CDO Valuation Using Stochastic Correlations in Structural Models

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CDO-s are credit derivatives created to transform the otherwise illiquid credit portfolio of a bank into tradable liquid assets.

CDO-s are the black sheeps of the financial industry, pinpointed as the instruments, responsible for the subprime crisis (joined by mathematicians and their methodologies for pricing CDO-s.)

Although once thriving, their trade nowadays is scaled back significantly.

But the contracts were for 5 to 10 years so the assets are still there on the market or in the portfolio of a bank or investor; hence one has to be able to valuate them.

The crucial element in evaluating CDO-s is the interdependence structure of the underlying assets i.e. the credit defaults of single names. (The credited company, institution or individual is called the Name.)
CDO-s in the literature

- The creation of the instrument itself goes back to the ’80s.
- Cash CDOs have more complex path-dependent payouts (”waterfalls”) and very little research and literature is available on them.
- Synthetic CDOs have more simple and standardized payouts and are typically valued with more sophisticated models.
- I have no pretension on giving a full account of the rich literature on CDO’s, just mention Duffie, Singleton, Garleanu, Lando, Giesecke, Hardle, Gregory etc. as the authors of early works and add that recent publications of Das (2005), Kothari (2006), Rajan et al. (2007), Mounfield(2009), Brigo et al.(2012) can serve as good references for an overview.
Introduction
Single name calibration
Stochastic correlation
Base correlations

CDO-s in the literature

- Focusing more on the interdependence structures, the Gaussian Copula was introduced in the CDO world by Li (2000).
- Other multifactor copula models were considered by Laurent and Gregory (2003), Andersen and Sidenius (2004), Hull and White (2004)
- Hull and White (2006) invented the concept of implied copula.
- I have no knowledge about the origin of the idea of base correlations but Torresetti et al. (2006) give a detailed accounting on their merits and drawbacks.
- The payoffs of CDO tranches can be valued in terms of the expected tranched losses (ETLs), first presented in Torresetti et al. (2006). ETL-s are natural quantities that can be implied from market data and in general, the implied ETL surface can be used to value tranches with non-standard attachments and maturities as an alternative to implied correlation.
How CDO works

- In general, a CDO is a structured finance product in which the credit risk on a pool of assets is sold to investors.
- We are given a portfolio of names that may default thus generating losses to investors exposed to those names.
- In a (non-actuarial) sense, CDOs look like contracts selling (or buying) insurance on portions of the loss of that portfolio.
- The valuation determines the fair price of this insurance.
How CDO works

- Two parties enter into contract concerning a CDO tranche: a protection buyer and a protection seller. A tranche is a portion of the loss of the portfolio between two percentages.

- For example, the tranche with 3% attachment and 6% detachment collects all losses between 3% and 6% of the total notional.

- The protection seller agrees to pay to the buyer all notional default losses (minus the recoveries) in the portfolio whenever they occur due to one or more defaults of the entities, within 3% and 6% of the total pooled loss, until either final maturity arrives (at $T = T_b$) or the tranche is exhausted (notional defaults exceed 6%).

- In exchange for this, the buyer pays the seller a periodic fee on the notional given by the portion of the tranche that is still "alive" in each relevant period.
The tranches that are quoted on the market refer to standardized pools, meaning that standard attachment points are used.

The traded tranches are slightly different in Europe and the US: an equity tranche comprises all losses between 0% and 3%, Mezzanine and senior tranches for the DJ-iTRAXX Europe cover 3%-6%, 6%-9%, 9%-12% and 12%-22%. whereas for the main US index, the DJ CDX NA: 3%-7%, 7%-10%, 10%-15% and 15%-30%.

