CONSTRUCTIONS OF STRATIFIED ALGEBRAS

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Abstract. In this paper a construction to build recursively all basic finite dimensional standardly stratified algebras is given. In comparison to the construction described by Dlab and Ringel for the quasi-hereditary case ([DR3]) some new features appear here.

1. Introduction

The concept of standardly stratified algebras (or Δ-filtered algebras) appears as a natural generalization of the concept of quasi-hereditary algebras. The class of quasi-hereditary algebras was introduced by Cline, Parshall and Scott (see [CPS1], [PS]) in connection with their study of highest weight categories arising in the representation theory of semisimple complex Lie algebras and algebraic groups. The study of quasi-hereditary algebras grew into an extensive volume of contributions starting with the seminal papers [DR1], [R], [DR2]. The concept of standardly stratified algebras was introduced independently in [D1] and in the comprehensive study [CPS2] and further extended in [ADL1] and [ADL2]. It may be also pointed out that the concept of a stratifiying ideal of [CPS2] appeared already as a strongly idempotent ideal in [APT]. A particular type of standardly stratified algebras, namely properly stratified algebras of [D2] illustrates again a very close relationship to the representation theory of Lie algebras (see also [FM], [FKM]).

Ever since their introduction, standardly stratified algebras have drawn much attention; their structural and homological properties were investigated among others in [AHLU1], [AHLU2], [ADL3], [ChD], [ADL4], [M]. It is worth mentioning that the main body of results in this field is established for standardly stratified algebras and then easily generalized for particular types of these algebras such as quasi-hereditary and properly stratified algebras.

As in the case of quasi-hereditary algebras, the structure of standardly stratified algebras includes two recursive sequences of standardly stratified algebras. One sequence is obtained by taking consecutive quotients of the algebra modulo the respective idempotent trace ideals. The other sequence is obtained by taking centralizer algebras of the corresponding sequence of indecomposable projective

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modules. In most cases the first approach is used to study these algebras. On the other hand Dlab and Ringel ([DR3]) have shown that the sequence of centralizer algebras relates to the structure of the categories of perverse sheaves and provides a recursive construction of all finite dimensional basic quasi-hereditary algebras.

The goal of this paper is to extend this construction for the general situation. Much of the original results can be adopted for this case, however, the complexity of standard modules requires some extra precautions. Furthermore, to construct all standardly stratified algebras we need to introduce an extra step to make the procedure complete. Thus, the main result is the following theorem.

Theorem. *Let L be a local algebra, and C a basic algebra such that C_C is filtered by standard C-modules with respect to some order $(e_2, \dots, e_n)$ of a complete set of primitive orthogonal idempotents in C. Furthermore, let ${}_LE_C$ and ${}_CF_L$ be bimodules such that E_C is filtered by standard C-modules and ${}_CF$ is filtered by proper standard C^{opp}-modules. In addition, suppose that $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$ is a C-C bimodule homomorphism. Then $\tilde{A} = L \oplus (E \underset{C}{\otimes} F) \oplus E \oplus F \oplus C$ has an algebra structure such that $\tilde{A}_{\tilde{A}}$ is filtered by standard $\tilde{A}$-modules with respect to the order $(1_L, e_2, \dots, e_n)$. Moreover, one can get all basic standardly stratified algebras recursively, starting with a local algebra, by constructing and taking suitable quotients of algebras $\tilde{A}$ obtained this way.*

2. CPS-stratified algebras

Let $(A, \mathbf{e})$ be a basic finite dimensional K-algebra with a (linearly) ordered complete set $\mathbf{e} = (e_1, \dots, e_n)$ of primitive orthogonal idempotents. Here K denotes an arbitrary field. We also use the notation $\varepsilon_i = e_i + \dots + e_n$ throughout the paper.

Let us recall some of the characterizations of the so-called CPS-stratifying ideals.

