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ABOUT THE NON-INTEGER PROPERTY OF HYPERHARMONIC NUMBERS

By

ISTVÁN MEZŐ

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Abstract. It was proven in 1915 by Leopold Theisinger that the H_n harmonic numbers are never integers. In 1996 Conway and Guy have defined the concept of hyperharmonic numbers. The question naturally arises: are there any integer hyperharmonic numbers? The author gives a partial answer to this question and conjectures that the answer is "no".

The n-th harmonic number is the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Conway and Guy in [2] defined the harmonic numbers of higher orders, also known as the hyperharmonic numbers: $H_n^{(1)} := H_n$, and for all $r > 1$ let

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}$$

be the n-th harmonic number of order r. These numbers can be expressed by binomial coefficients and ordinary harmonic numbers:

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}).$$

The prominent role of these numbers has been realized recently in combinatory. The $\left[n \atop k \right]_r$ r-Stirling number is the number of the permutations of

AMS Subject Classification (2000): 11B83 the set $\{1, \ldots, n\}$ having k disjoint, non-empty cycles, in which the elements 1 through r are restricted to appear in different cycles.

In [7] one can find the following interesting equality:

$$H_n^{(r)} = \frac{\left[{n+r \atop r+1} \right]_r}{n!}.$$

Let us turn our attention to the main question of this paper. It is known that any number of consecutive terms not necessarily beginning with 1 will never sum to an integer (see [4]). As a corollary, we get that the H_n harmonic numbers are never integers ($n > 1$). Theisinger proved this latter result directly in 1915 [1] . The question appears obviously: are there any integer hyperharmonic numbers?

Theisinger's main tool was the 2-adic norm. We give a short summary of his method. Every rational number $x \neq 0$ can be represented by $x = \frac{p^\alpha r}{s}$, where p is a fixed prime number, r and s are relative prime integers to p. α is a unique integer. We can define the p-adic norm of x by

$$|x|_p = p^{-\alpha}, \text{ and let } |0|_p = 0.$$

This norm fulfills the properties of the usual norms, namely

$$|x|_p = 0 \Longleftrightarrow x = 0,$$

$$|xy|_p = |x|_p |y|_p \quad (x, y \in \mathbf{Q}),$$

$$|x+y|_p \leq |x|_p + |y|_p \quad (x, y \in \mathbf{Q}).$$

Furthermore, the so-called strong triangle inequality also holds:

$$|x+y|_p \leq \max\{|x|_p, |y|_p\} (\leq |x|_p + |y|_p).$$

We shall use the following property of integer numbers:

$$x \in \mathbf{Z} \Longrightarrow x = p^\alpha r \Longrightarrow |x|_p = \frac{1}{p^\alpha} \leq 1,$$

where r and the prime p are relative prime integers. This means that if the p-norm of a rational x is greater than 1 then x is necessarily non-integer.

Let us introduce the order of a natural number n: if $2^m \leq n < 2^{m+1}$, then $\mathrm{Ord}_2(n) := m$. It is obvious that $\mathrm{Ord}_2(n) = \lfloor \ln(n) / \ln(2) \rfloor$.

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THEOREM 1.

$$|H_n|_2 = 2^{\mathrm{Ord}_2(n)} \quad (n \in \mathbf{N}),$$

that is – by our observation above – H_n is never integer.

PROOF. First, let n be even. Since $|x|_2 = |-x|_2$ for all $x \in \mathbf{Q}$, by the strong triangle inequality we get

$$\max\{|H_n|_2, |1|_2\} = \max\left\{|H_n|_2, |1|_2, \left|\frac{1}{3}\right|_2, \left|\frac{1}{5}\right|_2, \dots, \left|\frac{1}{n-1}\right|_2\right\} \geq$$

$$\geq \left|H_n - 1 - \frac{1}{3} - \frac{1}{5} - \dots - \frac{1}{n-1}\right|_2 =$$

$$= \left|\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{n-2} + \frac{1}{n}\right|_2 =$$

$$= \left|\frac{1}{2}\right|_2 \left|1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n/2}\right|_2 = 2\left|H_{n/2}\right|_2.$$

If n is odd, the situation is the same:

$$\max\{|H_n|_2, |1|_2\} \geq 2\left|H_{(n-1)/2}\right|_2.$$

The reader may verify it.

