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A GENERALIZATION OF NEARLY CONTINUOUS MULTIFUNCTIONS

By

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Abstract. Recently Ekici [9] has introduced the notion of upper/lower nearly continuous multifunctions as a generalization of continuous multifunctions and N-continuous functions [16]. In this paper, we obtain the unified form of several generalizations of upper/lower nearly continuous multifunctions.

1. Introduction

The notion of N-closed sets in a topological space is introduced in [6] and studied in [20], [21] and other papers. Ekici [9] introduced and studied upper/lower nearly continuous multifunctions as a generalization of upper/lower semi-continuous multifunctions and N-continuous functions. In [26], the present authors introduced the notion of upper/lower m-continuous multifunctions.

In this paper we introduce and study the notion of upper/lower nearly m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. The multifunction is a generalization of upper/lower m-continuous multifunctions and upper/lower nearly continuous multifunctions. We obtain several characterizations and properties of such multifunctions by generalizing the results established in [9] and other results. In the last section, we recall some types of modifications of open sets and point out the possibility for new forms of nearly continuous multifunctions.

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2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $\mathrm{Int}(\mathrm{Cl}(A)) = A$ (resp. $\mathrm{Cl}(\mathrm{Int}(A)) = A$).

DEFINITION 2.1. A subset A of a topological space (X,τ) is said to be *N-closed relative to X* (briefly *N-closed*) [6] if every cover of A by regular open sets of X has a finite subcover.

DEFINITION 2.2. Let (X,τ) be a topological space. A subset A of X is said to be *α-open* [19] (resp. *semi-open* [14], *preopen* [17], *β-open* [1] or *semi-preopen* [3], *b-open* [4]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $SPO(X)$, $\mathrm{BO}(X)$).

DEFINITION 2.3. The complement of a semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) set is said to be *semi-closed* [8] (resp. *preclosed* [11], *α-closed* [18], *β-closed* [1], *semi-preclosed* [3], *b-closed* [4]).

DEFINITION 2.4. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, semi-preclosed, b-closed) sets of X containing A is called the *semi-closure* [8] (resp. *preclosure* [11], *α-closure* [18], *β-closure* [2], *semi-preclosure* [3], *b-closure* [4]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\,\mathrm{Cl}(A)$, $_\beta\,\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$, $\mathrm{bCl}(A)$.

DEFINITION 2.5. The union of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *semi-preinterior*, *b-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\,\mathrm{Int}(A)$, $_\beta\,\mathrm{Int}(A)$, $\mathrm{spInt}(A)$, $\mathrm{bInt}(A)$).

DEFINITION 2.6. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *N-continuous* at $x \in X$ [16] if for each open set V of Y containing $f(x)$ and having N-closed complement, there exists an open set U containing x such that $f(U) \subset V$. The function is said to be *N-continuous* if it has this property at each point of X.

Throughout the present paper, (X,τ) and (Y,σ) (briefly X and Y) always denote topological spaces and $F\colon X \to Y$ (resp. $f\colon X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F\colon X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$.

DEFINITION 2.7. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly continuous* (briefly *u.n.c.*) at a point $x \in X$ [9] if for each open set V containing $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(U) \subset V$,

(2) *lower nearly continuous* (briefly *l.n.c.*) at a point $x \in X$ [9] if for each open set V meeting $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly continuous* on X if it has this property at each point of X.

3. Nearly m-continuous multifunctions

DEFINITION 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [24], [25] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (briefly *m-closed*).

REMARK 3.1. Let (X,τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, BO(X) and PO(X) are all m-structures on X.

DEFINITION 3.2. Let (X, m_X) be an m-space. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [15] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F\colon A \subset F, X - F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

REMARK 3.2. Let (X,τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), PO(X)), then we have

(a) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\,\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$, $\mathrm{spCl}(A)$),

(b) mInt(A) = Int(A) (resp. sInt(A), pInt(A), α Int(A), bInt(A), spInt(A)).

