Theses of PhD Thesis

First-Order Logic Investigation of Relativity Theory with an Emphasis on Accelerated Observers

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1 Aims and introduction

Applying mathematical logic in foundations of relativity theories is not a new idea at all, it goes back to such leading mathematicians and philosophers as Hilbert, Reichenbach, Carnap, Gödel, Tarski, Suppes and Friedman among others. The work of our school of Logic and Relativity led by Andréka and Németi is continuation to their research. My thesis is a direct continuation of the works by Andréka, Madarász, Németi and their contributors [1].

Our research is strongly related to Hilbert's sixth problem of axiomatization of physics. Moreover, it goes beyond this problem since its general aim is not only to axiomatize physical theories but to investigate the relationship between basic assumptions (axioms) and predictions (theorems).

Our other general aims are to axiomatize relativity theories within pure first-order logic using simple, comprehensible and transparent basic assumptions only; to prove the surprising predictions of relativity theories from a few convincing axioms; to eliminate tacit assumptions from relativity by replacing them with explicit axioms formulated in first-order logic (in the spirit of the first-order logic foundation of mathematics and Tarski's axiomatization of geometry); and to provide a foundation for physics similar to that of mathematics.

For good reasons, the foundation of mathematics was performed strictly within first-order logic. One of the reasons is that staying within first-order logic helps to avoid tacit assumptions. Another reason is that first-order logic has a complete inference system while second-order logic (and thus any higher-order logic) cannot have one.

If we have an axiom system, we can ask which axioms are responsible for a certain consequence of our theory. This kind of reverse thinking can help us to answer the why-type questions of relativity. For example, we can take the twin paradox and check which axiom of special relativity was and which one was not needed to derive it. The weaker an axiom system is, the better answer it offers to the question: "Why is the twin paradox true?". For details on answering why-type questions of relativity by the methodology of the present work, see [14]. In my thesis there are several predictions of relativity investigated in this manner.

2 Methods used

The thesis was basically built by using standard techniques of first-order logic including some results of model theory, such as Gödel's completeness theorem; however, to achieve our results on accelerated observers, we had to redevelop some tools and methods of real analysis over arbitrary ordered fields in order to keep our investigation within first-order logic.

3 Results

To formulate our results, we present our first-order logic language that we used in our investigation. First we fix a natural number $d \geq 2$ for the dimension of spacetime. Our language contains the following non-logical symbols:

$$\{\, \mathrm{B}, \mathrm{Ob}, \mathrm{IOb}, \mathrm{Ph}, \mathrm{Q}, +, \cdot, <, \mathrm{W} \,\},$$

where B (bodies), Ob (observers), IOb (inertial observers), Ph (light signals or photons) and Q (quantities) are unary relation symbols; $+$ and $\cdot$ are binary function symbols and $<$ is a binary relation symbol (for field operations and ordering on Q); and W (world-view relation), the key relation of our theory, is a $2+d$-ary relation symbol.

$\mathrm{B}(x)$, $\mathrm{Ob}(x)$, $\mathrm{IOb}(x)$, $\mathrm{Ph}(x)$ and $\mathrm{Q}(x)$ are translated as "x is a body," "x is an observer," "x is an inertial observer," "x is a photon" and "x is a quantity," respectively. We use the world-view relation W to speak about coordinatization by translating $\mathrm{W}(x, y, z_1, \ldots, z_d)$ as "observer x coordinatizes body y at spacetime location $\langle z_1, \ldots, z_d\rangle$," (i.e., at space location $\langle z_2, \ldots, z_d\rangle$ and at instant z_1). Our first axiom expresses very basic assumptions, such as: both photons and observers are bodies, inertial observers are also observers, etc.

AxFrame $\mathrm{Ob} \cup \mathrm{Ph} \subseteq \mathrm{B}$, $\mathrm{IOb} \subseteq \mathrm{Ob}$, $\mathrm{W} \subseteq \mathrm{Ob} \times \mathrm{B} \times \mathrm{Q}^d$, $\mathrm{B} \cap \mathrm{Q} = \emptyset$; $+$ and $\cdot$ are binary operations, and $<$ is a binary relation on Q.

