1 - \frac{2}{5} \alpha_1 \right) \langle \tilde{x}, \tilde{s} \rangle,$$

where the second inequality is due to $r \geq 1$ and $\omega \geq 34$. This proves the lemma. \hfill \square

The above lemma also shows that with the determined bound for $\alpha_1$ in the predictor step, we get $\mu(\alpha_1) \leq \mu$.

Lemma 5.5. Let $(\Delta x_-, \Delta s_-)$ be the search directions in the predictor step, $(\tilde{x}, \tilde{s}) \in N(\tau, \beta)$, $\beta < \frac{1}{2}$, $\tau < \frac{1}{8}$ and $\alpha_1 \leq \frac{1}{(1+4\kappa)^3\sqrt{\ell\omega^2r}}$. Then, after performing the predictor step, we have $(\tilde{x}(\alpha_1), \tilde{s}(\alpha_1)) \in N(\tau, 2\beta)$.

Proof. Using the proof of Lemma 5.4, we have $\mu(\alpha_1) = (1 - \alpha_1) \mu + \alpha_1^2 \frac{\langle \Delta x_-, \Delta s_- \rangle}{r}$.
Assuming $\tilde{w}(\alpha_1) = Q_{\tilde{x}(\alpha_1)^{1/2}} \tilde{s}(\alpha_1)$, from (30) and Lemmas 4.11 and 4.12, we obtain

\begin{align*}
\| (\tau \mu(\alpha_1)e - \tilde{w}(\alpha_1))^+ \|_F & \leq \| (\tau \mu(\alpha_1)e - \tilde{x}(\alpha_1) \circ \tilde{s}(\alpha_1))^+ \|_F \\
& = \left\| \left( (1 - \alpha_1)(\tau e - \tilde{x} \circ \tilde{s}) + \alpha_1^2 \tau \frac{\langle \Delta x_-, \Delta s_- \rangle}{r} - \alpha_1^2 \Delta x_\circ \Delta s_\circ \right)^+ \right\|_F \\
& \leq (1 - \alpha_1) \| (\tau e - \tilde{x} \circ \tilde{s})^+ \|_F + \alpha_1^2 \left\| \frac{\langle \Delta x_-, \Delta s_- \rangle}{r} - \Delta x_\circ \Delta s_\circ \right\|_F^+. \hspace{1cm} (31)
\end{align*}

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On the other hand, we have
\[
\left\| \left( \frac{\Delta x_-, \Delta s_-}{\tau} e - \Delta x_- \circ \Delta s_- \right) \right\|_F^2 \leq \left\| \frac{\Delta x_-, \Delta s_-}{r} e - \Delta x_- \circ \Delta s_- \right\|_F^2 \leq \left\| \Delta x_- \circ \Delta s_- \right\|_F^2 - \tau (2 - \tau) \frac{\left\langle \frac{\Delta x_-, \Delta s_-}{r} \right\rangle^2}{\tau} \leq \left\| \Delta x_- \circ \Delta s_- \right\|_F^2.
\]

Therefore, we derive
\[
\left\| (\tau \mu(\alpha_1) e - \tilde{w}(\alpha_1))^+ \right\|_F \leq (1 - \alpha_1) \beta \tau \mu + \alpha_1^2 \left\| \Delta x_- \circ \Delta s_- \right\|_F.
\]

Moreover, we have
\[
\mu(\alpha_1) \geq (1 - \alpha_1) \mu - \alpha_1^2 \frac{\| \Delta x_- \|_F \| \Delta s_- \|_F}{r}.
\]

