CONFERENCES

Student Research Circle conference
Eötvös Loránd University, November 30, 2007

Kristóf Bérczi: Packings and covering in directed graphs

Zalán Gyenis: On finite substructures of certain stable strutures

Dávid Kunszenti-Kovács: Flows in networks—a probabilistic approach

Marcella Takáts: Vandermonde sets and super Vandermonde sets

Ramón Horváth: On lifting up an eversion

Tamás Horváth: The number of solutions of a boundary value problem containing a singular nonlinearity

János Geleji: Summing count matroids

Gabriella Pluhár: The free spectrum of the variety of bands

M60. Miklós Laczkovich is Sixty
A miniconference in Real Analysis. 22–23 February, 2008

`http://www.cs.elte.hu/buczo/M60/M60.html`

L. Lovász: Opening remarks

Á. Császár: Some results of Miklós Laczkovich

There are some common results of M. Laczkovich and the author from the years between 1975 and 1980. Let (f_n) be a sequence of real valued functions defined on a set X. We say that a function f is the *discrete* limit of this sequence iff, for each $x \in X$, there is an index n_0 such that $f_n(x) = f(x)$ whenever $n > n_0$. f is the *equal* limit of the sequence iff there is a sequence of positive numbers ϵ_n such that, for each $x \in X$, there exists an index n_0 such that $|f_n(x) - f(x)| < \epsilon_n$ whenever $n > n_0$. The common papers discuss results on these sorts of convergence, in particular, they consider Baire classes where the usual convergence is replaced with discrete or equal convergence.

P. Humke: A little bit of this and a little bit of that

Z. Daróczy: On a family of functional equations

Let $I \subset \mathbb{R}$ be a non-empty open interval. We consider the following family of functional equations

$$f(px + (1-p)y)\left[r(1-q)g(y) - (1-r)qg(x)\right] = \\ = \mu\,[p(1-q)f(x)g(y) - (1-p)qf(y)g(x)],$$

where $(p, q, r) \in (0, 1)^3$ and $\mu \neq 0, 1$ are constants, $f, g : I \to \mathbb{R}_+$ are unknown functions and the equation holds for all $x, y \in I$. We give a review on some special cases of this equation depending on the four parameters which are already solved.

Sz. Révész: Integral concentration of idempotent trigonometric polynomials on small sets

L. Székelyhidi: Spectral synthesis on hypergroups

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. As translation has a natural meaning on commutative hypergroups, too, it seems reasonable to consider similar problems on these structures. In this talk we present recent results on spectral analysis and spectral synthesis over discrete Abelian groups. The first problem is to define the basic building blocks of spectral analysis and synthesis: the exponential monomials. The reasonable definition is not straightforward. In the talk we show how to manage this in accordance with the group case on polynomial hypergroups and indicate possible extensions for Sturm–Liouville hypergroups, as well. The next problem is to study spectral analysis and synthesis. We deal with these problems again on polynomial hypergroups and solve them in the positive.

M. Balcerzak: On the Laczkovich–Komjáth property concerning sequences of analytic sets

The talk is based on the paper "On the Laczkovich-Komjáth property of sigma-ideals" (joint with Szymon G⁄ląb) accepted for publication in the Topology and Its Applications. In 1977 Laczkovich proved that, for each sequence (A_n) of Borel subsets of a Polish apace X, if $\limsup_{n\in H} A_n$ is uncountable for every $H \in [\mathbb{N}]^\omega$ then $\bigcap_{n\in G} A_n$ is uncountable for some $G \in [\mathbb{N}]^\omega$. In 1984 this result was generalized by Komjáth to the case when the sets A_n are analytic. His theorem, by our definition, means that the σ-ideal $[X]^{\le\omega}$ has the Laczkovich-Komjáth property (in short (LK)). We prove that every σ-ideal generated by X/E has property (LK), for an equivalence relation $E \subset X^2$ of type F_σ with uncountably many equivalence classes. We also show the parametric version of this result and we study the invariance of property (LK) with respect to various operations.

Zs. Páles: The extension of Clarke's generalized derivative of real-valued locally Lipschitz functions to the Radon–Nikodym space-valued setting

Locally Lipschitz functions acting between infinite dimensional normed spaces are considered. Clarke's generalized Jacobian is extended to the setting when the range is a dual space and satisfies the Radon–Nikodým property. Characterization and fundamental properties of this extended generalized Jacobian are established including nonemptiness, compactness and upper semicontinuity with respect to a relevant topology, and a mean-value theorem. Connection to known notions of differentiation and chain rules are provided. The generalized Jacobian introduced is shown to enjoy all the properties required of a derivative like-set.