Definition 2.1. (Cf. [CPS2], [APT], [ADL2]) An idempotent ideal AeA of the algebra A will be called *CPS-stratifying* (or *stratifying* for short) if it satisfies any of the equivalent conditions (S1), (S1′), (S2), (S3):

(S1) (i) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and
(ii) $\operatorname{Tor}_t^{eAe}(Ae, eA) = 0$ for all $t > 0$;

(S1′) (i) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and
(ii) $\operatorname{Ext}^t_{eAe}(Ae, D(eA)) = 0$ for all $t > 0$;

(S2) $\operatorname{Ext}^t_{A/AeA}(X, Y) = \operatorname{Ext}^t_A(X, Y)$ for all $t \geq 0$ and A/AeA-modules X and Y;

(S3) Each term in the minimal projective resolution of AeA_A is generated by eA.

Definition 2.2. $(A, \mathbf{e})$ is said to be *CPS-stratified* if either $n = 1$ (i. e. the algebra is local) or in case $n > 1$, the ideal Ae_nA is stratifying and $(A/Ae_nA, (e_1, \dots, e_{n-1}))$ is CPS-stratified.

In the later sections of the paper we shall use some simple facts about CPS-stratified algebras (cf. also [CPS2]). In particular, we will need the following lemma.

LEMMA 2.3. *Let $(A, \mathbf{e})$ be CPS-stratified. Then $A\varepsilon_i A$ is a stratifying ideal and $(\varepsilon_i A\varepsilon_i, (e_i, \ldots, e_n))$ is CPS-stratified for all $i = 1, \ldots, n$.*

Proof. We prove both statements by induction on $n - i$, where the $i = n$ case immediately follows from the definition of CPS-stratified algebras. Now suppose that $n - i > 0$. Then the algebra $\bar{A} = A/Ae_nA$ is CPS-stratified with respect to the idempotents $\bar{\mathbf{e}} = (e_1, \ldots, e_{n-1})$, so the lemma holds for $\bar{A}$ (for the same i) by the induction hypothesis.

To prove the first statement, we show that condition (S2) holds for $A\varepsilon_i A$. Let us take $X, Y \in$ mod-$A/A\varepsilon_i A$. Then we have $\mathrm{Ext}^t_{A/A\varepsilon_i A}(X, Y) = \mathrm{Ext}^t_{\bar{A}/\bar{A}\varepsilon_i\bar{A}}(X, Y) = \mathrm{Ext}^t_{\bar{A}}(X, Y) = \mathrm{Ext}^t_A(X, Y)$ for all $t > 0$, thus $A\varepsilon_i A$ is stratifying in A.

Next, the fact that Ae_nA is stratifying implies by (S3) that $\varepsilon_i Ae_nA\varepsilon_i$ is stratifying in $\varepsilon_i A\varepsilon_i$. On the other hand $\varepsilon_i A\varepsilon_i/\varepsilon_i Ae_nA\varepsilon_i \simeq \varepsilon_i\bar{A}\varepsilon_i$ is CPS-stratified by induction, hence $\varepsilon_i A\varepsilon_i$ is CPS-stratified. □

LEMMA 2.4. *Suppose that for some i the algebra $(\varepsilon_i A\varepsilon_i, (e_i, \ldots, e_n))$ is CPS-stratified, and $A\varepsilon_i A$ is a stratifying ideal in A. Then the multiplication map $Ae_n \underset{e_nAe_n}{\otimes} e_nA \to Ae_nA$ is bijective.*

Proof. Let us denote $\varepsilon_j A\varepsilon_j$ by C_j. Since the multiplication map is clearly surjective, it is enough to show that $Ae_n \underset{C_n}{\otimes} e_nA$ and Ae_nA are isomorphic. So the following succession of isomorphisms provides a proof.