To be able to add, multiply and compare measurements of observers, we provide an algebraic structure for the set of quantities with the help of our next axiom.

AxEOF The quantity part $\langle \mathrm{Q}; +, \cdot, < \rangle$ is a Euclidean ordered field, i.e., a linearly ordered field in which positive elements have square roots.

We treat AxFrame and AxEOF as part of our logic frame. Hence without any further mentioning, they are always assumed and are part of each axiom system we propose here.

3.1 Results on Clock Paradox

First we give a geometrical characterization of the clock paradox (a linear approximation of the twin paradox) within an axiom system of kinematics containing the following four axioms only:

AxSelf An *inertial* observer coordinatizes itself at a coordinate point iff its space component is the origin, i.e., space location $\langle 0, \ldots, 0 \rangle$.

AxLinTime The world-lines of *inertial* observers are lines on which time is elapsing uniformly.

AxEv Every *inertial* observer coordinatizes the very same events (meetings of bodies).

AxShift Any *inertial* observer observing another *inertial* observer with a certain time-unit vector also observes still another *inertial* observer, with the same time-unit vector, at each coordinate point of its coordinate domain.

$$\boxed{\mathsf{Kinem} := \{\, \mathsf{AxSelf}, \mathsf{AxLinTime}, \mathsf{AxEv}, \mathsf{AxShift} \,\}}$$

Situations in which clock paradox can occur (i.e., in which one *inertial* observer leaves another (stay-at-home observer), then meets a third one with whom it synchronizes its clock and who returns to the stay-at-home observer) are called clock paradox situations.

CP In every clock paradox situation, every *inertial* observer observes that the stay-at-home observer measures more time than the other two do together.

NoCP In every clock paradox situation, every *inertial* observer observes that the stay-at-home observer measures the same amount of time as the other two do together.

AntiCP In every clock paradox situation, every *inertial* observer observes that the stay-at-home observer measures less time than the other two do together.

Minkowski sphere of observer m, in symbols $MS^{\ddagger}_{m}$, is defined as the set of time-unit vectors (reflected to the origin if they point towards the past) of observers. We have given the following geometrical characterization of clock paradox and its variants:

Theorem 3.1. Assume Kinem. Then

$$\begin{aligned}
\mathsf{CP} &\iff \forall m \in \mathrm{IOb}\ \ MS^{\ddagger}_{m} \text{ is convex,}\\
\mathsf{NoCP} &\iff \forall m \in \mathrm{IOb}\ \ MS^{\ddagger}_{m} \text{ is flat,}\\
\mathsf{AntiCP} &\iff \forall m \in \mathrm{IOb}\ \ MS^{\ddagger}_{m} \text{ is concave.}
\end{aligned}$$

This characterization has several surprising consequences. To formulate them we need the following axioms:

AxThExp^{+} *Inertial* observers can move in any direction at any finite speed.

AxThExp* *Inertial* observers can move in any direction at a speed which is arbitrarily close to any finite speed.

AbsTime All *inertial* observers measure the same elapsed time between any two events.

By the following theorem, NoCP logically implies AbsTime if AxThExp^{+} and Kinem are assumed; however, if we assume the more experimental axiom AxThExp* instead of AxThExp^{+}, AbsTime does not follow from NoCP, which is an astonishing fact since it means that without the theoretical assumption AxThExp^{+} we would not be able to conclude that time is absolute in the Newtonian sense even if there were no clock paradox in our world.

Theorem 3.2.

$$\mathsf{AbsTime} \models \mathsf{NoCP}, \text{ and} \tag{1}$$

$$\mathsf{Kinem} + \mathsf{AxThExp}^{+} + \mathsf{NoCP} \models \mathsf{AbsTime}, \text{ but} \tag{2}$$

$$\mathsf{Kinem} + \mathsf{AxThExp}^{*} + \mathsf{NoCP} \not\models \mathsf{AbsTime}. \tag{3}$$

To formulate consequences of our characterization on special relativity, we introduce an axiom system of special relativity.

