MODELING PROPAGATION PROCESSES ON NETWORKS BY USING DIFFERENTIAL EQUATIONS

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A graph with $N$ nodes is given

The nodes can be susceptible ($S$) or infected ($I$)
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Transitions:

- $S \rightarrow I$, rate: $k\tau$, $k$ is the number of $I$ neighbours.
- $I \rightarrow S$, rate: $\gamma$
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Known models:
- Master equation
- Mean-field equation
- Pairwise model
- Compact pairwise model
- ...
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The nodes can be in the states $\{a_1, a_2, \ldots a_m\}$. 
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The state space of the graph has $m^N$ elements.
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The transitions between different states can be described by a Poisson process

Probability of a transition from state $a_i$ to state $a_j$ in a time interval of length $\Delta t$ is:

$$1 - \exp(-\lambda_{ij}\Delta t).$$
$SIS$ epidemic
SIS epidemic

States of the nodes: \( \{S, I\} \).
Network processes

SIS epidemic

States of the nodes: \{S, I\}.

Transitions and their rates

- \( S \rightarrow I, \lambda = k\tau \), \( k \) is the number of \( I \) neighbours.
- \( I \rightarrow S, \lambda = \gamma \)
SIR epidemic
**NETWORK PROCESSES**

**SIR epidemic**

States of the nodes: \( \{ S, I, R \} \).
SIR epidemic

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Transitions and their rates

- \( S \rightarrow I, \lambda = k\tau \), \( k \) is the number of \( I \) neighbours.
- \( I \rightarrow R, \lambda = \gamma \)
Rumour spreading
Network processes

Rumour spreading

States of the nodes: \( \{X, Y, Z\} \) (ignorant, spreader, stifler).
Rumour spreading

States of the nodes: \{X, Y, Z\} (ignorant, spreader, stifler).

Transitions and their rates

- \( X \rightarrow Y, \lambda = k\tau, \) \( k \) is the number of \( Y \) neighbours.
- \( Y \rightarrow Z, \lambda = \gamma + jp, \) \( j \) is the number of \( Y \) and \( Z \) neighbours.
Propagation of activity in neuronal networks
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States of the nodes: \( \{ E_+, E_-, I_+, I_- \} \) (active and inactive excitatory neurons, active and inactive inhibitory neurons).
Propagation of activity in neuronal networks

States of the nodes: \( \{ E_+, E_-, I_+, I_- \} \) (active and inactive excitatory neurons, active and inactive inhibitory neurons).

Transitions and their rates

- \( E_+ \rightarrow E_-, \lambda = \alpha \).
- \( E_- \rightarrow E_+ , \lambda = \tanh(iw_E - jw_I + h_E) \), \( i, j \) is the number of \( E_+ \) and \( I_+ \) neighbours.
- \( I_+ \rightarrow I_-, \lambda = \alpha \).
- \( I_- \rightarrow I_+ , \lambda = \tanh(iw_E - jw_I + h_I) \), \( i, j \) is the number of \( E_+ \) and \( I_+ \) neighbours.
Derive differential equations for different processes and for different types of graphs.
AIM OF THE RESEARCH

Derive differential equations for different processes and for different types of graphs.

Frequently used random graphs:

- Erdős-Rényi
- Configuration model (Bollobás)
- Small-world (Watts-Strogatz)
- Graphs with scale free degree distribution (Barabási-Albert)
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- Erdős-Rényi
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Examples for network processes:
- Epidemic propagation
- Rumour spreading
- Propagation of neuronal activity
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- $S \rightarrow I$, rate: $k\tau$, $k$ is the number of $I$ neighbours.
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A graph with $N$ nodes is given.