The market quotes either the periodic premium rate $S_{A:B}^0$ of these tranches or their upfront premium rate $U_{A:B}$ for maturities $T = 3y; 5y; 7y; 10y$. Tranches with low detachment points (3%) are usually quoted in terms of the upfront premium, while tranches with higher detachment points are quoted in terms of the periodic premium.
Structural models for CDO-s

- The single name value is given by a non-observable asset price process (Geometric BM, Exponential Lévy, etc).
- Default of a single name occurs when the value process associated to the given name hits a barrier curve.
- For illustration, we choose here simple single name (log-)value processes, e.g., a BM or an Ornstein–Uhlenbeck (OU) process.
- The barrier curve is determined by the hazard rate inferred from market and the value process structure of the name.
- CDO tranche prices are then determined by the interdependence structure of the single name processes.
- The interdependence structure is originated in the interactions of the components through their evolution dynamics; hence the model is a structural one.
- Copulas are only used to visualise these interdependences and not for specifying it.
Our goals

- The barrier curve for a single name is calculated from the survival probability curve, inferred from market quotes of CDS written on the name.
- Our aim is to create an interdependence structure so that its parameters can be changed without distorting the marginal distributions.
- This way the marginals, the names, can be calibrated from CDS quotes and the interdependence remains freely variable in order to calibrate them by CDO tranche quotes.
- The names are furnished with suitable interdependence structure by modifying the simple one factor model.
- We achieve it by specifying the driving forces as non-linearly dependent (so called stochastically correlated) Wiener processes.
First Passage Time and Inverse First Passage Time problem

Continuous version:
- For a given barrier, what is the probability that the process does not hit the barrier before $t$. **Direct problem, FPT.**
- For a given survival probability curve, what is the corresponding barrier. **Inverse problem, IFPT.**

Sampled version:
- $0 < t_1 < t_2 < \ldots$ are the sampling time points.
- For a given $n$, what is the probability that the process is below the barrier for $t_1, \ldots, t_n$. **Direct problem.**
- For a given survival probability curve, what is the corresponding (sampled) barrier. **Inverse problem.**
Solution for the sampled IFPT problem

- For single name calibration we need to solve the IFPT problem.
- Explicit solution for the continuous version exists for some cases.
- Applying this curve to the sampled version results in a bias that is considerable given the small default probabilities and the fixed sampling frequency.
- We need to reconsider the solution and use Monte Carlo
When using Monte Carlo, we must keep in mind that defaults are rare events as typical hazard rates are between 0.01 and 0.1.

In order to
- see defaults more often,
- decrease the number of simulations needed and
- reduce the variance of the estimation,
we may use importance sampling.

We add a deterministic drift to the trajectories so that they reach the barrier earlier with higher probability.

To keep track of the change in default probabilities the Radon-Nikodym derivative can be determined using Girsanov’s theorem.
The choice of drift

The drift should be chosen to

- minimize the standard deviation of the estimation;
- minimize the relative hazard rate error;
- remain moderate as too large drift creates simultaneous defaults at the same time making estimations worse or useless.

We do not find the optimal drift but give an upper bound on the standard deviation, which is easier to minimize, and whose minimum point is close to the optimal one.
Creating interdependence structure

- We create an interdependence structure of Wiener processes that keeps marginal distribution intact, while varies the joint distribution and creates tail dependence.
- Interdependence should be parameterized by a few parameters only.
- Should be flexible enough to calibrate to tranche prices.
- Should create proper base correlation skew.
- Ideally, should be structural - created by dynamics.

Our suggestion: use stochastic correlations.
Suppose the (log-)value or credit worthiness $V_i(t)$ of name (company) $i$ to be described by Ornstein-Uhlenbeck (OU) processes.

$$dV_i(t) = -\alpha_i \cdot V_i(t) dt + \sigma_i \cdot dW_i(t). \tag{1}$$

We are interested in the default time of names, and their interdependence. Interdependence stems from the OU processes and ultimately from the generating Wiener $W_i(t)$ processes. It is created by a one factor model, the factor $Z(t)$ representing the economic environment.
One factor model for the noises

Both factor and idiosynchratic components are Wiener processes. Once $\rho(t)$ is given, $W_i(t)$ is obtained as

$$dW_i(t) = \rho(t) \cdot dZ(t) + \sqrt{1 - \rho^2(t)} d\varepsilon_i(t).$$ (2)

