ANNALES UNIV. SCI. BUDAPEST., **50** (2007), 1–10

A GENERAL DECOMPOSITION OF CONTINUITY

By

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(Received June 5, 2008)

Abstract. Quite recently, Sheik and Sundaram [22] have obtained the following theorem: a function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if f is ω-continuous and slc^*-continuous. In this paper, by using the notion of mg^*-closed sets, we obtain the unified theory for the above decomposition of continuity in topological spaces.

1. Introduction

In 1970, Levine [10] introduced the notion of generalized closed (g-closed) sets in topological spaces. As modifications of g-closed sets, Murugalingam [13] introduced the notions of sg^*-closed (resp. αg^*-closed, pg^*-closed βg^*-closed) sets by using semi-open (resp. α-open, preopen, β-open) sets and studied their basic properties and characterizations. In [16], these notions are unified by the notion of mg^*-closed sets. The notion of sg^*-closed sets is also called ω-closed [23], semi-star-closed [19], or $\hat{g}$-closed [8]. Recently, by using the notion of ω-closed sets, Sheik and Sundaram [22] obtained the following theorem: a function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if f is ω-continuous and slc^*-continuous.

The present authors [17], [18] introduced and investigated the notions of m-structures, m-spaces and m-continuity. In this paper, we introduce the notion of mlc-sets as a general form of locally closed sets. By using mg^*-closed sets and mlc-sets, we introduce the notions of mg^*-continuity and mlc-continuity, respectively, and obtain a general decomposition of continuity for the above theorem. Furthermore, we provide a sufficient condition for an mg^*-continuous function to be continuous. In the last section, we consider new forms of decomposition of continuity.

AMS Subject Classification (2000): 54A05, 54C08.

2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively.

DEFINITION 2.1. A subset A of a topological space (X,τ) is said to be *semi-open* [9] (resp. *preopen* [12], *α-open* [14], *b-open* [3], *β-open* [1]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup$ $\cup\, \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(Cl(A)))$).

For the subsets defined in Definition 2.1, the following relations are well-known:

DIAGRAM I

$$\begin{array}{ccccccc} \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & & \\ & & \Downarrow & & \Downarrow & & \\ & & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow & \beta\text{-open} \end{array}$$

The family of all semi-open (resp. preopen, α-open, b-open, β-open) sets in (X,τ) is denoted by SO(X) (resp. PO(X), $\alpha(X)$, BO(X), $\beta(X)$).

DEFINITION 2.2. Let (X,τ) be a topological space. A subset A of X is said to be *g-closed* [10] (resp. *sg^*-closed*, *pg^*-closed*, *αg^*-closed*, *bg^*-closed*, *βg^*-closed* [13]) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open (resp. semi-open, preopen, α-open, b-open, β-open) in (X,τ).

REMARK 2.1.

(1) An sg^*-closed set is also called ω-closed [23], semi-star-closed [19], or $\hat{g}$-closed [8].

(2) By the definitions, we obtain the following diagram:

DIAGRAM II

$$\begin{array}{ccccccc} g\text{-closed} & \Leftarrow & \alpha g^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & & \\ & & \Uparrow & & \Uparrow & & \\ & & sg^*\text{-closed} & \Leftarrow & bg^*\text{-closed} & \Leftarrow & \beta g^*\text{-closed} \end{array}$$

Throughout the present paper, (X,τ) and (Y,σ) always denote topological spaces and $f:(X,\tau) \to (Y,\sigma)$ presents a function.

DEFINITION 2.3. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *g-continuous* [4] (resp. *ω-continuous* [21] or *$\hat{g}$-continuous* [8]) if $f^{-1}(F)$ is g-closed (resp. ω-closed) in (X,τ) for each closed set F of (Y,σ).

REMARK 2.2. It is known in [22] that the following implications hold and the converses are not necessarily true: continuity $\Rightarrow$ ω-continuity $\Rightarrow$ g-continuity.

DEFINITION 2.4. A subset A of a topological space (X,τ) is called an $slc*$-set [22] or an slc-set [5] if $A = U\cap F$, where $U\in \mathrm{SO}(X)$ and F is closed in (X,τ).

