FACTORIZATION OF DISCRETE–TIME
ALL–PASS FUNCTIONS*

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Abstract. In this paper we analyse the problem of characterizing and factorizing rational all-pass functions with respect to the unit circle. We prove using a mixture of coordinate-free, "geometric" and algebraic approach that the left all-pass factors in minimal factorizations are determined via the solutions of two equations which in some cases can be transformed into algebraic Riccati equations. We also show that – possibly non-minimal – realizations of the factors can be defined using oblique projections.

1. Introduction

In the complex functional analysis, approximation theory and also in system theory the isometric operator and correspondingly the all-pass functions play an important role. In a Hilbert space endowed with a shift operator the all-pass functions are just Fourier transforms of unitary (isometric) operators commuting with the shift operator.

The factorization theory of rational inner functions considered on the left (right) half plane were analysed in Finesso and Picci [FP81], Picci and Pinzoni [PP94]. Fuhrmann [Fuh95] considered an extension of this study in connection to the characterization of all minimal square spectral factors of a coercive rational spectral density function without any specification on the pole structure.

It was proved that there is a one-to-one correspondence between the left inner factors of an inner function Φ, the invariant subspaces of the state transition operator in any minimal realization of Φ and positive semidefinite solutions of an algebraic Riccati equation.

In Fuhrmann and Hoffmann [FH97] the discrete-time analog of this problem was investigated. Their main theorem was reproved and extended in Michaletzky [Mic97] using ”geometric” approach.

In this paper – based on the state-space realization theory of acausal spectral factors developed in Michaletzky and Ferrante [MF95] – we consider the factorization theory of all-pass functions in the discrete-time case.

In Section 3 we prove that the equation characterizing the all-pass functions is equivalent to the orthogonality of some transformation. Taking the Redheffer-transform of the orthogonal matrix we obtain a J-unitary matrix.

In Section 4 we consider so-called minimal all-pass factorizations of the all-pass functions. For special type of all-pass functions, for the phase functions this was analysed in Fuhrmann [Fuh95]. Generalizing his results we show that the minimal factorizations are determined by two invariant subspaces. Gluing together these subspaces we define

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an oblique projection. The factorization equation obtained this way is similar to those in Bart et al. [BGK84].

2. Notation

A $p \times q$ matrix valued function is called \textbf{all-pass}, if the following equation holds:

$$\Phi \Phi^* = I,$$  

(2.1)

where $\Phi^*(z) = \Phi(\bar{z}^{-1})^*$. Let us remark, that if equation (2.1) holds and $\Phi$ is analytic inside of the complex unit circle, i.e. on the region \{ $z$ \mid $|z| < 1$ \}, then it is called \textbf{inner} function. We shall consider only rational all-pass functions. Assume that

$$\Phi(z) = D + C(z^{-1}I - A)^{-1}B$$  

(2.2)

is a realization of the function $\Phi$, where the matrix $A$ has no eigenvalues of modulus 1. In the case when $\Phi$ is an inner function we assume that the matrix $A$ is a (discrete-time) stable matrix, i.e. its eigenvalues are of modulus less then 1.

We shall associate a state space system to the realization (2.2). Applying a nonsingular transformation the matrix $A$ can be transformed into a block-diagonal form

$$A = \begin{bmatrix} A_1 & \phantom{C} \\ A_2 \end{bmatrix}$$

where $A_1$ and $A_2^{-1}$ are stable matrices (i.e. with eigenvalues of modulus less than 1). Denote $n_1$ and $n_2$ the number of stable and unstable eigenvalues of $A$, respectively, $n = n_1 + n_2$. Let

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

be the corresponding partitions.

In this case the function $\Phi$ has the following form:

$$\Phi(z) = D + C_1(z^{-1}I - A_1)^{-1}B_1 + C_2(zI - \bar{A}_2)^{-1}\bar{B}_2$$  

(2.3)

Assume that $w(t), t \in \mathbb{Z}$ is a $q$-dimensional white noise, i.e. an uncorrelated sequence of $q$-dimensional random variables with unit covariances. Define the stationary purely nondeterministic $p$-dimensional process \{ $y(t); t \in \mathbb{Z}$ \} and an $n$-dimensional stationary process \{ $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; t \in \mathbb{Z}$ \} via the equations

$$x_1(t+1) = A_1x_1(t) + B_1w(t)$$

$$x_2(t) = A_2^{-1}x_2(t+1) - A_2^{-1}B_2w(t)$$

$$y(t) = C_1x_1(t) + C_2A_2^{-1}x_2(t) + (D - C_2A_2^{-1}B_2)w(t).$$  

(2.4)

Observe that the condition on the spectrum of $A$ implies that the random variables $w(t), t > s$ are uncorrelated to the random variables $x_1(\tau), \tau \leq s$ and $w(t), t \leq s$ are uncorrelated to the random variables $x_2(\tau), \tau > s$. The system in equation (2.4) can be written shortly as follows:

$$\begin{cases} x(t + 1) = Ax(t) + Bw(t) \\ y(t) = Cx(t) + Dw(t). \end{cases}$$  

(2.5)

The arguments in this paper are based on elementary facts from the geometric theory of stochastic realization theory developed in [LP85] and extended to the acausal case in
Michaletzky and Ferrante [MF95]. Let us briefly recall the corresponding notions and notations we need later.

Consider the Hilbert space $H$ generated by the random variables

$$\{w(i)(t) \mid i = 1, 2, \ldots, q; t \in \mathbb{Z}\},$$

where $w(i)(t)$ denotes the $i$-th coordinate of $w(t)$ with the inner product

$$\langle \xi, \eta \rangle = E\{\xi, \eta\}.$$ 

The unitary operator $U : H \to H$ is the shift determined by

$$U w(i)(t) = w(i)(t + 1).$$

Then $U$ also acts as the shift for all processes in the system, i.e., $U y(i)(t) = y(i)(t + 1)$ and $U x(i)(t) = x(i)(t + 1)$. In the case when we would like to emphasize that the Hilbert space was defined by the white noise $w$ we shall use the notation $H(w)$.

For any subspace $Y \subset H$ we shall write $E^Y \lambda$ to denote the orthogonal projection of $\lambda \in H$ onto $Y$. Occasionally we shall misuse notations somewhat by writing $E^Y z$ when $z$ is a random vector to denote the vector with components $\{E^Y z(i)\}$. By $E^YZ$ we shall mean the closure of $\{E^Y \zeta \mid \zeta \in Z\}$.

For any pair of subspaces $Y$ and $Z$ we write $Y + Z$ to denote direct sum (implying that $Y \cap Z = 0$), $Y \oplus Z$ for orthogonal direct sum, and $Y \vee Z$ for the vector sum in the general case, i.e., for closure$\{\eta + \zeta \mid \eta \in Y, \zeta \in Z\}$. Moreover, we write $Z \perp$ to denote the orthogonal complement $H \oplus Z$ of $Z$ in the Hilbert space $H$. Finally, we write $Z \perp Y \mid X$ to denote that $Z$ and $Y$ are conditionally orthogonal given $X$, i.e., that

$$\langle \eta - E^X \eta, \zeta - E^X \zeta \rangle = 0 \quad \text{for all } \eta \in Y, \zeta \in Z.$$ 

Observe that the function $\Phi = D + C(zI - A)^{-1}B$ is all-pass if and only if the output process in equation (2.5) $y$ is also white noise. Geometrically speaking, an all-pass function defines an isometric transformation on the Hilbert space $H$. This transformation is not necessarily unitary due to the fact that the Hilbert space generated by the process $y$ is not necessarily equals to $H$, or in other words the all-pass function $\Phi$ may not be a square all-pass function.