P. Holický: Decompositions of Borel bimeasurable mappings

We stated the nonseparable analogue of the classical results by Luzin, Novikov and Purves on the decomposition of Borel measurable mappings which map Borel sets to Borel sets. We demonstrated the method of proof on the classical result on countable decompositions of Borel bimeasurable mappings between Polish spaces, which reads as follows:

Let $X,\ Y$ be Polish spaces, $f : X \to Y$ be a Borel measurable mapping. Then the following statements are equivalent:

(a) $f(Borel(X)) \subset Borel(Y)$;

(b) $f(G_\delta(X)) \subset Borel(Y)$;

(c) there are Borel subsets $X_0, X_1, \ldots$ of X such that $\bigcup_{i=0}^{\infty} X_i = X, f(X_0)$ is countable, and $f \restriction X_n$, $n = 1, 2, \ldots$, are injective.

J. Lindenstrauss, D. Preiss, J. Tišer: Fréchet differentiability of Lipschitz functions via a variational principle

In the talk (presented by the third author) we indicated how a new variational principle which in particular does not assume the completeness of the domain, can give a new, more natural, proof of the fact that a real valued Lipschitz function on an Asplund space has points of Fréchet differentiability. In more details, as an illustration, we showed that everywhere Gâteaux differentiable Lipschitz function on a a space with separable dual is somewhere Fréchet differentiable.

M. Csörnyei: Lipschitz image of sets of positive measure

We present two constructions to Laczkovich's problem about Lipschitz mappings of planar sets of positive measures onto balls. The first one uses discrete techniques, the second one (due to Khrushchev) is based on complex analytic methods and the Hahn-Banach theorem. The higher dimensional problem remains open.

B. Kirchheim: Convexity notions in the Calculus of Variations

42. Jahrestagung der Gesellschaft für Didaktik der Mathematik 13.–18. März 2008 BUDAPEST

Kurzfassungen

A) HAUPTVORTRÄGE

COHORS-FRESENBORG, Elmar: Mechanismen von Metakognition und Diskursivität im Mathematikunterricht

Es soll die Bedeutung von metakognitiven und diskursiven Aktivitäten von Lehrenden und Lernenden für die Qualität von Mathematikunterricht herausgearbeitet werden. Dazu werden die Konstrukte ”Metakognition” und ”Diskursivität” im Hinblick auf ihre Bedeutung für das Lehren und Lernen von Mathematik dekomponiert und ein Kategoriensystem zur Unterrichtsanalyse vorgestellt.
Im Einzelnen wird über den Effekt von metakognitiven Aktivitäten für den Lernerfolg und die Problemlösekompetenz berichtet und herausgearbeitet, inwieweit Diskursivität als Instrument geeignet ist, inhaltliche Klarheit (bzw. Unklarheit) aufzudecken. Anhand von videographierten Unterrichtsszenen wird exemplarisch dargelegt, wie Wirkmechanismen von Metakognition und Diskursivität funktionieren. Schließlich wird dargelegt, welche Rolle die Kategorisierung von Unterrichtstranskripten nach metakognitiven und diskursiven Aktivitäten bei der Lehreraus- und -weiterbildung sowie der Qualitätsanalyse von Mathematikunterricht spielen kann.

HERBER, Hans-Jörg: Psychologische Hintergrundsparadigmen von Innerer Differenzierung und Individualisierung

Wissenschaftlich fundierter Schulunterricht muss sich - wie die aktuelle Bildungsdiskussion zeigt - zunehmend mehr der individuellen Lernvoraussetzungen der Schüler in lern-, motivations- und entwicklungspsychologischer Hinsicht annehmen. Unter der Annahme der Abhängigkeit des schulischen Lernverhaltens von solchen Bedingungen, kann die Optimierung der Lehrer-Schüler-Interaktion durch rationale Analyse der relevanten Bedingungszusammenhänge und deren praktische Berücksichtigung verbessert werden: In unserem Begriffsverständnis von schülergerechtem Unterricht soll durch solcherart fundierte Maßnahmen der Inneren Differenzierung und Individualisierung dem heranwachsenden Menschen gemäß seiner je individuellen kognitiven und emotional-motivationalen Entwicklungsvoraussetzungen eine pädagogische Hilfestellung angeboten werden, durch die er seine Kompetenzen (intellektuelle Fähigkeiten, sachbezogne und soziale Motivationen, etc.) optimal entfalten kann. Kurz gesagt: Schulischer Unterricht soll

