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ON SUBSETS DEFINED IN TERMS OF WEAK ENVELOPES AND ENVELOPES

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Abstract. We define and discuss the various characterizations and properties of some kind of sets in monotonic spaces, weak envelope spaces and envelope spaces which are similar to that of dense sets, nowheredense sets and δ-sets in topological spaces.

1. Introduction and preliminaries

Let X be a nonempty set and $\gamma\colon \wp(X) \to \wp(X)$. We say that $\gamma \in \Gamma(X)$ or simply $\gamma \in \Gamma$ if $\gamma(A) \subset \gamma(B)$ whenever $A \subset B$ where A and B are subsets of X. If $\gamma \in \Gamma$, we call the pair (X, γ), a *monotonic space*. A subset A of X is γ-open [1] if $A \subset \gamma(A)$. The complement of a γ-open set is said to be a γ-closed set. If $\gamma \in \Gamma$, then $\gamma^\star\colon \wp(X) \to \wp(X)$ is defined by $\gamma^\star(A) = X - \gamma(X - A)$ and $\gamma^\star \in \Gamma$ [1]. A subset A of X is $\gamma^\star$-closed if and only if $\gamma(A) \subset A$ [1, Proposition 1.8]. Let $\xi = \{A \subset X \mid A = \gamma(A)\}$. ξ is called the family of all γ-regularclosed (γ-regular [1])sets. Therefore, a subset A of X is γ-regularclosed if and only if A is γ-open and $\gamma^\star$-closed. The complement of a γ-regularclosed set is called a γ-regularopen set. Let μ be the family of all γ-regularopen sets. Then, $A \in \mu$ if and only if $A = \gamma^\star(A)$ if and only if A is γ-closed and $\gamma^\star$-open. We have the following subclasses of Γ.

$\Gamma_0 = \{\gamma \in \Gamma \mid \gamma(\emptyset) = \emptyset\}$,

$\Gamma_1 = \{\gamma \in \Gamma \mid \gamma(X) = X\}$,

$\Gamma_2 = \{\gamma \in \Gamma \mid \gamma(\gamma(A)) = \gamma(A)$ for every subset A of $X\}$,

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$\Gamma_- = \{\gamma \in \Gamma \mid \gamma(A) \subset A$ for every subset A of $X\}$ and
$\Gamma_+ = \{\gamma \in \Gamma \mid A \subset \gamma(A)$ for every subset A of $X\}$. If $I = \{0,1,2,+,-\}$ and $A \subset I$, then $\gamma \in \Gamma_A$ if and only if $\gamma \in \Gamma_i$ for every $i \in A$. If $\gamma \in \Gamma_+$, then γ is called a *weak envelope* [3]. The pair (X,γ) is called a *weak envelope* space. If $\gamma \in \Gamma_{2+}$, then γ is called an *envelope* [3]. The pair (X,γ) is called an *envelope* space. Weak envelope and envelope operations are further studied by Á. CSÁSZÁR, in [4].

A subset μ of $\wp(X)$ is called a *generalized topology* (briefly GT)[2] if $\emptyset \in \mu$ and arbitrary union of members of μ is again in μ. Elements of μ are called μ*-open* sets. Complements of μ-open sets are called μ*-closed* sets. In this paper, we define and discuss the various characterizations and properties of sets in monotonic spaces, weak envelope spaces and envelope spaces which are similar to that of dense sets, nowheredense sets and δ-sets in topological spaces.

2. rc-dense and rc-nwdense sets

Let (X,γ) be a monotonic space. A subset A of X is said to be *rc-dense* if $\gamma(A) = X$. It is clear that $\gamma(A) = X$ if and only if $\gamma^\star(X - A) = \emptyset$. Since $\gamma \in \Gamma$, it follows that every superset of an rc-dense set is rc-dense and so the existence of a rc-dense set implies that $\gamma(X) = X$ and so $\gamma \in \Gamma_1$ which says that, by Proposition 1.7(b) of [1], $\gamma^\star \in \Gamma_0$. Equivalently, if $\gamma^\star \notin \Gamma_0$, then no rc-dense sets exist. The following Theorem 2.1 gives a property of rc-dense sets. Example 2.2 shows that the converse of Theorem 2.1 is not true.

THEOREM 2.1. *Let (X,γ) be a monotonic space. If a subset A of X is rc-dense, then $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

PROOF. Suppose $A \cap V = \emptyset$ for some nonempty γ-regularopen set V. $A \cap V = \emptyset$ implies that $V \subset X - A$ and so $V \subset \gamma^\star(X - A)$. Therefore, $X - \gamma^\star(X - A) \subset X - V$ and so $X = \gamma(A) \subset X - V$ which implies that $V = \emptyset$, a contradiction to the hypothesis. Therefore, $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V. ∎

EXAMPLE 2.2. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{b\}$, $\gamma(\{a\}) = \{a,b\}$, $\gamma(\{b\}) = \{b,c\}$, $\gamma(\{c\}) = \{b,c\}$, $\gamma(\{a,b\}) = X$, $\gamma(\{a,c\}) = X$, $\gamma(\{b,c\}) = \{b,c\}$, $\gamma(X) = X$. Then $\gamma \in \Gamma$, $\gamma \notin \Gamma_2$, $\xi = \{\{b,c\}, X\}$ and $\mu = \{\emptyset, \{a\}\}$. Now $V = \{a\}$ is the only nonempty

γ-regularopen set such that $V \cap \{a\} \neq \emptyset$ but $\{a\}$ is not rc-dense. Note that $\{a,b\}$, $\{a,c\}$ and X are rc-dense sets.

THEOREM 2.3. *Let (X,γ) be a weak envelope space. If A is a subset of X such that $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V, then $\gamma(A) \cap V \neq \emptyset$ for every non empty γ-regularopen set V.*

PROOF. The proof follows from the fact that $\gamma \in \Gamma_+$. ∎

The following Example 2.4 shows that the condition $\gamma \in \Gamma_+$ in Theorem 2.3 cannot be dropped.

EXAMPLE 2.4. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{a\}$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{a\}$, $\gamma(\{c\}) = X$, $\gamma(\{a,b\}) = \{a\}$, $\gamma(\{a,c\}) = X$, $\gamma(\{b,c\}) = X$, $\gamma(X) = X$. Then $\gamma \in \Gamma$, $\gamma \notin \Gamma_+$, $\xi = \{\{a\}, X\}$ and $\mu = \{\emptyset, \{b,c\}\}$. Let $A = \{a,b\}$. $V = \{b,c\}$ is the only nonempty γ-regularopen set such that $V \cap A = \{b\} \neq \emptyset$. But $\gamma(A) = \{a\}$ and so $\gamma(A) \cap V = \emptyset$.

THEOREM 2.5. *Let (X,γ) be a monotonic space and $A \subset X$. Then the following hold.*

(a) *If A is rc-dense, then $\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

(b) *If $\gamma \in \Gamma_2$ and $\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V, then A is rc-dense.*

PROOF. (a) The proof of (a) is clear.

(b) Suppose $\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V and A is not rc-dense. Then $X - \gamma(A) \neq \emptyset$. If $V = X - \gamma(A)$, then $\gamma^\star(V) = \gamma^\star(X - \gamma(A)) = X - \gamma(X - (X - \gamma(A))) = X - \gamma(\gamma(A)) = X - \gamma(A) = V$ and so V is γ-regularopen. But $V \cap \gamma(A) = (X - \gamma(A)) \cap \gamma(A) = \emptyset$, a contradiction to the hypothesis. Therefore, A is rc-dense. ∎

The following Example 2.6 shows that the condition $\gamma \in \Gamma_2$ in the above Theorem 2.5(b) cannot be dropped. The proof of the Corollary 2.7 below follows from Theorems 2.1, 2.3 and 2.5.