$$\begin{aligned}
Ae_n \underset{C_n}{\otimes} e_nA &= \\
A\varepsilon_i Ae_n \underset{C_n}{\otimes} e_nA\varepsilon_i A &\simeq && \text{(since } A\varepsilon_i A \text{ is stratifying in } A\text{)}\\
(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A)e_n \underset{C_n}{\otimes} e_n(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A) &\simeq \\
A\varepsilon_i \underset{C_i}{\otimes}(\varepsilon_i Ae_n \underset{C_n}{\otimes} e_nA\varepsilon_i) \underset{C_i}{\otimes} \varepsilon_i A &\simeq && \text{(since } \varepsilon_i A\varepsilon_i \text{ is CPS-stratified)}\\
A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i Ae_nA\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A &= \\
(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A)e_nA\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A &\simeq && \text{(since } A\varepsilon_i A \text{ is stratifying in } A\text{)}\\
A\varepsilon_i Ae_nA\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A &= \\
A\varepsilon_i Ae_n(A\varepsilon_i \underset{C_i}{\otimes} \varepsilon_i A) &\simeq && \text{(since } A\varepsilon_i A \text{ is stratifying in } A\text{)}\\
A\varepsilon_i Ae_nA\varepsilon_i A &= \\
Ae_nA.
\end{aligned}$$

□

3. Δ-filtered algebras

For the reader's convenience let us recall some basic definitions and results.

For a given algebra $(A, \mathbf{e})$ the *standard modules* are defined by $\Delta(i) = e_iA/e_iA\varepsilon_{i+1}A$ and the *proper standard modules* by $\bar{\Delta}(i) = e_iA/e_i \operatorname{rad} A\varepsilon_iA$ for $1 \leq i \leq n$. Similarly one can define the left standard and left proper standard modules $\Delta^\circ(i)$ and $\bar{\Delta}^\circ(i)$. The (full) subcategories $\mathcal{F}(\Delta_A)$ and $\mathcal{F}(\bar{\Delta}_A)$ of the category mod-A of all finite dimensional right A-modules consist of those A-modules which have a filtration by standard modules (or proper standard modules, respectively).

It is well known (cf. for example [ADL1]) that a module $M \in$ mod-A belongs to $\mathcal{F}(\Delta_A)$ (in this case we will say that M is Δ-*filtered*) if and only if the trace Me_nA is projective (i. e. it is $\Delta(n)$-filtered) and M/Me_nA is Δ-filtered as an $\bar{A} = A/Ae_nA$-module. Similarly, $M \in \mathcal{F}(\bar{\Delta}_A)$ if and only if Me_nA has a filtration by $\bar{\Delta}(n)$ and M/Me_nA belongs to $\mathcal{F}(\bar{\Delta}_{\bar{A}})$.

Definition 3.1. An algebra $(A, \mathbf{e})$ is said to be Δ-*filtered* if the regular module A_A belongs to $\mathcal{F}(\Delta_A)$. Similarly, $(A, \mathbf{e})$ is said to be $\bar{\Delta}$-*filtered* if $A_A \in \mathcal{F}(\bar{\Delta}_A)$. An algebra is called *standardly stratified* if it is either Δ or $\bar{\Delta}$-filtered.

By a result of Dlab (cf. [D1]) $(A, \mathbf{e})$ is Δ-filtered if and only if $(A^{opp}, \mathbf{e})$ is $\bar{\Delta}^\circ$-filtered. Furthermore it is straightforward that Δ-filtered algebras are also CPS-stratified (cf. condition (S3) or [CPS2]). Hence the above result of Dlab implies that $\bar{\Delta}$-filtered algebras are also CPS-stratified algebras (since condition (S1) is obviously left-right symmetric).

In the following we want to describe the property that $(A, \mathbf{e})$ is Δ-filtered in terms of its centralizer algebra $\varepsilon_2A\varepsilon_2$, and the corresponding subalgebra and bimodules e_1Ae_1, $e_1A\varepsilon_2$ and ε_2Ae_1.