AxPh For every *inertial* observer, the speed of photons is 1.

AxSymDist *Inertial* observers agree as to the spatial distance between events if they are simultaneous for both of them.

AxThExp *Inertial* observers can move in any direction at any speed slower than 1, i.e., the speed of light.

$$\boxed{\mathsf{SpecRel} := \{\, \mathsf{AxSelf}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSymDist} \,\}}$$

The following statement is a consequence of our axiom system SpecRel:

SlowTime Relatively moving observers' clocks slow down.

We cannot logically compare SlowTime and CP within SpecRel since both of them are its consequences. Therefore we compare them within Kinem extended with AxPh. The following theorem shows that SlowTime is logically stronger than CP.

Theorem 3.3. Let $d \geq 3$. Then

$$\mathsf{Kinem} + \mathsf{AxPh} + \mathsf{SlowTime} \models \mathsf{CP}, \text{ but} \tag{4}$$

$$\mathsf{Kinem} + \mathsf{AxPh} + \mathsf{AxThExp} + \mathsf{CP} \not\models \mathsf{SlowTime}. \tag{5}$$

Like the similar results of [12] and [13], the following theorem also answers Question 4.2.17 of Andréka–Madarász–Németi [1]. It shows that CP is logically weaker than axiom AxSymDist of SpecRel.

Theorem 3.4. Let $d \geq 3$. Then

$$\mathsf{Kinem} + \mathsf{AxPh} + \mathsf{AxSymDist} \models \mathsf{CP}, \text{ but} \tag{6}$$

$$\mathsf{Kinem} + \mathsf{AxPh} + \mathsf{AxThExp} + \mathsf{CP} \not\models \mathsf{AxSymDist}. \tag{7}$$

These results are based on [12], [13] and [10].

3.2 Results on Relativistic Dynamics

Here we extend our approach to dynamics. The idea is that we use collisions for measuring relativistic mass. We could say that the relativistic mass of a body is a quantity that shows the magnitude of its influence on the state of motion of the other bodies it collides with. The bigger the relativistic mass of a body is, the more it changes the motion of the bodies colliding with it. To be able to formulate that, let us extend our first-order logic language by a new $(d+3)$-ary relation M for relativistic mass. We use this relation to speak about the relativistic masses of bodies according to observers by translating $\mathrm{M}(b, \vec{p}, x, k)$ as "the relativistic mass of body b at coordinate point $\vec{p}$ is x according to observer k." Since there can be more than one x which is M-related to b, $\vec{p}$ and k, we introduce the following definition: the relativistic mass of body b at $\vec{p} \in \mathrm{Q}^d$ according to observer k, in symbols $\mathsf{m}_k(b, \vec{p})$, is defined as x if $\mathrm{M}(b, \vec{p}, x, k)$ holds and there is only one such $x \in \mathrm{Q}$; otherwise $\mathsf{m}_k(b, \vec{p})$ is undefined.

AxMass According to any observer, the relativistic mass of a body at any coordinate point is defined and nonnegative, and it is zero iff the body is not present at the point.

AxCenter The world-line of the *inertial* body originated by an inelastic collision of two *inertial* bodies is the continuation of the center-line of the masses of the colliding *inertial* bodies according to every *inertial* observer.

The rest mass $\mathsf{m}_0(b)$ of body b is defined as $\lambda \in \mathrm{Q}$ if (1) there is an observer according to which b is at rest and the relativistic mass of b is λ, and (2) the relativistic mass of b is λ for every observer according to which b is at rest.

AxSpeed According to any *inertial* observer, the relativistic masses of two *inertial* bodies are the same if both of their rest masses and speeds are equal.

AxSpeed justifies the notation $\mathsf{m}_0(b)$.