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State space for a triangle:
A graph with $N$ nodes is given

The nodes can be susceptible ($S$) or infected ($I$)

State space for a triangle

- Infection: $SIS \rightarrow SII$, $IIS$
- Recovery: $SIS \rightarrow SSS$
Master equations

\[
\begin{align*}
\dot{X}_{SSS} &= \gamma (X_{SSI} + X_{SIS} + X_{ISS}), \\
\dot{X}_{SSI} &= \gamma (X_{SII} + X_{ISI}) - (2\tau + \gamma) X_{SSI}, \\
\dot{X}_{SIS} &= \gamma (X_{SII} + X_{IIS}) - (2\tau + \gamma) X_{SIS}, \\
\dot{X}_{ISS} &= \gamma (X_{ISI} + X_{IIS}) - (2\tau + \gamma) X_{ISS}, \\
\dot{X}_{SII} &= \gamma X_{III} + \tau (X_{SSI} + X_{SIS}) - 2(\tau + \gamma) X_{SII}, \\
\dot{X}_{ISI} &= \gamma X_{III} + \tau (X_{SSI} + X_{ISS}) - 2(\tau + \gamma) X_{ISI}, \\
\dot{X}_{IIS} &= \gamma X_{III} + \tau (X_{SIS} + X_{ISS}) - 2(\tau + \gamma) X_{IIS}, \\
\dot{X}_{III} &= -3\gamma X_{III} + 2\tau (X_{SII} + X_{ISI} + X_{IIS}),
\end{align*}
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Master equations

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\dot{X}_{SSS} &= \gamma (X_{SSI} + X_{SIS} + X_{ISS}), \\
\dot{X}_{SSI} &= \gamma (X_{SII} + X_{ISI}) - (2\tau + \gamma) X_{SSI}, \\
\dot{X}_{SIS} &= \gamma (X_{SII} + X_{IIS}) - (2\tau + \gamma) X_{SIS}, \\
\dot{X}_{ISS} &= \gamma (X_{ISI} + X_{IIS}) - (2\tau + \gamma) X_{ISS}, \\
\dot{X}_{SII} &= \gamma X_{III} + \tau (X_{SSI} + X_{SIS}) - 2(\tau + \gamma) X_{SII}, \\
\dot{X}_{ISI} &= \gamma X_{III} + \tau (X_{SSI} + X_{ISS}) - 2(\tau + \gamma) X_{ISI}, \\
\dot{X}_{IIS} &= \gamma X_{III} + \tau (X_{SIS} + X_{ISS}) - 2(\tau + \gamma) X_{IIS}, \\
\dot{X}_{III} &= -3\gamma X_{III} + 2\tau (X_{SII} + X_{ISI} + X_{IIS}),
\end{align*}
\]

2\(N\) equations for a graph with \(N\) nodes
Master equations

\[ \begin{align*}
\dot{X}_{SSS} &= \gamma(X_{SSI} + X_{SIS} + X_{ISS}), \\
\dot{X}_{SSI} &= \gamma(X_{SII} + X_{ISI}) - (2\tau + \gamma)X_{SSI}, \\
\dot{X}_{SIS} &= \gamma(X_{SII} + X_{IIS}) - (2\tau + \gamma)X_{SIS}, \\
\dot{X}_{ISS} &= \gamma(X_{ISI} + X_{IIS}) - (2\tau + \gamma)X_{ISS}, \\
\dot{X}_{SII} &= \gamma X_{III} + \tau(X_{SSI} + X_{SIS}) - 2(\tau + \gamma)X_{SII}, \\
\dot{X}_{ISI} &= \gamma X_{III} + \tau(X_{SSI} + X_{ISS}) - 2(\tau + \gamma)X_{ISI}, \\
\dot{X}_{IIS} &= \gamma X_{III} + \tau(X_{SIS} + X_{ISS}) - 2(\tau + \gamma)X_{IIS}, \\
\dot{X}_{III} &= -3\gamma X_{III} + 2\tau(X_{SII} + X_{ISI} + X_{IIS}).
\end{align*} \]

The size of the system can be reduced by using the automorphisms of the graph:

Exact equation: $\dot{I} = \tau [SI] - \gamma [I]$
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$[SI](t)$: expected number of SI edges