LEMMA 3.1. (Maki et al. [15]). *Let* (X, m_X) *be an* m*-space. For subsets* A *and* B *of* X*, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$*,*

(2) If $(X - A) \in m_X$*, then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$*, then* $\mathrm{mInt}(A) = A$*,*

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$*,*

(4) If $A \subset B$*, then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$*,*

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$*,*

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

LEMMA 3.2. (Popa and Noiri [25]). *Let* (X, m_X) *be an* m*-space and* A *a subset of* X*. Then* $x \in \mathrm{mCl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *containing* x*.*

DEFINITION 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathscr{B}$ [15] if the union of any family of subsets belonging to m_X belongs to m_X.

REMARK 3.3. Let (X, τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, BO(X) and PO(X) have property $\mathscr{B}$.

LEMMA 3.3. (Popa and Noiri [27]). *For an* m*-structure* m_X *on a nonempty set* X*, the following properties are equivalent:*

(1) m_X *has property* $\mathscr{B}$*;*

(2) If $m_X - \mathrm{Int}(A) = A$*, then* $A \in m_X$*;*

(3) If $m_X - \mathrm{Cl}(A) = A$*, then* A *is* m_X*-closed.*

DEFINITION 3.4. *Let* (X, m_X) *be an* m*-space and* (Y, σ) *a topological space.* A multifunction $F\colon (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper* m*-continuous* (briefly *u.m.c.*) [26] at a point $x \in X$ if for each open set V containing $F(x)$, there exists an m_X-open set U containing x such that $F(U) \subset V$,

(2) *lower* m*-continuous* (briefly *l.m.c.*) [26] at a point $x \in X$ if for each open set V meeting $F(x)$, there exists an m_X-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower* m*-continuous* on X if it has this property at every point of X.

DEFINITION 3.5. *Let* (X, m_X) *be an* m*-space and* (Y,σ) *a topological space.* A multifunction $F\colon (X, m_X) \to (Y,\sigma)$ is said to be

(1) *upper nearly* m*-continuous* (briefly *u.n.m.c.*) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $F(U) \subset V$,

(2) *lower nearly* m*-continuous* (briefly *l.n.m.c.*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly* m*-continuous* on X if it has this property at every point of X.

REMARK 3.4. Every upper/lower m-continuous multifunction is upper/lower nearly m-continuous. The converse is not true by Example 4 of [9].

THEOREM 3.1. *For a multifunction* $F\colon (X, m_X) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) F *is u.n.m.c.;*

(2) $F^+(V) = \mathrm{mInt}(F^+(V))$ *for each open set* V *of* Y *having* N*-closed complement;*

(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ *for every* N*-closed and closed set* K *of* Y*;*

(4) $\mathrm{mCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having the* N*-closed closure;*

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ *for every subset* B *of* Y *such that* $Y - \mathrm{Int}(B)$ *is* N*-closed.*

PROOF. (1) ⇒ (2): Let V be any open set of Y having N-closed complement and $x \in F^+(V)$. Then $F(x) \subset V$ and there exists $U \in m_X$ containing x such that $F(U) \subset V$. Therefore, $x \in U \subset F^+(V)$ and hence $x \in \mathrm{mInt}(F^+(V)$. This shows that $F^+(V) \subset \mathrm{mInt}(F^+(V))$. Therefore, by Lemma 3.1 we obtain $F^+(V) = \mathrm{mInt}(F^+(V))$.

(2) ⇒ (3): Let K be any N-closed and closed set of Y. Then, by Lemma 3.1 we have $X - F^-(K) = F^+(Y - K) = \mathrm{mInt}(F^+(Y - K)) = \mathrm{mInt}(X - F^-(K)) = X - \mathrm{mCl}(F^-(K))$. Therefore, we obtain $F^-(K) = \mathrm{mCl}(F^-(K))$.