Ax∀inecoll For any *inertial* observer, any possible kind of inelastic collision of *inertial* bodies can be realized.

AxThExp$^\uparrow$ For any *inertial* observer, in any spacetime location, in any direction, at any speed slower than that of light it is possible to "send out" an observer whose time flows "forwards."

$$\boxed{\mathsf{SpecRelDyn} := \left\{ \mathsf{AxMass}, \mathsf{AxCenter}, \mathsf{AxSpeed}, \mathsf{Ax\forall inecoll}, \mathsf{AxThExp}^{\uparrow} \right\} \cup \mathsf{SpecRel}}$$

Theorem 3.5. Let $d \geq 3$. Assume SpecRelDyn and let k be an *inertial* observer and b be an *inertial* body having rest mass. Then

$$\mathsf{m}_0(b) = \sqrt{1 - v_k(b)^2} \cdot \mathsf{m}_k(b).$$

This theorem is stronger than corresponding results in the literature since it requires fewer assumptions, and it also leads to the Einsteinian insight $E = mc^2$.

Proposition 3.6. $\mathsf{SpecRelDyn} \not\models \mathsf{ConsMass}$, and $\mathsf{SpecRelDyn} \not\models \mathsf{ConsMomentum}$.

In the spirit of AxCenter, we also formulate a geometrical axiom $\mathsf{AxCenter}^+$ which is equivalent to the conservation of mass and momentum (ConsMass and ConsMomentum). These results are based on [2], [3] and [9].

3.3 Results on Twin Paradox

It is clear that SpecRel is too weak to answer any nontrivial question about acceleration since none of its axioms mentions non-*inertial* observers. So we extend it by the following natural axiom on accelerated observers:

AxCmv Any accelerated observer at any event encountered coordinatizes the nearby world for a short while as an *inertial* observer does.

If we add AxCmv and two auxiliary axioms (AxSelf_0^+, AxEvTr) to our axiom system SpecRel, we get its following natural extension to accelerated observers:

$$\boxed{\mathsf{AccRel}_0 := \mathsf{SpecRel} \cup \{\, \mathsf{AxCmv}, \mathsf{AxSelf}_0^+, \mathsf{AxEvTr} \,\}}$$

AxSelf_0^+ An observer coordinatizes itself at a coordinate point iff its space component is the origin and the observer coordinatizes something there; and the set of time-instances in which an observer encounters an event is connected and has at least two distinct elements.

AxEvTr Every observer encounters the events in which it is observed.

Surprisingly, AccRel_0 does not imply the formulated version of the twin paradox (TwP). Moreover, by the following theorem, even adding the whole first-order theory of real numbers to this natural extension is not enough to get a theory that implies TwP.

Theorem 3.7. $Th(\mathbb{R}) + \mathsf{AccRel}_0 \not\models \mathsf{TwP}$.

At first sight this result suggests that our task cannot be carried out within first-order logic, which would be depressing as there are weighty methodological reasons for staying within first-order logic, see, e.g., [1] or Chapter 11. However, by using a trick from the methodology of approximating second-order theories by first-order ones, we can formulate an axiom schema (CONT) that says that every nonempty bounded subset of Q that is definable in our language has a supremum. By the following theorem, if we add CONT to our theory AccRel_0, we get a theory strong enough to imply TwP.

$$\boxed{\mathsf{AccRel} := \mathsf{AccRel} \cup \mathsf{CONT}}$$

Theorem 3.8. $\mathsf{AccRel} \models \mathsf{TwP}$ if $d \geq 3$.

These results are based on [6] and [13].

3.4 Results on Tower Paradox

Now we show that our theory AccRel is also strong enough to make predictions about gravitation by proving theorems about gravitational time dilation, which is roughly the same as "gravitation makes time flow slower," that is to say, clocks in the bottom of a tower run slower than clocks in its top.