EXAMPLE 2.6. Let (X,γ) be the monotonic space of Example 2.2. $\gamma \notin \Gamma_2$, $V = \{a\}$ is the only nonempty γ-regularopen set such that $V \cap \gamma(\{a\}) \neq \emptyset$ but $\{a\}$ is not rc-dense.

COROLLARY 2.7. *Let (X,γ) be an envelope space and $A \subset X$. Then the following are equivalent.*
(a) *A is rc-dense.*
(b) *$A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*
(c) *$\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

Let (X,γ) be a monotonic space. We say that a subset A of X is said to be *rc-nowheredense* (in short, *rc-nwdense*) if $\gamma^{\bigstar}\gamma(A) = \emptyset$. It is clear that $\gamma^{\bigstar}\gamma(A) = \emptyset$ if and only if $\gamma\gamma^{\bigstar}(X - A) = X$. We will denote the family of all rc-nwdense sets in a monotonic space (X,γ) by $\mathcal{N}$. Since $\gamma \in \Gamma$, it follows that every subset of an rc-nwdense set is an rc-nwdense set and so the existence of an rc-nwdense set implies that $\emptyset$ is rc-nwdense and $\gamma^{\bigstar}\gamma \in \Gamma_0$. In other words, if $\gamma^{\bigstar}\gamma \notin \Gamma_0$, then rc-nwdense sets will not exist and so $\mathcal{N} = \emptyset$. The following Example 2.8 shows that in monotonic spaces, we can have either $\mathcal{N} = \emptyset$ or $\mathcal{N} = \{\emptyset\}$ or $\mathcal{N}$ has more than one element. Theorem 2.9 below gives a property of rc-nwdense sets.

EXAMPLE 2.8. (a) [1, Example 1.12] Let $X = \mathbf{R}$ be the set of all real numbers and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(A) = \{0\}$ if $0 \in A$ and $\emptyset$ otherwise. In this space, $\mathcal{N} = \emptyset$.
(b) Consider the monotonic space of Example 2.2. In this space, $\mathcal{N} = \{\emptyset\}$.
(c) Consider the monotonic space of Example 2.4. In this space,

$$\mathcal{N} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}.$$

THEOREM 2.9. *Let (X,γ) be a monotonic space and $A \subset X$ be rc-nwdense. If V is a nonempty γ-regularopen set, then V is not a subset of $\gamma(A)$.*

PROOF. Since A is rc-nwdense, $\gamma^{\bigstar}\gamma(A) = \emptyset$ and so $\gamma\gamma^{\bigstar}(X - A) = X$ which implies that $\gamma^{\bigstar}(X - A) = X - \gamma(A)$ is rc-dense. By Theorem 2.1, $V \cap (X - \gamma(A)) \neq \emptyset$ for every nonempty γ-regularopen set V. Therefore, $V - \gamma(A) \neq \emptyset$ which implies that $V \not\subset \gamma(A)$ for every nonempty γ-regularopen set V. This completes the proof. ∎

The following Example 2.10 shows that the converse of the above Theorem 2.9 is not true even if the space is an weak envelope space. Theorem 2.11 below shows that the converse is true if (X,γ) is an envelope space. Example 2.12 shows that either the condition $\gamma \in \Gamma_2$ or the the condition $\gamma \in \Gamma_+$ cannot be dropped from Theorem 2.11.

EXAMPLE 2.10. Let (X,γ) be the monotonic space of Example 2.2. If $A = \{b,c\}$, then $V = \{a\}$ is the only nonempty γ-regularopen set such that $V \not\subset \gamma(A)$. But $\gamma^\bigstar\gamma(A) = \gamma^\bigstar(\{b,c\}) = \{c\} \neq \emptyset$ and so A is not rc-nwdense.

THEOREM 2.11. *Let (X,γ) be an envelope space and $A \subset X$. If for every nonempty γ-regularopen set V, V is not a subset of $\gamma(A)$, then A is rc-nwdense.*

PROOF. If V is a nonempty γ-regularopen set such that V is not a subset of $\gamma(A)$, then $V - \gamma(A) \neq \emptyset$ which implies that $V \cap (X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_+$, by Theorem 2.3, $V \cap \gamma(X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_2$, by Theorem 2.5, $X - \gamma(A)$ is rc-dense and so $\gamma(X - \gamma(A)) = X$ which implies that $X - \gamma^\bigstar\gamma(A) = X$. Therefore, $\gamma^\bigstar\gamma(A) = \emptyset$ and so A is rc-nwdense. ∎

EXAMPLE 2.12. (a) Example 2.10 shows that the condition Γ_2 cannot be dropped in Theorem 2.11.

(b) Consider the monotonic space of Example 2.8(a). Then $\mu = \{\mathbf{R}, \mathbf{R} - \{0\}\}$. Clearly, $\gamma \in \Gamma_2$. If $B \subset X$ such that $0 \in B$ and B has more than one point, then $\gamma(B) = \{0\} \not\supset B$ and so $\gamma \notin \Gamma_+$. If A is a nonempty subset of $\mathbf{R}$ not containing 0, then $\mathbf{R} \not\subset \gamma(A)$. But A is not rc-nwdense.

Let (X,γ) be a monotonic space. A subset A of X is said to be *weak rc-nwdense* (in short, *wrc-nwdense*) if for every nonempty $V \in \mu$, there exists a nonempty $W \in \mu$ with $W \subset V$ such that $W \cap A = \emptyset$. The following Examples 2.13 and 2.14 shows that rc-nwdenseness and wrc-nwdenseness are independent concepts.

EXAMPLE 2.13. Consider the monotonic space of Example 2.8(a). Then $\mu = \{\mathbf{R}, \mathbf{R} - \{0\}\}$ and $\{0\}$ is wrc-nwdense but not rc-nwdense. Therefore, a wrc-nwdense set need not be an rc-nwdense set.

EXAMPLE 2.14. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset, \gamma(\{a\}) = \{a\}, \gamma(\{b\}) = \{a,b\}, \gamma(\{c\}) = X, \gamma(\{a,b\}) = X, \gamma(\{a,c\}) = X, \gamma(\{b,c\}) = X, \gamma(X) = X$. Then $\mu = \{\emptyset, X, \{b,c\}\}$. If $A = \{b\}$, then A is rc-nwdense but not wrc-nwdense.

Let (X,γ) be a monotonic space. We say that γ is *subadditive* if $\gamma(A \cup B) \subset \gamma(A) \cup \gamma(B)$ for every subsets A and B of X. Since γ is monotonic, if γ is subadditive, then γ is additive. That is, $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$ for every subsets A and B of X. The following Lemma 2.15 is essential to characterize rc-nwdense sets in Theorem 2.17 below.

LEMMA 2.15. *Let (X,γ) be a monotonic space and $A \subset X$.*
(a) $\gamma \in \Gamma_2$ *if and only if* $\gamma(A)$ *is* γ*-regularclosed for every subset A of X.*
(b) *If γ is subadditive, then the intersection of two γ-regularopen sets is a γ-regularopen set.*
(c) *If $G \cap A = \emptyset$, then $G \cap \gamma(A) = \emptyset$ for every nonempty γ-regularopen set G. The reverse direction is true, if $\gamma \in \Gamma_+$.*
(d) *If $x \in \gamma(A)$, then $G \cap A \neq \emptyset$ for every γ-regularopen set G containing x.*
(e) *If $\gamma \in \Gamma_{2+}$ and $G \cap A \neq \emptyset$ for every γ-regularopen set G containing x, then $x \in \gamma(A)$.*
(f) *A is rc-nwdense if and only if $X - \gamma(A)$ is rc-dense.*