REMARK 2.3. It is known in [22] that every closed set is an $slc*$-set but not conversely and that an ω-closed set and an $slc*$-set are independent.

DEFINITION 2.5. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *$slc*$-continuous* [22] if $f^{-1}(F)$ is an $slc*$-set of (X,τ) for each closed set F of (Y,σ).

REMARK 2.4. It is known in [22] that every continuous function is $slc*$-continuous but not conversely and that ω-continuity and $slc*$-continuity are independent.

THEOREM 2.1. (Sheik and Sundaram [22]). *A function $f:(X,\tau)\to(Y,\sigma)$ is continuous if and only if f is ω-continuous and slc^*-continuous.*

3. m-Structures and mg^*-closed sets

DEFINITION 3.1. A subfamily m_X of the power set $\mathcal{P}(\mathcal{X})$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [17], [18] on X if $\emptyset\in m_X$ and $X\in m_X$.

By (X,m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed.*

DEFINITION 3.2. A minimal structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [11] if the union of any family of subsets belonging to m_X belongs to m_X.

REMARK 3.1. Let (X,τ) be a topological space. Then the families $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ are all m-structures with property $\mathcal{B}$.

DEFINITION 3.3. Let (X, τ) be a topological space and m_X an m-structure on X. A subset A is said to be *mg^*-closed* [16] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$.

REMARK 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), $\beta(X)$) and A is mg^*-closed, then A is g-closed (resp. sg^*-closed, pg^*-closed, αg^*-closed, bg^*-closed, βg^*-closed).

LEMMA 3.1. (Noiri and Popa [16]). *Let (X, τ) be a topological space and m_X an m-structure on X such that $\tau \subset m_X$. Then the following implications hold:*

$$\textit{closed} \;\Rightarrow\; mg^*\textit{-closed} \;\Rightarrow\; g\textit{-closed}$$

LEMMA 3.2. (Noiri and Popa [16]). *If A is mg^*-closed and m_X-open, then A is closed.*

DEFINITION 3.4. Let (X, τ) be a topological space and m_X an m-structure on X. A subset A is called an *mlc-set* if $A = U \cap F$, where $U \in m_X$ and F is closed in (X, τ).

REMARK 3.3. Let (X, τ) be a topological space and A a subset of X.

(1) if $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), $\beta(X)$) and A is an mlc-set, then A is called a locally closed set [6] (briefly lc-set) (resp. an slc^*-set [22] or an slc-set [5], a plc-set [5], an αlc-set [2], a blc-set, a βlc-set [5]),

(2) every closed set is an mlc-set but not conversely by Remark 2.3,

(3) an mg^*-closed set and an mlc-set are independent by Remark 2.3,

(4) by the definitions, we obtain the following diagram:

DIAGRAM III

$$\begin{array}{ccccccc}
lc\text{-sets} & \Rightarrow & \alpha lc\text{-sets} & \Rightarrow & plc\text{-sets} & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & slc\text{-sets} & \Rightarrow & blc\text{-sets} & \Rightarrow & \beta lc\text{-sets}
\end{array}$$

4. Decompositions of continuity

THEOREM 4.1. *Let (X,τ) be a topological space and m_X a minimal structure on X such that $\tau \subset m_X$. Then a subset A of X is closed if and only if it is mg^*-closed and an mlc-set.*

PROOF. *Necessity.* Suppose that A is closed in (X,τ). Then, by Lemma 3.1, A is mg^*-closed. Furthermore, $A = X \cap A$, where $X \in m_X$ and A is closed and hence A is an mlc-set.

Sufficiency. Suppose that A is mg^*-closed and an mlc-set. Since A is an mlc-set, $A = U \cap F$, where $U \in m_X$ and F is closed in (X,τ). Therefore, we have $A \subset U$ and $A \subset F$. By the hypothesis, we obtain $\mathrm{Cl}(A) \subset U$ and $\mathrm{Cl}(A) \subset F$ and hence $\mathrm{Cl}(A) \subset U \cap F = A$. Thus, $\mathrm{Cl}(A) = A$ and A is closed.