This differential equation holds for any graph

Exact equation: $\dot{I} = \tau[S\dot{I}] - \gamma[I]$

Approximation $[S\dot{I}] \approx n\frac{[I]}{N}[S]$, where the average degree is $n$
Exact equation: $\dot{I} = \tau[S] - \gamma[I]$

Approximation $[SI] \approx n \frac{[I]}{N} [S]$, where the average degree is $n$

Approximating differential equation for $[I]$

$$\dot{I} = \tau \frac{n}{N} (N - \bar{I}) - \gamma \bar{I}.$$
Mean-field approximation for SIS epidemic

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Approximating differential equation for $[I]$

$$\dot{I} = \tau \frac{n}{N} \bar{I}(N - \bar{I}) - \gamma \bar{I}.$$ 

This is the well-known compartmental model, which does not give accurate result for networks. Reason: the approximation assumes random distribution of infected nodes.
Mean-field approximation for SIS epidemic

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Approximation $[SI] \approx n \frac{[I]}{N} [S]$, where the average degree is $n$

Approximating differential equation for $[I]$

$$\dot{I} = \tau \frac{n}{N} I (N - I) - \gamma I.$$

This is the well-known compartmental model, which does not give accurate result for networks.
Reason: the approximation assumes random distribution of infected nodes.

Better idea: derive a differential equation for $[SI]$, this leaded to the pairwise model.

PAIRWISE APPROXIMATION

Keep the exact equation

\[ \dot{I} = \tau[S/I] - \gamma[I] \]

and derive a differential equation for \([S/I]\).
Keep the exact equation \[ \dot{I} = \tau[S] - \gamma[I] \]
and derive a differential equation for \( [S] \).

Exact differential equations:

\[
\begin{align*}
\dot{I} & = \tau[S] - \gamma[I], \\
\dot{S}[I] & = \gamma([II] - [SI]) + \tau([SSI] - [ISI] - [SI]), \\
\dot{I}[I] & = -2\gamma[II] + 2\tau([ISI] + [SI]), \\
\dot{SS} & = 2\gamma[S][I] - 2\tau[SSI].
\end{align*}
\]
Keep the exact equation \[ \dot{I} = \tau[S] - \gamma[I] \]

and derive a differential equation for \([S]\).

Exact differential equations:

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\begin{align*}
\dot{I} &= \tau[S] - \gamma[I], \\
\dot{S} &= \gamma([II] - [S]) + \tau([SSI] - [ISI] - [SI]), \\
\dot{I} &= -2\gamma[I] + 2\tau([ISI] + [SI]), \\
\dot{S} &= 2\gamma[S] - 2\tau[SSI].
\end{align*}
\]

Approximation:

\[ [ABC] \approx \frac{n - 1}{n} \frac{[AB][BC]}{[B]}, \quad n \text{ average degree} \]
Keep the exact equation \[ \dot{I} = \tau [SI] - \gamma [I] \]
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Exact differential equations:

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\dot{SI} &= \gamma ([II] - [SI]) + \tau ([SSI] - [ISI] - [SI]), \\
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\dot{SS} &= 2\gamma [SI] - 2\tau [SSI].
\end{align*}
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Approximation:

\[ [ABC] \approx \frac{n - 1}{n} \frac{[AB][BC]}{[B]}, \quad n \text{ average degree} \]

Regular random graph with $N = 1000$ nodes, average degree $n = 20$, $\gamma = 1$, critical value of $\tau$ from compartmental model: $\tau_{cr} = \gamma / n$
Comparison of ODE models to simulation

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Mean-field: dashed, Pairwise: continuous
Simulation (average of 200 runs): grey thick curve
Regular random graph with $N = 1000$ nodes, average degree $n = 20$, $\gamma = 1$, critical value of $\tau$ from compartmental model: $\tau_{cr} = \gamma/n$

$\tau = \tau_{cr} \iff$ basic reproduction number $R_0 = 1$. 
Bimodal random graph with $N = 1000$ nodes, average degree $n = 20$, $\gamma = 1$, $\tau = 2\tau_{cr} = 2\gamma/n$.