nach Möglichkeit die Selbstbildungsprozesse des Individuums behutsam unterstützen und vor allem nicht behindern. Dies erfordert einen strukturierten Lernraum, in dem wechselseitiges Vertrauen herrscht und selbständiges sowie kooperatives Lernen möglich ist. Ziel ist die Bildung von selbstverantwortlichen, lernmotivierten, autonomen Menschen mit hoher sozialer Kompetenz. Das über jahrzehntelange Forschung entwickelte Grundmodell der Inneren Differenzierung und Individualisierung (z.B. Herber & Vásárhelyi 2002) stützt sich auf die wichtigsten Theoreme zeitgemäßer psychologischer Hintergrundsparadigmen individueller und sozialer Lernprozesse und entsprechende Feldforschung im Zusammenhang schulischen Lernens.
Im aktuellen Vortrag werden durch prototypische Schlaglichter die wichtigsten Argumente für Innere Differenzierung und Individualisierung – theorienbezogen und empirisch gestützt – zusammengefasst und kritisch diskutiert.

Literatur: Herber, H.-J. & Vásárhelyi, É. (2002). Das Unterrichtsmodell ”Innere Differenzierung einschließlich Analogiebildung” - Aspekte einer empirisch veranlassten Modellentwicklung. Salzburger Beiträge zur Erziehungswissenschaft 6, Heft 2, 5-19

LOVÁSZ, László: Trends in Mathematics, and how they Change Education

Mathematical activity has changed a lot in the last 50 years. Some of these changes, like the use of computers, are very visible and are being implemented in mathematical education quite extensively. There are other, more subtle trends that may not be so obvious. We discuss some of these trends and how they could, or should, influence the future of mathematical education.

MEVARECH, Zemira R.: Why teaching facts is just not enough? The effects of meta-cognitive instruction on mathematics achievement

No child left behind is one of the most challenging issues of the 21st century. The fact that all children attend schools and the rate of dropout is quite low, raises the question of how to provide effective education to ALL: lower and higher achievers, LD as well as gifted children, and of course, “ordinary” children.
The challenge of ”no child left behind” is particularly applicable to mathematics education because on one hand a large proportion of school time is devoted to the studying of mathematics, and on the other hand it is considered to be one of the most difficult subjects taught in school.
Along the developments in the theoretical and empirical studies of cognition

and meta-cognition, major changes have been suggested also in intervention programs attempting to enhance mathematics reasoning via metacognitive guidance. The first intervention programs were based on meta-memory and the explicit teaching of facts, strategies, and algorithms. Although these methods have many advantages, mainly with regard to the easiness of its implementation in classes with a large number of students, recent studies have started to question its effectiveness. These findings raise three basic research questions: first, how to transform recent meta-cognitive theories into effective instructional methods? Second, who benefits from this kind of innovative instructional methods? And finally, at what age this kind of teaching methods are needed? The present presentation focuses on these issues with regard to mathematics education.
The presentation includes four parts:
(a) Metacognitive Framework - an overview and rationale;
(b) IMPROVE - an effective metacognitive teaching method in which no child left behind;
(c) Results of experimental and quasi-experimental studies showing the impact of IMPROVE on various measurements of mathematics reasoning and meta-cognitive skills of students at different age groups, and
(d) Metacognitive instructional methods - restructuring mathematics education.
The theoretical and practical implications of these studies will be discussed at the conference.

PLÉH, Csaba: Two traditions and two strategies of cognitive science

The talk shall outline a formal and a more content oriented strategy of cognitive science. During the late 19th century these two strategies were first outlined by Wilhelm Wundt and Gottlob Frege as the sensualistic and the propositional theory of thought processes. Frege in this regard treated his propositions as Platonic entities, thus denying their reality in individual minds.
The second half of the twentieth century can be seen as a renewal of Frege where propositions are treated as actual characterizations of human thought process. This lead to the victorious computational theories of modern cognitive science illustrated by names like Noam Chomsky and David Marr.
Not only computers but humans were to be subjected to the Turing test. The last decades of twentieth century however realized that propositions allocated to individual minds must have an origin in themselves too. This has lead to different levels of the Turing test and to present day neural network and evolution anchored theories of cognition.

