Tail behaviour of $\beta$-TARCH models

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1 Introduction

We examine here the tail behaviour of the stationary distribution of certain ARCH-type models defined by the equation

$$X_t = \left( \omega + \alpha_+ \left( X_{t-1}^+ \right)^{2\beta} + \alpha_- \left( X_{t-1}^- \right)^{2\beta} \right)^{1/2} Z_t, \quad (1)$$

where we apply the usual notations $x^+ = \max (x, 0)$ and $x^- = -\min (x, 0)$. The model parameters $\omega > 0$, $\min (\alpha_+, \alpha_-) \geq 0$, $\max (\alpha_+, \alpha_-) > 0$ and $Z_t$ is an i.i.d. sequence with zero mean and unit variance.

An important feature of this process is that it is an uncorrelated – when the autocorrelation function exists – but not an independent sequence; the conditional variance changes over time (conditional heteroscedasticity).
If $\beta = 1$ and $\alpha_+ = \alpha_- > 0$, we obtain the well-known ARCH (autoregressive conditionally heteroscedastic) model (Engle, 1982), where the conditional variance has a quadratic functional form.

Since this process can reproduce the stylised facts of financial time series, it has become a basic tool in financial econometrics in the past two decades, and gave rise to various generalisations.

For instance, in order to model the fact that the variance of stock returns responds more strongly to negative shocks than to positive ones, Glosten et al. (1993) defined the TARCH (threshold ARCH) process by allowing $\alpha_+ \neq \alpha_-$ in the equation with $\beta = 1$. (Hence $\alpha_+ < \alpha_-$ generally holds in financial applications.)

Due to the popularity of the quadratic ARCH models in finance, their probabilistic properties are quite much studied and well understood. It is a well-known fact for $\beta = 1$ that not all choices of $(\omega, \alpha_+, \alpha_-)$ and of the distribution of $Z_t$ permit a stationary solution of equation (1).
For instance, if $Z_t$ is normally distributed, the quadratic ARCH model has a stationary solution if and only if $\alpha_+ = \alpha_- < 2 \cdot e^\delta \approx 3.562$.

Also, much is known about the tail behaviour of the stationary distribution if $\beta = 1$.

It was proven two decades ago (Goldie, 1991) that the simple ARCH process has regularly varying (roughly speaking: polynomially decaying) tail even when $Z_t$ is normally distributed. This phenomenon is often summarised as: "light-tailed input generates heavy-tailed output".

More generally, Borkovec and Klüppelberg (2001) proved that even an AR(1) model driven by a quadratic ARCH(1) innovation has regularly varying tail for a wide class of noise distributions.

Using the concepts of extreme value theory (EVT) it follows that the stationary distribution of quadratic ARCH processes belongs to the maximum domain of attraction of the Fréchet extreme value distribution. Equivalently, their tail can be approximated by a generalised Pareto distribution (GPD) with shape parameter $\xi > 0$. 
The $0 < \beta < 1$ case – where the conditional variance is increasing slower than a quadratic function of the lagged values – is very different from the usual $\beta = 1$ parameter choice, and is much less studied in the literature. This model is called the $\beta$-TARCH process and was analysed e.g. by Guegan and Diebolt (1994).

It follows relatively easily from the drift condition for Markov chains (Meyn and Tweedie, 1993) that in the $0 < \beta < 1$ case the $X_t$ process defined by (1) is stationary irrespective of the choice for the parameters and for the distribution of $Z_t$ (provided that the latter has a finite second moment).

Moreover, if the $m$th moment of $Z_t$ is finite, the $m$th moment of the stationary distribution of $X_t$ will be finite, too (see Guegan and Diebolt (1994), or in a more general setting Elek and Márkus (2008)).

Hence, if all moments of $Z_t$ is finite and its distribution has infinite support, the distribution of $X_t$ may only belong to the maximum domain of attraction of the Gumbel law and, equivalently, the shape parameter of the GPD fitted to it may only be zero – if the distribution belongs to the maximum domain of attraction of an extreme value law at all.
This result already yields that the $\beta$-TARCH model is lighter tailed than the usual, quadratic specification: for light-tailed $Z_t$ noises the tail of $X_t$ decays faster to zero than a polynomial function.