$N/2$ nodes have degree $d_1$, $N/2$ nodes have degree $d_2$. 
Bimodal random graph with $N = 1000$ nodes, average degree $n = 20$, $\gamma = 1$, $\tau = 2\tau_{cr} = 2\gamma/n$

$N/2$ nodes have degree $d_1$, $N/2$ nodes have degree $d_2$. 

![Graph showing prevalence over time for two different degree distributions with $d_1 = 18$, $d_2 = 22$ and $d_1 = 5$, $d_2 = 35$.]
Bimodal random graph with $N = 1000$ nodes, average degree $n = 20$, $\gamma = 1$, $\tau = 2\tau_{cr} = 2\gamma/n$

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Bimodal random graph with $N = 1000$ nodes, average degree $n = 20$, $\gamma = 1$, $\tau = 2\tau_{cr} = 2\gamma/n$

$N/2$ nodes have degree $d_1$, $N/2$ nodes have degree $d_2$.

Reason of inaccuracy: in the closure $[ABC] \approx \frac{n-1}{n} \frac{[AB][BC]}{[B]}$ it is assumed that each node has the same degree $n$. 
There are $N_k$ nodes with degree $d_k$ for $k = 1, 2, \ldots, K$. 
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$$[ASI] = \sum_{k=1}^{K} [AS_k I], \quad [AS_k I] \approx \frac{d_k - 1}{d_k} \frac{[AS_k][S_k I]}{[S_k]}$$
There are $N_k$ nodes with degree $d_k$ for $k = 1, 2, \ldots, K$.

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$[S_k]$: expected number of susceptible nodes of degree $d_k$,

$[S_k I]$: expected number of edges connecting an infected node to a susceptible node of degree $d_k$
There are $N_k$ nodes with degree $d_k$ for $k = 1, 2, \ldots, K$.

$$[ASI] = \sum_{k=1}^{K} [AS_k I], \quad [AS_k I] \approx \frac{d_k - 1}{d_k} \frac{[AS_k][S_k \ I]}{[S_k]}$$

Differential equations are needed for the new unknowns.
There are $N_k$ nodes with degree $d_k$ for $k = 1, 2, \ldots, K$. 

$$[A_S I] = \sum_{k=1}^{K} [A S_k I], \quad [A S_k I] \approx \frac{d_k - 1}{d_k} \frac{[A S_k][S_k I]}{[S_k]}$$

$$[\dot{S}_k] = \gamma [I_k] - \tau [S_k I], \quad k = 1, 2, \ldots, K.$$
There are $N_k$ nodes with degree $d_k$ for $k = 1, 2, \ldots, K$.

\[ [ASI] = \sum_{k=1}^{K} [AS_k I], \quad [AS_k I] \approx \frac{d_k - 1}{d_k} \frac{[AS_k][S_k I]}{[S_k]} \]

\[ \dot{[S_k]} = \gamma [I_k] - \tau[S_k I], \quad k = 1, 2, \ldots, K. \]

\[ [S_k A] \approx [SA] \frac{d_k[S_k]}{\sum_{l=1}^{K} d_l[S_l]} \]
There are $N_k$ nodes with degree $d_k$ for $k = 1, 2, \ldots, K$.