ZIMMERMANN, Bernd: György Pólya, 1887-1985 – Zur Biographie, zum Lebenswerk und zu seiner Wirkung auf die Mathematikdidaktik

György Pólya gehört zweifellos zu den bedeutendsten Persönlichkeiten, die bis heute einen sehr starken Einfluss auf die internationale Diskussion über Mathematikunterricht haben. Pólyas Weg zur Mathematik war keineswegs gradlinig und zeugt von einem vielseitigem Talent und Engagement. Schon mit Beginn seiner beruflichen Tätigkeit in Mathematik befasste er sich auch mit Fragen des Unterrichtens von Mathematik. Seine Arbeiten in der Mathematik reichen von der Analysis über die Zahlentheorie und Geometrie bis zur Wahrscheinlichkeitsrechnung und Kombinatorik. Auch hierin erkennt man z. T. schon sein Interesse an Methoden des Entdeckens und des Lösens von Problemen. Hierauf konzentrieren sich seine mathematikdidaktischen Arbeiten, insbesondere seine unübertroffenen Werke zum mathematischen Problemlösen. Schließlich werden ein Ausblick auf die heutige Situation des Mathematikunterrichts insbesondere in Deutschland gegeben sowie mögliche oder wünschenswerte Wirkungen der Ideen von Pólya präsentiert.

First ELTE–NUS Science Forum between the Faculty of Science of Eötvös Loránd University and National University of Singapore Budapest, 21–22 May 2008 Mathematics Session

Chengbo Zhu:
Local theta correspondence: introduction and recent results

We give a brief introduction to local theta correspondence, which links representations of certain classical groups. We then explain some of its recent development as well as applications to the theory of invariant distributions.

István Ágoston:
On homological properties of quasi-hereditary algebras

Quasi-hereditary algebras were introduced in the late 1980's to deal with certain problems arising in the representation theory of complex semisimple Lie algebras and algebraic groups. A number of 'universality' results shows their importance also within the class of associative algebras. For example, every finite dimensional associative algebra is the endomorphism algebra of a projective module over a suitable quasi-hereditary algebra. The recursive construction of quasi-hereditary and more generally, of standardly stratified algebras makes it possible to give an explicit bound on the finitistic dimension of standardly stratified algebras. Results about the quasi-heredity of the Koszul dual of a quasi-hereditary algebra were also presented.

Hung Yean Loke: The smallest representation of non-linear covers of odd orthogonal groups

In this talk, I will first explain and motivate the definition of small representations of real reductive Lie groups. Then I will describe the construction of the smallest representation of the indefinite orthogonal groups. The latter is a joint work with Gordan Savin.

László Verhóczki: Cohomogenenity one isometric actions on compact symmetric spaces of type E_6/K

As is well-known, the exceptional compact Lie group E_6 has four symmetric subgroups up to isomorphisms. In this talk we discuss the four Riemannian symmetric spaces of type E_6/K and show that three of them admit a cohomogeneity one isometric action with a totally geodesic singular orbit. This implies that these symmetric spaces can be thought of as compact tubes. We describe the shape operators and the volumes of the principal orbits of the considered isometric actions. Hence, we obtain a simple method to compute the volumes of these exceptional symmetric spaces.

Zoltán Buczolich:
Pointwise convergence and divergence of ergodic averages

We discuss almost everywhere convergence results concerning the non-conventional ergodic averages

(∗) $\frac{1}{N}\sum_{k=1}^{N} f(T^{n_k}x)$ as $N \to \infty$. Motivated by questions of A. Bellow and H. Furstenberg, J. Bourgain showed that for the sequence $n_k = k^2$ for $f \in L^p$, $p > 1$ the averages (∗) converge μ almost everywhere and he raised the question of almost everywhere convergence of (∗) for $f \in L^1$. In a joint paper with D. Mauldin we showed that there are $f \in L^1$ for which the averages (∗) along the squares do not converge μ almost everywhere. Answering another related well-known problem I have managed to construct a sequence n_k with gaps $n_{k+1} - n_k \to \infty$ for which for any ergodic dynamical system (X, Σ, μ, T) and $f \in L^1(\mu)$ the averages (∗) converge μ almost everywhere to the integral of f and this result disproves a conjecture of J. Rosenblatt and M. Wierdl.