PROOF. (a) The proof is clear.
(b) Let U and V be γ-regularopen. Now

$$\gamma^\star(U \cap V) = X - \gamma(X - (U \cap V)) = X - \gamma((X - U) \cup (X - V)) =$$
$$= X - (\gamma(X - U) \cup \gamma(X - V)) = (X - \gamma(X - U)) \cap (X - \gamma(X - V)) =$$
$$= \gamma^\star(U) \cap \gamma^\star(V) = U \cap V$$

and so $U \cap V$ is γ-regularopen.
(c) If $G \cap A = \emptyset$, then $A \subset X - G$ and so $\gamma(A) \subset \gamma(X - G) = X - \gamma^\star(G) = X - G$. Therefore, $G \cap \gamma(A) = \emptyset$. The proof of the converse is clear.
(d) If G is a γ-regularopen set containing x, then $\gamma(A) \cap G \neq \emptyset$. By (c), $G \cap A \neq \emptyset$.
(e) Suppose $x \notin \gamma(A)$. Since $\gamma \in \Gamma_2$, $G = X - \gamma(A)$ is a γ-regularopen set containing x by (a), such that $G \cap \gamma(A) = \emptyset$. Since $\gamma \in \Gamma_+$, by (c), $G \cap A = \emptyset$, a contradiction to the hypothesis which proves (e).
(f) A is rc-nwdense if and only if $\gamma^\star\gamma(A) = \emptyset$ if and only if $X - \gamma^\star\gamma(A) = X$ if and only if $X - (X - \gamma(X - \gamma(A))) = X$ if and only if $\gamma(X - \gamma(A)) = X$ if and only if $X - \gamma(A)$ is rc-dense. ∎

Example 2.16(a) shows that the condition $\gamma \in \Gamma_+$ cannot be dropped to prove the reverse direction in Lemma 2.15(c). Example 2.16(b) shows that the subadditivity cannot be dropped in the above Lemma 2.15(b). Also, it shows that *in an envelope space the intersection of two γ-regularopen sets need not be a γ-regularopen set.* Example 2.16(c) shows that the condition $\gamma \in \Gamma_2$ cannot be dropped in Lemma 2.15(e). That is, Lemma 2.15(e) is not true in a weak envelope space.

EXAMPLE 2.16. (a) Let $X = \{a,b,c\}$ and define $\gamma : \wp(X) \to \wp(X)$ by $\gamma(A) = \{a\}$ for every subset A of X. Then $\mu = \{\{b,c\}\}$ and $\gamma \notin \Gamma_+$. If $A = \{b\}$, then $G \cap \gamma(A) = \emptyset$ for every $G \in \mu$ but $G \cap A \neq \emptyset$.

(b) Consider $X = \{a,b,c\}$ and define $\gamma : \wp(X) \to \wp(X)$ by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{b\}$, $\gamma(\{c\}) = \{c\}$, $\gamma(\{a,b\}) = \{a,b\}$, $\gamma(\{a,c\}) = \{a,c\}$, $\gamma(\{b,c\}) = \gamma(X) = X$. Then $\gamma \in \Gamma_+$ and $\gamma \in \Gamma_2$. If $A = \{b\}$ and $B = \{c\}$, then $\gamma(A) \cup \gamma(B) = \{b,c\}$. But $\gamma(A \cup B) = \gamma(\{b,c\}) = X \nsubseteq \{b,c\} = \gamma(A) \cup \gamma(B)$. Therefore, γ is not subadditive. Here

$$\xi = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, X\}$$

and

$$\mu = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}.$$

If $U = \{a,b\}$ and $V = \{a,c\}$, then U and V are γ-regularopen sets but $U \cap V = \{a\}$ is not a γ-regularopen set.

(c) Consider the monotonic space (X,γ) of Example 2.14. Then $\gamma \in \Gamma_+$ +, $\gamma \notin \Gamma_2$ and $\mu = \{\emptyset, X, \{b,c\}\}$. If $A = \{b\}$, then $G \cap A \neq \emptyset$ for every nonempty γ-regularopen set G containing c. Since $\gamma(A) = \gamma(\{b\}) = \{a,b\}$, $c \notin \gamma(A)$.

THEOREM 2.17. *Let (X,γ) be an envelope space and $A \subset X$. Then the following hold.*

(a) *If A is wrc-nwdense, then A is rc-nwdense.*

(b) *If γ is subadditive and A is rc-nwdense, then A is wrc-nwdense.*

PROOF. (a) Suppose A is not rc-nwdense. Then $G = \gamma^\star\gamma(A) \neq \emptyset$. Since $\gamma \in \Gamma_2$, by Proposition 1.7(c) of [1], $\gamma^\star \in \Gamma_2$ and so $\gamma^\star(G) = \gamma^\star(\gamma^\star\gamma(A)) = \gamma^\star\gamma(A) = G$ and so G is a nonempty γ-regularopen set. Since $\gamma \in \Gamma_+$, by Proposition 1.7(d) of [1], $\gamma^\star \in \Gamma_-$ and so $\gamma^\star\gamma(A) \subset \gamma(A)$ which implies that $G \subset \gamma(A)$. Then for every nonempty $W \in \mu$ with $W \subset G$, $W \cap \gamma(A) \neq \emptyset$ and so by Lemma 2.15(c), $W \cap A \neq \emptyset$, a contradiction to the hypothesis.

(b) Suppose A is rc-nwdense. Then by Lemma 2.15(a), $X - \gamma(A)$ is a γ-regularopen set. By Lemma 2.15(f), $X - \gamma(A)$ is rc-dense. By Theorem 2.1, $(X - \gamma(A)) \cap V = W$ is nonempty and $W \subset V$ for every nonempty γ-regularopen set V. By Lemma 2.15(b), W is γ-regularopen. Since $\gamma \in \Gamma_+$, $A \cap W = A \cap ((X - \gamma(A)) \cap V) = \emptyset$. Therefore, A is wrc-nwdense. ∎

COROLLARY 2.18. *Let (X,γ) be an envelope space, $A \subset X$ and γ be subadditive. Then the following are equivalent.*

(a) *A is wrc-nwdense.*

(b) *A is rc-nwdense.*

(c) *If V is a nonempty γ-regularopen set, then V is not a subset of $\gamma(A)$.*

PROOF. The proof follows from Theorems 2.17, 2.9 and 2.11. ∎

The following Example 2.19 shows that in Theorem 2.17(a), the condition *envelope* cannot be replaced by *weak envelope*. Example 2.20 shows that in Theorem 2.17(b), the condition *subadditive* on γ cannot be dropped.

EXAMPLE 2.19. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{a\}, \gamma(\{a\}) = \{a,b\}, \gamma(\{b\}) = \{a,b\}, \gamma(\{c\}) = X, \gamma(\{a,b\}) = = \gamma(\{a,c\}) = \gamma(\{b,c\}) = \gamma(X) = X$. Then (X,γ) is a weak envelope space. Since $\mu = \{\emptyset\}$, every nonempty subset of X is wrc-nwdense and so $\{c\}$ is wrc-nwdense but is not rc-nwdense.

EXAMPLE 2.20. Let (X,γ) be an envelope space where $X = \{a,b,c,d\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by

$$\gamma(\emptyset) = \emptyset,\ \gamma(\{a\}) = \{a\},\ \gamma(\{b\}) = \{b\},\ \gamma(\{c\}) = \{a,b,c\},\ \gamma(\{d\}) = \{a,d\},$$
$$\gamma(\{a,b\}) = \gamma(\{a,c\}) = \gamma(\{b,c\}) = \{a,b,c\},$$
$$\gamma(\{a,d\}) = \{a,d\}, \quad \gamma(\{b,d\}) = \gamma(\{c,d\}) = \gamma(X) = X,$$
$$\gamma(\{a,b,c\}) = \{a,b,c\}, \quad \gamma(\{a,b,d\}) = \gamma(\{a,c,d\}) = \gamma(\{b,c,d\}) = X.$$

Then $\mu = \{\emptyset, \{d\}, \{b,c\}, \{a,c,d\}, \{b,c,d\}, X\}$. We show that γ is not subadditive. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(A \cup B) = \{a,b,c\}$ and $\gamma(A) \cup \gamma(B) = \{a,b\}$. Therefore, γ is not subadditive. If $A = \{b\}$, then $\gamma^\bigstar\gamma(A) = \gamma^\bigstar(\{b\}) = \emptyset$ and so A is rc-nwdense. If $V = \{b,c\}$, then $V \cap A \neq \emptyset$ and so A is not wrc-nwdense.