So we get that the 2-adic norm of the harmonic numbers is monotone increasing. Since $|H_2|_2 = \left|\frac{3}{2}\right|_2 = 2$, the 2-adic norm of all the harmonic numbers are greater than 1. As a corollary, this means that the harmonic numbers are not integers because of the property of the 2-adic norm mentioned above.

We can continue the calculations on $H_{n/2}$ (or on $H_{(n-1)/2}$) instead of H_n. For instance let us consider that $n/2$ is even. Then the method described above gives that

$$|H_{n/2}|_2 \geq \left|\frac{1}{2}\right|_2 |H_{n/4}|_2 = 2|H_{n/4}|_2.$$

This and the previous estimation implies that

$$|H_n|_2 \geq \left|\frac{1}{2}\right|_2 |H_{n/2}|_2 \geq \left|\frac{1}{2}\right|_2 \left|\frac{1}{2}\right|_2 |H_{n/4}|_2 = 4|H_{n/4}|_2.$$

And so on. If $n/2$ is odd then we choose $(n/2-1)/2$ instead of $n/4$. We can perform these steps exactly $\mathrm{Ord}_2(n)$ times.

After all, we shall have the following:

$$|H_n|_2 \geq \left|\frac{1}{2^{\mathrm{Ord}_2(n)}}\right|_2 |H_1|_2 = 2^{\mathrm{Ord}_2(n)}.$$

On the other hand,

$$|H_n|_2 \leq \max\left\{|1|_2, \left|\frac{1}{2}\right|_2, \cdots, \left|\frac{1}{n}\right|_2\right\} = \left|\frac{1}{2^{\mathrm{Ord}_2(n)}}\right|_2 = 2^{\mathrm{Ord}_2(n)},$$

because the greatest 2-power occuring between 1 and n is $\mathrm{Ord}_2(n)$.

The inequalities detailed above give the statement. ∎

A different approach can be found in [3] , [5] and in their references. The next proof comes from these sources.

PROOF. Let us fix the order of n, i.e.: $\mathrm{Ord}_2(n) := m$. This implies that the denominator of $\frac{2^{m-1}}{n}$ is odd, unless $n = 2^m$. We get that the number

$$2^{m-1}H_n - \frac{1}{2}$$

can be represented by the sum of rationals with odd denominators. Write

$$2^{m-1}H_n - \frac{1}{2} = \frac{a_1}{b_1} + \cdots + \frac{a_s}{b_s} = \frac{c}{\mathrm{lcm}(b_1, \ldots, b_s)},$$

where b_i is odd for all $i = 1, \ldots, s$. It means that $b := \mathrm{lcm}(b_1, \ldots, b_s)$ is odd. The last formula gives the result

$$H_n = \frac{\frac{c}{b} + \frac{1}{2}}{2^{m-1}} = \frac{2c + b}{2^m b}.$$ ∎

Let us turn our attention to hyperharmonic numbers. We need a lemma which can be found in [6] :

LEMMA 2.

$$|n!|_p = p^{(A_p(n)-n)/(p-1)},$$

where $A_p(n)$ is the sum of the digits of the p-adic expansion of n.

EXAMPLE 3. Let $p = 2$ and $n = 11$. Then $n = 1011_2$, that is, $A_2(n) = 3$.

$$|n!|_2 = |39916800|_2 = |256 \cdot 155925|_2 = |256|_2|155925|_2 = |2^8|_2 \cdot 1 = 2^{-8}.$$

We can apply the lemma: $A_2(n) - n = 3 - 11 = -8$, whence $|n!|_2 = 2^{-8}$.