Together with the Hilbert space $H$ we shall relate to any sequence of $l$-dimensional random variables $z(t), t \in \mathbb{Z}$ a subspace $H^-(z)$ defined as the subspace generated by the random variables

$$\{z(i)(t) \mid i = 1, 2, \ldots, l; t = -1, -2, -3, \ldots \},$$

called the past space of $z$, and the future space $H^+(z)$ as the subspace generated by

$$\{z(i)(t) \mid i = 1, 2, \ldots, l; t = 0, 1, 2, \ldots \}.$$ 

Also denote by $X$ the subspace generated by $\{x(i)(0) \mid i = 1, 2, \ldots, n\}$. $X$ is the so-called state space of the realization (2.5). Let us point out that $H^+(y) \perp H^-(y) \mid X$ as it was proven in [MF95]

If the system (2.5) defines a minimal realization of the all-pass function $\Phi$ then the state space $X$ can be split into two parts (cf. Theorem 2.1 in Michaletzky and Ferrante [MF95])

$$X_s = E^{H^-(w)}H^+(y), \quad X_u = E^{H^+(w)}H^-(y)$$

(2.6)
defining the stable and unstable parts. I.e.

\[ X = X_s \oplus X_u. \]  \hfill (2.7)

3. Characterization of all-pass functions

In this section we consider a characterization of rational all-pass functions:

**Theorem 3.1.** Assume that the \( p \times q \) transfer function \( \Phi \) has a realization

\[ \Phi(z) = D + C \left( z^{-1}I - A \right)^{-1} B \]

where the matrix \( A \) has no eigenvalues of modulus 1. If there exists a solution of the equation

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
P & I \\
B^T & D^T
\end{bmatrix}
= \begin{bmatrix}
P & I
\end{bmatrix}
\hfill (3.1)
\]

where \( P = P^T \), then \( \Phi \) is an all-pass function.

**Proof.** By definition

\[ \Phi^*(z) = D^T + B^T \left( zI - A^T \right)^{-1} C^T. \]

Define

\[
\begin{align*}
\Lambda(z) &= \left( z^{-1}I - A \right)^{-1} B \\
\Delta(z) &= \left( zI - A^T \right)^{-1} C^T \\
T(z) &= C \left( z^{-1}I - A \right)^{-1}.
\end{align*}
\]

Using that

\[
P - APA^T = (I - zA)P + AP \left( zI - A^T \right)
\]

we obtain that

\[
\begin{align*}
\Lambda(z)\Phi^*(z) &= (z^{-1}I - A)^{-1} B \left[ B^T \left( zI - A^T \right)^{-1} C^T + D^T \right] \\
&= (z^{-1}I - A)^{-1} \left[ BB^T + APA^T - P \right] \left( zI - A^T \right)^{-1} C^T \\
&+ zP \left( zI - A^T \right)^{-1} C^T + \left( zI - A^T \right)^{-1} \left( APC^T + BD^T \right)
\end{align*}
\]

and

\[
\begin{align*}
\Phi(z)\Phi^*(z) &= (D + C\Lambda(z))\Phi^*(z) \\
&= T(z) \left( BB^T + APA^T - P \right) \Delta(z) + T(z) \left( APC^T + BD^T \right) \\
&+ DD^T + DB^T \left( zI - A^T \right)^{-1} C^T + CPz \left( zI - A^T \right)^{-1} C^T \\
&= T(z) \left( BB^T + APA^T - P \right) \Delta(z) + T(z) \left( APC^T + BD^T \right) \\
&+ \left( CPA^T + DB^T \right) \Delta(z) + \left( CPC^T + DD^T \right).
\end{align*}
\]
So, under the assumption (3.1),
\[ \Phi(z) \Phi^*(z) = I_p. \]  
(3.2)
Consequently, \( p \) cannot be greater than \( q \) and \( \Phi \) is an all-pass transfer function. \( \square \)

**Remark 3.2.** As a side result we obtained that under the assumptions of the previous theorem
\[ \Lambda(z) \Phi^*(z) = zP\Delta(z) \]
If \( p = q \) then
\[ \Phi^{-1}(z) = \Phi^*(z) \]  
(3.3)
thus
\[ \Lambda(z) = zP\Delta(z)\Phi(z) \]
Expressing this in terms of the realizations of \( \Phi^* \) and \( \Phi \)
\[ \overline{x}(t) = A^T\overline{x}(t + 1) + C^T y(t) \]
\[ w(t) = B^T\overline{x}(t + 1) + D^T y(t) \]
\[ x(t + 1) = Ax(t) + Bw(t) \]
\[ y(t) = Cx(t) + Dw(t) \]
(where \( w(t), t \in \mathbb{Z} \) is the output of the first system in view of (3.3) and \( \overline{x} \) denotes the corresponding state process) we get that
\[ P \overline{x}(t) = x(t). \]
This observation is the corner-stone of the converse theorem.

**Theorem 3.3.** Assume that
\[ x(t + 1) = Ax(t) + Bw(t) \]
\[ y(t) = Cx(t) + Dw(t) \]  
(3.4)
defines a minimal realization of a \( p \times p \) all-pass transfer function, where the matrix \( A \) has no eigenvalues of modulus 1.
Then there exists a nonsingular matrix \( P, P = P^T \), such that
\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & I \\ 0 & I \end{bmatrix} \begin{bmatrix} P^T & C^T \\ B^T & D^T \end{bmatrix} = \begin{bmatrix} P & I \end{bmatrix}. \]  
(3.5)

**Proof.** We may assume that
\[ A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \]
where \( A_1 \) and \( A_2^{-1} \) are stable matrices (i.e. with eigenvalues of modulus less than 1). \( n_1 \) and \( n_2 \) denote the number of stable and unstable eigenvalues of \( A \), respectively. \( n = n_1 + n_2 \) is the dimension of \( x(t) \). The minimality implies that \((A_1, B_1), (A_2, B_2)\) are controllable and \((C_1, A_1), (C_2, A_2)\) are observable, where
\[ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \]
are the corresponding partitions. Consider the realization defined in equation (2.4)
\[
\begin{align*}
x_1(t+1) &= A_1x_1(t) + B_1w(t) \\
x_2(t) &= A_2^{-1}x_2(t+1) - A_2^{-1}B_2w(t) \\
y(t) &= C_1x_1(t) + C_2A_2^{-1}x_2(t) + (D - C_2A_2^{-1}B_2)w(t).
\end{align*}
\]
This determines an acausal spectral factor of \( y \). Its state space \( X \) has the orthogonal decomposition
\[
X = X_s \oplus X_u, \tag{3.6}
\]
where
\[
X_s = E^{H^-(w)H^+(y)}, \quad X_u = E^{H^+(w)H^-(y)}. \tag{3.7}
\]