A nonempty collection $\mathcal{I}$ of subsets of X is said to be an *ideal* [6] if it satisfies the following. (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$ and (ii) $A \cup B \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and $B \in \mathcal{I}$. In the rest of this section, we discuss some properties of rc-nwdense sets and analyze under what additional conditions on γ, $\mathcal{N}$ is an ideal on X.

THEOREM 2.21. *Let (X,γ) be a monotonic space and $A \subset X$. If A is the union of a γ-regularopen set and an rc-nwdense set, then $A \cap \gamma(X - A)$ is an rc-nwdense set.*

PROOF. Let $A = G \cup N$ where G is γ-regularopen and N is rc-nwdense. If $M = N - G$, then $\gamma^\bigstar\gamma(M) = \gamma^\bigstar\gamma(N - G) \subset \gamma^\bigstar(\gamma(N) \cap \gamma(X - G)) \subset \subset \gamma^\bigstar(\gamma(N)) \cap \gamma^\bigstar(\gamma(X - G)) = \emptyset \cap \gamma^\bigstar(\gamma(X - G)) = \emptyset$ and so M is rc-nwdense.

Again, $M \cup G = (N - G) \cup G = N \cup G = A$. Since G is γ-regularopen such that $G \subset A$, we have $G \subset \gamma^{\bigstar}(A)$. Now

$A \cap \gamma(X - A) = (G \cup N) \cap \gamma((X - G) \cap (X - N)) \subset (G \cup N) \cap \gamma(X - G) =$

$= (G \cup N) \cap (X - \gamma^{\bigstar}(G)) = (G \cup N) \cap (X - G) = N \cap (X - G) = N - G = M.$

Since $A \cap \gamma(X - A)$ is a subset of an rc-nwdense set M, $A \cap \gamma(X - A)$ is rc-nwdense. ∎

The following Theorem 2.22 shows that the converse of Theorem 2.21 is true if the space (X,γ) is an envelope space. Example 2.23 below shows that the condition envelope on the space cannot be dropped in Theorem 2.22. Theorem 2.24 gives a characterization of rc-nwdense sets in an envelope space.

THEOREM 2.22. *Let* (X,γ) *be an envelope space and* $A \subset X$. *If* $A \cap \cap \gamma(X - A)$ *is an rc-nwdense set, then* A *is the union of a* γ*-regularopen set and an rc-nwdense set.*

PROOF. If $A \cap \gamma(X - A) = \emptyset$, then $A \subset X - \gamma(X - A) = \gamma^{\bigstar}(A)$ which implies that $A = \gamma^{\bigstar}(A)$, since by Proposition 1.7(d) of [1] $\gamma \in \Gamma_+$ if and only if $\gamma^{\bigstar} \in \Gamma_-$. Now $\gamma^{\bigstar}(A) \cup (A \cap \gamma(X - A)) = A \cup (A \cap (X - \gamma^{\bigstar}(A)) = A$. Suppose $A \cap \gamma(X - A) \neq \emptyset$. Then

$$\gamma^{\bigstar}(A) \cup (A \cap \gamma(X - A)) = (\gamma^{\bigstar}(A) \cup A)) \cap (\gamma^{\bigstar}(A) \cup (X - \gamma^{\bigstar}(A))) = \\ = \gamma^{\bigstar}(A) \cup A.$$

Since $\gamma \in \Gamma_2, \gamma^{\bigstar}(A) \in \mu$ and since $\gamma \in \Gamma_+, \gamma^{\bigstar}(A) \subset A$. Therefore, $\gamma^{\bigstar}(A) \cup (A \cap \gamma(X - A)) = A$. This completes the proof. ∎

EXAMPLE 2.23. (a) Let $X = \mathbf{R}$ be the set of all real numbers and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(A) = A$ if $0 \in A$ and $\emptyset$ otherwise. Then (X,γ) is a monotonic space, $\gamma \in \Gamma_2, \mathcal{N} = \{\emptyset\} \cup \{A \mid 0 \notin A\}$ and $\mu = \{\emptyset, \mathbf{R}\} \cup \{A \mid 0 \notin A\}$. If B is a nonempty subset of $\mathbf{R}$ such that $0 \notin B$, then $\gamma(B) = \emptyset$ and so $\gamma \notin \Gamma_+$. If $A = [0, 1)$, then $A \cap \gamma(X - A) = \emptyset$ and so $A \cap \gamma(X - A)$ is rc-nwdense but A is not the union of an rc-nwdense set and a γ-regularopen set.

(b) Let $X = \{a, b, c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\}) = = \{a\}$, $\gamma(\{b\}) = \{a, b\}$, $\gamma(\{c\}) = X$, $\gamma(\{a, b\}) = \gamma(\{a, c\}) = \gamma(\{b, c\}) = = \gamma(X) = X$. Then $\gamma \in \Gamma_+$, $\mu = \{\emptyset, \{b, c\}, X\}$ and $\mathcal{N} = \{\emptyset, \{a\}, \{b\}\}$. Since $\gamma(\gamma(\{b\})) \neq \gamma(\{b\}), \gamma \notin \Gamma_2$. If $A = \{a, c\}$, then $A \cap \gamma(X - A) = \{a\}$, which is rc-nwdense but A cannot be written as the union of an rc-nwdense set and a γ-regularopen set.

THEOREM 2.24. *Let* (X,γ) *be an envelope space and* $A \subset X$. *Then* A *is rc-nwdense if and only if* $A \subset \gamma(X - \gamma(A))$.

PROOF. If A is *rc – nwdense*, then $\gamma^{\star}\gamma(A) = \emptyset$. Now,

$$\gamma(X - \gamma(A)) = X - \gamma^{\star}\gamma(A) = X \supset A.$$

Conversely, if $A \subset \gamma(X - \gamma(A))$, then $A \subset X - \gamma^{\star}\gamma(A)$ and so

$$\gamma^{\star}\gamma(A) \subset \gamma^{\star}\gamma(X - \gamma^{\star}\gamma(A)) = \gamma^{\star}(X - \gamma^{\star}\gamma^{\star}\gamma(A)) =$$
$$= \gamma^{\star}(X - \gamma^{\star}\gamma(A)) \subset X - \gamma^{\star}\gamma(A). \text{ (cf. [1], 1.7).}$$

Therefore, $\gamma^{\star}\gamma(A) = \emptyset$ which implies that A is rc-nwdense. ∎

THEOREM 2.25. *Let* (X,γ) *be an envelope space and* γ *be subadditive. Then the union of two rc-nwdense sets is again an rc-nwdense set and so, if* $\mathcal{N}$ *is nonempty, then* $\mathcal{N}$ *is an ideal.*

PROOF. Let A and B be two rc-nwdense subsets of X. Then $\gamma^{\star}\gamma(A) = \emptyset$ and $\gamma^{\star}\gamma(B) = \emptyset$ and so $X - \gamma^{\star}\gamma(A) = X$ and $X - \gamma^{\star}\gamma(B) = X$. This implies that $\gamma(X - \gamma(A)) = X$ and $\gamma(X - \gamma(B)) = X$ and so $X - \gamma(A)$ and $X - \gamma(B)$ are rc-dense sets. Let $\emptyset \neq G \in \mu$. Since $X - \gamma(A)$ is rc-dense, by Theorem 2.1, $G \cap (X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_2$, $X - \gamma(A) \in \mu$ by Lemma 2.15(a). By Lemma 2.15(b), $G \cap (X - \gamma(A)) \in \mu$. Since $X - \gamma(B)$ is rc-dense, $(G \cap (X - \gamma(A))) \cap (X - \gamma(B)) \neq \emptyset$ which implies that $G \cap (X - (\gamma(A) \cup \gamma(B))) \neq \emptyset$ and so $G \cap (X - (\gamma(A \cup B))) \neq \emptyset$, since γ is additive. By Corollary 2.7, $X - (\gamma(A \cup B))$ is rc-dense and so $\gamma(X - (\gamma(A \cup B))) = X$ which implies that $X - \gamma(X - (\gamma(A \cup B))) = \emptyset$. Therefore, $\gamma^{\star}\gamma(A \cup B) = \emptyset$ and so $A \cup B$ is rc-nwdense. This completes the proof. ∎

Example 2.23(a) above shows that if the finite union of rc-nwdense subsets of the space (X,γ) is rc-nwdense, then γ need not be additive. The following Example 2.26 shows that the condition subadditive on γ in Theorem 2.25 cannot be dropped.