(3) ⇒ (4): Let B be any subset of Y having the N-closed closure. By Lemma 3.1, we have $F^-(B) \subset F^-(\mathrm{Cl}(B)) = \mathrm{mCl}(F^-(\mathrm{Cl}(B)))$. Hence $\mathrm{mCl}(F^-(B)) \subset \mathrm{mCl}(F^-(\mathrm{Cl}(B))) = F^-(\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be a subset of Y such that $Y - \mathrm{Int}(B)$ is N-closed. Then by Lemma 3.1 we have

$$X - \mathrm{mInt}(F^{+}(B)) = \mathrm{mCl}(X - F^{+}(B)) = \mathrm{mCl}(F^{-}(Y - B)) \subset$$
$$\subset F^{-}(\mathrm{Cl}(Y - B)) \subset F^{-}(Y - \mathrm{Int}(B)) = X - F^{+}(\mathrm{Int}(B)).$$

Therefore, we obtain $F^{+}(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^{+}(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y containing $F(x)$ and having N-closed complement. Then $x \in F^{+}(V) = F^{+}(\mathrm{Int}(V)) \subset \mathrm{mInt}(F^{+}$ $+(V))$. There exists $U \in m_X$ containing x such that $U \subset F^{+}(V)$; hence $F(U) \subset V$. This shows that F is *u.n.m.c.*

THEOREM 3.2. *For a multifunction $F\colon (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.n.m.c.;

(2) $F^{-}(V) = \mathrm{mInt}(F^{-} V))$ for each open set V of Y having N-closed complement;

(3) $F^{+}(K) = \mathrm{mCl}(F^{+}(K))$ is for every N-closed and closed set K of Y;

(4) $\mathrm{mCl}(F^{+}(B)) \subset F^{+}(\mathrm{Cl}(B))$ for every subset B of Y having the N-closed closure;

(5) $F^{-}(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^{-}(B))$ for every subset B of Y such that $Y -$ $- \mathrm{Int}(B)$ is N-closed.

PROOF. The proof is similar to that of Theorem 3.1.

COROLLARY 3.1. *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. For a multifunction $F\colon (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.n.m.c. (resp. l.n.m.c.);

(2) $F^{+}(V)$ (resp. $F^{-}(V)$) is m_X-open for each open set V of Y having N-closed complement;

(3) $F^{-}(K)$ (resp. $F^{+}(K)$) is m_X-closed for every N-closed and closed set K of Y.

PROOF. This is an immediate consequence of Theorems 3.1 and 3.2 and Lemma 3.3.

REMARK 3.5. Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$ and $F\colon (X, m_X) \to (Y, \sigma)$ is upper/lower nearly m-continuous, then by Theorems 3.1 and 3.2 we obtain the results established in Theorem 2 of [9].

COROLLARY 3.2. *A multifunction* $F:(X,m_X)\to(Y,\sigma)$ *is u.n.m.c. (resp. l.n.m.c.) if* $F^-(K)=\mathrm{mCl}(F^-(K))$ *(resp.* $F^+(K)=\mathrm{mCl}(F^+(K))$*) for every* N*-closed set* K *of* Y.

PROOF. Let G be any open set of Y having N-closed complement. Then $Y-G$ is N-closed and closed. By the hypothesis, $X-F^+(G)=F^-(Y-G)=\mathrm{mCl}(F^-(Y-G))=\mathrm{mCl}(X-F^+(G))=X-\mathrm{mInt}(F^+(G))$ and hence, $F^+(G)=\mathrm{mInt}(F^+(G))$. It follows from Theorem 3.1 that F is *u.n.m.c.* The proof of lower near m-continuity is entirely similar.

DEFINITION 3.6. A function $f:(X,m_X)\to(Y,\sigma)$ is said to be *nearly* m*-continuous* if for each point $x\in X$ and each open set V containing $f(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $f(U)\subset V$.

COROLLARY 3.3. *For a function* $f:(X,m_X)\to(Y,\sigma)$*, the following properties are equivalent:*

(1) f *is nearly* m*-continuous;*

(2) $f^{-1}(V)=\mathrm{mInt}(f^{-1}(V))$ *for each open set* V *of* Y *having* N*-closed complement;*

(3) $f^{-1}(K)=\mathrm{mCl}(f^{-1}(K))$ *for every* N*-closed and closed set* K *of* Y*;*

(4) $\mathrm{mCl}(f^{-1}(B))\subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having the* N*-closed closure;*

(5) $f^{-1}(\mathrm{Int}(B))\subset\mathrm{mInt}(f^{-1}(B))$ *for every subset* B *of* Y *such that* $Y-\mathrm{Int}(B)$ *is* N*-closed.*

COROLLARY 3.4. *For a function* $f:(X,m_X)\to(Y,\sigma)$*, where* m_X *has property* $\mathcal{B}$*, the following properties are equivalent:*

(1) f *is nearly* m*-continuous;*

(2) $f^{-1}(V)$ *is* m_X*-open for each open set* V *of* Y *having N-closed complement;*

(3) $f^{-1}(K)$ *is* m_X*-closed for every N-closed and closed set* K *of* Y*.*

REMARK 3.6. Let (X,τ) and (Y,σ) be topological spaces. If $m_X=\tau$ and $f:(X,m_X)\to(Y,\sigma)$ is nearly m-continuous, then by Corollary 3.4 we obtain the results established in Theorem 1 of [16].