\[ [ASI] = \sum_{k=1}^{K} [AS_k] I, \quad [AS_k] I \approx \frac{d_k - 1}{d_k} \frac{[AS_k][S_k] I}{[S_k]} \]

\[ [\dot{S}_k] = \gamma[l_k] - \tau[S_k] I, \quad k = 1, 2, \ldots, K. \]

\[ [S_k A] \approx [SA] \frac{d_k[S_k]}{\sum_{l=1}^{K} d_l[S_l]} \]

\[ [AS_k I] \approx \frac{[AS][S] I d_k(d_k - 1)[S_k]}{S_2^2} \Rightarrow [ASI] \approx [AS][S] I \frac{S_2 - S_1}{S_1^2} \]

\[ S_1 = \sum_{k=1}^{N} d_k[S_k], \quad S_2 = \sum_{k=1}^{K} d_k^2[S_k]. \]
COMPACT PAIRWISE MODEL

\[ \dot{[S_k]}_c = \gamma[l_k]_c - \tau d_k [S_k]_c \frac{[S]_c}{S_s}, \]

\[ \dot{[S]}_c = \gamma([l]_c - [S]_c) + \tau ([S]+c - [S]_c)[S]_c p - \tau [S]_c, \]

\[ \dot{[S]}_c = 2\gamma[S]_c - 2\tau [S]_c [S]_c p, \]

\[ \dot{[l]}_c = 2\tau[S]_c - 2\gamma[l]_c + 2\tau [S]^2_c p, \]
\[ [S_k]_c = \gamma [l_k]_c - \tau d_k [S_k]_c \frac{[S|]_c}{S_s}, \]
\[ [S|]_c = \gamma ([|l|]_c - [S]_c) + \tau ([SS]_c - [S|]_c)[S|]_c P - \tau [S]_c, \]
\[ [SS]_c = 2\gamma [S|]_c - 2\tau [SS]_c [S|]_c P, \]
\[ [|l|]_c = 2\tau [S|]_c - 2\gamma [|l|]_c + 2\tau [S|]_c^2 P, \]

with \( S_s = \sum_{k=1}^{K} d_k [S_k]_c \) and \( P = \frac{1}{S_s^2} \sum_{k=1}^{K} (d_k - 1) d_k [S_k]_c. \)
Compact pairwise model:

\[
\begin{align*}
\dot{S}_k &= \gamma l_k - \tau d_k S_k \frac{[Sl]_c}{S_s}, \\
\dot{SI} &= \gamma ([II]_c - [SI]_c) + \tau ([SS]_c - [SI]_c)[Sl]_c P - \tau [SI]_c, \\
\dot{SS} &= 2\gamma [Sl]_c - 2\tau [SS]_c[Sl]_c P, \\
\dot{II} &= 2\tau [Sl]_c - 2\gamma [II]_c + 2\tau [Sl]_c^2 P,
\end{align*}
\]

Compact pairwise model: \( K + 3 \) equations
\[
\begin{align*}
[S_k]_c &= \gamma [l_k]_c - \tau d_k [S_k]_c \frac{[Sl]_c}{S_s}, \\
[Sl]_c &= \gamma ([ll]_c - [Sl]_c) + \tau ([SS]_c - [Sl]_c)[Sl]_c P - \tau [Sl]_c, \\
[SS]_c &= 2 \gamma [Sl]_c - 2 \tau [SS]_c[Sl]_c P, \\
[ll]_c &= 2 \tau [Sl]_c - 2 \gamma [ll]_c + 2 \tau [Sl]_c^2 P,
\end{align*}
\]

Compact pairwise model: \(K + 3\) equations

More complex and accurate models:

Pre-compact pairwise model: \(5K\) equations
**Compact pairwise model**:

\[
\begin{align*}
[S_k]'_c &= \gamma[I_k]_c - \tau d_k [S_k]_c \frac{[S/l]_c}{S_s}, \\
[S/l]'_c &= \gamma([I/l]_c - [S/l]_c) + \tau ([S/S]_c - [S/l]_c)[S/l]_c P - \tau [S/l]_c, \\
[S/S]'_c &= 2\gamma [S/l]_c - 2\tau [S/S]_c [S/l]_c P, \\
[I/I]'_c &= 2\tau [S/l]_c - 2\gamma [I/l]_c + 2\tau [S/l]^2_c P,
\end{align*}
\]

Compact pairwise model: \(K + 3\) equations

More complex and accurate models:

Pre-compact pairwise model: \(5K\) equations

Heterogeneous pairwise model: \(2K^2 + K\) equations
Bimodal random graph with $N = 1000$ nodes, average degree $n_1 = 20$, $\gamma = 1$, $\tau = 3\gamma n_1/n_2$, $n_i = \sum d_i^k p_k$

$N/2$ nodes have degree $d_1 = 5$, $N/2$ nodes have degree $d_2 = 35$. 
Bimodal random graph with $N = 1000$ nodes, average degree $n_1 = 20$, $\gamma = 1$, $\tau = 3\gamma n_1/n_2$, $n_i = \sum d_k^i p_k$

$N/2$ nodes have degree $d_1 = 5$, $N/2$ nodes have degree $d_2 = 35$. 

![Graph showing the comparison of ODE models to simulation](#)
Comparison of ODE models to simulation

Bimodal random graph with $N = 1000$ nodes, average degree $n_1 = 20$, $\gamma = 1$, $\tau = 3\gamma n_1 / n_2$, $n_i = \sum d_k^i p_k$

$N/2$ nodes have degree $d_1 = 5$, $N/2$ nodes have degree $d_2 = 35$.

Pairwise: dashed, Compact pairwise: continuous black, Heterogeneous pairwise: continuous red, Simulation (average of 200 runs): grey thick curve
Compact pairwise model is accurate for heterogeneous networks, but it contains $K + 3$ differential equations.
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For power-law random graphs or Barabási-Albert networks $K$ is large.
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AIM: derive a system of 4 differential equations performing well for strongly heterogeneous networks.
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**AIM:** derive a system of 4 differential equations performing well for strongly heterogeneous networks.

\[
\begin{align*}
\dot{[S_k]}_c &= \gamma [l_k]_c - \tau d_k [S_k]_c \frac{[S_l]_c}{S_s}, \\
\dot{[S_l]}_c &= \gamma ([l_l]_c - [S_l]_c) + \tau ([S_S]_c - [S_l]_c)[S_l]_c P - \tau [S_l]_c, \\
\dot{[S_S]}_c &= 2\gamma [S_l]_c - 2\tau [S_S]_c [S_l]_c P, \\
\dot{[l_l]}_c &= 2\tau [S_l]_c - 2\gamma [l_l]_c + 2\tau [S_l]_c^2 P,
\end{align*}
\]

with $S_s = \sum_{k=1}^K d_k [S_k]_c$ and $P = \frac{1}{S_s^2} \sum_{k=1}^K (d_k - 1) d_k [S_k]_c$. 
Compact pairwise model is accurate for heterogeneous networks, but it contains $K + 3$ differential equations.

AIM: derive a system of 4 differential equations performing well for strongly heterogeneous networks.

IDEA: Approximate $S_2 = \sum_{k=1}^{K} d_k^2 [S_k]$ without having differential equations for $[S_k]$ and use the simple pairwise model with the closure $[ASI] \approx [AS][SI] \frac{S_2 - S_1}{S_1^2}$. 
Compact pairwise model is accurate for heterogeneous networks, but it contains $K + 3$ differential equations.

AIM: derive a system of 4 differential equations performing well for strongly heterogeneous networks.

IDEA: Approximate $S_2 = \sum_{k=1}^{K} d_k^2 [S_k]$ without having differential equations for $[S_k]$ and use the simple pairwise model with the closure $[ASI] \approx [AS][SI] \frac{S_2 - S_1}{S_1^2}$.

\[
\begin{align*}
\dot{S} &= \gamma[I] - \tau[SI], \\
\dot{SI} &= \gamma([II] - [SI]) + \tau([SSI] - [ISI] - [SI]), \\
\dot{II} &= -2\gamma[II] + 2\tau([ISI] + [SI]), \\
\dot{SS} &= 2\gamma[SI] - 2\tau[SSI].
\end{align*}
\]
Approximation of the second moment $S_2$

$$[S] = \sum_{k=1}^{K} [S_k], \quad \sum_{k=1}^{K} d_k [S_k] = [SI] + [SS]$$
Approximation of the second moment $S_2$