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THEOREM 4. *If* $\mathrm{Ord}_2(n+r-1) > \mathrm{Ord}_2(r-1)$ *then*

$$|H_n^{(r)}|_2 = 2^{A_2(n+r-1)-A_2(n)-A_2(r-1)+\mathrm{Ord}_2(n+r-1)},$$

else

$$|H_n^{(r)}|_2 = 2^{A_2(n+r-1)-A_2(n)-A_2(r-1)+\max\left\{\left|\frac{1}{r}\right|_2,\left|\frac{1}{r+1}\right|_2,\dots,\left|\frac{1}{n+r-1}\right|_2\right\}}.$$

PROOF. Let a, b, c, d be odd numbers. Then

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$$\left|H_n^{(r)}\right|_2 = \left|\binom{n+r-1}{r-1}(H_{n+r-1}-H_{r-1})\right|_2 =$$

$$= \left|\binom{n+r-1}{r-1}\right|_2 \left|\frac{a}{2^{\mathrm{Ord}_2(n+r-1)}b} - \frac{c}{2^{\mathrm{Ord}_2(r-1)}d}\right|_2 =$$

$$= \left|\binom{n+r-1}{r-1}\right|_2 \left|\frac{2^{\mathrm{Ord}_2(r-1)}ad - 2^{\mathrm{Ord}_2(n+r-1)}bc}{2^{\mathrm{Ord}_2(n+r-1)+\mathrm{Ord}_2(r-1)}bd}\right|_2.$$

Because of the condition $\mathrm{Ord}_2(n+r-1) > \mathrm{Ord}_2(r-1)$ we get

$$\left|H_n^{(r)}\right|_2 = \left|\binom{n+r-1}{r-1}\right|_2 \left|\frac{ad - 2^{\mathrm{Ord}_2(n+r-1)-\mathrm{Ord}_2(r-1)}bc}{2^{\mathrm{Ord}_2(n+r-1)}bd}\right|_2.$$

Since the nominator is odd, we get the following:

$$\left|\frac{ad - 2^{\mathrm{Ord}_2(n+r-1)-\mathrm{Ord}_2(r-1)}bc}{2^{\mathrm{Ord}_2(n+r-1)}bd}\right|_2 = 2^{\mathrm{Ord}_2(n+r-1)}.$$

To compute the 2-adic norm of the binomial coefficient, we use the previous lemma.

$$\left|\binom{n+r-1}{r-1}\right|_2 = \left|\frac{(n+r-1)!}{(r-1)!n!}\right|_2 =$$

$$= \frac{2^{A_2(n+r-1)-n-r+1}}{2^{A_2(r-1)-r+1}2^{A_2(n)-n}} = 2^{A_2(n+r-1)-A_2(n)-A_2(r-1)}.$$

This, and the previous equality give the result with respect to the condition $\mathrm{Ord}_2(n+r-1) > \mathrm{Ord}_2(r-1)$.

Let us fix an arbitrary n for which $\mathrm{Ord}_2(n+r-1) = \mathrm{Ord}_2(r-1)$.

$$H_{n+r-1} - H_{r-1} = \frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n+r-1}.$$

Let us substract all of the fractions with odd denominators. Then we can take $\frac{1}{2}$ out of the remainder and continue the recursive method described in the first proof of Theorem 1. We can make such substraction steps

$$\max\left\{\left|\frac{1}{r}\right|_2, \left|\frac{1}{r+1}\right|_2, \dots, \left|\frac{1}{n+r-1}\right|_2\right\}$$

times. The result:

$$|H_{n+r-1} - H_{r-1}|_2 \geq \max\left\{\left|\frac{1}{r}\right|_2, \left|\frac{1}{r+1}\right|_2, \dots, \left|\frac{1}{n+r-1}\right|_2\right\}.$$

On the other hand, by the strong triangle inequality

$$|H_{n+r-1} - H_{r-1}|_2 \leq \max\left\{\left|\frac{1}{r}\right|_2, \left|\frac{1}{r+1}\right|_2, \dots, \left|\frac{1}{n+r-1}\right|_2\right\}. \qquad \blacksquare$$