DEFINITION 3.7. A subset A of a topological space (X,τ) is said to be

(1) α*-paracompact* [30] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,

(2) α*-regular* [13] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

For a multifunction $F\colon X \to (Y,\sigma)$, a multifunction $\mathrm{Cl}\,F\colon X \to (Y,\sigma)$ is defined in [5] as follows: $(\mathrm{Cl}\,F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\alpha\,\mathrm{Cl}\,F$, $\mathrm{sCl}\,F$, $\mathrm{pCl}\,F$, $\mathrm{spCl}\,F$, and $\mathrm{bCl}\,F$.

LEMMA 3.4. (Popa and Noiri [26]) *If $F\colon (X,m_X) \to (Y,\sigma)$ is a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$, then for each open set V of Y $F^+(V) = G^+(V)$, where G denotes* $\mathrm{Cl}\,F$, $\alpha\,\mathrm{Cl}\,F$, $\mathrm{sCl}\,F$, $\mathrm{pCl}\,F$, $\mathrm{bCl}\,F$ *or* $\mathrm{spCl}\,F$.

PROOF. The proof is similar to that of Lemma 3.3 of [24].

THEOREM 3.3. *Let $F\colon (X,m_X) \to (Y,\sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is u.n.m.c. if and only if $G\colon (X,m_X) \to (Y,\sigma)$ is u.n.m.c., where G denotes* $\mathrm{Cl}\,F$, $\alpha\,\mathrm{Cl}\,F$, $\mathrm{sCl}\,F$, $\mathrm{pCl}\,F$, $\mathrm{bCl}\,F$ *or* $\mathrm{spCl}\,F$.

PROOF. *Necessity*. Suppose that F is *u.n.m.c.* Let V be any open set of Y having N-closed complement. By Lemma 3.4 and Theorem 3.1, we have $G^+(V) = F^+(V) = \mathrm{mInt}(F^+(V)) = \mathrm{mInt}(G^+(V))$. This shows that G is *u.n.m.c.*

Sufficiency. Suppose that G is *u.n.m.c.* Let V be any open set of Y having N-closed complement. By Lemma 3.4 and Theorem 3.1, $F^+(V) = G^+(V) = \mathrm{mInt}(G^+(V)) = \mathrm{mInt}(F^+(V))$. By Theorem 3.1, F is *u.n.m.c.*

LEMMA 3.5. (Popa and Noiri [26]) *If $F\colon (X,m_X) \to (Y,\sigma)$ is a multifunction, then for each open set V of Y $F^-(V) = G^-(V)$, where G denotes* $\mathrm{Cl}\,F$, $\alpha\,\mathrm{Cl}\,F$, $\mathrm{sCl}\,F$, $\mathrm{pCl}\,F$, $\mathrm{bCl}\,F$ *or* $\mathrm{spCl}\,F$.

THEOREM 3.4. *A multifunction $F\colon (X,m_X) \to (Y,\sigma)$ is l.n.m.c. if and only if $G\colon (X,m_X) \to (Y,\sigma)$ is l.n.m.c., where G denotes* $\mathrm{Cl}\,F$, $\alpha\,\mathrm{Cl}\,F$, $\mathrm{sCl}\,F$, $\mathrm{pCl}\,F$, $\mathrm{bCl}\,F$ *or* $\mathrm{spCl}\,F$.

PROOF. By using Lemma 3.5, this is shown similarly as in Theorem 3.3.