Similarly, taking the inverse transfer function, which maps \( y(t), t \in \mathbb{Z} \) to \( w(t), t \in \mathbb{Z} \), since in our case \( y(t) \) is also a white noise, the usual construction can be applied, thus its state space is generated by
\[
\overline{X}_s = E^{H^-(y)H^+(w)}, \quad \overline{X}_u = E^{H^+(y)H^-(w)}, \tag{3.8}
\]
and
\[
\overline{X} = \overline{X}_s \oplus \overline{X}_u. \tag{3.9}
\]
First we prove that
\[
X = \overline{X}. \tag{3.10}
\]
This is immediate from the identities
\[
\begin{align*}
H^-(w) &= X_s \oplus (H^-(w) \cap H^-(y)) \\
H^+(w) &= X_u \oplus (H^+(w) \cap H^+(y)) \\
H^-(y) &= \overline{X}_s \oplus (H^-(w) \cap H^-(y)) \\
H^+(y) &= \overline{X}_u \oplus (H^+(w) \cap H^+(y))
\end{align*}
\]
giving that
\[
X = H \ominus (N^- \oplus N^+) = \overline{X}
\]
where
\[
N^- = H^-(w) \cap H^-(y), \quad N^+ = H^+(w) \cap H^+(y).
\]
Observe that
\[
\begin{align*}
n_1 &= \dim X_s = \dim \overline{X}_u, \tag{3.11} \\
n_2 &= \dim X_u = \dim \overline{X}_s. \tag{3.12}
\end{align*}
\]
Denote by \( Y_0 \) and \( W_0 \) the subspace spanned by the coordinates of \( y(0) \) and \( w(0) \), respectively. Then the subspaces \( W_0, X_s, UX_u \) are pairwise orthogonal. Similarly, using that \( y(t), t \in \mathbb{Z} \) is a white noise, we obtain that \( Y_0, \overline{X}_s, UX_u \) are pairwise orthogonal. Also,
\[
\dim [W_0 \oplus X_s \oplus UX_u] = p + n_1 + n_2 = \dim [Y_0 \oplus U\overline{X}_u \oplus \overline{X}_s].
\]
Obviously
\[
Y_0, UX_s, X_u \subset W_0 \oplus X_s \oplus UX_u
\]
and also
\[
W_0, \overline{X}_u, U\overline{X}_s \subset Y_0 \oplus U\overline{X}_u \oplus \overline{X}_s.
This implies that
\[ W_0 \oplus X_s \oplus UX_u = Y_0 \oplus U\overline{X}_u \oplus \overline{X}_s = (W(0) \vee X = Y(0) \vee U\overline{X}) . \]

Consequently we can define an orthogonal transformation on the space \( W_0 \vee X \) which maps the subspaces \( W_0, \ X_s, \ UX_u \) onto \( Y_0, \ U\overline{X}_u, \ \overline{X}_s \), respectively.

In order to construct such a rotation explicitly first consider an orthogonal transformation on \( X \) which maps \( X_s \) onto \( \overline{X}_u \), and \( X_u \) onto \( \overline{X}_s \), respectively. Assume that the coordinates \( \tilde{z}_1(0), \tilde{z}_2(0), \ z_1(0), \ z_2(0) \) generate \( \overline{X}_u, \ \overline{X}_s, \ X_s, \ X_u \), respectively, and moreover they have unit covariance matrices. In view of equation (3.10) there exists an orthonormal matrix \( \mathcal{P} \) such that
\[
\mathcal{P} z(0) = \begin{bmatrix} p_1 & p_{12} \\ p_{21} & p_2 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} .
\]

On the other hand
\[
z_1(0) = q_1 x_1(0) \quad \quad z_2(0) = q_2 x_2(0)
\]

for appropriate nonsingular matrices \( q_1, \ q_2 \). Using \( \mathcal{P} \) the corresponding rotation on \( W_0 \vee X \) can be defined as
\[
\mathcal{P} \tilde{z}(0) = \begin{bmatrix} \tilde{z}_1(0) \\ \tilde{z}_2(0) \end{bmatrix} = \mathcal{P} z(0) = \begin{bmatrix} p_1 & p_{12} \\ p_{21} & p_2 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} .
\]

Consequently
\[
\begin{bmatrix} p_1 q_1 A_1 q_1^{-1} & p_{12} & p_1 q_1 B_1 \\ p_{21} & p_2 q_2 A_2 q_2^{-1} & -p_2 q_2 A_2^{-1} B_2 \\ C_1 q_1^{-1} & C_2 A_2^{-1} q_2^{-1} & D - C_2 A_2^{-1} B_2 \end{bmatrix} \quad (3.13)
\]
is an orthonormal matrix. We show that \( p_2 \) is nonsingular. This is equivalent to saying that
\[
X_u^\perp \cap \overline{X}_s = 0
\]
or in other words
\[
X_s \cap \overline{X}_s = 0.
\]

But
\[
H^-(w) = X_s \oplus (H^-(y) \cap H^-(w))
\]
\[
H^-(y) = \overline{X}_s \oplus (H^-(y) \cap H^-(w))
\]
proving the previous identity. Symmetrically, the relationship
\[
X_u \cap \overline{X}_u = 0
\]
implies that the matrix \( p_1 \) is nonsingular.

The orthogonality of (3.11) implies that its Redheffer-transform is J-unitary (see [Dym88]). Denoting this transform by \( \mathcal{R} \):
\[
\mathcal{R} = \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 \end{bmatrix}
\]
where

$$\mathcal{R}_1 = \begin{bmatrix} p_1 q_1 A_1 q_1^{-1} - p_{12} \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} p_{21} \\ - \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} p_{21} \\ C_1 q_1^{-1} - \left( C_2 A_2^{-1} q_2^{-1} \right) \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} p_{21} \end{bmatrix},$$

$$\mathcal{R}_2 = \begin{bmatrix} p_{12} \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} p_{21} \\ \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} p_{21} \\ C_2 A_2^{-1} q_2^{-1} \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} \end{bmatrix},$$

$$\mathcal{R}_3 = \begin{bmatrix} p_1 q_1 B_1 - p_{12} \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} \left( - \frac{1}{p_2 q_2 A_2 q_2^{-1}} B_2 \right) \\ \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} \left( - \frac{1}{p_2 q_2 A_2 q_2^{-1}} B_2 \right) \\ D - C_2 A_2^{-1} B_2 + C_2 A_2^{-1} q_2^{-1} \left( \frac{1}{p_2 q_2 A_2 q_2^{-1}} \right)^{-1} p_2 q_2 A_2^{-1} B_2 \end{bmatrix}.$$
But the orthogonality of $P$ also implies that

$$p_1 p_{21}^T + p_{12} p_2^T = 0$$

i.e.