COROLLARY 5. *The sum of the harmonic numbers cannot be integer:*

$$H_1 + H_2 + \cdots + H_n \notin \mathbf{N} \quad (n > 1).$$

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PROOF. $H_1 + H_2 + \cdots + H_n = H_n^{(2)}$. The condition with respect to the order of n and r holds because $\mathrm{Ord}_2(n+2-1) > \mathrm{Ord}_2(2-1) = 0$ for all $n \geq 1$. Furthermore,

$$\left|H_n^{(2)}\right|_2 = 2^{A_2(n+1) - A_2(n) - A_2(1) + \mathrm{Ord}_2(n+1)}.$$

Let $m = \mathrm{Ord}_2(n+1)$. Our goal is to minimize the power of 2. $\mathrm{Ord}_2(n+1) = m$ implies that $n+1 < 2^{m+1}$, therefore $1 \leq A_2(n+1) \leq m+1$ and $1 \leq A_2(n) \leq m$. The minimum in the power is taken when $A_2(n) = m$ and $A_2(n+1) = 1$. It is possible if and only if $n = 2^m - 1$. In this case

$$A_2(n+1) - A_2(n) - A_2(1) + \mathrm{Ord}_2(n+1) = 1 - m - 1 + m = 0.$$

We get that if $n \neq 2^m - 1$ for some m, then $|H_n^{(2)}|_2 > 1$, that is, $H_n^{(2)} \notin \mathbf{N}$. On the other hand, let us assume that n has the form $2^m - 1$. This implies that

$$H_n^{(2)} = \binom{n+2-1}{2-1}(H_{n+2-1} - H_{2-1}) =$$

$$= (n+1)(H_{n+1} - 1) = 2^m\left(\frac{a}{2^{\mathrm{Ord}_2(n+1)}b} - 1\right) = \frac{a}{b} - 2^m \notin \mathbf{N}. \qquad \blacksquare$$

One can easily prove the following, using the method in the previous proof.

COROLLARY 6. *$H_n^{(3)} \notin \mathbf{N}$ for all $n > 1$.*

As we can see, the method to prove the non-integer property of harmonic numbers does not work for hyperharmonic numbers, because there are n and r integers for which $|H_n^{(r)}|_2 = 1$. In spite of this fact, we believe that Theisinger's theorem holds for all hyperharmonic numbers, too.

CONJECTURE 7. *None of the hyperharmonic numbers can be integers ($r, n \geq 2$).*

EXAMPLE 8. We demonstrate that the theorem described above simplifies the calculation of the 2-norm of hyperharmonic numbers.

For instance,

$$H_{18}^{(8)} = \binom{18+8-1}{8-1}(H_{18+8-1} - H_{8-1}) =$$

$$= 480700 \left(\frac{34052522467}{8923714800} - \frac{363}{140}\right) = \frac{10914604807}{18564}.$$

Since $|18564|_2 = 2^{-2}$, we get that $|H_{18}^{(8)}|_2 = 2^2 = 4$.

On the other hand, $A_2(18+8-1) = A_2(16+8+1) = 3$, $A_2(8-1) = A_2(4+ +2+1) = 3$, $A_2(18) = A_2(16+2) = 2$ and $\mathrm{Ord}_2(18+8-1) = \mathrm{Ord}_2(16+9) = 4$. By theorem 5,

$$|H_{18}^{(8)}|_2 = 2^{3-3-2+4} = 2^2 = 4.$$

Finally, we pose an interesting question:

PROBLEM 9. For which $n_1 \neq n_2$ and $r_1 \neq r_2$ does the equality

$$H_{n_1}^{(r_1)} = H_{n_2}^{(r_2)}$$

stand?

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István Mező

Department of Algebra and Number Theory,
Institute of Mathematics,
University of Debrecen, Hungary
imezo@math.klte.hu
http://www.math.klte.hu/algebra/mezo.htm