Theorem 3.2. *Given an algebra $(A, \mathbf{e})$ let us consider the local algebra $L = e_1Ae_1$, the centralizer algebra $C = \varepsilon_2A\varepsilon_2$ together with the order $\mathbf{e}' = (e_2, \dots, e_n)$ and the bimodules $E = e_1A\varepsilon_2$ and $F = \varepsilon_2Ae_1$. Then $(A, \mathbf{e})$ is Δ-filtered if and only if the following conditions hold:*
(1) *$C_C \in \mathcal{F}(\Delta_C)$;*
(2) *$E_C \in \mathcal{F}(\Delta_C)$;*
(3) *${}_CF \in \mathcal{F}({}_C\bar{\Delta}^\circ)$;*
(4) *the multiplication map $E \underset{C}{\otimes} F \to L$ is injective.*

Proof. Let us note that the condition (4) is equivalent to the condition (4′) stating that the multiplication map $A\varepsilon_2 \underset{C}{\otimes} \varepsilon_2A \to A\varepsilon_2A$ is injective (in fact, bijective), since $A\varepsilon_2 = E \oplus C$, $\varepsilon_2A = F \oplus C$, and the injectivity of the multiplication map on the other three components is obvious.

First assume that A is Δ-filtered. Then it is also CPS-stratified and thus by Lemma 2.3, (4′) and hence (4) of the theorem holds. The conditions (1) and (2) follow from the fact that $A\varepsilon_2A$ is Δ-filtered. Hence $e_1A\varepsilon_2A$ and ε_2A are also Δ-filtered, and therefore $e_1A\varepsilon_2$ and $\varepsilon_2A\varepsilon_2$ are Δ-filtered over $\varepsilon_2A\varepsilon_2$. Similarly, (3) holds because $A\varepsilon_2A$ is $\bar{\Delta}^\circ$-filtered.

The opposite statement will be proved by induction on n. Thus, assume that the conditions (1)–(4) hold for A. We will show that Ae_nA_A is projective and that the conditions (1)–(4) hold for the factor algebra $\bar{A} = A/Ae_nA$.

First, let us prove that $A\varepsilon_2 A$ is a stratifying ideal. By the condition (4′) the map $A\varepsilon_2 \underset{C}{\otimes} \varepsilon_2 A \to A\varepsilon_2 A$ is injective. On the other hand, using the condition (1) and the dual of Theorem 3.1 of [ADL1], we conclude that $\mathrm{Ext}^t_C(\Delta_C, D({}_C\bar{\Delta}^{\circ})) = 0$ for all $t > 0$. Thus (2) and (3) imply that $\mathrm{Ext}^t_C(A\varepsilon_2, D(\varepsilon_2 A)) = \mathrm{Ext}^t_C(E \oplus C, D(F \oplus C)) = 0$ for all $t > 0$. Hence the condition (S1′) implies that $A\varepsilon_2 A$ is a stratifying ideal in A.

By the conditions (1) and (2), $A\varepsilon_2 = E \oplus C \in \mathcal{F}(\Delta_C)$, so the trace of $\Delta_C(n)$ on $A\varepsilon_2$ is projective: $Ae_n A\varepsilon_2 \simeq \oplus\, e_n A\varepsilon_2$. Thus $Ae_n \simeq \oplus\, e_n Ae_n$ as $e_n Ae_n$-modules. Hence we get that $Ae_n \underset{e_n Ae_n}{\otimes} e_n A \simeq (\oplus\, e_n Ae_n) \underset{e_n Ae_n}{\otimes} e_n A \simeq \oplus\, e_n A$ is a projective A-module. Finally, by Lemma 2.4 (using that C is Δ-filtered, thus CPS-stratified as well), $Ae_n \underset{e_n Ae_n}{\otimes} e_n A \simeq Ae_n A$, and so $Ae_n A$ is a projective right A-module.