EXAMPLE 2.26. Consider the monotonic space (X,γ) of Example 2.19. Then $\gamma \in \Gamma_+$ and $\gamma \notin \Gamma_2$. If $A = \{a\}$and $B = \{b\}$, then $\gamma(A) \cup \gamma(B) = \{a, b\}$. But $\gamma(A \cup B) = \gamma(\{a, b\}) = X \not\subseteq \{a, b\} = \gamma(A) \cup \gamma(B)$. Therefore, γ is not subadditive. Here $\xi = \{X\}$ and $\mu = \{\emptyset\}$. Also, $\mathcal{N} = \{\emptyset, \{a\}, \{b\}\}$. If $C = \{a\}$

and $D = \{b\}$, then $C \cup D = \{a,b\}$ and $\gamma^{\bigstar}\gamma(C \cup D) = \gamma^{\bigstar}(X) = X - \gamma(\emptyset) = = X - \{a\} = \{b,c\} \neq \emptyset$ and so $C \cup D$ is not an rc-nwdense set.

THEOREM 2.27. *Let (X,γ) be an envelope space, $A \subset X$ and γ be subadditive. Then the following hold.*

(a) *$\gamma(A) \cap V \subset \gamma(A \cap V)$ for every γ-regularopen set V.*

(b) *$\gamma(\gamma(A) \cap V) = \gamma(A \cap V)$ for every γ-regularopen set V.*

(c) *If A is rc-dense, then $\gamma(V) = \gamma(A \cap V)$ for every γ-regularopen set V.*

PROOF. (a) Suppose $x \in \gamma(A) \cap V$. Then $x \in \gamma(A)$ and $x \in V$. If G is a γ-regularopen set containing x, by Lemma 2.15(b), $G \cap V$ is a γ-regularopen set containing x and so by Lemma 2.15(d), $(G \cap V) \cap A = G \cap (V \cap A) \neq \emptyset$. By Lemma 2.15(e), $x \in \gamma(V \cap A)$. Therefore, $\gamma(A) \cap V \subset \gamma(A \cap V)$.

(b) Since $\gamma \in \Gamma_+$, $A \cap V \subset \gamma(A) \cap V$ and so $\gamma(A \cap V) \subset \gamma(\gamma(A) \cap V) \subset \subset \gamma\gamma(A \cap V) = \gamma(A \cap V)$ and so $\gamma(\gamma(A) \cap V) = \gamma(A \cap V)$.

(c) By (b), if V is γ-regularopen, then

$$\gamma(A \cap V) = \gamma(\gamma(A) \cap V) = \gamma(X \cap V) = \gamma(V)$$

and so (c) follows. ∎

Let $\lambda \subset \wp(X)$. $\gamma \in \Gamma$ is said to be *λ-friendly* [5] if $\gamma(A) \cap V \subset \gamma(A \cap V)$ for every subset A of X and $V \in \lambda$. A generalized topology ψ is said to be a *quasi-topology* [5] if $M \cap M_1 \in \psi$ whenever $M \in \psi$ and $M_1 \in \psi$.

THEOREM 2.28. *Let (X,γ) be a monotonic space. Then the following hold.*

(a) *If γ is a weak envelope, then μ is a generalized topology. In addition, if γ is subadditive, then μ is a quasi-topology.*

(b) *If γ is an envelope, then γ is subadditive if and only if γ is μ-friendly.*

(c) *γ is subadditive if and only if $\gamma^{\bigstar}(A \cap B) = \gamma^{\bigstar}(A) \cap \gamma^{\bigstar}(B)$ for every subsets A and B of X.*

(d) *If γ is an envelope, γ is subadditive, G is γ-regularopen and $A \subset X$, then $G \cap \gamma^{\bigstar}(A) = \gamma^{\bigstar}(G \cap A)$.*

(e) *If γ is an envelope, γ is subadditive, F is γ-regularclosed and $A \subset X$, then $\gamma^{\bigstar}(A \cup F) \subset \gamma^{\bigstar}(A) \cup F$.*

(f) *If γ is an envelope, γ is subadditive, F is γ-regularclosed and $A \subset X$, then $\gamma(A \cup F) = \gamma(A) \cup F$.*

PROOF. (a) By Lemmas 1.3 and 1.4 of [3], μ is a generalized topology. By Lemma 2.15(b), μ is a quasi-topology.

(b) If γ is subadditive, then by Theorem 2.27(a), γ is μ-friendly. Conversely, suppose γ is μ-friendly. For subsets A and B of X,

$$\gamma(A \cup B) - \gamma(B) = \gamma(A \cup B) \cap (X - \gamma(B)).$$

Since $X - \gamma(B) \in \mu$ by Lemma 2.15(a), by Theorem 2.27(a),

$$\gamma(A \cup B) - \gamma(B) \subset \gamma((A \cup B) \cap (X - \gamma(B))) = \gamma(A \cap (X - \gamma(B))) \subset \gamma(A)$$

and so $\gamma(A \cup B) \subset \gamma(A) \cup \gamma(B)$.

(c) The proof follows from the definition of $\gamma^\bigstar$.

(d) Let G be γ-regularopen and A be any subset of X. Then $G \cap \gamma^\bigstar(A)$ is a γ-regularopen set by Lemma 2.15(b), such that $G \cap \gamma^\bigstar(A) \subset G \cap A$, since $\gamma^\bigstar \in \Gamma_-$. Therefore, $G \cap \gamma^\bigstar(A) \subset \gamma^\bigstar(G \cap A) = \gamma^\bigstar(G) \cap \gamma^\bigstar(A) = G \cap \gamma^\bigstar(A)$, by (c). Therefore, $G \cap \gamma^\bigstar(A) = \gamma^\bigstar(G \cap A)$.

(e) Now

$$X - \gamma^\bigstar(A \cup F) = \gamma(X - (A \cup F)) = \gamma((X - A) \cap (X - F)) \supset \gamma(X - A) \cap (X - F),$$

by Theorem 2.27(a). Therefore,

$$X - \gamma^\bigstar(A \cup F) \supset (X - \gamma^\bigstar(A)) \cap (X - F) = X - (\gamma^\bigstar(A) \cup F)$$

and so $\gamma^\bigstar(A \cup F) \subset \gamma^\bigstar(A) \cup F$.

(f) Now

$$X - \gamma(A \cup F) = \gamma^\bigstar(X - (A \cup F)) = \gamma^\bigstar((X - A) \cap (X - F)) =$$
$$= \gamma^\bigstar(X - A) \cap (X - F) = (X - \gamma(A)) \cap (X - F) = X - (\gamma(A) \cup F)$$

and so $\gamma(A \cup F) = \gamma(A) \cup F$. ∎

The following Example 2.29 shows that in a weak envelope space (X, γ), if γ is μ-friendly, then γ need not be subadditive. Also, this example shows that the reverse direction of Lemma 2.15(b) is not true.