- The loadings $\rho(t), \sqrt{1 - \rho^2(t)}$ are time dependent, random and adapted to the Wiener filtration.
- Were $\rho(t) \equiv \rho$ constant, it would equal $\text{corr}(W_i(t), Z(t))$.
- Hence $\rho(t)$ is called stochastic correlation
- $\rho(t) \in L_2(\Omega \times [0, T])$ as it is bounded, and the time horizon is finite. So, its integral by a Wiener process is a martingale.
- As the quadratic variation of $W_i(t)$ is $\int_0^T (\rho^2(t)) + (1 - \rho^2(t)) dt = t$, Lévy’s characterization theorem yields that all $W_i(t)$-s are Wiener processes.
- So, marginal distributions are preserved as requested.
Creating stochastic correlation

- Stochastic correlation - given as the product of two components:
  \[ \rho(t) = h(Z(t)) \cdot \zeta(t). \]

- Various methods may specify \( h(Z(t)) \) and \( \zeta(t) \).
- One may be based on an arbitrary distribution function, e.g. that of the standard normal law \( \Phi \), as
  \[ h(Z(t)) = \Phi(c \cdot \frac{Z(t)}{\sqrt{t}} + a). \]

  where the constants \( c \) and \( a \) regulate the interdependence.

- Small common factor values - close-to-zero ”correlation”
- Large common factor values - close-to-one ”correlation”
Creating stochastic correlation

\( \zeta(t) \) is more of stochastic character.

- The normalised factor-increments \( \frac{\Delta Z(t)}{\sqrt{\Delta t}} = X(t) \) in discrete time are standard normal, their squares are \( \Gamma\left(\frac{1}{2}, \frac{1}{2}\right) \) distributed.

- The fraction \( \frac{X^2}{X^2 + Y} \) is suitable for correlation value, as it is Beta distributed when \( Y \) is \( \Gamma\left(d, \frac{1}{2}\right) \), independent of \( Z \).

- This definition depends on the scale, that is the magnitude of \( \Delta t \). We can get rid of this inconvenient scale dependence by considering a fixed time length for the Wiener increment used.

- In this context the \( \Gamma\left(d, \frac{1}{2}\right) \) distributed \( Y \) variable can be interpreted as the increment on the same fixed time length of a gamma process of independent, stationary increments.

- Lévy’s theorem remains in force and the mentioned marginal distributions remain intact. The interdependence varies with the shape parameter \( d \) of \( Y \).
An alternative stochastic correlation

Alternatively $\rho_t$ may follow a Jacobi process, (cf. Gourieroux Valery 2002) generated from the common factor, $Z_t$, as:

$$d\rho_t = \kappa (R - \rho_t) \, dt + \sigma_3 \cdot \sqrt{(R_+ - \rho_t)(\rho(t) - R_-)} \, dZ_t$$  \hspace{1cm} (3)

Here $\kappa$ represents the mean-reverting parameter, $R$ the mean of the process and $\sigma_3$ the volatility coefficient.

- This process is known to be geometrically ergodic.
- It takes values between $R_-$ and $R_+$, specifically, between 0 and 1 when the term under the square root reduces to $\rho - \rho^2$, and between -1 and 1 when it reduces to $1 - \rho^2$. Hence it is a meaningful correlation coefficient.
- The marginal distribution of the stationary Jacobi process with $\rho - \rho^2$ under the square root is beta.
Copulas for interdependence

The variability of the suggested interdependence structure is of main concern. Copulas are known to determine interdependence in full and independently of the marginal distributions.