$$p_{1}^{-1} p_{12} = -p_{21}^T p_2^{-1}.$$ 

Thus we obtained the following equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix} \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix}$$

proving the theorem. □

**Remark 3.4.** Introducing another coordinate system defined as

$$\begin{bmatrix} \pi_1(0) \\ \pi_2(0) \end{bmatrix} = \begin{bmatrix} q_1^T p_1^T \\ -q_2^T p_2^T \end{bmatrix} \begin{bmatrix} \tilde{z}_1(0) \\ \tilde{z}_2(0) \end{bmatrix}$$

we get that

$$P \pi(0) = \begin{bmatrix} q_1^{-1} q_{11}^{-1} & -q_1^{-1} p_{21}^T p_2^{-1} q_2^{-1} \\ q_2^{-1} p_{12}^T p_1^{-1} q_1^{-1} & -q_2^{-1} q_2^{-1} \end{bmatrix} \times \begin{bmatrix} q_1^T p_1^T \\ -q_2^T p_2^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_{21} \\ p_{12} \\ p_2 \end{bmatrix} \begin{bmatrix} q_1 x_1(0) \\ q_2 x_2(0) \end{bmatrix} = x(0).$$

Consequently, applying both sides of (3.5) to $[ \pi(1) \ y(0) ]^T$ and solving the equation to $[ \pi(0) \ w(0) ]^T$ we obtain that

$$\pi(0) = A^T \pi(1) + C^T y(0)$$

$$w(0) = B^T \pi(1) + D^T y(0)$$

(3.14)

which gives a “backward” realization of the function $\Phi^*$. 

Observe that the coordinates of $\pi_1$ and that of $\pi_2$ generate the subspaces $X_u$ and $X_s$, respectively.

### 4. All-pass factorization of all-pass functions

Let us consider now the question of factorization of all-pass functions in the form

$$\Phi = \Phi_1 \Phi_2$$

where $\Phi_1$ and $\Phi_2$ are also all-pass functions. Since $\Phi_1^* \Phi$ is all-pass, the problem formulated in this way is too general. Instead of this we may consider the so-called minimal factorizations (see Bart et al. [BGK88]).

**Definition 4.1.** Assume that $\Phi$, $\Phi_1$, $\Phi_2$ are rational transfer functions for which $\Phi = \Phi_1 \Phi_2$. This factorization is called minimal, if

$$n = n_1 + n_2$$

where $n$, $n_1$, $n_2$ are the dimensions of the state spaces of minimal realizations of $\Phi$, $\Phi_1$ and $\Phi_2$, respectively.
Theorem 4.2. Let \( \Phi, \Phi_1, \Phi_2 \) be \( p \times p \) rational all-pass transfer functions for which
\[
\Phi = \Phi_1 \Phi_2. \tag{4.1}
\]
Consider a \( p \)-dimensional white noise sequence \( w(t), t \in \mathbb{Z} \) and apply the transformation determined by \( \Phi \) and \( \Phi_2 \) to the sequence \( w \), defining in this way the stationary sequences \( y(t), u(t), t \in \mathbb{Z} \), respectively. Denote by \( X, Z, R \) the state spaces of the minimal realizations of \( \Phi, \Phi_1, \Phi_2 \) in the Hilbert-space \( H(w) \).

Then
(i) \( X \subset Z \vee R \)
(ii) the factorization (4.1) is minimal, if and only if
\[
H^+(u) \supset H^+(y) \cap H^+(w) \tag{4.2a}
\]
\[
H^+(u) = [H^+(y) \cap H^+(u)] \vee [H^+(w) \cap H^+(u)]. \tag{4.2b}
\]

Proof. Since \( \Phi \) and \( \Phi_2 \) are square \( p \times p \) all-pass functions we have that
\[
H = H(y) = H(w) = H(u).
\]
At the same time
\[
H \ominus X = (H^-(y) \cap H^-(w)) \oplus (H^+(y) \cap H^+(w)), \tag{4.3a}
\]
\[
H \ominus Z = (H^-(y) \cap H^-(u)) \oplus (H^+(y) \cap H^+(u)), \tag{4.3b}
\]
\[
H \ominus R = (H^-(u) \cap H^-(w)) \oplus (H^+(u) \cap H^+(w)). \tag{4.3c}
\]

The inclusions
\[
[H^-(y) \cap H^-(u)] \cap [H^-(w) \cap H^-(w)] \subset H^-(y) \cap H^-(w)
\]
\[
[H^+(y) \cap H^+(u)] \cap [H^+(w) \cap H^+(w)] \subset H^+(y) \cap H^+(w)
\]
implies that
\[
X \subset Z \vee R
\]
proving (i) and the inequality \( n \leq n_1 + n_2 \).

It follows that the factorization is minimal, if and only if
\[
X = Z \vee R, \tag{4.4}
\]
and
\[
Z \cap R = 0. \tag{4.5}
\]

The previous argument gives that
\[
X = Z \vee R,
\]
is equivalent to
\[
H^-(u) \supset H^-(y) \cap H^-(w)
\]
\[
H^+(u) \supset H^+(y) \cap H^+(w). \tag{4.6}
\]

On the other hand
\[
Z \cap R = 0
\]
means that
\[
Z^\perp \vee R^\perp = H
\]
We prove that (4.6) and (4.7) imply (4.6) and (4.7). Obviously, from (4.7) it follows that
\[ H^-(u) = (H^-(y) \cap H^-(w)) \cap \big( H^-(w) \cap H^-(u) \big) \supset H^-(y) \cap H^-(w) \]
so (4.6) is fulfilled. Also, taking the intersection of both sides with \( H^-(y) \) we obtain that
\[ H^-(u) \cap H^-(y) = H^-(y) \cap \big( H^-(w) \cup H^-(u) \big). \]
But equation (4.6) implies that
\[ H^-(u) \subset H^-(y) \cup H^-(w). \]
If \( a \in H^-(u) \) then in view of the rationality of \( \Phi_1, \Phi_2 \) there exist \( b \in H^-(y), c \in H^-(w) \) such that \( a = b + c \). But in this case
\[ b = a - c \in H^-(y) \cap \big( H^-(u) \cup H^-(w) \big). \]
Consequently, \( b \in H^-(u) \cap H^-(y) \). Similarly, \( c \in H^-(w) \cap H^-(y) \), proving equation (4.7). This concludes the proof of (ii). □

**Remark 4.3.** Symmetric argument gives that the minimality of the factorization is equivalent to
\[ H^-(u) \supset H^-(y) \cap H^-(w) \]
\[ H^-(u) = [H^-(y) \cap H^-(u)] \cup [H^-(w) \cap H^-(u)]. \]

**Remark 4.4.** Using the notation
\[ N^+ = H^+(y) \cap H^+(w), \quad N^- = H^-(y) \cap H^-(w) \]
we see that an arbitrary minimal factorization is characterized by the pair of subspaces
\[ Z_+ = (H^+(y) \cap H^+(u)) \ominus N^+ \]
\[ R_+ = (H^+(w) \cap H^+(u)) \ominus N^+. \]
Obviously
\[ Z_+ \subset H^+(y) \ominus N^+ = \overline{X}_u \]
\[ R_+ \subset H^+(w) \ominus N^+ = X_u. \]

But we may write that
\[ \overline{X}_u = [H^-(y) \cup H^-(w)] \ominus H^-(y), \]
\[ Z_+ = [H^-(y) \cup H^-(w)] \ominus [H^-(y) \cup H^-(u)]; \]
\[ X_u = [H^-(y) \cup H^-(w)] \ominus H^-(w), \]
\[ R_+ = [H^-(y) \cup H^-(w)] \ominus [H^-(w) \cup H^-(u)]. \]
Thus these subspaces are invariant under the operator 
\[ E^{H^-(y)\vee H^-(w)} U|_{H^-(y)\vee H^-(w)} \cdot \]

Since \( Z_+ \subset \overline{X}_u \) is an invariant subspace this gives the possibility of characterizing it via a left inner factor of an inner function, or as a solution of a matrix equation.