So, we may write
\[
(1 - \alpha_1) \beta \tau \mu + \alpha_1^2 \left\| \Delta x_- \right\|_F \left\| \Delta s_- \right\|_F - 2 \beta \tau \mu(\alpha_1)
\]
\[
\leq (1 - \alpha_1) \beta \tau \mu + \alpha_1^2 \left\| \Delta x_- \right\|_F \left\| \Delta s_- \right\|_F - 2(1 - \alpha_1) \beta \tau \mu + 2 \beta \tau \alpha_1 \left\| \Delta x_- \right\|_F \left\| \Delta s_- \right\|_F
\]
\[
= \left( 1 + \frac{2 \beta \tau}{r} \right) \alpha_1^2 \left\| \Delta x_- \right\|_F \left\| \Delta s_- \right\|_F - (1 - \alpha_1) \beta \tau \mu
\]
\[
\leq \left( 1 + \frac{2 \beta \tau}{r} \right) \alpha_1^2 \frac{\sqrt{l}}{2} (1 + 4 \kappa)^3 \left( \frac{1}{\sqrt{r}} + \omega \right)^2 r^2 \mu - (1 - \alpha_1) \beta \tau \mu
\]
\[
= \mu \left( \left( 1 + \frac{2 \beta \tau}{r} \right) \alpha_1^2 \frac{\sqrt{l}}{2} (1 + 4 \kappa)^3 \left( \frac{1}{\sqrt{r}} + \omega \right)^2 r^2 - (1 - \alpha_1) \beta \tau \right)
\]
\[
\leq \mu \left( \frac{1}{2 \omega^4} \left( 1 + \frac{2 \beta \tau}{r} \right) \left( \frac{1}{\sqrt{r}} + \omega \right)^2 - \left( 1 - \frac{1}{\omega^2} \right) \beta \right) \leq 0,
\]
where the last inequality relies on the facts that \( \omega \geq 34, \beta < \frac{1}{2}, \tau < \frac{1}{8} \) and \( r \geq 1 \). Consequently, we have \( \|(\tau \mu(\alpha_1) e - \tilde{w}(\alpha_1))^+\|_F \leq 2 \beta \tau \mu(\alpha_1) \). From the
above discussion, we have
\[ \lambda_{\min}(\tilde{x}(\alpha_1) \circ \tilde{s}(\alpha_1)) \geq (1 - 2\beta)\tau \mu(\alpha_1). \]  
(33)
Noticing (32), using (29) and the upper bound of \( \alpha_1 \), we may derive \( \mu(\alpha_1) > 0 \). So, from (33) and the fact that \( 1 - 2\beta > 0 \), we get \( \lambda_{\min}(\tilde{x}(\alpha_1) \circ \tilde{s}(\alpha_1)) > 0 \). Now, by a similar argument as the one given at the end of the proof of Lemma 4.13, we have \( \tilde{x}(\alpha_1) \in \text{int} \ K \) and \( \tilde{s}(\alpha_1) \in \text{int} \ K \). This ends the proof. \( \square \)

6 Complexity Analysis

Based on the results of the two previous sections, here we give the iteration bound of our algorithm. Let \( \alpha_{\text{cor}} := (\alpha_{1\text{cor}}, \alpha_{2\text{cor}}) \) and \( \alpha_{1\text{pre}} \) be the step lengths in the corrector and predictor steps, respectively. Corresponding to Sections 4 and 5, we can take \( \alpha_{\text{cor}} = \left(\sqrt{\frac{\beta r}{r}} \alpha_{2\text{cor}}, \frac{\sqrt{r}}{(1+4\kappa)^3\omega^2 r \sqrt{l}}\right) \) and \( \alpha_{1\text{pre}} = \frac{1}{(1+4\kappa)^3\omega^2 r \sqrt{l}} \). Note that, in each main iteration of the algorithm, we perform a corrector and a predictor step. The algorithm terminates after \( k \) iteration, with \( \mu^k \leq \mu^0 \varepsilon \). Using Lemmas 4.8 and 5.4, we may write
\[
\mu^k \leq \left(1 - \frac{1}{3} \alpha_{1\text{cor}}\right) \left(1 - \frac{2}{5} \alpha_{1\text{pre}}\right) \mu^{k-1} \leq \ldots \\
\leq \left(1 - \frac{1}{3} \alpha_{1\text{cor}}\right)^k \left(1 - \frac{2}{5} \alpha_{1\text{pre}}\right)^k \mu^0 \leq \mu^0 \varepsilon.
\]
Therefore, it suffices to have
\[
\left(1 - \frac{1}{3} \alpha_{1\text{cor}}\right)^k \left(1 - \frac{2}{5} \alpha_{1\text{pre}}\right)^k \leq \varepsilon.
\]
Noticing \( \alpha_{1\text{pre}} \geq \alpha_{1\text{cor}} \), the above condition is satisfied, whenever
\[
\left(1 - \frac{1}{3} \alpha_{1\text{cor}}\right)^{2k} \leq \varepsilon.
\]
Taking logarithms at both sides, noting the inequality \( \log(1 - \theta) \leq -\theta \), for \( \theta > -1 \), and substituting the value of \( \alpha_{1\text{cor}} \), we find the following main result of the paper.