Now, take the factor algebra $\bar{A} = A/Ae_n A$. The corresponding objects to consider are $\bar{C} \simeq \varepsilon_2 A\varepsilon_2/\varepsilon_2 Ae_n A\varepsilon_2$, $\bar{E} \simeq e_1 A\varepsilon_2/e_1 Ae_n A\varepsilon_2$ and $\bar{F} \simeq \varepsilon_2 Ae_1/\varepsilon_2 Ae_n Ae_1$. The remarks preceding Definiton 3.1 show that the conditions (1), (2) and (3) also hold for $\bar{A}$. Finally, since $A\varepsilon_2 A$ and $Ae_n A$ are stratifying, (S2) implies that $\bar{A}\bar{\varepsilon}_2\bar{A}$ is stratifying in $\bar{A}$: for any $X, Y \in \text{mod-}\bar{A}/\bar{A}\bar{\varepsilon}_2\bar{A}$ we have $\mathrm{Ext}^t_{\bar{A}/\bar{A}\bar{\varepsilon}_2\bar{A}}(X,Y) = \mathrm{Ext}^t_{A/A\varepsilon_2 A}(X,Y) = \mathrm{Ext}^t_A(X,Y) = \mathrm{Ext}^t_{\bar{A}}(X,Y)$. Thus by (S1), the condition (4) also holds for $\bar{A}$. By induction we get that $\bar{A}$ is Δ-filtered, so A is also Δ-filtered.

□

Note that the data above correspond to the Peirce decomposition of the algebra $A \simeq \begin{pmatrix} L & E \\ F & C \end{pmatrix}$.

4. Construction of Δ-filtered algebras

In this chapter we proceed in the opposite direction and construct all Δ-filtered algebras from "smaller" algebras, using a recursive process.

Suppose L and C are algebras together with an L-C-bimodule E and a C-L-bimodule F and a C-C-bimodule homomorphism $\mu : F \underset{L}{\otimes} E \to \mathrm{rad}(C)$. Then it is easy to see that the map

$$(E \underset{C}{\otimes} F) \underset{L}{\otimes} (E \underset{C}{\otimes} F) \simeq E \underset{C}{\otimes} (F \underset{L}{\otimes} E) \underset{C}{\otimes} F \overset{\mathrm{id}_E \otimes \mu \otimes \mathrm{id}_F}{\longrightarrow} E \underset{C}{\otimes} C \underset{C}{\otimes} F \simeq E \underset{C}{\otimes} F$$

defines an algebra multiplication on the L-L-bimodule $E \underset{C}{\otimes} F$. Thus:

$$(e \otimes f)(e' \otimes f') = e\mu(f \otimes e') \otimes f' = e \otimes \mu(f \otimes e')f'.$$

The split extension $\tilde{L} = L \ltimes (E \underset{C}{\otimes} F)$ of the algebra L by $E \underset{C}{\otimes} F$ is defined, in the usual way, on the cartesian product with multiplication:

$$(l, u)(l', u') = (ll', lu' + ul' + uu').$$

Now we can extend the L-C and C-L bimodule structure of E and F respectively to $\tilde{L}$-C and C-$\tilde{L}$ structure, using the maps

$$(E \underset{C}{\otimes} F) \underset{L}{\otimes} E \simeq E \underset{C}{\otimes} (F \underset{L}{\otimes} E) \overset{\mathrm{id}_E \otimes \mu}{\longrightarrow} E \underset{C}{\otimes} C \simeq E$$

and

$$F \underset{L}{\otimes} (E \underset{C}{\otimes} F) \simeq (F \underset{L}{\otimes} E) \underset{C}{\otimes} F \overset{\mu \otimes \mathrm{id}_F}{\longrightarrow} C \underset{C}{\otimes} F \simeq F.$$