In order to be able to formulate this statement precisely let us first analyse the factorizations of \( \Phi = \Phi_1 \Phi_2 \) when \( \Phi_1 \) is an inner function. Obviously, this is equivalent to the fact that the unstable part of the state space \( Z \) is trivial, i.e.

\[ Z_u = E^{H^+(u)} H^-(y) = 0. \]

Thus

\[ H^+(u) \subset H^+(y). \]

Let us observe that in this case the minimality condition (4.2) is reduced to (4.2a), i.e.

\[ H^+(u) \supset N^+. \]

Let us denote by \( \Phi_s \) the left inner factor of determined by the selection

\[ H^+(u) = N^+. \]

In this case the corresponding state space is

\[ Z = E^{H^-(u)} H^+(y) = \overline{X}_u. \]

In view of Theorem 3.4 in [Mic97] or Theorem 3.1 in [FH97] the invariant subspaces of \( \overline{X}_u \) can be characterized in terms of the left inner functions of \( \Phi_s \).

Now denote the right all-pass factor of \( \Phi \) in the factorization determined by \( \Phi_s \) as \( \Phi_r \). Since \( H^+(u) \subset H^+(w) \) this is a coinner function, i.e. the stable part of its state space is trivial.

\[ R = R_u = E^{H^+(w)} H^-(u) \]

Taking the adjoint transfer function, \( \Phi_r^* \), this is inner, it has the same state space

\[ \overline{R} = R. \]

Thus in this case

\[ E^{H^+(w)} H^-(u) = H^+(w) \cup \left[ H^+(w) \cap H^+(u) \right] = H^+(w) \cup N^+ = X_u. \]

and moreover,

\[ X_u = E^{H^-(y)\vee H^-(w)} H^-(w). \]

Invoking again Theorem 3.4 in [Mic97] we have the possibility of characterizing the invariant subspaces of \( X_u \) in terms of left inner factors of \( \Phi_r^* \). The following picture shows the connections between the various subspaces.
Theorem 4.5. Assume that
\[ x(t + 1) = A x(t) + B w(t) \]
\[ y(t) = C x(t) + D w(t) \]
(4.8)
defines a minimal realization of a \( p \times p \) all-pass transfer function \( \Phi(z) \), where the matrix \( A \) has no eigenvalues of modulus 1. Assume that \( A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \)
where \( A_1 \) and \( A_2^{-1} \) are stable matrices (i.e. with eigenvalues of modulus less than 1) and
\[ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \]
are the corresponding partitions. Then left all-pass factors defining minimal factorizations of the all-pass function \( \Phi, \) are determined via solutions of the equations
\[ \begin{bmatrix} A_1 & b_1 \\ C_1 & d_1 \end{bmatrix} \begin{bmatrix} p_1 \\ I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A_1^T & C_1^T \end{bmatrix} \\ \begin{bmatrix} b_1^T & d_1^T \end{bmatrix} \end{bmatrix} = \begin{bmatrix} p_1 \\ I \end{bmatrix} \]
(4.9) and
\[ \begin{bmatrix} A_2^T \\ B_2^T A_2^{-1} \\ c_2 \end{bmatrix} \begin{bmatrix} p_2 \\ I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A_2^{-1} & A_2^{-1} B_2 \end{bmatrix} \\ \begin{bmatrix} c_2^T & d_2^T \end{bmatrix} \end{bmatrix} = \begin{bmatrix} p_2 \\ I \end{bmatrix} \]
(4.10)
in terms of \( b_1, d_1, c_2, d_2, \) and \( p_1, p_2 \) \( (p_1 = p_1^T, p_1 \geq 0, p_2 = p_2^T, p_2 \geq 0). \)

Proof. Let us partition the variable \( x \) according to the partition of the matrix \( A \) considered above.
\[ x(t) = \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} \]
Let us point out that \( x_1 \) and \( x_2 \) used in Section 3 are the same quantities as \( x_s \) and \( x_u. \) The latter expresses the fact that the coordinates of \( x_s(0) \) generate \( X_s \) and similarly the coordinates of \( x_u(0) \) generate \( X_u. \)

Similar partitioning can be applied to the system considered in (3.14). Here again we shall use the notations \( \overline{x}_s \) and \( \overline{x}_u. \)
Let us determine first the "A" and "C" matrices in some minimal realizations of $\Phi_s$ and $\Phi^*_r$. Their state space is $\overline{X_u}$ and $X_u$, respectively. Obviously

$$E^{H(w)}Ux_s(0) = A_1x_s(0)$$
$$E^{H(w)}y(0) = C_1x_s(0)$$

Since

$$\overline{X_u} \cap X_s^\perp = 0$$

consequently for the matrix form of the operators

$$E^{H(y)\vee H(w)}U|_{\overline{X_u}} \text{ and } E^{H(y)\vee H(w)}|_{\{y(0)\}}$$

in an appropriate coordinate system we may use the matrices $A_1$ and $C_1$, respectively.

Similarly

$$E^{H(y)}U\overline{x_s}(0) = A_2^{T^{-1}}\overline{x_s}(0)$$
$$E^{H(y)}w(0) = B_2^TA_2^{T^{-1}}\overline{x_s}(0)$$

where the coordinates of $\overline{x_s}(0)$ generate $X_s$

$$X_u \cap X_s^\perp = 0$$

so the operators

$$E^{H(y)\vee H(w)}U|_{\overline{X_u}} \text{ and } E^{H(y)\vee H(w)}|_{\{w(0)\}}$$

may have the matrix representation $A_2^{T^{-1}}$ and $B_2^TA_2^{T^{-1}}$, respectively.

As we have pointed out in Remark 4.4 the minimal factorizations of $\Phi$ are characterized by the pair of subspaces $Z_+$ and $R_+$, where $Z_+ \subset \overline{X_u}$ and $R_+ \subset X_u$ and they are invariant under $E^{H(y)\vee H(w)}U|_{\text{H}^{-1}}$. Consequently, using Theorem 3.1 in [FH97] they determine an inner factor of $\Phi_s$ and $\Phi^*_r$, respectively. The same theorem also characterizes the inner factors in terms of solutions of the equations given in (4.9) and (4.10), respectively. (Cf. also Theorem 3.4 in [Mic97].)