- We analyze our interdependence structures through their copulas.
- For illustrative purposes we confine ourselves to 2 dimensions.
- We generate 1000 samples of two Wiener processes and create the empirical copulas of $W_1(t)$, $W_2(t)$ at an arbitrarily chosen but fixed time point $t$.
- Formal tests and numerical methods work well in higher dimensions, and results are available in terms of those, too.
Constant correlation creates Gaussian copulas
Copula for stochastic correlation

- Functional dependence only \( \rho(t) = h(Z(t)) \),
- At low values—wide spread —independence
- at large values—strong concentration —tail dependence
- Such a strong synchronisation may be too excessive.
\[ \rho(t) = h(Z(t)) \cdot \zeta(t), \]

- The presence of the stochastic \( \zeta(t) \) decreases concentration in the upper right corner.
Known copula families are rejected

- Formal tests (e.g., Cramer-von Mises statistics) reject any well-known copula family:
  
<table>
<thead>
<tr>
<th>Copula</th>
<th>Statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gumbel</td>
<td>0.1629</td>
<td>&lt; 0.001</td>
</tr>
<tr>
<td>Gaussian</td>
<td>0.1626</td>
<td>&lt; 0.001</td>
</tr>
<tr>
<td>Galambos</td>
<td>0.1686</td>
<td>&lt; 0.001</td>
</tr>
<tr>
<td>Hüsler–Reiss</td>
<td>0.2108</td>
<td>&lt; 0.001</td>
</tr>
</tbody>
</table>

- Density plots show significant difference from fitted copula families.
The Clayton family is the closest

The smallest difference can be seen in the density plots generated from the Clayton family. It is in line what we got from formal tests: value of Cramer-von Mises statistics 0.0316, p-value = 0.064.
Jump processes for name values

- We created a multidimensional process with Wiener marginals, but the joint distribution at any time point $t$ is not even multivariate normal.

- Using a subordinator (e.g. an Inverse Gaussian process) we can extend our model to time changed Brownian Motions as marginals.

- Interpreting the subordinator as business time, it is common for every name, meaning that the arrival of trading information is instantaneously available for everyone.

- This is still the "cheap" generalisation.
Copulas of stochastically correlated Wiener and the derived NIG processes

Tail dependence of the NIG processes is stronger.
Generalisation for $\alpha$-stable processes

- We may have $\alpha$-stable processes for factor and idiosynchratic components.
- Given the result of Wang, Stoev and Roy (2011) if for

$\sum_{k=0}^{K} |\rho_k(s)|^\alpha = 1$ and $X_k(t) = \int_S \rho_k(s)f_k(s)dM_\alpha s$.

- This is a genuine stochastic correlation generalisation.
- Both factor and idiosynchratic components are $\alpha$-stable as are the value processes. Requiring this coincidence of distributions, it is clearly not possible to go beyond stable processes. But this is not a necessary requirement.
Computing Base Correlations

- Wishing to put to the test our suggested interdependence structure we have to expose it to market data.
- Despite its shortcomings the Gaussian copula method for pricing is still there on the market as industry standard for pricing CDO-s.
- The situation is very much the same as with the celebrated Black-Scholes pricing of options.
- Just as volatility in Black-Scholes, correlation is constant in the Gaussian copula model.
- So, similarly to implied volatility, correlations can be inferred "back" from market data, though problems of uniqueness for implied correlations appear.
- We wish to check whether the base correlations obtained from our models are rich enough to be able to match market data.
Computing Base Correlations

- The CDO **default leg** $DL$ is the cash flow of protection payments corresponding to the defaulted names of the pool.
- The cash flow of the periodic premium paid from the protection buyer to the seller is the CDO’s **premium leg** $PL$.

We denote by $L_t$ the portfolio cumulated loss and with $C_t$ the notional value of defaults up to time $t$. (Contracts are written on notionals and not on spot or present values!) The discounted payoff of the default leg is given as follows:

$$DL(0) = \int_0^T D(0; T) dL_t$$

$$PL(0) = \sum_{i=1}^d \delta_i D(0; T_i)(1 - C_{T_i})$$

where $D(s; t)$ is the discount factor (often assumed to be deterministic) between times $s$ and $t$, $T_i$-s are the premium payment times $i = 1, ..., b$ and $\delta_i = T_i - T_{i-1}$ is the year fraction.
Computing Base Correlations