The finding, however, does not determine the exact tail behaviour: the maximum domain of attraction of the Gumbel law contains many different types of distributions (e.g. normally, exponentially or lognormally decaying ones).

Our intention was to give a more precise estimate for the tail decay by showing that $X_t$ has approximately Weibull-like tail provided that $Z_t$ has a Weibull-like distribution.

Our research is motivated by the fact that $\beta$-TARCH models proved useful to model conditional heteroscedasticity in areas where the quadratic ARCH model was considered too heavy-tailed, such as in the analysis of water discharge series of rivers with large catchments. (See e.g. Elek and Márukus (2008) or in a broader context Szilágyi et al. (2006).)

Throughout the rest of the talk we will use the notations $\bar{F}_X(u) = 1 - F_X(u)$ for the survival function and $f_X(u)$ for the density function of the random variable $X$. 
2 Tail behaviour

To examine the tail behaviour of $X_t$ let us introduce an assumption on the tail of $Z_t$:

**Assumption 1.** $Z_t$ is an i.i.d. sequence with an absolutely continuous probability distribution. Moreover, there exist $u_0 > 0$, $\gamma > 0$, $\kappa > 0$, $K_1 > 0$ and $K_2$ such that its probability density satisfies

$$f_{Z_t}(u) = K_1 |u|^{K_2} \exp (-\kappa |u|^{\gamma})$$

(2)

for every $|u| > u_0$.

According to this assumption, $Z_t$ is symmetric and has a Weibull-like tail with exponent $\gamma$. The Gaussian ($\gamma = 2$) or the Laplace ($\gamma = 1$) distributions are obtained as special cases.

Guegan and Diebolt (1994) showed under the assumption $\min (\alpha_+, \alpha_-) > 0$ that if $\beta > (\gamma - 1)/\gamma$, $X_t$ has no exponential moment (i.e. it is heavier tailed than the exponential distribution) while if $\beta < (\gamma - 1)/\gamma$, then $X_t$ does have a moment generating function defined around the neighbourhood of zero.
Here we prove the following theorem showing that $X_t$ has approximately Weibull-like tail.

**Theorem 1.** Assume equation (1), Assumption 1, $\omega > 0$, $\min(\alpha_+, \alpha_-) > 0$ and $0 < \beta < 1$. Then, using the notation $\alpha = \max(\alpha_+, \alpha_-)$,

$$\exp \left( -\frac{\alpha^{-\gamma/2} \kappa \gamma \beta^{-\frac{1-\beta}{\gamma}}}{2} u^{\gamma(1-\beta)} + O \left( u^{\gamma(1-\beta)/2} \right) \right) \leq \bar{F}_{X_t}(u) \leq \exp \left( -\frac{(\alpha + \omega)^{-\gamma/2} \kappa \gamma \beta^{-\frac{1-\beta}{\gamma}}}{2} u^{\gamma(1-\beta)} + O \left( u^{\gamma(1-\beta)/2} \right) \right).$$

(3)

**Proof.** We may assume without loss of generality that $\alpha = \alpha_+ \geq \alpha_-$. Let $Y_t = \log(X_t^2)$, $U_{t,1} = \log(\alpha_+ Z_t^2)$, and $U_{t,2} = \log(\alpha_- Z_t^2)$. Furthermore, let us introduce the functions

$$h_1(y) = \log(\omega/\alpha_+ + \exp(\beta y)),$$

$$h_2(y) = \log(\omega/\alpha_- + \exp(\beta y))$$

and the random variables $V_{t,i} = h_i(Y_{t-1}) - \beta Y_{t-1} (i = 1, 2)$. Then

$$Y_t = h_1(Y_{t-1}) + U_{t,1} = \beta Y_{t-1} + U_{t,1} + V_{t,1} \quad \text{if} \quad Z_{t-1} > 0,$$

$$Y_t = h_2(Y_{t-1}) + U_{t,2} = \beta Y_{t-1} + U_{t,2} + V_{t,2} \quad \text{if} \quad Z_{t-1} \leq 0.$$
Since \( h_i(y) \geq \beta y (i = 1, 2) \), \( V_{t,i} \geq 0 \) a.s. Moreover, as \( Z_t \) is a symmetrically distributed i.i.d. sequence, \( Y_t \) can be written as

\[
Y_t = \beta Y_{t-1} + U_t + V_t
\]

where \( U_t \) is an independent 1/2-1/2 mixture of \( U_{t,1} \) and \( U_{t,2} \) and is itself and i.i.d. process. Similarly \( V_t \) is an independent 1/2-1/2 mixture of \( V_{t,1} \) and \( V_{t,2} \).