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[S] = \sum_{k=1}^{K} [S_k], \quad \sum_{k=1}^{K} d_k [S_k] = [SI] + [SS]
\]

Let $s_k = [S_k]/[S]$, then

\[
s_1 + s_2 + \ldots + s_K = 1,
\]

\[
d_1 s_1 + d_2 s_2 + \ldots + d_K s_K = n_S := \frac{[SI] + [SS]}{[S]}.
\]
Approximation of the second moment $S_2$

$$[S] = \sum_{k=1}^{K} [S_k], \quad \sum_{k=1}^{K} d_k [S_k] = [SI] + [SS]$$

Let $s_k = [S_k]/[S]$, then

$$s_1 + s_2 + \ldots + s_K = 1,$$

$$d_1 s_1 + d_2 s_2 + \ldots + d_K s_K = n_S := \frac{[SI] + [SS]}{[S]}.$$

Degree distribution of the network: $p_k = N_k/N$. 

19/24
Approximation of the second moment $S_2$

$$[S] = \sum_{k=1}^{K} [S_k], \quad \sum_{k=1}^{K} d_k[S_k] = [SL] + [SS]$$

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Approximation of the second moment \( S_2 \)

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Degree distribution of the network: \( p_k = N_k/N \).

Numerical observation: \( s_k/p_k \) is a linear function of the degree \( d_k \).

Then some calculation yields

\[
(n_2 - n_1^2)\frac{s_k}{p_k} = n_2 - n_S n_1 + d_k(n_S - n_1),
\]

where \( n_i = \sum d_k^i p_k \). Leading to
Approximation of the second moment $S_2$

$$[S] = \sum_{k=1}^{K} [S_k], \quad \sum_{k=1}^{K} d_k[S_k] = [Sl] + [SS]$$

Let $s_k = [S_k]/[S]$, then

$$s_1 + s_2 + \ldots + s_K = 1,$$
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Degree distribution of the network: $p_k = N_k/N$.

Numerical observation: $s_k/p_k$ is a linear function of the degree $d_k$.

$$(n_2 - n_1^2) \sum_{k=1}^{K} d_k^2 s_k = n_2(n_2 - n_S n_1) + n_3(n_S - n_1),$$

where $n_i = \sum d_k^i p_k$. 

Approximation of the second moment $S_2$

\[
S_2 = \sum_{k=1}^{K} d_k^2 [S_k] \approx \sum_{k=1}^{K} d_k^2 [S] s_k = [S] \frac{n_2(n_2 - n_S n_1) + n_3(n_S - n_1)}{n_2 - n_1^2}.
\]
Approximation of the second moment $S_2$

$$S_2 = \sum_{k=1}^{K} d_k^2 S_k \approx \sum_{k=1}^{K} d_k^2 S k = [S] \frac{n_2(n_2 - n_S n_1) + n_3(n_S - n_1)}{n_2 - n_1^2}.$$ 

$S_1 = [SI] + [SS] = n_S[S]$ implies

$$\frac{S_2 - S_1}{S_1^2} \approx \frac{1}{n_S^2[S]} \left( \frac{n_2(n_2 - n_S n_1) + n_3(n_S - n_1)}{n_2 - n_1^2} - n_S \right).$$
Approximation of the second moment $S_2$

\[
S_2 = \sum_{k=1}^{K} d_k^2 [S_k] \approx \sum_{k=1}^{K} d_k^2 [S] s_k = [S] \frac{n_2(n_2 - n_S n_1) + n_3(n_S - n_1)}{n_2 - n_1}.
\]

$S_1 = [SI] + [SS] = n_S [S]$ implies

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\frac{S_2 - S_1}{S_1^2} \approx \frac{1}{n_S^2 [S]} \left( \frac{n_2(n_2 - n_S n_1) + n_3(n_S - n_1)}{n_2 - n_1} - n_S \right).
\]