EXAMPLE 2.29. Let (X, γ) be the space of Example 2.19. Then $\mu = \{\emptyset\}$. Since $\gamma(A) \cap V \subset \gamma(A \cap V)$ for every γ-regularopen set V and $A \subset X$, γ is μ-friendly. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(\{a\}) = \{a, b\}$ and $\gamma(\{b\}) = \{a, b\}$ so that $\gamma(A) \cup \gamma(B) = \{a, b\}$. But $\gamma(A \cup B) = X$ and so γ is not subadditive.

3. $r\delta$-sets and $r\delta^{\star}$-sets

A subset A of a monotonic space (X,γ) is said to be an *$r\delta$-set* if $\gamma^{\star}\gamma(A) \subset \gamma\gamma^{\star}(A)$. Clearly, every rc-nwdense set is an $r\delta$-set. The following Theorem 3.1 shows that γ-regularopen sets are $r\delta$-sets, if $\gamma \in \Gamma_{+}$.

THEOREM 3.1. *Let (X,γ) be a monotonic space. If $\gamma \in \Gamma_{+}$, then every γ-regularopen set is an $r\delta$-set. In particular, $\emptyset$ is an $r\delta$-set.*

PROOF. Suppose A is a γ-regularopen set. Then $\gamma^{\star}(A) = A$. Now, since $\gamma \in \Gamma_{+}$, $\gamma^{\star}\gamma(A) \subset \gamma(A) = \gamma\gamma^{\star}(A)$. Therefore, A is an $r\delta$-set. ∎

THEOREM 3.2. *Let (X,γ) be a monotonic space and $\gamma \in \Gamma_2$. If $A \subset B \subset \subset \gamma(A)$ and A is an $r\delta$-set, then B, $\gamma(A)$ and $\gamma(B)$ are $r\delta$-sets.*

PROOF. Since A is an $r\delta$-set, $\gamma^{\star}\gamma(A) \subset \gamma\gamma^{\star}(A)$. Again, $A \subset B$ implies that $\gamma^{\star}(A) \subset \gamma^{\star}(B)$ and so $\gamma\gamma^{\star}(A) \subset \gamma\gamma^{\star}(B)$. Since $\gamma \in \Gamma_2$, $A \subset B \subset \subset \gamma(A)$ implies that $\gamma(A) = \gamma(B)$. Now $\gamma^{\star}\gamma(B) = \gamma^{\star}\gamma(A) \subset \gamma\gamma^{\star}(A) \subset \subset \gamma\gamma^{\star}(B)$ and so B is an $r\delta$-set. Clearly, $\gamma(A)$ and $\gamma(B)$ are $r\delta$-sets. ∎

COROLLARY 3.3. *Let (X,γ) be a monotonic space and $\gamma \in \Gamma_2$. If A is an $r\delta$-set which is also an rc-dense set, then every superset of A is an $r\delta$-set.*

The following Example 3.4 shows that the condition $\gamma \in \Gamma_2$ in Theorem 3.2 cannot be dropped.

EXAMPLE 3.4. Consider the monotonic space (X,γ) of Example 2.2. Then $\gamma \notin \Gamma_2$. If $A = \{a\}$, then $\gamma^{\star}\gamma(A) = \{a\}$ and $\gamma\gamma^{\star}(A) = \{a,b\}$. Therefore, A is an $r\delta$-set. If $B = \{a,b\}$, then $A \subset B \subset \gamma(A) = \{a,b\}, \gamma^{\star}\gamma(B) = = \{a,c\}$ and $\gamma\gamma^{\star}(B) = \{a,b\}$. Therefore, B is not an $r\delta$-set.

THEOREM 3.5. *Let (X,γ) be a monotonic space and $A \subset X$. If A is an $r\delta$-set, then $X - A$ is also an $r\delta$-set.*

PROOF. If A is an $r\delta$-set, then $\gamma^{\star}\gamma(A) \subset \gamma\gamma^{\star}(A)$ which implies that $X - \gamma\gamma^{\star}(A) \subset X - \gamma^{\star}\gamma(A)$ and so $\gamma^{\star}(X - \gamma^{\star}(A)) \subset \gamma(X - \gamma(A))$. Therefore, $\gamma^{\star}\gamma(X - A) \subset \gamma\gamma^{\star}(X - A)$ and so $X - A$ is an $r\delta$-set. ∎

THEOREM 3.6. *Let (X,γ) be a monotonic space. If there exists a singleton set which is both γ-regularopen and rc-dense in X, then every singleton set is an $r\delta$-set.*

PROOF. Suppose $\{x\}$ is both γ-regularopen and rc-dense. Then $\gamma^{\bigstar}(\{x\}) = \{x\}$ and $\gamma(\{x\}) = X$. Let $y \in X$ be arbitrary. If $y = x$, then $\gamma\gamma^{\bigstar}(\{y\}) = \gamma\gamma^{\bigstar}(\{x\}) = \gamma(\{x\}) = X$ and $\gamma^{\bigstar}\gamma(\{y\}) = \gamma^{\bigstar}\gamma(\{x\}) = \gamma^{\bigstar}(X)$. Therefore, $\gamma^{\bigstar}\gamma(\{y\}) \subset \gamma\gamma^{\bigstar}(\{y\})$ and so $\{y\}$ is an $r\delta$-set. If $y \neq x$, then $y \in X - \{x\}$ and so $\gamma^{\bigstar}\gamma(\{y\}) \subset \gamma^{\bigstar}\gamma(X - \{x\}) = \gamma^{\bigstar}(X - \gamma^{\bigstar}(\{x\})) = \gamma^{\bigstar}(X - \{x\}) = X - \gamma(\{x\}) = \emptyset$. Therefore, $\gamma^{\bigstar}\gamma(\{y\}) \subset \gamma\gamma^{\bigstar}(\{y\})$ and so $\{y\}$ is an $r\delta$-set in this case also. ∎

THEOREM 3.7. *Let (X,γ) be a monotonic space and $\mu = r\delta(X)$ where $r\delta(X)$ is the family of all $r\delta$-sets in (X,γ). Then the following hold.*

(a) *If $A \in \mu$, then $A \in \xi$.*

(b) *If $\gamma \in \Gamma_+$, then $\gamma^{\bigstar}\gamma(A) \neq \emptyset$ for every nonempty subset A of X.*

PROOF. (a) Suppose $A \in \mu$ such that $A \notin \xi$. $A \notin \xi$ implies that $X - A \notin \mu$. By hypothesis, $X - A \notin r\delta(X)$ which implies that $A \notin r\delta(X)$, by Theorem 3.5, a contradiction to the hypothesis. This completes the proof.

(b) Suppose $\gamma^{\bigstar}\gamma(A) = \emptyset$ for some nonempty subset A of X. Then A is an $r\delta$-set and so by hypothesis, $A \in \mu$ which implies that $\gamma^{\bigstar}(A) = A$. Since $\gamma \in \Gamma_+$, $A \subset \gamma(A)$ which implies that $\gamma^{\bigstar}(A) \subset \gamma^{\bigstar}\gamma(A) = \emptyset$ and so $A = \emptyset$, a contradiction to the hypothesis. Therefore, $\gamma^{\bigstar}\gamma(A) \neq \emptyset$ for every nonempty subset A of X. ∎

The following Theorem 3.8 gives a characterization of $r\delta$-sets in envelope spaces. Theorem 3.9 below gives a necessary condition for a set to be an $r\delta$-set.