4. Some properties

LEMMA 4.1. (Popa and Noiri [26]) *A multifunction* $F: (X, m_X) \to (Y, \sigma)$ *is u.m.c. (resp. l.m.c.) if and only if* $F^+(V) = \mathrm{mInt}(F^+(V))$ *(resp.* $F^-(V) = = \mathrm{mInt}(F^-(V))$*) for every open set* V *of* Y.

THEOREM 4.1. *Let* $F: (X, m_X) \to (Y, \sigma)$ *be a multifunction such that* (Y, σ) *has a base of open sets having* N*-closed complements and* m_X *has property* $\mathcal{B}$*. If F is l.n.m.c., then F is l.m.c.*

PROOF. Let V be any open set of Y. By the hypothesis, $V = \cup_{i \in I} V_i$, where V_i is an open set having N-closed complement for each $i \in I$. Since m_X has property $\mathcal{B}$, by Corollary 3.1 $F^-(V_i) \in m_X$ for each $i \in I$. Moreover, $F^-(V) = F^-(\cup\{V_i : i \in I\}) = \cup\{F^-(V_i) : i \in I\}$. Therefore, we have $F^-(V) \in m_X$. Then by Lemma 4.1 and Lemma 3.3 F is *l.m.c.*

REMARK 4.1. Let (X, τ) and (Y, σ) be topological spaces and

$$F: (X, m_X) \to (Y, \sigma)$$

be *l.n.m.c.* If $m_X = \tau$, then by Theorem 4.1 we obtain the result established in Theorem 5 of [9].

THEOREM 4.2. *Let* $F: (X, m_X) \to (Y, \sigma)$ *and* $G: (Y, \sigma) \to (Z, \theta)$ *be multifunctions. If F is u.m.c. (resp. l.m.c.) and G is u.n.c. (resp. l.n.c.), then* $G \circ F: (X, m_X) \to (Z, \theta)$ *is u.n.m.c. (resp. l.n.m.c.).*

PROOF. Let V be any open set of V having N-clsoed complement. Since G is *u.n.c.* (resp. *l.n.c.*), by Theorem 2 of [9] $F^+(V)$ (resp. $F^-(V)$) is an open set of Y. Since F is *u.m.c.* (resp. *l.m.c.*), by Lemma 4.1 $(G \circ F)^+(V) = F^+ +(G^+(V)) = \mathrm{mInt}(F^+(G^+(V))) = \mathrm{mInt}((G \circ F)^+(V))$ (resp. $(G \circ F)^-(V) = = F^-(G^-(V)) = \mathrm{mInt}(F^-(G^-(V))) = \mathrm{mInt}((G \circ F)^-(V))$). By Theorem 3.1 (resp. Theorem 3.2) F is *u.n.m.c.* (resp. *l.n.m.c.*).

REMARK 4.2. If $F: (X, \tau) \to (Y, \sigma)$ is a multifunction and $m_X = \tau$, then by Theorem 4.2 we obtain the result established in Theorem 6 of [9].

DEFINITION 4.1. A topological space (Y, σ) is said to be N*-normal* [9] if for each disjoint closed sets K and H of Y, there exist open sets U and V having N-closed complement such that $K \subset U$, $H \subset V$ and $U \cap V = \emptyset$.

DEFINITION 4.2. An m-space (X, m_X) is said to be m-T_2 [24] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

THEOREM 4.3. *If* $F\colon (X, m_X) \to (Y, \sigma)$ *is an u.n.m.c. multifunction satisfying the following conditions:*

(1) $F(x)$ *is closed in* Y *for each* $x \in X$,

(2) $F(x) \cap F(y) = \emptyset$ *for each distinct points* $x, y \in X$,

(3) m_X *has property* $\mathcal{B}$, *and*

(4) (Y, σ) *is an* N*-normal space,*

then (X, m_X) *is* m-T_2.

PROOF. Let x and y be distinct points of X. Then, we have $F(x) \cap F(y) = \emptyset$. Since $F(x)$ and $F(y)$ are closed and Y is N-normal, there exist disjoint open sets U and V having N-closed complement such that $F(x) \subset U$ and $F(y) \subset V$. By Corollary 3.1, we obtain $x \in F^+(U) \in m_X$, $y \in F^+(V) \in m_X$ and $F^+(U) \cap F^+(V) = \emptyset$. This shows that X is m-T_2.