In the continuous time case Fuhrmann [Fuh95] gives explicit formulae for the minimal left all-pass factors of a special all-pass function, the so-called phase function.

In that paper the left factors are characterized and computed explicitly in terms of the solutions of two Riccati-equations, which are natural continuous–time counterparts of equations (4.9) and (4.10).

The next theorem contains the corresponding state-space equations for the discrete-time case.

**Theorem 4.6.** Assume that $\Phi(z) = D + C(z^{-1}I - A)^{-1}B$ a $p \times p$, rational all-pass function where the matrix $A$ has no eigenvalues of modulus 1. Furthermore, assume that $(C, A)$ is observable, $(A, B)$ is controllable. Let $p_1$, $p_2$ be arbitrary solutions of (4.9) and (4.10) and $P$ be a nonsingular solution of (3.5). Set $Q = P^{-1}$. Define the projection $\Pi$ determining its range and kernel space as follows

$$\text{range}(\Pi) = \text{range} \begin{bmatrix} p_1 & P_2 - P_2p_2P_2 \\ P_2 - P_2p_2P_2 & 0 \end{bmatrix}$$

$$\text{ker}(\Pi) = \text{ker} \begin{bmatrix} p_1 \\ P_2 - P_2p_2P_2 \end{bmatrix} Q$$.
Then the corresponding minimal factorization $\Phi = \Phi_1\Phi_2$ is defined as

$$\Phi_1(z) = d_1 + C(z^{-1}I - A)^{-1}b_1$$
$$\Phi_2(z) = d_2 + c_2(z^{-1}I - A)^{-1}B$$

where

$$\begin{bmatrix} b_1 \\ d_1 \end{bmatrix} \begin{bmatrix} c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \Pi A (I - \Pi) & \Pi B \\ C (I - \Pi) & D \end{bmatrix}.$$ 

**Proof.** The proof of this theorem is similar to that of Theorem 3.6 in [Mic97]. We know already that in case of minimal factorization $X = Z \vee R$, $Z \cap R = 0$ so using a coordinate transformation in the state space we split the state vector into two parts, spanning $Z$ and $R$, separately. Define the transformation $T$ for which

$$T x(t) = \begin{bmatrix} T_1 x(t) \\ T_2 x(t) \end{bmatrix} = \begin{bmatrix} \bar{z}(t) \\ r(t) \end{bmatrix}.$$ 

Denote $U = [U_1, U_2] = T^{-1}$. Then

$$\begin{bmatrix} \bar{z}(t+1) \\ r(t+1) \end{bmatrix} = TAT^{-1} \begin{bmatrix} \bar{z}(t) \\ r(t) \end{bmatrix} + TBw(t)$$

$$y(t) = CT^{-1} \begin{bmatrix} \bar{z}(t) \\ r(t) \end{bmatrix} + Dw(t). \quad (4.11)$$ 

Since $R_s \subset H^-(w)$ is invariant under $E^{|H^-(w)}_{H^-(w)}$, and $R_u \subset H^+(w)$ is invariant under $E^{|H^+(w)}_{H^+(w)}$ we obtain that in the state space equation (4.11) $r(t+1)$ is expressed using only $r(t)$ and $w(t)$, i.e.

$$T_2 A U_1 = 0.$$ 

Define the the projection $\Pi$ as follows

$$\Pi = U_1 T_1.$$ 

Let us point out that

$$\text{range}(\Pi) = \ker(T_2)$$
$$\ker(\Pi) = \ker(T_1).$$

Consequently, the argument used in the proof of Theorem 3.6 in [Mic97] can be repeated here leading to the factorization

$$D + C \left( z^{-1}I - A \right)^{-1} B = \left( d_1 + C \left( z^{-1}I - A \right)^{-1} d_1 \right) \times \left( d_2 + c_2 \left( z^{-1}I - A \right)^{-1} B \right)$$

where

$$\begin{bmatrix} b_1 \\ d_1 \end{bmatrix} \begin{bmatrix} c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \Pi A (I - \Pi) & \Pi B \\ C (I - \Pi) & D \end{bmatrix}.$$ 

For the sake of readers convenience we outline the main steps of the proof.
Observe that \( z(t) \) and \( r(t) \) may serve as state processes of (minimal) realizations of \( \Phi_1 \) and \( \Phi_2 \). Thus with appropriate \( B_z, D_z, C_r, D_r \) matrices the following state space system descriptions hold:

\[
\begin{align*}
\frac{z(t+1)}{y(t)} &= T_1 AU_1 z(t) + B_z u(t) & \text{(4.12)} \\
\frac{y(t)}{z(t)} &= C U_1 z(t) + D_z u(t) \\
\frac{r(t+1)}{u(t)} &= T_2 AU_2 r(t) + T_2 B w(t) & \text{(4.14)} \\
\frac{u(t)}{r(t)} &= C_r r(t) + D_r w(t) \\
\end{align*}
\]

In other words
\[
\Phi_1(z) = D_z + C U_1 (z^{-1} I - T_1 A U_1)^{-1} B_z
\]
and
\[
\Phi_2(z) = D_r + C_r (z^{-1} I - T_2 A U_2)^{-1} T_2 B
\]
But the identity \( T_2 A U_1 = 0 \) implies that \( U_1 T_1 A U_1 = A U_1 \) and \( T_2 A U_2 T_2 = T_2 A \), consequently
\[
\Phi_1(z) = D_z + C (z^{-1} I - A)^{-1} U_1 B_z
\]
and
\[
\Phi_2(z) = D_r + C_r T_2 (z^{-1} I - A)^{-1} B
\]
Let us introduce the notations \( d_1 = D_z, b_1 = U_1 B_z, d_2 = D_r \) and \( c_2 = C_r T_2 \).

Now compare the equations (4.12), (4.14) to (4.11). We obtain that
\[
B_z C_r r(t) + B_z D_r w(t) = T_1 A U_2 r(t) + T_1 B w(t)
\]
and
\[
D_z C_r r(t) + D_z D_r w(t) = C U_2 r(t) + D w(t).
\]
Using that the coordinates of \( r(0) \) and \( w(0) \) are linearly independent we obtain that
\[
\begin{bmatrix}
B_z \\
D_z
\end{bmatrix}
\begin{bmatrix}
C_r \\
D_r
\end{bmatrix} = \begin{bmatrix}
T_1 A U_2 & T_1 B \\
C U_2 & D
\end{bmatrix},
\]
and consequently the equations
\[
\begin{bmatrix}
b_1 \\
d_1
\end{bmatrix}
\begin{bmatrix}
c_2 d_2
\end{bmatrix} = \begin{bmatrix}
\Pi A (I - \Pi) & \Pi B \\
C (I - \Pi) & D
\end{bmatrix}
\]
are fulfilled.