Péter Komjáth: Paradoxical decompositions of Euclidean spaces

Ferenc Izsák: Error estimations in the numerical solutions of the Maxwell equations

An a posteriori error estimation technique was presented for the (finite element) numerical solution of the time harmonic Maxwell equations. One can prove that the estimate is a lower bound of the exact error and we exhibited a strong correlation with this as shown in some numerical experiments.

Kwok Pui Choi:
Asymptotics of the average of functions of order stateistics

László Márkus:
On the extremes of nonlinear time series models describing river flows

Jialiang Li, Xiao-Hua Andrew Zhou: Nonparametric and semiparametric estimations of the three way receiver operating characteristic surface

Tamás Király: An introduction to iterative relaxation

Iterative relaxation is a new method for obtaining approximate solutions to combinatorial optimization problems with degree constraints, when the corresponding feasibility problem is already NP-complete. The method allows a slight violation of the degree constraints, and finds a solution of this relaxation that has small cost. A prime example of this approach is the Minimum Bounded Degree Spanning Tree problem, where we have upper (and possibly lower) bounds on the degree of the spanning tree at each node. Singh and Lau showed that if the value of the optimal solution is OPT, then an iterative relaxation algorithm can find a spanning tree of cost at most OPT that violates the degree bounds by at most 1. In this talk we show how this technique can be extended to problems involving arbitrary matroids.

SOME ASPECTS OF GEOMETRY THROUGH THE SCHOOL YEARS

By

RICHARD ASKEY

(Received November 30, 2008)

Abstract. We will focus on some aspects of triangles and some quadrilaterals which are both of interest for their own sake and because of the ideas which are involved. We start with early primary school and get to some results which in the United States have disappeared from school geometry and are not known by most high school teachers.

After students learn what a triangle is, a very important property should be introduced: a triangle is rigid. This can be illustrated with fingers, and contrasted with the fact that a quadrilateral is not rigid. By this I mean that when the sides are given, a triangle is determined, but a quadrilateral is not. Children like to show this with their fingers.

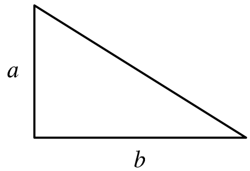
After students learn that a triangle is determined by knowing its sides, by knowing two sides and the included angle, and by knowing two angles and the included side, it is time to show that the sum of the angles of a triangle is two right angles, or 180°. First, students should learn how to form a right angle by folding paper, and then continue folding to make a rectangle so they develop a feel for right angles and learn that a rectangle has four right angles, and its angles add to 360°.

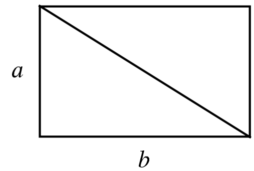
One way to study general triangles is to start with right triangles.

For a right triangle,

one can use it to form a rectangle by drawing lines perpendicular to the shorter sides of the triangle.

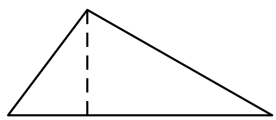
Plenary talk of Tamás Varga Conference on Mathematics Education held at ELTE, Budapest, November 7–8, 2008.





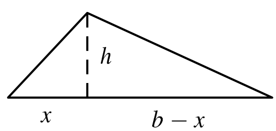
The two triangles are congruent since they have sides of the same length, so we know that they have the same area and the sum of the angles is the same. Since the area of the rectangle is ab, and the sum of the angles is 360°, the original right triangle has area ab/2 and the sum of its angles is 180°.

Take a general triangle, first with the longest side used as the base, and decompose it into two right triangles by dropping a perpendicular to the longest side from the opposite vertex.



The previous result for right triangles gives $180° + 180°$ for the two right triangles. Subtract 180° for the two right angles to get 180° for the sum of the angles in the triangle.

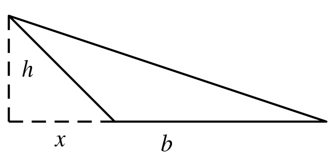
For the area, a different argument completes the derivation of a formula for the area of a triangle.



The area of the triangle is $\frac{xh}{2} + \frac{(b-x)h}{2} = \frac{bh}{2}$.

There is another case where a similar argument works.

The area of the triangle comes from $\frac{(x+b)h}{2} - \frac{xh}{2} = \frac{bh}{2}$.