Let us introduce the auxiliary sequence

\[
Y^*_t = \beta Y^*_{t-1} + U_t = \sum_{i=0}^{\infty} \beta^i U_{t-i}. 
\]

\( Z_t \) has finite variance and its distribution is absolutely continuous (at zero as well), thus \( E \left( U_t^2 \right) < \infty \), hence \( Y^*_t \) exists and \( E \left( Y^*_t \right)^2 < \infty \). It is clear that \( Y^*_t \leq Y_t \), therefore by examining the tail behaviour of \( Y^*_t \) we obtain a lower bound for the tail of \( Y_t \) as well.

To determine the tail of \( Y^*_t \), we will apply the framework of Klüppelberg and Lindner (2005) who examined the tail behaviour of linear moving average processes with increments lighter tailed than the exponential distribution.
Let $\sum_{-\infty}^{\infty} c_i W_{i-t}$ be the examined process and assume that the probability density of the i.i.d. sequence $W_t$ satisfies

$$f(u) = \nu(u) \exp(-\psi(u)), \quad u \geq u_0$$

for some $u_0$, and $\psi(u)$ is $C^2$, $\psi'(u_0) = 0$, $\psi'(\infty) = \infty$, $\psi''$ is strictly positive on $[u_0, \infty]$ and $\phi = 1/\sqrt{\psi''}$ is self-neglecting, i.e.

$$\lim_{u \to \infty} \frac{\phi(u + x\phi(u))}{\phi(u)} = 1$$

uniformly on bounded $x$-intervals. The function $\nu$ is assumed to be flat for $\phi$, i.e.

$$\lim_{u \to \infty} \frac{\nu(u + x\phi(u))}{\nu(u)} = 1$$

uniformly on bounded $x$-intervals. (Roughly speaking, these assumptions require that $\psi(u)$ should be strictly convex and $\nu(u)$ should behave approximately as a constant as $u \to \infty$.)

Furthermore, following the original notations of Klüppelberg and Lindner (2005), define

$$q(\tau) = \psi'^{-1}(\tau), \quad q_i(\tau) = c_i q(c_i \tau), \quad Q(\tau) = \sum_{i=-\infty}^{\infty} q_i(\tau),$$

$$\sigma_i^2(\tau) = q_i'(\tau), \quad \sigma_\infty^2(\tau) = \sum_{i=-\infty}^{\infty} \sigma_i^2(\tau).$$

It follows from the conditions that $Q$ is a strictly increasing function. Then, provided that $c_i$ is a summable sequence of
non-negative real numbers, not all zero, and assuming that the two conditions below hold:

\[
\lim_{m \to \infty} \limsup_{\tau \to \infty} \frac{\sum_{|j| > m} \sigma_j^2(\tau)}{\sigma_\infty^2(\tau)} = 0, \tag{5}
\]

\[
\lim_{m \to \infty} \limsup_{\tau \to \infty} \frac{\sum_{|j| > m} \sigma_j(\tau)}{\sigma_\infty^2(\tau)} = 0, \tag{6}
\]

the following theorem is true:

**Theorem 2.** \((\text{Klüppelberg and Lindner, 2005})\) Under the above conditions, as \(u \to \infty\),

\[
P \left( \sum_{i=-\infty}^{\infty} c_i W_{t-i} > u \right) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{Q^{-1}(u) \sigma_\infty(Q^{-1}(u))} \exp \left( -\int_{u_0}^{u} (Q^{-1}(v) + \rho(Q^{-1}(v))) \, dv \right)
\]

where \(\rho(\tau) = o(1/\sigma_\infty(\tau))\). It is also true that \(1/\sigma_\infty(\tau) = o(\tau)\) so the first term in the integral is the leading term.

In our case, this theorem will be used with the choice \(U_t = W_t\) so the conditions of the theorem should be checked first.