- A model for the loss and the number of defaults plugged in to the two legs, leads to the same risk neutral expectation. Thus the price of a tranche the so called fair spread, $S_0$, equalizes the two legs in terms of risk neutral expectation:

$$S_0 = \frac{\int_0^T D(0; T)dE_0L_t}{\sum_{i=1}^d \delta_i D(0; T_i)(1 - E_0 C_{T_i})}$$

- The market quotes the values of the fair spread.
- Once we have the default and premium legs we can calculate the base correlations as well. These are correlation values assigned to all, but the last detachment point. Roughly, these are the correlations that reproduce the spreads in the Gaussian copula model.
Computing Base Correlations

- The Gaussian copula model for correlation $\rho$ gives the default leg $DL(\rho, d)$ for the equity tranche $[0, d]$.
- Similarly $PL(\rho, d)$ denotes the premium leg for the Gaussian copula model.
- The detachment points are denoted by $d_0 = 0 < d_1 < d_2 < \cdots < d_n = 1$. Then the $k$th tranche has attachment point $d_{k-1}$, detachment point $d_k$ and spread $s_k$.
- With the introduced notations the base correlations are defined as the solution of the system of equations:

$$s_k \cdot PL(\rho_k, d_k) - PL(\rho_{k-1}, d_{k-1}) = DL(\rho_k, d_k) - DL(\rho_{k-1}, d_{k-1})$$

- This equation can be solved recursively, since for $d_0 = 0$ both legs are zero.
- For the last tranche, $k = n$ and $d_n = 1$, the expression does not depend on $\rho_n$, so we cannot define the base correlation for the very last (super senior) tranche.
By calibrating directly base correlations:

- one gets a monotonic mapping of the base correlation parameter into tranche spreads;
- one can invert a wider range of tranche spreads into a base correlation parameter;
- one can price bespoke detachments by interpolating the base correlation across detachments.

Even so, base correlation needs to be handled with care, since it may lead to negative loss distributions, thus violating basic no-arbitrage constraints.
Spreads and Base Correlations under Constant Correlation

Simulated names, Constant rho, positive sign
rho in 0.9, 0.95, 0.975, 0.99

CDO Base Correlations

Simulated names, Constant rho, positive sign
rho in 0.9, 0.95, 0.975, 0.99

CDO Spreads

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Introduction
Single name calibration
Stochastic correlation
Base correlations

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Spreads and Base Correlations under Jacobi Process driven Stochastic Correlation

IG9, Homogeneous, 2014, Stoch. rho, Jacobi generation
k=1, Theta=0.5, Sigma=0.5, 2, 8, 16
CDO Base Correlations
20% 40% 60% 80%

IG9, Inhomogeneous, 2014, Stoch. rho, Jacobi generation
k=1, Theta=0.5, Sigma=0.5, 2, 8, 16
CDO Base Correlations
20% 40% 60% 80%

IG9, Homogeneous, 2017, Stoch. rho, Jacobi generation
k=5, Theta=0.5, Sigma=0.5, 2, 8, 16
CDO Base Correlations
20% 40% 60% 80%

IG9, Inhomogeneous, 2017, Stoch. rho, Jacobi generation
k=5, Theta=0.5, Sigma=0.5, 2, 8, 16
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IG9, Homogeneous, 2014, Stoch. rho, Jacobi generation
k=1, Theta=0.5, Sigma=0.5, 2, 8, 16
CDO Spreads
0−20% 20−40% 40−60% 60−80% 80−100%

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Conclusions

- Essentially, we create interdependent Brownian motions, where the origin of non-linear interdependence is a common “factor” Brownian motion. These interdependent Brownian motions are the driving forces in the structural equations for the names in CDO.
- The predictable creation of the interdependence leaves marginal distributions intact and allows for free variability in the interdependence structure.
- This feature enables separate calibration of individual names and CDO tranches, in particular by the rich base correlation structures the model provides.
- Further immediate studies may concern the maturity dependence of the model and the calibration of the tranche model by expected tranche loss surfaces.
- A further goal may be the creation of a similar interdependence scheme for jump measures in the case of Lévy processes.
Thank you for your attention!