There are many ways to prove the Pythagorean theorem. Here is the URL to a beautiful way:

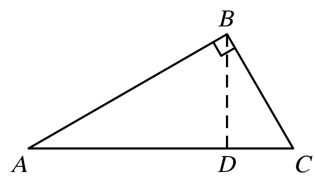
http://math.berkeley.edu/∼giventh/papers/eu.pdf

This article was written for mathematicians, but if you skip the parts which use words you do not know, the essence is still there.

Here is a sketch.

The Pythagorean theorem says that the area of the square drawn on the hypotenuse is equal to the sum of the areas of squares drawn on the other sides of a right triangle. Squares are similar, and a little thought along with the knowledge that areas of similar figures scale as the square of the corresponding sides will be used.

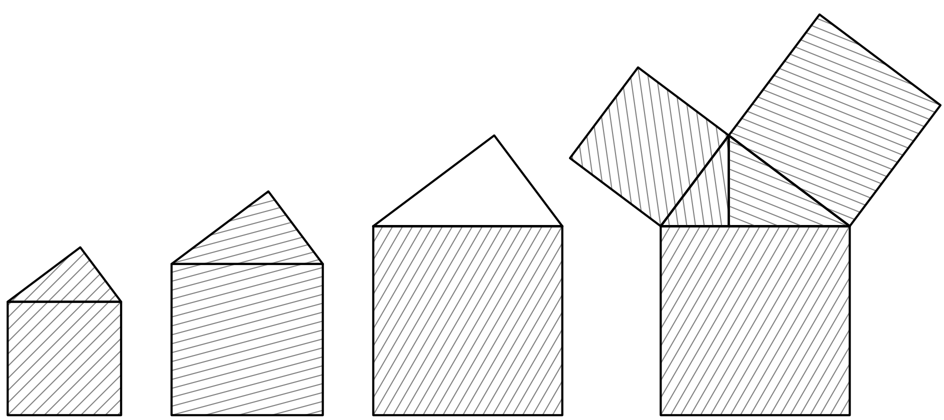
This is true in general when side is interpreted appropriately, but we just use this for triangles and squares so side means what you think it does). Thus it suffices to find similar figures placed on the sides of a right triangle so that their areas add as they should. Here is the picture.



Triangle ABC is a right triangle and it is similar to triangles ADB and BDC. Clearly the area of triangle ADB plus the area of triangle BDC is the area of triangle ABC.

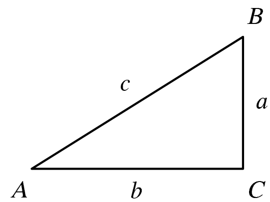
The area of triangle ABC is a constant k times the area of the square on AC, and this is true with the same constant k for the other two triangles and the corresponding squares of AB and BC.

Thus $k|AC|^2 = k|AB|^2 + k|BC|^2$ and this is the Pythagorean theorem.



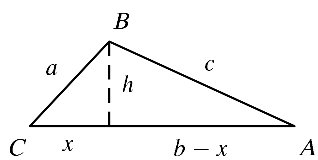
With some knowledge of similarity one can do much more, including introducing trigonometric functions.

$$\sin(A) = \frac{a}{c} \qquad \cos(A) = \frac{b}{c}$$



There is an extension of the Pythagorean theorem due to Euclid. It holds for a general triangle, but following Euclid we only give a proof in one of the two cases and leave the other one to you.

Drop a perpendicular from B to AC and use the



Pythagorean theorem twice.

$$a^2 = h^2 + x^2, \qquad c^2 = h^2 + (b - x)^2$$

Subtract and do a little algebra to get Euclid's version:

$$c^2 = a^2 - x^2 + b^2 + x^2 - 2bx, \qquad c^2 = a^2 + b^2 - 2bx$$

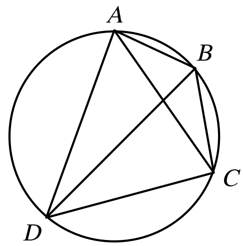
We now rewrite this as $c^2 = a^2 + b^2 - 2ab\cos(C)$.

This is usually called the law of cosines or the cosine rule.