THEOREM 3.8. *Let (X,γ) be an envelope space. Then a subset A of X is an $r\delta$-set if and only if $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$.*

PROOF. Suppose A is an $r\delta$-set. Then, $\gamma^{\bigstar}\gamma(A) \subset \gamma\gamma^{\bigstar}(A)$. Since $\gamma \in \Gamma_2$, we have $\gamma^{\bigstar}\gamma(A) \subset \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$. Since $\gamma \in \Gamma_+$, we have $\gamma^{\bigstar}(A) \subset A$ which implies that $\gamma^{\bigstar}\gamma\gamma^{\bigstar}(A) \subset \gamma^{\bigstar}\gamma(A)$. Therefore, $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$. Conversely, suppose $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$. Since $\gamma \in \Gamma_+$, $\gamma^{\bigstar}\gamma(A) \subset \gamma\gamma^{\bigstar}(A)$ and so A is an $r\delta$-set. ∎

THEOREM 3.9. *Let (X,γ) be an envelope space and $A \subset X$ be an $r\delta$-set. Then the following hold.*

(a) $A = B \cup C$ *where* B *is* γ*-regularopen,* C *is rc-nwdense and* $B \cap C = \emptyset$.

(b) *If* γ *is subadditive, then* $\gamma^\bigstar\gamma(A \cap B) = \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)$ *for every subset* B *of* X.

PROOF. (a) Suppose that A is an $r\delta$-set. Then $\gamma^\bigstar\gamma(A) \subset \gamma\gamma^\bigstar(A)$. If $B = \gamma^\bigstar(A)$, then $\gamma^\bigstar(B) = \gamma^\bigstar\gamma^\bigstar(A) = \gamma^\bigstar(A) = B$ and so B is γ-regularopen. If $C = A - \gamma^\bigstar(A)$, then $B \cup C = \gamma^\bigstar(A) \cup (A - \gamma^\bigstar(A)) = = A \cup \gamma^\bigstar(A) = A$. Now $C \subset A$ implies that $\gamma(C) \subset \gamma(A)$ and so $\gamma^\bigstar\gamma(C) \subset \subset \gamma^\bigstar\gamma(A) \subset \gamma\gamma^\bigstar(A)$. Clearly, $B \cap C = \emptyset$. By Lemma 2.15(c), $B \cap \gamma(C) = \emptyset$ which implies that $B \cap \gamma^\bigstar\gamma(C) = \emptyset$. Since $\gamma^\bigstar\gamma(C)$ is γ-regularopen, by Lemma 2.15(c), $\gamma(B) \cap \gamma^\bigstar\gamma(C) = \emptyset$ and so $\gamma\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(C) = \emptyset$ which implies that $\gamma^\bigstar\gamma(C) = \emptyset$. Therefore, C is rc-nwdense.

(b) Suppose that A is an $r\delta$-set. Clearly, $\gamma^\bigstar\gamma(A \cap B) \subset \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)$. Since $\gamma \in \Gamma_2$ and γ is subadditive,

$$\begin{aligned} &\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B) = \\ &= \gamma^\bigstar(\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)) \subset \\ &\subset \gamma^\bigstar\gamma(\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)) \subset \\ &\subset \gamma^\bigstar\gamma(\gamma\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B)) = \\ &= \gamma^\bigstar\gamma(\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B)), \end{aligned}$$

by Theorem 2.27(b). Therefore, $\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B) \subset \gamma^\bigstar\gamma(\gamma^\bigstar(A) \cap \gamma(B)) \subset \subset \gamma^\bigstar\gamma\gamma(\gamma^\bigstar(A) \cap B) = \gamma^\bigstar\gamma(\gamma^\bigstar(A) \cap B) \subset \gamma^\bigstar\gamma(A \cap B)$. Therefore, $\gamma^\bigstar\gamma(A \cap B) = \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)$. ∎

The following Theorem 3.10 shows that in an envelope space (X,γ), if γ is subadditive, then the finite intersection of $r\delta$-sets is an $r\delta$-set and the finite union of $r\delta$-sets is again an $r\delta$-set.

THEOREM 3.10. *Let* (X,γ) *be an envelope space and* γ *be subadditive. If* A *and* B *are* $r\delta$*-sets of* X*, then the following hold.*

(a) $A \cap B$ *is an* $r\delta$*-set.*

(b) $A \cup B$ *is an* $r\delta$*-set.*

PROOF. (a) Suppose A and B are $r\delta$-sets. Now

$$\gamma^\bigstar\gamma(A \cap B) \subset \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B) \subset \gamma\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B) \subset \gamma(\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B)),$$

by Theorem 2.27(a). Since B is an $r\delta$-set,

$$\gamma^{\bigstar}\gamma(A\cap B)\subset\gamma(\gamma^{\bigstar}(A)\cap\gamma\gamma^{\bigstar}(B))\subset\gamma\gamma(\gamma^{\bigstar}(A)\cap\gamma^{\bigstar}(B))=$$
$$=\gamma(\gamma^{\bigstar}(A)\cap\gamma^{\bigstar}(B))=\gamma\gamma^{\bigstar}(A\cap B),$$

by Theorem 2.28(c). Hence $A\cap B$ is an $r\delta$-set.

(b) Now $A\subset A\cup B$ implies that $\gamma^{\bigstar}(A)\subset\gamma^{\bigstar}(A\cup B)$ which in turn implies that $\gamma\gamma^{\bigstar}(A)\subset\gamma\gamma^{\bigstar}(A\cup B)$. Similarly, $\gamma\gamma^{\bigstar}(B)\subset\gamma\gamma^{\bigstar}(A\cup B)$ and so $\gamma\gamma^{\bigstar}(A)\cup\gamma\gamma^{\bigstar}(B)\subset\gamma\gamma^{\bigstar}(A\cup B)$. Since γ is subadditive and hence additive, $\gamma^{\bigstar}\gamma(A\cup B)=\gamma^{\bigstar}(\gamma(A)\cup\gamma(B))=\gamma^{\bigstar}\gamma^{\bigstar}(\gamma(A)\cup\gamma(B))\subset\gamma^{\bigstar}(\gamma(A)\cup\gamma^{\bigstar}\gamma(B))$, by Lemma 2.15(a) and Theorem 2.28(e). Since B is an $r\delta$-set, $\gamma^{\bigstar}\gamma(A\cup B)\subset$ $\subset\gamma^{\bigstar}(\gamma(A)\cup\gamma\gamma^{\bigstar}(B))\subset\gamma^{\bigstar}\gamma(A)\cup\gamma\gamma^{\bigstar}(B)\subset\gamma\gamma^{\bigstar}(A)\cup\gamma\gamma^{\bigstar}(B)$, since A is an $r\delta$-set. Therefore, $\gamma^{\bigstar}\gamma(A\cup B)\subset\gamma\gamma^{\bigstar}(A\cup B)$ and so $A\cup B$ is an $r\delta$-set. ∎

The following Example 3.11 shows that the condition subadditive on γ cannot be dropped in the above Theorem 3.10. Also, it shows that subsets of an $r\delta$-set need not be an $r\delta$-set.

EXAMPLE 3.11. Consider the monotonic space of Example 2.19. If $A=$ $=\{a\}$ and $B=\{b\}$, then

$$\gamma^{\bigstar}\gamma(A)=\gamma^{\bigstar}(\{a,b\})=X-\gamma(\{c\})=X-X=\emptyset\subset\gamma(\gamma^{\bigstar}(A)).$$

Also, $\gamma^{\bigstar}\gamma(B)=\gamma^{\bigstar}(\{a,b\})=\emptyset\subset\gamma\gamma^{\bigstar}(B)$. Therefore, A and B are $r\delta$-sets. But $A\cup B=\{a,b\}$ is not an $r\delta$-set. For,

$$\gamma^{\bigstar}\gamma(A\cup B)=\gamma^{\bigstar}(X)=X-\gamma(\emptyset)=X-\{a\}=\{b,c\}$$

and

$$\gamma\gamma^{\bigstar}(A\cup B)=\gamma(X-\gamma(\{c\}))=\gamma(X-X)=\gamma(\emptyset)=\{a\}\not\supset\gamma^{\bigstar}\gamma(A\cup B)$$

and so $A\cup B$ is not an $r\delta$-set.