COROLLARY 4.1. *Let (X,τ) be a topological space. Then, for a subset A of X, the following properties are equivalent:*

(1) A is closed;

(2) A is g-closed and a locally closed set;

(3) A is αg^*-closed and an αlc-set;

(4) A is pg^*-closed and a plc-set;

(5) A is sg^*-closed (resp. ω-closed) and slc-closed (resp. sl^*c-set);

(6) A is bg^*-closed and a blc-set;

(7) A is βg^*-closed and a βlc-set.

PROOF. This is an immediate consequence of Theorem 4.1.

DEFINITION 4.1. Let (X,τ) be a topological space and m_X a minimal structure on X. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be mg^*-*continuous* (resp. *mlc-continuous*) if $f^{-1}(F)$ is mg^*-closed (resp. an mlc-set) in (X,τ) for each closed set F of (Y,σ).

REMARK 4.1. Let (X,τ) be a topological space and m_X an m-structure on X such that $\tau \subset m_X$. Then,

(1) by Lemma 3.1, the following implications hold: continuity $\Rightarrow$ mg^*-continuity $\Rightarrow$ g-continuity,

(2) every continuous function is mlc-continuous but not conversely by Remark 2.4,

(3) mg^*-continuity and mlc-continuity are independent by Remark 2.4.

THEOREM 4.2. *Let (X,τ) be a topological space and m_X a minimal structure on X such that $\tau \subset m_X$. Then a function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if it is mg^*-continuous and mlc-continuous.*

PROOF. This is an immediate consequence of Theorem 4.1

DEFINITION 4.2. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *g-continuous* [4] (resp. *αg^*-continuous*, *pg^*-continuous*, *sg^*-continuous* or *ω-continuous* [21], *bg^*-continuous*, *βg^*-continuous*) if $f^{-1}(F)$ is g-closed (resp. αg^*-closed, pg^*-closed, sg^*-closed, bg^*-closed, βg^*-closed) in (X,τ) for each closed set F of (Y,σ).

DEFINITION 4.3. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *LC-continuous* [6] (resp. *αlc-continuous*, *plc-continuous*, *slc-continuous* or *slc^*-continuous* [22], *blc-continuous*, *βlc-continuous*) if $f^{-1}(F)$ is a locally closed set (resp. αlc-set, plc-set, slc-set, blc-set, βlc-set) of (X,τ) for each closed set F of (Y,σ).

COROLLARY 4.2. *For a function $f:(X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) f is continuous;

(2) f is g-continuous and LC-continuous;

(3) f is αg^*-continuous and αlc-continuous;

(4) f is pg^*-continuous and plc-continuous;

(5) f is ω-continuous and slc^*-continuous;

(6) f is bg^*-continuous and blc-continuous;

(7) f is βg^*-continuous and βlc-continuous.

Proof. This is an immediate consequence of Corollary 4.1.

DEFINITION 4.4. Let (X,τ) be a topological space and m_X a minimal structure on X. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *contra-m-continuous* [15] at $x \in X$ if for each closed set F of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset F$. f is said to be *contra-m-continuous* if it has this property at each point $x \in X$.

LEMMA 4.1. (Noiri and Popa [15]). *Let (X,m_X) be an m-space such that m_X has property $\mathcal{B}$. Then a function $f:(X,m_X) \to (Y,\sigma)$ is contra-m-continuous if and only if $f^{-1}(F)$ is m_X-open for every closed set F of (Y,σ).*

THEOREM 4.3. *Let* (X,τ) *be a topological space and* m_X *a minimal structure on* X *such that* $\tau \subset m_X$ *and* m_X *has property* $\mathscr{B}$. *If a function* $f:(X,\tau) \to (Y,\sigma)$ *is* mg^**-continuous and contra-*m*-continuous, then* f *is continuous.*

PROOF. Let F be any closed set of (Y,σ). Since f is contra-m-continuous and m_X has property $\mathscr{B}$, by Lemma 4.1 $f^{-1}(F)$ is m_X-open. Since f is mg^*-continuous, $f^{-1}(F)$ is mg^*-closed and hence, by Lemma 3.2, $f^{-1}(F)$ is closed. Therefore, f is continuous.