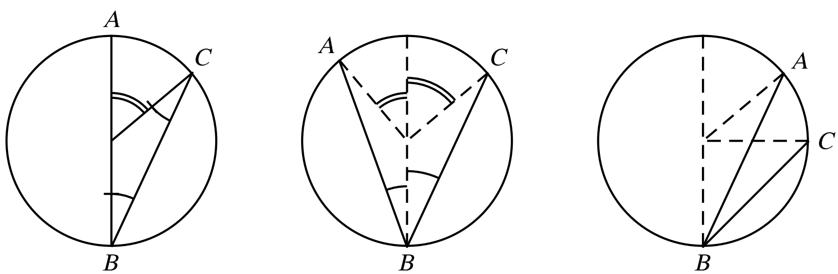
For the Greeks, trigonometry did not deal with right triangles, but with chords in circles. This was because of their serious interest in astronomy. When this was developed enough, they had to construct tables of lengths of chords for different angles. About 150 AD (or CE depending on your preference and possibly age), Ptolemy found a beautiful theorem which allowed him to construct such a table. Rather than give Ptolemy's proof, I will give one which was likely the proof found by the Indian mathematician

Brahmagupta [1] before 630. However, he did not give a proof, he just stated the result.

Given a circle, mark four points A, B, C, D on it and connect them with line segments.



AB, BC, CD, DA are the sides of this quadrilateral and AC and BD are diagonals. Quadrilaterals whose vertices lie on a circle are called cyclic quadrilaterals. Like triangles, they are rigid.



A condition which tells if a quadrilateral is cyclic was given in Euclid, the angles DAC and BCD add to 180°, and since a quadrilateral has 360° in its angles, the other angles ABC and CDA also add to 180°. Here is a proof without words that the inscribed angle ABC is half of the central angle AOC.

Given a cyclic quadrilateral as above, one should be able to find the length of the diagonal AC in terms of the lengths of the sides of the quadrilateral. Here is one way to do this. Use the cosine rule twice. To simplify the typing and reading, we will use $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$, $|AC| = x$, $|BD| = y$.

$$x^2 = a^2 + b^2 - 2ab\cos(B) =$$

$$= c^2 + d^2 - 2cd\cos(D) =$$

$$= c^2 + d^2 + 2cd\cos(B) =$$

since $B + D = 180^\circ$ and $\cos(180^\circ - B) = -\cos(B)$.

Multiply the first equation by cd, the third by ab and add them together. The result is

$$(ab+cd)x^2 = cd(a^2+b^2)+ab(c^2+d^2).$$

The left side is nice, but the right hand side is not. The mixture of squares and first powers makes this side less attractive, so let us fix it by taking each squared term, split it into two linear factors and put one with each of the other two factors. Thus $cd(a^2)$ becomes $(ac)(ad)$ which looks nicer.

When this is done with each term, the right hand side becomes

$$(ac)(ad)+(bc)(bd)+(ac)(bc)+(ad)(bd)$$

Put the first and third terms together by factoring out $(ac)$and factor (bd) from the other two terms. This gives

$$(ac)[(ad)+(bc)]+(bd)[(ad)+(bc)] = [(ac)+(bd)][(ad)+(bc)]$$

so

$$x^2 = \frac{[ac+bd][ad+bc]}{ab+cd}$$

By symmetry,

$$y^2 = \frac{[ac+bd][ab+cd]}{ad+bc}$$

Multiply these together and take a square root to get

$$xy = ac+bd,$$

which is Ptolemy's theorem. Notice the importance of factoring. Without it, the cancellation which was immediate would have been hard to see, and might have been missed.

Similarly, divide the expressions for x^2 and y^2 to get

$$\frac{x}{y} = \frac{ad+bc}{ab+cd}.$$

There are many things which can be done after getting Ptolemy's theorem. One which is little known is to find other parts of a cyclic quadrilateral. What parts, you ask? Brahmagupta tells us what else can be found.

If you call the intersection of the diagonals P, then the lengths of AP, BP, CP, and DP can be found. He wrote that these can be found by proportion, and they can. He also mentioned the needles. These are formed by extending the sides of the quadrilateral until they meet. He does not seem to mention the third diagonal, the line segment connecting the terminal points of the two needles. That can also be found.

There are two books on trigonometry published in England, one by Hobson [5] and one by Durell and Robson [2], which contain the formula for the length of the third diagonal, but both proofs are more complicated than necessary.