If $C=\{a,c\}$ and $D=\{b,c\}$, then

$$\gamma^{\bigstar}\gamma(C)=\gamma^{\bigstar}(X)=X-\gamma(\emptyset)=X-\{a\}=\{b,c\}$$

and

$$\gamma\gamma^{\bigstar}(C)=\gamma(X-\gamma(\{b\}))=\gamma(X-\{a,b\})=\gamma(\{c\})=X\supset\gamma^{\bigstar}\gamma(C).$$

Also,

$$\gamma^{\bigstar}\gamma(D)=\gamma^{\bigstar}(X)=X-\gamma(\emptyset)=X-\{a\}=\{b,c\}$$

and

$$\gamma\gamma^{\bigstar}(D) = \gamma(X - \gamma(\{a\})) = \gamma(X - \{a,b\}) = \gamma(\{c\}) = X \supset \gamma^{\bigstar}\gamma(D)$$

and so C and D are $r\delta$-sets. But $C \cap D = \{c\}$ is not an $r\delta$-set. For,

$$\gamma^{\bigstar}\gamma(C \cap D) = \gamma^{\bigstar}(X) = X - \gamma(\emptyset) = X - \{a\} = \{b,c\}$$

and

$$\gamma\gamma^{\bigstar}(C \cap D) = \gamma(X - \gamma(\{a,b\})) = \gamma(X - X) = \gamma(\emptyset) = \{a\} \not\supset \gamma^{\bigstar}\gamma(C \cap D)$$

and so $C \cap D$ is not an $r\delta$-set.

COROLLARY 3.12. *Let (X,γ) be an envelope space and γ be subadditive. If $A = B \cup C$ where B is γ-regularopen and C is rc-nwdense, then A is an $r\delta - set$.*

PROOF. The proof follows from Theorem 3.1 and Theorem 3.10(b). ∎

A subset A of a monotonic space (X,γ) is said to be an *$r\delta^{\bigstar}$-set* if A is an *$r\delta$-set* and every subset of A is also an *$r\delta$-set*. Clearly, every rc-nwdense set is an *$r\delta^{\bigstar}$-set*. In fact, the following Theorem 3.13, the proof of which follows from Theorem 3.10(b), shows that the family of all *$r\delta^{\bigstar}$-sets* is an ideal in an envelope space (X,γ), if γ is subadditive.

THEOREM 3.13. *Let (X,γ) be an envelope space and γ be subadditive. Then the family of all $r\delta^{\bigstar}$-sets of X is an ideal.*

A subset A of a monotonic space (X,γ) is said to be *locally rc-closed* if $A = G \cap F$ where G is a γ-regularopen set and F is a γ-regularclosed set. If γ is a weak envelope, then X is γ-regularclosed. Therefore, every γ-regularopen set is a locally rc-closed set. Clearly, A is locally rc-closed if and only if $X - A$ is the union of a γ-regularopen set and a γ-regularclosed set. The following Theorem 3.14, gives characterizations of locally rc-closed sets.

THEOREM 3.14. *Let (X,γ) be an envelope space, γ be subadditive and $A \subset X$. Then the following are equivalent.*

(a) *A is locally rc-closed.*

(b) *$A = G \cap \gamma(A)$ for some γ-regularopen set G.*

(c) *$\gamma(A) - A$ is γ-regularclosed.*

(d) *$A \cup (X - \gamma(A))$ is γ-regularopen.*

PROOF. (a)⇒(b). Suppose $A = G \cap F$ where G is γ-regularopen and F is γ-regularclosed. $A = G \cap F$ implies that $A \subset F$ and so $\gamma(A) \subset \gamma(F) = F$.

By Theorem 2.27(a), $\gamma(A) = \gamma(G \cap F) \supset G \cap \gamma(F) = G \cap F = A$ and so $A \subset \gamma(A)$. Now, $A \subset \gamma(A)$ implies that

$$A = A \cap \gamma(A) = (G \cap F) \cap \gamma(A) = G \cap (F \cap \gamma(A)) = G \cap \gamma(A).$$

(b)⇒(c). Suppose that $A = G \cap \gamma(A)$ for some γ-regularopen set G. Now, $\gamma(A) - A = \gamma(A) \cap (X - A) = \gamma(A) \cap (X - (G \cap \gamma(A))) = \gamma(A) \cap ((X - G) \cup$ $\cup (X - \gamma(A))) = \gamma(A) \cap (X - G)$. By Lemma 2.15(a) and Theorem 2.28(a), $\gamma(A) \cap (X - G)$is γ-regularclosed. Hence $\gamma(A) - A$ is γ-regularclosed.

(c)⇒(d). $\gamma(A) - A$ is γ-regularclosed implies that $X - (\gamma(A) - A)$ is γ-regularopen which in turn implies that $X - (\gamma(A) \cap (X - A))$ is γ-regularopen and so $(X - \gamma(A)) \cup (X - (X - A))$ is γ-regularopen. Therefore, $(X - \gamma(A)) \cup A$ is γ-regularopen.

(d)⇒(a). $(A \cup (X - \gamma(A))) \cap \gamma(A) = (A \cap \gamma(A)) \cup ((X - \gamma(A)) \cap \gamma(A)) = A \cap$ $\cap \gamma(A) = A$, since $A \subset \gamma(A)$. Hence A is locally rc-closed. ∎

The following Example 3.15 shows that γ-regularopen sets and locally rc-closed sets are independent concepts. Theorem 3.16 below shows that for rc-dense sets, γ-regularopen sets and locally rc-closed sets coincide in an envelope space (X, γ) where γ is subadditive. Theorem 3.17 below gives a property of locally rc-closed sets.

EXAMPLE 3.15. Let $X = \{a, b, c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by

$$\gamma(\emptyset) = \gamma(\{a\}) = \gamma(\{b\}) = \gamma(\{c\}) = \gamma(\{a, c\}) = \gamma(\{b, c\}) = \{a\}$$

and

$$\gamma(\{a, b\}) = \gamma(X) = \{a, b\}.$$

Then $\gamma \in \Gamma$, $\xi = \{\{a\}, \{a, b\}\}$ and $\mu = \{\{c\}, \{b, c\}\}$. If $A = \{b\}$, then $A = \{a, b\} \cap \{b, c\}$ and so A is locally rc-closed but not γ-regularopen. If $B = \{b, c\}$, then B is γ-regularopen but cannot be written as the intersection of a γ-regularopen set and a γ-reularclosed set.

THEOREM 3.16. *Let (X, γ) be a weak envelope space and $A \subset X$. Then the following hold.*

(a) *Every γ-regularopen set is locally rc-closed.*

(b) *If $\gamma \in \Gamma_2$, γ is subadditive and A is an rc-dense subset of X which is also locally rc-closed, then A is γ-regularopen.*

PROOF. (a) The proof follows from the fact that if $\gamma \in \Gamma_+$, then X is γ-regularclosed.

(b) Suppose that A is both rc-dense and locally rc-closed. Then $A = G \cap \gamma(A)$ for some γ-regularopen set G (cf. 3.14(b)). Now, $A = G \cap \gamma(A)$ implies that $A = G \cap X = G$ and so A is a γ-regularopen set. ∎

THEOREM 3.17. *Let (X,γ) be a monotonic space and γ be subadditive. If G is γ-regularopen and A is locally rc-closed, then $A \cap G$ is also locally rc-closed.*

PROOF. A is locally rc-closed implies that $A = U \cap F$ where U is γ-regularopen and F is γ-regularclosed. $A \cap G = (U \cap F) \cap G = (U \cap G) \cap F$. Since γ is subadditive, $U \cap G$ is γ-regularopen, by Lemma 2.15(b). Hence $A \cap G$ is locally rc-closed. ∎

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