REMARK 4.3. If $F\colon (X, \tau) \to (Y, \sigma)$ is a multifunction and $m_X = \tau$, then by Theorem 4.3 we obtain the result established in Theorem 19 of [9].

THEOREM 4.4. *Let* (X, m_X) *be an* m*-space. If for each pair of distinct points* x_1 *and* x_2 *in* X, *there exists a multifunction* F *from* (X, m_X) *into an* N*-normal space* (Y, σ) *satisfying the following conditions:*

(1) $F(x_1)$ *and* $F(x_2)$ *are closed in* Y,

(2) F *is u.n.m.c. at* x_1 *and* x_2, *and*

(3) $F(x_1) \cap F(x_2) = \emptyset$,

then (X, m_X) *is* m-T_2.

PROOF. Let x_1 and x_2 be distinct points of X. Then, we have $F(x_1) \cap F(x_2) = \emptyset$. Since $F(x_1)$ and $F(_2)$ are closed and Y is N-normal, there exist disjoint open sets V_1 and V_2 having N-closed complement such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Since F is *u.n.m.c.* at x_1 and x_2, there exist $U_1 \in m_X$ and $U_2 \in m_X$ containing x_1 and x_2, respectively, such that $F(U_1) \subset V_1$ and $F(U_2) \subset V_2$. This implies that $U_1 \cap U_2 = \emptyset$. Hence (X, m_X) is an m-T_2-space.

DEFINITION 4.3. A subset A of an m-space (X, m_X) is said to be *m-dense* on X if $\mathrm{mCl}(A) = X$.

THEOREM 4.5. *Let* X *be a nonempty set with two minimal structures* m_1 *and* m_2 *such that* $U \cap V \in m_2$ *whenever* $U \in m_1$ *and* $V \in m_2$ *and* (Y, σ) *be an* N*-normal space. If the following conditions are satisfied:*

(1) $F\colon (X, m_1) \to (Y, \sigma)$ *is u.n.m.c.,*

(2) $G\colon (X, m_2) \to (Y, \sigma)$ *is u.n.m.c.,*

(3) $F(x)$ and $G(x)$ are closed in Y for each $x \in X$, and

(4) $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,

then $A = \mathrm{m}_2\,\mathrm{Cl}(A)$. If $F(x) \cap G(x) \neq \emptyset$ for each point in an m-dense set D of (X, m_2), then $F(x) \cap G(x) \neq \emptyset$ for each point $x \in X$.

PROOF. Suppose that $x \in X - A$. Then $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are closed sets and Y is N-normal, there exist disjoint open sets V and W in Y having N-closed complement such that $F(x) \subset V$ and $G(x) \subset W$. Since F is *u.n.m.c.* at x, there exists $U_1 \in m_1$ containing x such that $F(U_1) \subset V$. Since G is *u.n.m.c.* at x, there exists $U_2 \in m_2$ containing x such that $G(U_2) \subset W$. Now set $U = U_1 \cap U_2$, then $U \in m_2$ and $U \cap A = \emptyset$. Therefore, by Lemma 3.2 we have $x \in X - \mathrm{m}_2\,\mathrm{Cl}(A)$ and hence $A = \mathrm{m}_2\,\mathrm{Cl}(A)$. On the other hand, if $F(x) \cap G(x) \neq \emptyset$ on an m-dense set D of an m-space (X, m_2), then we have $X = \mathrm{m}_2\,\mathrm{Cl}(D) \subset \mathrm{m}_2\,\mathrm{Cl}(A) = A$. Therefore, $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

COROLLARY 4.1. (Ekici [9]) *Let F and G be upper nearly continuous and point closed multifunctions from a topological space (X, τ) into a N-normal space (Y, σ). Then the set $\{x : F(x) \cap G(x) \neq \emptyset\}$ is closed in X.*

DEFINITION 4.4. A topological space (X, τ) is said to be *N-connected* [10] if X cannot be written as the union of two disjoint nonempty open sets having N-closed complements.

DEFINITION 4.5. An m-space (X, m_X) is said to be *m-connected* [22] if X cannot be written as the union of two disjoint nonempty m_X-open sets.