Here,

\[
f_{U_t}(u) = \frac{1}{2} \left( K_1 \exp(K_2 u) \exp(-\kappa_{\alpha}^{\gamma/2} e^{\gamma u/2}) \right) + \frac{1}{2} \left( K_3 \exp(K_4 u) \exp(-\kappa_{\alpha_-}^{\gamma/2} e^{\gamma u/2}) \right) = \nu(u) \exp(-\psi(u))
\]
with appropriate constants $K_1 > 0$, $K_3 > 0$ and $K_2$, $K_4$. To satisfy the necessary assumptions with $u_0 = 0$ in (4), $\psi(u)$ can be defined as

$$
\psi(u) = \kappa \alpha^{-\gamma/2} e^{\gamma u} - e \kappa \alpha^{-\gamma/2} / 2 \quad \text{if } u \geq 2/\gamma,
$$

$$
\psi(u) = e \kappa \alpha^{-\gamma/2} (\gamma/2)^2 u^2 / 2 \quad \text{if } u < 2/\gamma.
$$

Then it is a matter of routine to check that the resulting $\nu(u)$ function is flat for $\psi(u)$ and that $\phi(u)$ is self-neglecting (see also Example 2.4. (c) in Klüppelberg and Lindner (2005)), so the tail of $Y^*_t$ can be in principle approximated using $c_i = \beta^i$ for $i \geq 0$ and $c_i = 0$ for $i < 0$. (Conditions (5)-(6) will be checked later, see below.) Using the notation $\tau_0 = e \kappa \alpha^{-\gamma/2} \gamma/2$, we obtain

$$
\psi'(u) = \kappa \alpha^{-\gamma/2} (\gamma/2) e^{\gamma u} = e^{-1} \tau_0 e^{\gamma u} \quad \text{if } u \geq 2/\gamma,
$$

$$
\psi'(u) = e \kappa \alpha^{-\gamma/2} (\gamma/2)^2 u = \tau_0 (\gamma/2) u \quad \text{if } u < 2/\gamma
$$

and hence

$$
q(\tau) = 2 \gamma^{-1} \log(e \tau / \tau_0) = 2 \gamma^{-1} (\log \tau - \log \tau_0 + 1) \quad \text{if } \tau \geq \tau_0,
$$

$$
q(\tau) = 2 \gamma^{-1} \tau / \tau_0 \quad \text{if } \tau < \tau_0.
$$
Then
\[
Q(\tau) = \sum_{j=0}^{\infty} \beta^j q(\beta^j \tau) = 2\gamma^{-1} \sum_{j=0}^{\infty} \beta^j \log(\beta^j e^{\tau/\tau_0}) \\
+ 2\gamma^{-1} \sum_{j: \beta^j \tau < \tau_0} \beta^j (\beta^j \tau/\tau_0 - \log(\beta^j e^{\tau/\tau_0})) .
\] (7)

Nevertheless, for any \(0 < \theta < 1\), the sum in the second term can be written as
\[
\sum_{j: \beta^j \tau < \tau_0} \beta^j (\beta^j \tau/\tau_0 - \log(\beta^j e^{\tau/\tau_0})) \\
= (e^{\tau/\tau_0})^{-\theta} \sum_{j: \beta^j \tau < \tau_0} (\beta^{1-\theta} e^{\theta(\beta^j \tau/\tau_0)})^{1+\theta} \\
- (\beta^j e^{\tau/\tau_0})^\theta \log(\beta^j e^{\tau/\tau_0}) .
\] (8)

For \(0 < \theta < 1\) the function \(g(x) = x^\theta \log x\) is bounded on \((0, e]\), hence \((\beta^j e^{\tau/\tau_0})^\theta \log(\beta^j e^{\tau/\tau_0})\) and trivially \(e^\theta (\beta^j \tau/\tau_0)^{1+\theta}\) are bounded if \(0 < \beta^j \tau/\tau_0 < 1\). Thus, as \(\tau \to \infty\),
\[
\sum_{j: \beta^j \tau < \tau_0} (\beta^{1-\theta} e^{\theta(\beta^j \tau/\tau_0)})^{1+\theta} \\
- (\beta^j e^{\tau/\tau_0})^\theta \log(\beta^j e^{\tau/\tau_0}) \\
= O(1) \sum_{j: \beta^j \tau < \tau_0} (\beta^{1-\theta})^j = o(1)
\]
because \(\tau \to \infty\) implies that \(j \to \infty\) in the summation condition.
Therefore the sum in (8) (and so the second term in (7)) is $o\left(\tau^{-\theta}\right)$, thus