To finish the proof of the theorem we have to determine the range and the kernel subspace of the projection \( \Pi \). Let us assume that the matrix \( A \) is partitioned into the block diagonal from defining the stable and unstable part of \( \Phi \), and consider the corresponding partitions of the other element in the equations. For the state variables we shall use the indices \( s \) and \( u \) to indicate this partition. Now let us look at the finer structure of the transformation \( T \).

\[
T = \begin{bmatrix}
T_1 \\
T_2
\end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} \\
T_{13} & T_{14} \\
T_{21} & T_{22} \\
T_{23} & T_{24}
\end{bmatrix}
\]
according to the decomposition
\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{13} & T_{14} \\
T_{21} & T_{22} \\
T_{23} & T_{24}
\end{bmatrix}
\begin{bmatrix}
x_s(0) \\
x_u(0)
\end{bmatrix}
= \begin{bmatrix}
\bar{x}_u(0) \\
\bar{x}_s(0) \\
r_s(0) \\
r_u(0)
\end{bmatrix}.
\]

Then \( T_{22} = 0, \ T_{23} = 0 \).

Let us consider first the transformations \( T_{21}, \ T_{11}, \ T_{12} \) determined by the equations
\[
T_{21}x_s(0) = r_s(0) \quad (4.18)
\]
\[
T_{11}x_s(0) + T_{12}x_u(0) = z_u(0) \quad (4.19)
\]

The subspace \( \mathfrak{X}_u \) is the state space of the maximal left inner factor \( \Phi_s \) of \( \Phi \) (Recall that \( \Phi = \Phi_s\Phi_r \)) On the other hand \( \mathfrak{X}_u \ominus \mathfrak{Z}_u = \mathfrak{Z}_u \cap H^+(u) \) is a \( E^{H^-(y)\vee H^-(w)}U \)-invariant subspace of \( \mathfrak{X}_u \). Theorem 3.4 in [Mic97] connects this subspace to a solution of equation (4.9). Furthermore, Theorem 3.6 in [Mic97] shows that this solution determines a factorization of \( \Phi_s \) which is intimately connected to a projection. First we determine this oblique projection. Since
\[
\bar{x}_u(0) = Q_1x_s(0) + Q_{12}x_u(0) \quad (4.20)
\]

where
\[
Q = \begin{bmatrix}
Q_1 & Q_{12} \\
Q_{21} & Q_2
\end{bmatrix} = P^{-1} = \begin{bmatrix}
P_1 & P_{12} \\
P_{21} & P_2
\end{bmatrix}^{-1}
\]

choosing \( Q_1^{-1}\mathfrak{x}_u(t), \ t \in \mathbb{Z} \) as the state vector process of a realization of \( \Phi_r \) the equations
\[
\mathfrak{X}_u = E^{H^-(y)\vee H^-(w)}H^+(y)
\]
\[
X_s = E^{H^-(w)}H^+(y)
\]

assure that the matrix form of the operators \( E^{X_u}U|_{\mathfrak{X}_u} \) and \( E^{X_u}_{\{y(0)\}} \) is just \( A_1, C_1 \).

Furthermore
\[
\mathfrak{X}_u \cap H^+(u) = E^{H^-(y)\vee H^-(w)}(H^+(y) \cap H^+(u))
\]

and in view of the minimality of the factorization
\[
R_s = E^{H^-(w)}(H^+(y) \cap H^+(u))
\]

so we obtain that
\[
X_s = E^{H^-(w)}\mathfrak{X}_u
\]
\[
R_s = E^{H^-(w)}(\mathfrak{X}_u \cap H^+(u)).
\]

Now consider the solution \( p_1 \) of equation (4.9). Introduce a state-space transformation in \( \mathfrak{X}_u \) in coherence to the decomposition
\[
\mathfrak{X}_u = \mathfrak{Z}_u \oplus (\mathfrak{X}_u \cap H^+(u))
\]

defined by
\[
Q_1^{-1}\mathfrak{x}_u(0) \mapsto \begin{bmatrix}
t_1Q_1^{-1}\mathfrak{x}_u(0) \\
t_2Q_1^{-1}\mathfrak{x}_u(0)
\end{bmatrix}
\]

where
\[
t_1Q_1^{-1}\mathfrak{x}_u(0) = \bar{x}_u(0)
\]
the coordinates of $t_2Q_1^{-1}x_u(0)$ form a basis in $X_u \cap H^+(u)$. Moreover projecting $t_2Q_1^{-1}x_u(0)$ onto $H^-(w)$ we obtain a basis in $R_u$, i.e. we may assume that
\[ r_s(0) = E^{H^-(w)}t_2Q_1^{-1}x_u(0) = t_2x_s(0). \]
using (4.20) and the fact that $x_u(0)$ is orthogonal to $H^-(w)$. Consequently
\[ T_{21} = t_2 \]
and
\[ \begin{bmatrix} T_{11} & T_{12} \end{bmatrix} = t_1Q_1^{-1} \begin{bmatrix} Q_1 & Q_{12} \end{bmatrix}. \]
According to Theorem 3.6 in [Mic97] the kernel subspaces of $t_1$, $t_2$ can be determined as follows:
\[ \Pi_1 = p_1 \left[ \text{Var} \left( Q_1^{-1}x_u(0) \right) \right]^{-1} = p_1Q_1 \]
is an oblique projection and
\[ \ker(t_1) = \ker(\Pi_1) = \ker p_1Q_1 \]
and
\[ \ker(t_2) = \text{range}(\Pi_1) = \text{range} p_1Q_1. \]
Similar analysis can be carried out for the other parts of the transformation matrix $T$.
The subspace $X_u$ is the state-space of $\Phi^r$, where $\Phi^r$ is the maximal right co-inner factor of $\Phi$. On the other hand $X_u \ominus R_u = R_+ = X_u \cap H^+(u)$ is a $E^{H^-(y)\vee H^-(w)}U$-invariant subspace of $X_u$.
\[ X_u = E^{H^-(y)\vee H^-(w)}H^+(w) \]
\[ \overline{X}_s = E^{H^-(y)}H^+(w) \]
consequently the equation
\[ x_u(0) = P_{21}\overline{x_u}(0) + P_2\overline{x_s}(0) \]
gives that — introducing the state-vector process
\[ P_2^{-1}x_u(0) \]
— the matrix form of the operators $E^{H^-(y)\vee H^-(w)}U|_{X_u}$ and $E^{H^-(y)\vee H^-(w)}|_{\{x_u(0)\}}$ are $A_2T_2^{-1}$, $A_2T_2^{-1}B_2T_2$. Considering the solution $p_2$ of equation (4.10) the coordinate transformation
\[ P_2^{-1}x_u(0) \mapsto \begin{bmatrix} l_1P_2^{-1}x_u(0) \\ l_2P_2^{-1}x_u(0) \end{bmatrix} \]
determined by the decomposition
\[ X_u = R_u \oplus \left( X_u \cap H^+(u) \right) \]
is connected to the oblique projection
\[ \Pi_2 = p_2 \left[ \text{Var} \left( P_2^{-1}x_u(0) \right) \right]^{-1} = p_2P_2. \]
Especially, let
\[ r_u(0) = l_1P_2^{-1}x_u(0) \]
so
\[ T_{24} = l_1P_2^{-1}. \]
Also
\[ X_u \cap H^+(u) = E^{H^-(y) \vee H^-(w)} (H^+(u) \cap H^+(w)) \]
\[ Z_s = E^{H^-(y)} (H^+(u) \cap H^+(w)) \]
consequently
\[ Z_s = E^{H^-(y)} [X_u \cap H^+(u)] . \]
The coordinates of \( l_2 P_2^{-1} x_u(0) \) form a basis in \( X_u \cap H^+(u) \). Projecting this to \( H^-(y) \) we get a basis in \( Z_s \). So we may assume that
\[ Z_s(0) = E^{H^-(y)} l_2 P_2^{-1} x_u(0) = l_2 \pi_s(0) . \]
using (4.21) and the orthogonality of \( \pi_u(0) \) and \( H^-(y) \).
This implies that
\[ \begin{bmatrix} T_{13} & T_{14} \end{bmatrix} = l_2 \begin{bmatrix} Q_{21} & Q_2 \end{bmatrix} . \]
Furthermore
\[ \ker(l_1) = \ker(\Pi_2) = \text{range} (I - p_2 P_2) \]
\[ \ker(l_2) = \text{range}(\Pi_2) = \ker (I - p_2 P_2) \]
Summing up
\[ \text{range}(\Pi) = \ker(T_2) = \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \middle| \alpha \in \text{range} p_1 Q_1, \beta \in P_2 \text{range}(I - p_2 P_2) \right\} \]
and
\[ \begin{bmatrix} T_{11} & T_{12} \\ T_{13} & T_{14} \end{bmatrix} = \begin{bmatrix} t_1 Q_1^{-1} \\ l_2 \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{21} & Q_2 \end{bmatrix} \]
thus
\[ \ker(\Pi) = \ker(T_1) = Q^{-1} \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \middle| \alpha \in Q_1 \ker p_1 Q_1, \beta \in \ker(I - p_2 P_2) \right\} \]
\[ = Q^{-1} \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \middle| \alpha \in \ker p_1, \beta \in \ker(I - p_2 P_2) \right\} . \]
In other words
\[ \text{range}(\Pi) = \text{range} \left( \begin{bmatrix} p_1 & P_2 - P_2 p_2 P_2 \end{bmatrix} \right) \]
\[ \ker(\Pi) = \ker \left( \begin{bmatrix} p_1 & P_2 - P_2 p_2 P_2 \end{bmatrix} Q \right) . \]