THEOREM 4.6. *Let (X, m_X) be an m-space, where m_X has property $\mathcal{B}$. If $F: (X, m_X) \to (Y, \sigma)$ is an u.n.m.c. or l.n.m.c. surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m_X) is m-connected, then (Y, σ) is N-connected.*

PROOF. Suppose that (Y, σ) is not N-connected. There exist nonempty open sets U and V of Y having N-closed complement such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+$ $+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,

(2) $F^+(U) \cap F^+(V) = \emptyset$,

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are m_X-open sets. (i) In case F is *u.n.m.c.* by Corollary 3.1 $F^+(U)$ and $F^+(V)$ are m_X-open sets. (ii) In case F is *l.n.m.c.* by Corollary 3.1 $F^+(V)$ is m_X-closed because U is clopen in (Y,σ), therefore, $F^+(V)$ is m_X-open. Similarly $F^+(U)$ is m_X-open. Therefore, (X, m_X) is not m-connected.

DEFINITION 4.6. Let (X, m_X) be an m-space and A a subset of X. The *m-frontier* of A [25], denoted by mFr(A), is defined as follows:

$$\mathrm{mFr}(A) = mCl(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A).$$

THEOREM 4.7. *The set of all points $x \in X$ at which a multifunction $F:(X, m_X) \to (Y,\sigma)$ is not u.n.m.c. (resp. l.n.m.c.) is identical with the union of the m-frontiers of the u.n.m.c. (resp. l.n.m.c.) inverse images of open sets containing (resp. meeting) $F(x)$ and having N-closed complement.*

PROOF. Let x be a point of X at which F is not *u.n.m.c.* Then, there exists an open set V of Y containing $F(x)$ and having N-closed complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every m_X-open set U containing x. Hence, by Lemma 3.2 we have $x \in \mathrm{mCl}(X - F^+(V))$. On the other hand, we have $x \in F^+(V) \subset \mathrm{mCl}(F^+(V))$ and hence $x \in \mathrm{mFr}(F^+(V))$.

Conversely, suppose that F is *u.n.m.c.* Then for each open set V having N-closed complement and containing $F(x)$, we have $x \in \mathrm{mInt}(F^+(V))$. This is a contradiction. In case F is *l.n.m.c.*, the proof is similar.

5. New forms of nearly m-continuous multifunctions

For modifications of open sets defined in Definition 2.1, the following relationships are known:

$$\begin{array}{ccccccc} \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & & \\ & & \Downarrow & & \Downarrow & & \\ & & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow & \text{semi-preopen} \end{array}$$

First, we can define the following modifications of upper/lower nearly continuous multifunctions.

DEFINITION 5.1. A multifunction $F:(X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly α-continuous* (resp. *upper nearly precontinuous*, *upper nearly semi-continuous*, *upper nearly b-continuous*, *upper nearly sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and

having N-closed complement, there exists an α-open (resp. preopen, semi-open, b-open, semi-preopen) set U containing x such that $F(U) \subset V$,

(2) *lower nearly α-continuous* (resp. *lower nearly precontinuous*, *lower nearly semi-continuous*, *lower nearly b-continuous*, *lower nearly sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an α-open (resp. preopen, semi-open, b-open, semi-preopen) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly α-continuous* (resp. *upper/lower nearly precontinuous*, *upper/lower nearly semi-continuous*, *upper/lower nearly b-continuous*, *upper/lower nearly sp-continuous*) on X if it has this property at each $x \in X$.

For multifunctions defined in Definition 5.1, the following relationships hold:

$$\begin{array}{ccccc} \text{upper } n\text{-con.} \Rightarrow & \text{upper } n\text{-}\alpha\text{-con.} & \Rightarrow \text{upper } n\text{-precon.} & & \\ & \Downarrow & \Downarrow & & \\ & \text{upper } n\text{-semi-con.} \Rightarrow & \text{upper } n\text{-}b\text{-con.} & \Rightarrow \text{upper } n\text{-}sp\text{-con.} & \end{array}$$

REMARK 5.1. In the diagram above, "n" and "con." means near and continuity, respectively. And also the analogous diagram holds for the case "lower".