The new closure relation:

\[
[ASI] \approx [AS][SI] \frac{S_2 - S_1}{S_1^2} = [AS][SI] \frac{n_S^2 [S]}{n_S} \left( \frac{n_2(n_2 - n_S n_1) + n_3(n_S - n_1)}{n_S(n_2 - n_1)} - 1 \right).
\]
\[
\begin{align*}
\dot{\mathbf{S}}_s &= \gamma[I]_s - \tau[S]_s, \\
\dot{\mathbf{SI}}_s &= \gamma([II]_s - [SI]_s) + \tau[SI]_s([SS]_s - [SI]_s)Q - \tau[SI]_s, \\
\dot{SS}_s &= 2\gamma[SI]_s - 2\tau[SI]_s[SS]_sQ, \\
\dot{II}_s &= -2\gamma[II]_s + 2\tau[SI]^2_sQ + 2\tau[SI]_s,
\end{align*}
\]

where
\[
Q = \frac{1}{n_S[S]} \left( \frac{n_2(n_2 - n_Sn_1) + n_3(n_S - n_1)}{n_S(n_2 - n_1)} - 1 \right), \quad n_S := \frac{[SI] + [SS]}{[S]},
\]
\[
n_i = \sum d_k^i p_k.
\]
Bimodal random graph with \( N = 1000 \) nodes, average degree \( n_1 = 20, \gamma = 1, \tau = 3\gamma n_1/n_2, \quad n_i = \sum d_k^i p_k \).

\( N_1 \) nodes have degree \( d_1 = 5 \), \( N_2 \) nodes have degree \( d_2 = 35 \).
Bimodal random graph with $N = 1000$ nodes, average degree $n_1 = 20$, $\gamma = 1$, $\tau = 3\gamma n_1/n_2$, $n_i = \sum d_k^i p_k$

$N_1$ nodes have degree $d_1 = 5$, $N_2$ nodes have degree $d_2 = 35$. 

![Graph showing the performance of the model over time](image)
Performance of the Super Compact Pairwise Model

Bimodal random graph with $N = 1000$ nodes, average degree $n_1 = 20$, $\gamma = 1$, $\tau = 3\gamma n_1/n_2$, $n_i = \sum d_k^i p_k$.

$N_1$ nodes have degree $d_1 = 5$, $N_2$ nodes have degree $d_2 = 35$.

PW: dashed, compact PW: continuous, super compact PW: circles, upper curves: $N_1 = 0.1N$, $N_2 = 0.9N$, middle curves: $N_1 = 0.5N$, $N_2 = 0.5N$, lower curves: $N_1 = 0.9N$, $N_2 = 0.1N$.
Power-law random graph with $N = 1000$ nodes, $p_k = C k^{-\alpha}$ for $k = k_{\text{min}}, k_{\text{min}} + 1, \ldots, k_{\text{max}}$, $\gamma = 1$, $\tau = 3^{\gamma \frac{n_1}{n_2}}$, $n_i = \sum d_i^k p_k$
Performance of the Super Compact Pairwise Model

Power-law random graph with $N = 1000$ nodes, $p_k = Ck^{-\alpha}$ for $k = k_{\text{min}}, k_{\text{min}} + 1, \ldots, k_{\text{max}}$, $\gamma = 1$, $\tau = 3\gamma n_1/n_2$, $n_i = \sum d_i^k p_k$
Power-law random graph with $N = 1000$ nodes, $p_k = Ck^{-\alpha}$ for $k = k_{\text{min}}, k_{\text{min}} + 1, \ldots, k_{\text{max}}$, $\gamma = 1$, $\tau = 3\gamma n_1/n_2$, $n_i = \sum d_k^n p_k$

PW: dashed, compact PW: continuous, super compact PW: circles, upper curves: $k_{\text{min}} = 10$, $k_{\text{max}} = 140$, lower curves: $k_{\text{min}} = 5$, $k_{\text{max}} = 30$
Thank you for your attention!