\[
Q(\tau) = 2\gamma^{-1} \sum_{j=0}^{\infty} \beta^j \left( \log (\beta^j \tau) + 1 - \log \tau_0 \right) + o\left(\tau^{-\theta}\right)
\]

\[
= 2\gamma^{-1} (1 - \beta)^{-1} \left( \log \tau + \beta (1 - \beta)^{-1} \log \beta + 1 - \log \tau_0 \right) + o\left(\tau^{-\theta}\right)
\]

\[
= A (\log \tau + B) + o\left(\tau^{-\theta}\right)
\]

as $\tau \to \infty$, using the notations

\[
A = 2\gamma^{-1} (1 - \beta)^{-1} \quad \text{and} \quad B = \beta (1 - \beta)^{-1} \log \beta + 1 - \log \tau_0.
\]

Trivially, $Q^{-1}(u) \to \infty$ and hence

\[
\exp \left( -o \left( (Q^{-1}(u))^{-\theta} \right) \right) = 1 + o \left( (Q^{-1}(u))^{-\theta} \right) = 1 + o(1)
\]

as $u \to \infty$, therefore

\[
Q^{-1}(u) = \exp \left( A^{-1} u - B \right) \exp \left( -o \left( (Q^{-1}(u))^{-\theta} \right) \right)
\]

\[
= \exp \left( A^{-1} u - B \right) (1 + o(1)).
\]

Using these we obtain a better estimate for $Q^{-1}(u)$ in the
second round:

\[
Q^{-1}(u) = \exp \left( A^{-1}u - B \left( 1 + o \left( (Q^{-1}(u))^{-\theta} \right) \right) \right) = \exp \left( A^{-1}u - B \left( 1 + o \left( \exp \left( -\theta A^{-1}u \right) \right) \right) \right) \]

\[
= \frac{\kappa \gamma \alpha^{-\gamma/2} \beta^{-\beta}}{2} \exp \left( \frac{\gamma (1 - \beta)}{2}u \right) + o \left( \exp \left( \frac{\gamma (1 - \beta)(1 - \theta)}{2}u \right) \right).
\]

Let us also check the conditions (5)-(6). We obtain that

\[
q' (\tau) = 2\gamma^{-1}/\tau_0 \text{ if } \tau < \tau_0 \text{ and } q' (\tau) = 2\gamma^{-1}/\tau \text{ if } \tau \geq \tau_0,
\]

hence \( \sigma^2_i (\tau) = 2\gamma^{-1} \beta^{2i}/\tau_0 \text{ if } \beta^{i}\tau < \tau_0 \) and \( \sigma^2_i (\tau) = 2\gamma^{-1} \beta^i/\tau \text{ if } \beta^{i}\tau \geq \tau_0. \) Thus

\[
\sigma^2_\infty (\tau) = 2\gamma^{-1} \left( \sum_{j \geq \tau \geq \tau_0} \beta^j/\tau + \sum_{j < \beta^j \tau < \tau_0} \beta^{2j}/\tau_0 \right)
\]

\[
\sim 2\gamma^{-1} (1 - \beta)^{-1}/\tau,
\]

so (5)-(6) is easily seen to hold for the \( c_i = \beta^i \) sequence.
Thus Theorem 2. can be applied. We obtain

$$\bar{F}_{Y_t^*}(u) = \exp \left( - \log Q^{-1}(u) - \log \sigma_\infty \left( Q^{-1}(u) \right) \right)$$

$$- \int_0^u \left( Q^{-1}(v) + O \left( e^{\frac{\gamma(1-\beta)}{4}v} \right) \right) dv$$

$$= \exp \left( - \frac{\kappa \alpha - \gamma/2 \beta - \frac{\beta}{1-\beta}}{1-\beta} e^{\frac{\gamma(1-\beta)}{2}u} + O \left( e^{\frac{\gamma(1-\beta)}{4}u} \right) \right). \quad (9)$$

Taking into account that \( Y_t^* \leq Y_t = \log \left( X_t^2 \right) \), the lower bound is obtained for \( \bar{F}_{X_t}(u) \) in (3).

To show the upper bound for the tail, let us first observe that, trivially, the increase of either \( \alpha_+ \) or \( \alpha_- \) does not make the tail of \( Y_t \) lighter. Therefore, we can assume that \( \alpha = \alpha_+ = \alpha_- \) and get an upper bound for the tail of this restricted model.