Finally, the C-C-bimodule map $\mu : F \underset{L}{\otimes} E \to \mathrm{rad}(C)$ induces naturally a C-C-bimodule map $\tilde{\mu} : F \underset{\tilde{L}}{\otimes} E \to \mathrm{rad}\, C$, since $\mu(f(e' \underset{C}{\otimes} f') \underset{L}{\otimes} e) = \mu(\mu(f \underset{L}{\otimes} e')f' \underset{L}{\otimes} e) = \mu(f \underset{L}{\otimes} e')\mu(f' \underset{L}{\otimes} e) = \mu(f \underset{L}{\otimes} e' \mu(f' \underset{L}{\otimes} e)) = \mu(f \underset{L}{\otimes} (e' \underset{C}{\otimes} f')e)$.

It is easy to show that if L is a local algebra then $\tilde{L}$ is also local. Indeed, $E \underset{C}{\otimes} F$ is a nilpotent ideal in $\tilde{L}$, since $(E \underset{C}{\otimes} F)^k = E \underset{C}{\otimes} (\mu(F \underset{L}{\otimes} E))^{k-1} F \subseteq E \underset{C}{\otimes} (\mathrm{rad}\, C)^{k-1} F$. Thus, $\mathrm{rad}\, L + (E \underset{C}{\otimes} F)$ is a nilpotent ideal of $\tilde{L}$, and furthermore, $\tilde{L}/(\mathrm{rad}\, L + (E \underset{C}{\otimes} F)) \simeq L/\mathrm{rad}\, L$ is a simple $\tilde{L}$ module.

Now we can consider the matrix algebra $\tilde{A} = \begin{pmatrix} \tilde{L} & E \\ F & C \end{pmatrix}$ with the natural multiplication structure:

$$\begin{pmatrix} x & e \\ f & c \end{pmatrix} \begin{pmatrix} x' & e' \\ f' & c' \end{pmatrix} = \begin{pmatrix} xx' + e \underset{C}{\otimes} f' & xe' + ec' \\ fx' + cf' & \tilde{\mu}(f \underset{\tilde{L}}{\otimes} e') + cc' \end{pmatrix}$$

for arbitrary $x, x' \in \tilde{L}$, $e, e' \in E$, $f, f' \in F$ and $c, c' \in C$. The associativity of the multiplication follows directly from the definition of the bimodule structures ${}_{\tilde{L}}E_C$ and ${}_C F_{\tilde{L}}$. (Note that the algebra $\tilde{A}$ is usually called the *Morita ring* corresponding to the Morita context $(\tilde{L}, C, E, F, \iota, \tilde{\mu})$ where $\iota : E \underset{C}{\otimes} F \to \tilde{L}$ is the natural embedding.)

Theorem 4.1. *Let L be a local algebra with identity element denoted by e_1 and let $(C, (e_2, \ldots, e_n))$ be a (basic) Δ-filtered algebra. Let ${}_L E_C$ and ${}_C F_L$ be two bimodules such that $E_C \in \mathcal{F}(\Delta_C)$ and ${}_C F \in \mathcal{F}({}_C\bar{\Delta}^{\circ})$, together with a bimodule map $\mu : F \underset{L}{\otimes} E \to \mathrm{rad}\, C$. Then the algebra $\tilde{A}$ constructed above is a Δ-filtered algebra with respect to the sequence of idempotents $\mathbf{e} = (e_1, e_2, \ldots, e_n)$.*

Proof. We have seen that $\tilde{L} = e_1 \tilde{A} e_1$ is local, so $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ is a complete sequence of primitive orthogonal idempotents. It is also clear that $\varepsilon_2 \tilde{A} \varepsilon_2 = C$ and that $e_1 \tilde{A} \varepsilon_2 = E$ and $\varepsilon_2 \tilde{A} e_1 = F$ satisfy the filtration conditons of Theorem 3.2 by the assumptions on C, E and F. Moreover the multiplication map $\iota : E \underset{C}{\otimes} F \to \tilde{L}$ is injective by definition, so $(\tilde{A}, \mathbf{e})$ is Δ-filtered. □

To construct all Δ-filtered algebras, we need the following concept.