REMARK 4.1. Let (X,τ) be a topological space and $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), $\beta(X)$). Then by Theorem 4.3, we can obtain several sufficient conditions for a function to be continuous. For example, in case $m_X = \tau$ we have the following.

COROLLARY 4.3. *If a function* $f:(X,\tau) \to (Y,\sigma)$ *is* g*-continuous and contra-continuous, then* f *is continuous.*

5. New forms of decomposition of continuity

First, we recall the θ-closure and the δ-closure of a subset in a topological space. Let (X,τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the θ*-closure* (resp. δ*-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$) [24].

DEFINITION 5.1. A subset of a topological space (X,τ) is said to be

(1) δ*-preopen* [20] (resp. θ*-preopen* [16]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),

(2) δ*-*β*-open* [7] (resp. θ*-*β*-open* [16]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$).

By δ PO(X) (resp. $\delta\beta(X)$, θ PO(X), $\theta\beta(X)$), we denote the collection of all δ-preopen (resp. δ-β-open, θ-preopen, θ-β-open) sets of a topological space (X,τ). These four collections are m-structures with property $\mathscr{B}$. In [16], the following diagram is known:

DIAGRAM IV

$$\begin{array}{ccccccc}
\alpha\text{-open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{semi-open} & \Rightarrow & \beta\text{-open} & \Rightarrow & \delta\text{-}\beta\text{-open} & \Rightarrow & \theta\text{-}\beta\text{-open}
\end{array}$$

For subsets of a topological space (X,τ), we can define many new variations of g-closed sets. For example, in case $m_X = \delta\ \mathrm{PO}(X)$, $\delta\beta(X)$, $\theta\ \mathrm{PO}(X)$, $\theta\beta(X)$, we can define new types of g-closed sets as follows:

DEFINITION 5.2. A subset A of a topological space (X,τ) is said to be *δpg^*-closed* (resp. *θpg^*-closed*, *$\delta\beta g^*$-closed*, *$\theta\beta g^*$-closed*) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is δ-preopen (resp. θ-preopen, δ-β-open, θ-β-open) in (X,τ).

By DIAGRAM IV and Definitions 5.2, we have the following diagram:

DIAGRAM V

$$\begin{array}{ccccccccc}
g\text{-closed} \Leftarrow & \alpha pg^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & \Leftarrow & \delta pg^*\text{-closed} & \Leftarrow & \theta pg^*\text{-closed} & \\
& \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & \\
& sg^*\text{-closed} & \Leftarrow & \beta g^*\text{-closed} & \Leftarrow & \delta\beta g^*\text{-closed} & \Leftarrow & \theta\beta g^*\text{-closed} & \Leftarrow \text{closed}
\end{array}$$

DEFINITION 5.3. A subset A of a topological space (X,τ) is called a *δplc-set* (resp. *θplc-set*, *$\delta\beta lc$-set*, *$\theta\beta lc$-set*) if $A = U \cap F$, where U is δ-preopen (resp. θ-preopen, δ-β-open, θ-β-open) in (X,τ) and F is closed in (X,τ).

COROLLARY 5.1. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is closed;

(2) A is δpg^*-closed and a δplc-set;

(3) A is θpg^*-closed and a θplc-set;

(4) A is $\delta\beta g^*$-closed and a $\delta\beta lc$-set;

(5) A is $\theta\beta g^*$-closed and a $\theta\beta lc$-set.

PROOF. Let $m_X = \delta\ \mathrm{PO}(X)$, $\theta\ \mathrm{PO}(X)$, $\delta\beta(X)$ and $\theta\beta(X)$. Then this is an immediate consequence of Theorem 4.1.

By defining functions similarly to Definitions 4.2 and 4.3, we obtain the following decompositions of continuity:

COROLLARY 5.2. *For a function $f:(X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) f is continuous;

(2) f is δpg^*-continuous and δplc-continuous;

(3) f is θpg^*-continuous and θplc-continuous;

(4) f is $\delta\beta g^*$-continuous and $\delta\beta lc$-continuous;

(5) f is $\theta\beta g^*$-continuous and $\theta\beta lc$-continuous.

PROOF. This is an immediate consequence of Theorem 4.2.

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