Let define the further modifications of upper/lower nearly continuous multifunctions. For the purpose, we recall the definitions of the θ-closure and the δ-closure due to Veličko [29]. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$) [29]. A subset A is said to be *θ-closed* (resp. *δ-closed*) if $\mathrm{Cl}_\theta(A) = A$ (resp. $\mathrm{Cl}_\delta(A) = A$). The complement of a θ-closed (resp. δ-closed) set is said to be *θ-open* (resp. *δ-open*). The union of all θ-open (resp. δ-open) sets contained in the subset A is called the θ-interior (resp. δ-interior) of A and is denoted by $\mathrm{Int}_\theta(A)$ (resp. $\mathrm{Int}_\delta(A)$).

DEFINITION 5.2. A subset A of a topological space (X, τ) is said to be

(1) *δ-semiopen* [23] (resp. *θ-semiopen* [7]) if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\theta(A))$),

(2) *δ-preopen* [28] (resp. *θ-preopen* [22]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),

(3) δ-*sp-open* [12] (resp. θ-*sp-open* [22]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A))))$.

By δ SO(X) (resp. δ PO(X), δ PO(X), θ SO(X), θ PO(X), θ PO(X)), we denote the collection of all δ-semiopen (resp. δ-preopen, δ-sp-open, θ-semiopen, θ-preopen, θ-sp-open) sets of a topological space (X,τ). These six collections are all m-structures with property $\mathscr{B}$. It is known that the families of all θ-open sets and δ-open sets of (X,τ) are topologies for X, respectively. In [22] and [7], the following relationships are known:

$$\begin{array}{ccccccccccc}
\theta\text{-open} & \Rightarrow & \delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\theta\text{-semiopen} & \Rightarrow & \delta\text{-semiopen} & \Rightarrow & \text{semi-open} & \Rightarrow & sp\text{-open} & \Rightarrow & \delta\text{-}sp\text{-open} & \Rightarrow & \theta\text{-}sp\text{-open}
\end{array}$$

DEFINITION 5.3. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly θ-continuous* (resp. *upper nearly θ-precontinuous*, *upper nearly θ-semi-continuous*, *upper nearly θ-sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-sp-open) set U containing x such that $F(U) \subset V$,

(2) *lower nearly θ-continuous* (resp. *lower nearly θ-precontinuous*, *lower nearly θ-semi-continuous*, *lower nearly θ-sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-sp-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly θ-continuous* (resp. *upper/lower nearly θ-precontinuous*, *upper/lower nearly θ-semi-continuous*, *upper/lower nearly θ-sp-continuous*) on X if it has this property at each $x \in X$.

DEFINITION 5.4. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly δ-continuous* (resp. *upper nearly δ-precontinuous*, *upper nearly δ-semi-continuous*, *upper nearly δ-sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-sp-open) set U containing x such that $F(U) \subset V$,

(2) *lower nearly δ-continuous* (resp. *lower nearly δ-precontinuous*, *lower nearly δ-semi-continuous*, *lower nearly δ-sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-sp-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly δ-continuous* (resp. *upper/lower nearly δ-pre-continuous*, *upper/lower nearly δ-semi-continuous*, *upper/lower nearly δ-sp-continuous*) on X if it has this property at each $x \in X$.

For the multifunctions defined above, the following diagram hold, where u., n. and c. mean upper, near and continuity, respectively. And also the analogous diagram holds for the case "lower".

$$\begin{array}{ccccccccccc}
\text{u.n.}\theta\text{-c.} & \Rightarrow & \text{u.n.}\delta\text{-c.} & \Rightarrow & \text{u.n.c.} & \Rightarrow & \text{u.n.p.c.} & \Rightarrow & \text{u.n.}\delta\text{-p.c.} & \Rightarrow & \text{u.n.}\theta\text{-p.c.} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{u.n.}\theta\text{-s.c.} & \Rightarrow & \text{u.n.}\delta\text{-s.c.} & \Rightarrow & \text{u.n.s.c.} & \Rightarrow & \text{u.n.sp.c.} & \Rightarrow & \text{u.n.}\delta\text{-sp.c.} & \Rightarrow & \text{u.n.}\theta\text{-sp.c.}
\end{array}$$

CONCLUSION. We can apply the results established in Sections 3 and 4 to all multifunctions defined in Definitions 5.1, 5.2 and 5.3.

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