Definition 4.2. Let $(A, \mathbf{e})$ be a Δ-filtered algebra. An ideal $H \lhd A$ will be called *auxiliary* if $H \subseteq e_1(\mathrm{rad}\, A)e_1$ and $H \cap e_1 A \varepsilon_2 A e_1 = 0$.

LEMMA 4.3. *Let $(A, \mathbf{e})$ be a Δ-filtered algebra and $H \lhd A$ an auxiliary ideal. Then $(A/H, \mathbf{e})$ is also Δ-filtered.*

Proof. The conditions imply that $A\varepsilon_2 A \cap H = 0$, hence the trace ideal $A\varepsilon_2 A$ maps injectively into $\bar{A} = A/H$. Thus the Δ_A-filtration of $A\varepsilon_2 A_A$ gives a $\Delta_{\bar{A}}$-filtration of $\bar{A}\varepsilon_2\bar{A}_{\bar{A}}$. Since $\bar{A}/\bar{A}\varepsilon_2\bar{A} \simeq \Delta_{\bar{A}}(1)$, the algebra $(\bar{A}, \mathbf{e})$ is also Δ-filtered. □

Finally, we show that all Δ-filtered algebras can be obtained using the construction of Theorem 4.1, followed by factoring out an auxiliary ideal.

THEOREM 4.4. *Let $(A, \mathbf{e})$ be a basic Δ-filtered algebra. Take $L = e_1Ae_1$, $C = \varepsilon_2 A\varepsilon_2$, $E = e_1A\varepsilon_2$, $F = \varepsilon_2 Ae_1$ and let $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$ and $\nu : E \underset{C}{\otimes} F \to L$ be the multiplication maps in A. Construct the algebras $\tilde{L} = L \ltimes (E \underset{C}{\otimes} F)$ and $\tilde{A} = \begin{pmatrix} \tilde{L} & E \\ F & C \end{pmatrix}$ as in Theorem 4.1. Then $H = \left\{ \nu(u) - u \,|\, u \in E \underset{C}{\otimes} F \right\} \subseteq \tilde{L}$ is an auxiliary ideal of $\tilde{A}$ and the algebra $\tilde{A}/H$ is isomorphic to A.*

Proof. First, let us observe that Theorem 3.2 implies that L, C, E and F satisify the conditions of Theorem 4.1, hence the algebra $\tilde{A}$ is Δ-filtered.

In order to show that H is an ideal in $\tilde{A}$, note first that for any $u, u' \in E \underset{C}{\otimes} F \subseteq \tilde{L}$ we have $u'u = u'\nu(u)$. Indeed, for $e, e' \in E$ and $f, f' \in F$ we get $(e' \otimes f')(e \otimes f) = e' \otimes \mu(f' \otimes e)f = e' \otimes (f'e)f = e' \otimes f'(ef) = (e' \otimes f')(ef) = (e' \otimes f')\nu(e \otimes f)$. Similarly, for any $f' \in F$ and $u \in E \underset{C}{\otimes} F$, we have $f'u = f'\nu(u)$, since $f'(e \otimes f) = \mu(f' \otimes e)f = (f'e)f = f'(ef) = f'\nu(e \otimes f)$. Thus, for $\tilde{a} \in \tilde{A}$ and $u \in E \underset{C}{\otimes} F$:

$$\begin{aligned} \tilde{a}(u - \nu(u)) &= \begin{pmatrix} l' + u' & e' \\ f' & c' \end{pmatrix} \begin{pmatrix} u - \nu(u) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} l'(u - \nu(u)) + u'(u - \nu(u)) & 0 \\ f'(u - \nu(u)) & 0 \end{pmatrix} \\ &= \begin{pmatrix} l'u - \nu(l'u) & 0 \\ 0 & 0 \end{pmatrix} \in H. \end{aligned}$$

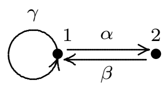
Similarly, one can show that $(u - \nu(u))\tilde{a} \in H$. Since H is clearly closed under addition, $H \lhd \tilde{A}$.