\[ \square \]

**Remark 4.7.** Let us remark that
\[ p_1 = \Pi_1 P_1 = P_1 \Pi_1^T \]
\[ P_2 - P_2 p_2 P_2 = P_2 (I - \Pi_2) = (I - \Pi_2^T) P_2 . \]

Our construction in a sense generalizes the theorem in [Fuh95] because not every all-pass function can be considered as phase functions. Green and Anderson [GA87a] proved that a necessary and sufficient condition of this is the all-pass function should be a “minimal” one. This notion is defined as follows. Consider the rational all-pass function \( \Phi \).
**Definition 4.8.** The all-pass function is called minimal all-pass if
\[ \dim X_s = \dim X_u + \dim \left[ H^+(y) \cap H^-(w) \right]. \]

**Remark 4.9.** The original formulation in Green and Anderson [GA87a] is given in terms of the stable and unstable poles of \( \Phi \) and the number of Hankel singular values which are equal to 1. But since this Hankel–operator is defined as \( E^{H^-(w)} : H^+(y) \to H^-(w) \) these numbers coincide with \( \dim X_s, \dim X_u, \dim \left[ H^+(y) \cap H^-(w) \right] \), respectively.

Let us remark that in case of phase functions the minimal factorization is connected to the minimal realizations. Namely, assume that now \( y \) is a purely-nondeterministic full-rank stationary process with rational spectral density. Let \( w_-(t), t \in \mathbb{Z} \) be the innovation process of \( y \), and \( \overline{w_+}(t), t \in \mathbb{Z} \) be the backward innovation. Denote the corresponding transfer functions by \( W_- \) and \( W_+ \), respectively. \( (W_- \) is a stable, minimum phase function, while \( W_+ \) is unstable, maximum phase function.) Consider the all-pass function
\[ \Phi = \overline{W_+}^{-1} W_- \]
this the so-called phase function. Let \( \Phi = \Phi_1 \Phi_2 \) be an arbitrary all-pass factorization. Apply the orthogonal transformation defined by \( \Phi_2 \) to the sequence \( w_-(t), t \in \mathbb{Z} \) defining in this way the white noise process \( w(t), t \in \mathbb{Z} \). Since \( W_- \Phi_2^{-1} \) is obviously a (possibly acausal) spectral factor of \( y \) the subspace
\[ X = E^{H^-(w)} H^+(y) \oplus E^{H^+(w)} H^-(y) \]
is a Markovian splitting subspace.

**Proposition 4.10.** The factorization \( \Phi = \Phi_1 \Phi_2 \) is a minimal factorization if and only if \( X \) is a minimal splitting subspace.

**Proof.** Set \( S = H^-(w) \oplus E^{H^+(w)} H^-(y), \overline{S} = H^+(w) \oplus E^{H^-(w)} H^+(y) \). The minimality of \( X \) is equivalent to
\[ S = H^-(y) \vee \overline{S}^\perp, \quad \overline{S} = H^+(y) \vee S^\perp. \]
(See Lindquist and Picci [LP90].) Since the inclusions
\[ S \supset H^-(y) \vee \overline{S}^\perp, \quad \overline{S} \supset H^+(y) \vee S^\perp \]
are always fulfilled we have to examine the converse inclusions.

Observe that
\[ H^-(y)^\perp = H^+(w_-) \]
and
\[ H^+(y)^\perp = H^-(w_+). \]
Consequently
\[ \overline{S}^\perp = H^-(w) \cap H^-(\overline{w}_+), \quad S^\perp = H^+(w) \cap H^+(w_-). \]
On the other hand if
\[ \left[ H^-(w) \cap H^-(\overline{w}_+) \right] \vee H^- \supset H^-(w) \quad (4.22) \]
then it contains \( E^{H^+(w)} H^-(y) \), as well. In fact, if \( \alpha \in H^+(y) \) then (4.22) implies that \( \Pr_{H^-(y)} \alpha \) is int the set on the left hand side, thus it contains \( \Pr_{H^+(y)} \alpha = \alpha - \Pr_{H^-(y)} \alpha \). Symmetric argument works for \( \overline{S} \), too.
Summing up, until now we have proved that the minimality of $X$ is equivalent to the inclusions

$$[H^-(w) \cap H^-(w_+)] \lor H^-(y) \supset H^-(w)$$

and

$$[H^+(w) \cap H^+(w_-)] \lor H^+(y) \supset H^+(w).$$

Now taking intersections with $H^-(w)$ and $H^+(w)$, respectively we arrive at the equations

$$[H^-(w) \cap H^-(w_+)] \lor [H^-(w) \cap H^-(y)] = H^-(w)$$

$$[H^+(w) \cap H^+(w_-)] \lor [H^+(w) \cap H^+(y)] = H^+(w)$$

which are the identities of the minimal factorizations. 

□

REFERENCES


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