We use Einstein's equivalence principle to treat gravitation in AccRel. So instead of gravitation we talk about acceleration and instead of towers we talk about spaceships. This way the gravitational time dilation becomes the following statement: "Time flows more slowly in the back of a uniformly accelerated spaceship than in its front." This statement can be translated into our language in several ways depending on which distance and simultaneity concepts we choose.

We define a spaceship, in symbols $>\!\!|b, k, c\rangle$, as a triplet of observers b, k and c such that b, k and c are coplanar and b and c are at (not necessarily the same) constant distances from k according to k.
In the case of radar simultaneity and radar distance, we prove the following theorem:

Theorem 3.9. Let $d \geq 3$. Assume AccRel. Let $>\!\!|b, k, c\rangle_{rad}$ be a radar spaceship (i.e., a spaceship which is determined by radar distance and synchronizes by radar simultaneity) such that: (i) observer k is positively accelerated, and (ii) the direction of the spaceship is the same as that of the acceleration of observer k. Then (1) the clock of b runs slower than the clock of c as seen by k by radar; and (2) the clock of b runs slower than the clock of c as seen by each of k, b and c by photons.

In the case of Minkowski simultaneity and Minkowski distance, we prove the following theorem:

Theorem 3.10. Let $d \geq 3$. Assume AccRel. Let $>\!\!|b, k, c\rangle_{\mu}$ be a Minkowski spaceship (i.e., a spaceship which is determined by Minkowski distance and synchronizes by Minkowski simultaneity) such that: (i) observer k is positively accelerated, (ii) the direction of the spaceship is the same as that of the acceleration of observer k, and (iii) observer b is not too far behind k. Then (1) the clock of b runs slower than the clock of c as seen by k by Minkowski simultaneity; and (2) the clock of b runs slower than the clock of c as seen by each of k, b and c by photons.

These results are based on [8], [7] and [11].

3.5 Results on General Relativity

We derive an axiomatic theory of general relativity from AccRel by "eliminating the privileged class of inertial reference frames," which was Einstein's original recipe for obtaining general relativity from special relativity, see [5]. So we realized Einstein's original program formally and literally, see [4]. We modify the axioms SpecRel and AxCmv one by one using the following two guidelines:

- let the new axioms not speak about *inertial* observers, and
- let the new axioms be consequences of the old ones and our theory AccRel.

The modified axioms are marked by a minus sign and instead of AxCmv^- we introduce a series of axioms, which are variants of AxCmv^- and each of which ensures the smoothness of world-view transformations to some degree.

$\mathsf{AxDiff_n}$ The world-view transformations are n-times differentiable functions.

We introduce a finite axiom system of general relativity for each natural number n:

$$\boxed{\mathsf{GenRel}_\omega := \{\, \mathsf{AxSelf}^-, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSymDist}^-, \mathsf{AxDiff_n} \,\} \cup \mathsf{CONT}}$$

We also introduce a smooth version which contains infinitely many axioms:

$$\boxed{\mathsf{GenRel}_\omega := \{\, \mathsf{AxSelf}^-, \mathsf{AxPh}^-, \mathsf{AxEv}^-, \mathsf{AxSymDist}^- \,\} \cup \{\, \mathsf{AxDiff_n} : n \geq 1 \,\} \cup \mathsf{CONT}}$$

The following theorems show that our axiom systems capture general relativity well.

Theorem 3.11. Let $d \geq 3$. Then $\mathsf{GenRel_n}$ is complete with respect to n-times differentiable Lorentzian manifolds over real closed fields.

Theorem 3.12. Let $d \geq 3$. Then GenRel_ω is complete with respect to smooth Lorentzian manifolds over real closed fields.

4 Conclusion

This thesis is a good example that within our flexible first-order logic framework several questions of relativity can be treated in a nice way; and that there are no unsurmountable barriers to extending our axiomatizations to general relativity. Moreover, we can obtain a simple, comprehensible and transparent first-order axiom system of general relativity in one natural step from AccRel, which fills the gap between special and general relativity, and is strong enough to make predictions about gravitation.

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