Also, $H \subseteq E \underset{C}{\otimes} F + \operatorname{rad} L = \operatorname{rad} \tilde{L} = e_1(\operatorname{rad} \tilde{A})e_1$. Since the map ν is injective, $u - \nu(u) \neq 0$ implies that $u \neq 0$ and $\nu(u) \neq 0$. Then $L \cap E \underset{C}{\otimes} F = 0$ yields that $u - \nu(u) \notin E \underset{C}{\otimes} F$ and $\notin L$, and consequently $H \cap E \underset{C}{\otimes} F = 0$ and $H \cap L = 0$. It follows that the condition $H \cap e_1\tilde{A}\varepsilon_2\tilde{A}e_1 = H \cap (E \underset{C}{\otimes} F) = 0$ holds for the ideal H, hence the ideal H is auxiliary. Furthermore, $H \cap A = H \cap \tilde{L} \cap A = H \cap L = 0$. Also, it is straightforward that $A + H = \tilde{A}$, so $\tilde{A}/H \simeq A$. □

COROLLARY 4.5. *Let L, C, E, F and μ be given as in Theorem 4.1 and let I be an auxiliary ideal of the algebra $\tilde{A}$. Then $\tilde{A}/I$ is a Δ-filtered algebra, and every basic Δ-filtered algebra can be obtained in this way.*

While all (basic) quasi-hereditary algebras over a perfect field can be recursively obtained by applying the construction described in Theorem 4.1, the next example illustrates that for standardly stratified algebras in some cases one cannot avoid factorization modulo an auxiliary ideal.

Example 4.6. Consider the algebra $A = KQ/I$, where the quiver Q is



and the admissible ideal $I = \langle \gamma\alpha, \gamma^2 - \alpha\beta, \beta\alpha, \beta\gamma \rangle$. Thus, the right regular representation of A is

$$A_A = \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}.$$

Then the construction described in Theorem 4.4 results in $\tilde{A}$ with regular representation as follows:

$$\tilde{A}_{\tilde{A}} = \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 1 & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}.$$

Observe that the quiver of A and $\tilde{A}$ coincide, however the products γ^2 and $\alpha\beta$ are not yet indentified in $\tilde{A}$. This is done when we take the quotient modulo the auxiliary ideal $\langle \gamma^2 - \alpha\beta \rangle$. Note that $e_1 A \varepsilon_2 A e_1$ has no subalgebra complement in $e_1 A e_1$ hence A cannot be obtained directly in the form $\tilde{A}$ for a suitable local algebra L.

Example 4.7. Consider the algebra $A = KQ/I'$, where the quiver Q is the same as in Example 4.6 and $I' = \langle \gamma\alpha, \gamma^2, \beta\alpha, \beta\gamma \rangle$. Thus, the right regular representation of A is

$$A_A = \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}.$$

Here, the algebra $\tilde{A}$ constructed according to Theorem 4.4 has the following regular representation:

$$\tilde{A}_{\tilde{A}} = \begin{smallmatrix} & & 1 & \\ 1 & & 1 & \\ & & & 2 \\ & & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}.$$

Hence in this example the quiver of $\tilde{A}$ differs from the quiver of A, since we get a new arrow corresponding to $\delta = \alpha\beta \in L$. This element is different from the product of arrows α and β, taken in $\tilde{A}$. We get A as a quotient of $\tilde{A}$ modulo the auxiliary ideal $\langle \delta - \alpha\beta \rangle$. Unlike the previous example, in this case we could obtain A directly as an algebra $\tilde{A}$: we would have to start with the local subalgebra $L = \langle e_1, \gamma \rangle$ instead of $e_1 A e_1$.

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