In this case, let us introduce for each \( t \) a random variable \( U_{t}^{**} \geq 0 \) such that \( U_t \leq U_{t}^{**} \) a.s. and \( f_{U_{t}^{**}}(u) = K f_{U_t}(u) \) for all \( u > 0 \) with an appropriate \( K > 0 \). (Such a variable can be easily constructed.) Define also \( h(y) = \beta y \chi_{\{y \geq 0\}} + \log \left( 1 + \omega / \alpha \right) \). It follows from \( \alpha_+ = \alpha_- \) that \( h_i(y) \leq h(y) \) (\( i = 1, 2 \)) and thus it can be shown straightforwardly that \( \bar{F}_{Y_{t}^{**}}(u) \geq \bar{F}_{Y_t}(u) \) holds for the stationary distribution of the model defined by

$$Y_{t}^{**} = h \left( Y_{t-1}^{**} \right) + U_{t}^{**}.$$
The result of Klüppelberg and Lindner (2005) again gives that the tail of $\sum_{i=0}^{\infty} \beta^i U_{t-i}^{**}$ has the same form as the tail of $Y_t^*$ (equation (9)). Then it is easy to derive the upper bound for the tail of $X_t$ in (3).

In some applications (e.g. Elek and Márkus (2008)) the $\alpha_- = 0$ restriction is used. Then the upper bound in (3) certainly holds and a slightly weaker lower bound can also be proven easily:

**Proposition 1.** Assume the assumptions of Theorem 1 but allow $\alpha_- = 0$. Then for every $\delta > 0$ there exists a $K > 0$ such that

$$\exp \left( -K u^{(1+\delta)\gamma(1-\beta)} \right) \leq \bar{F}_{X_t}(u).$$

(10)

**Proof.** Let us use the same notations as in the proof of Theorem 1 and let $Y_t^{***}$ be defined by

$$Y_t^{***} = \beta Y_{t-1}^{***} + U_t^+ \quad \text{if} \quad Z_t \geq 0,$$

$$Y_t^{***} = U_0^t \quad \text{if} \quad Z_t < 0,$$

where $U_0^t = \log \left( \alpha_0 Z_0^2 \right)$. It is easily shown that $Y_t^{***} \leq Y_t$ stochastically in this case. Moreover, as $Z_t$ is symmetrically distributed, for every $n \in \mathbb{Z}^+$ with probability $2^{-n}$

$$Y_t^{***} = \sum_{i=0}^{n-1} \beta^i U_{t-i}^+ + \beta^n Y_{t-n}^{***}.$$
Therefore, using the notations $q = \bar{F}_{Y_{t-n}^*}(0)$ and $Y_{t,n} = \sum_{i=0}^{n-1} \beta^i U_{t-i}^+$, 

$$\bar{F}_{Y_{t}^*}(u) \geq q 2^{-n} \bar{F}_{Y_{t,n}}(u).$$

Moreover, similarly to the derivation of the tail of $Y_{t}^*$, it follows again from Theorem 2 that 

$$\bar{F}_{Y_{t,n}}(u) = \exp\left(-Ke^{\gamma(1-\beta)} + O\left(e^{\gamma(1-\beta)\frac{2}{(1-\beta^n)}}\right)\right)$$

with a suitable $K > 0$. Choosing $n$ such that $1/ (1 - \beta^n) < 1 + \delta$ we obtain 

$$\bar{F}_{Y_{t}^*}(u) \geq \exp\left(-Ke^{\frac{(1+\delta)\gamma(1-\beta)}{2}}\right)$$

with a possibly different $K > 0$, and transforming $Y_t$ to $X_t$ gives the statement of the proposition. \qed
3 Conclusions

In this paper we showed that the tail of a $\beta$-TARCH model can be approximated by Weibull-like distributions with exponent $\beta (1 - \gamma)$ if the generating noise has Weibull-like tail with exponent $\gamma$. It is a natural question to ask how the tail behaviour is modified when AR- or MA-terms are added to the simple uncorrelated $\beta$-TARCH model. Unfortunately, this question is not yet settled but we conjecture that the more general ARMA-$\beta$-TARCH model has approximately a Weibull-like tail, too.

References