**Theorem 6.1.** The algorithm terminates after \( O \left( (1 + 4\kappa)^3 r^{3/2} \sqrt{l} \log \varepsilon^{-1} \right) \) iterations.
We can specialize the algorithm further by prescribing the scaling element \( p \in \mathcal{C}(x, s) \). Choosing \( p \) such that \( \tilde{x} = \tilde{s} \), we obtain the NT search direction. The \( xs \) and \( sx \) search directions are obtained by choosing \( p = s^{1/2} \) and \( p = x^{-1/2} \), respectively. For the NT search direction we have \( l = 1 \), which can be proved in the same way as Lemma 36 of [5]. For \( xs \) and \( sx \) search directions, we have \( l \leq \frac{r}{(1-\beta)\tau} \). Substituting these bounds for \( l \) in Theorem 6.1, we obtain the following corollary.

**Corollary 6.1.** For the NT direction, the algorithm has

\[
O \left( (1 + 4\kappa)^3 r^{3/2} \log \varepsilon^{-1} \right)
\]

iteration complexity, while in the \( xs \) and \( sx \) search directions, the algorithm terminates after \( O \left( (1 + 4\kappa)^3 r^2 \log \varepsilon^{-1} \right) \) iterations.

### 6.1 An analysis of the algorithm for feasible starting points

Consider the corrector step. For the feasible starting points, by the definition of \( \zeta \), we have \( \zeta = 0 \). Consequently, instead of (28), we derive

\[
\| \tilde{x}(\alpha) \|_F \| \tilde{s}(\alpha) \|_F \leq \frac{\sqrt{l}}{2} (1 + 2\kappa) \left( \alpha_1 + \alpha_2 \sqrt{\frac{\beta \tau}{r}} \right)^2 r \mu.
\]

Therefore, using calculations similar to the ones given in the proof of Lemma 4.8, we conclude that for \( \alpha_1 = \sqrt{\frac{\beta \tau}{r}} \alpha_2 \) and \( \alpha_2 \leq \frac{\sqrt{\beta \tau}}{(1+2\kappa)\sqrt{l}} \) along with \( \beta \leq \frac{1}{2} \) and \( \tau \leq \frac{1}{8} \), we have \( \langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle \leq \langle \tilde{x}, \tilde{s} \rangle \left( 1 - \frac{3}{4} \alpha_1 \right) \). Also, with the selected values for \( \alpha_1 \) and \( \alpha_2 \), we can show in the same way as done in Lemma 4.13 that \( (x(\alpha), s(\alpha)) \in N(\tau, \beta) \).

Now, consider the predictor step. For the feasible starting points, we have, \( \bar{\zeta} = 0 \) and therefore, instead of (29), we obtain

\[
\| \tilde{x}(\alpha) \|_F \| \tilde{s}(\alpha) \|_F \leq \frac{\sqrt{l}}{2} (1 + 2\kappa) r \mu.
\]

Thus, similar to Lemma 5.4, we have, for \( \alpha_1 \leq \frac{\sqrt{\beta \tau}}{(1+2\kappa)\sqrt{\tau l}}, \beta \leq \frac{1}{2} \) and \( \tau \leq \frac{1}{8} \),

\[
\langle \tilde{x}(\alpha_1), \tilde{s}(\alpha_1) \rangle \leq \langle \tilde{x}, \tilde{s} \rangle \left( 1 - \frac{7}{8} \alpha_1 \right).
\]

Also, with this value for \( \alpha_1 \), we can find in a similar way as done in Lemma 5.5 that \( (x(\alpha_1), s(\alpha_1)) \in N(\tau, 2\beta) \). Therefore, using a precisely similar argument
as in the beginning of this section, we conclude the following corollary for feasible starting points.

**Corollary 6.2.** Let the starting point of the algorithm be feasible. For the NT search direction, the algorithm terminates after $O\left((1 + 2\kappa)\sqrt{r} \log \varepsilon^{-1}\right)$, while for the $xs$ and $sx$ search directions, after $O\left((1 + 2\kappa)r \log \varepsilon^{-1}\right)$ interior-point iterations.

### 7 Conclusion

We analyzed the first large-neighborhood interior-point algorithm for the Cartesian P-SCHLCP using the symmetrization of the commutative class of search directions. The algorithm considered the general case of the starting points, which were the points contained in the interior of the associated symmetric cone, not necessarily satisfying the equations of the problem. A summarized iteration complexity bound for the feasible starting points was presented. We specialized the complexity bounds for some well-known search directions.

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