Here is a trigonometric reformulation of Ptolemy's theorem. Take a circle of diameter 1 to simplify formulas. Mark the angles as follows

$$CAB = t, \quad ABD = u, \quad CBD = v, \quad ACB = w.$$

From the inscribed angle result mentioned above, angle CAB is the same size as an angle cutting off the chord BC with the diameter as one side so $\sin t = |BC|$. Doing this for not only the angles above, but all of the angles between diagonals and sides, and sides and sides, Ptolemy's theorem can be given as

$$\sin(v + t)\sin(u + v) = \sin(t)\sin(u) + \sin(v)\sin(w).$$

There seem to be four variables, but there are really only three since $t + u + v + w = 180^\circ$. Using this and $\sin(180^\circ - x) = \sin(x)$, Ptolemy's theorem is equivalent to

$$\sin(v + t)\sin(u + v) = \sin(t)\sin(u) + \sin(v)\sin(t + u + v).$$

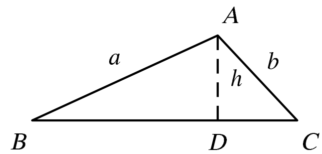
When $u + v = 90^\circ$, which is the same as having a diagonal be a diameter of the circle, this reduces to the well known addition formula

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Other choices will give other versions of addition formulas. Thus it is not surprising that Ptolemy could use his theorem to compute the lengths of chords in a circle when an inscribed or central angle is given.

Ptolemy's theorem contains the addition formulas, so it is natural to ask if the addition formulas can be used to prove Ptolemy's theorem. When I taught a course which was used primarily to help prospective high school teachers learn the mathematics they would teach, I would give this problem after deriving the addition formulas. Here is the way I would derive the addition formula for $\sin(x + y)$.

Draw the same picture we have used so often, this time for a general triangle.



Twice the area of triangle BAC is $ab\sin(BAC)$ and is also

$$ah\sin(BAD) + bh\sin(CAD).$$

Replace h by $b\cos(CAD)$ and by $a\cos(BAD)$.

$$ab\sin(BAC)ab\sin(BAD)\cos(CAD) + ab\cos(BAD)\sin(CAD)$$

The result is the addition formula for $\sin(x + y)$ which was given above. It is amazing how many important results can be obtained from this simple picture.

When students tried to use the addition formulas to prove the trigonometric identity given above as equivalent to Ptolemy's theorem, they would all start by expanding $\sin(t + u + v)$ and a few of them were able to get the result after about three pages of calculations. However, there is a much easier way.

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Subtract the second from the first to get

$$2\sin(x)\sin(y) = \cos(x - y) - \cos(x + y).$$

Use this in the three terms in the identity you want to prove and the next line completes the proof.

I told students that there are some things in mathematics which are of primary importance, and these need to be known backwards and forwards and in disguised forms. The addition formulas for trigonometric functions are examples. Then there are results of secondary importance. These you do not have to have in your working memory, but you have to know that something like this is true and be able to derive it easily from the results of primary importance. The linearization result above is such an example. Then there are results of lesser importance. These you might find easier to look up than to derive.

I recommend reading and working through Ptolemy's proof. You can find it on the web.

Here is one URL: http://www.cut-the-knot.org/proofs/ptolemy.shtml

This proof is also included in a great geometry text with a very interesting publication history. The most recent version is Kiselev's Geometry which was translated and edited by Alexander Givental [4]. He set up his own publishing firm in California with the most interesting name: Sumizdat. I highly recommend the book.

Here is an outline of one more proof of Ptolemy's theorem.

Take a circle with radius 1 and center at $(0,0)$. Then take a line with slope t which passes through the point $(-1,0)$. Find the point where these two curves intersect. The coordinates of this point are $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. The other point of intersection is $(-1,0)$, which students find surprising until they think a little and see why this has to be true. We are interested in the first of these two points.

Do the same thing with a line of slope s. The distance between these two points is

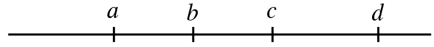
$$\frac{2|t-s|}{\left(1+t^2\right)^{1/2} \cdot \left(1+s^2\right)^{1/2}}.$$

Now compute the distances between each of the four points. There are six of them, since the lines connecting each pair of points is either a side or a diagonal of a cyclic quadrilateral.

Does this give a proof of Ptolemy's theorem? Of course, since I would not be writing about it if that did not happen. The question is what is needed to show that this gives such a proof. The answer is surprising.

Consider a very large circle and draw a tangent line to it. Place four points on the circle so that they are relatively close to the point of tangency. Then let the radius increase to infinity and have the points approach the tangent line. Here is what Ptolemy's theorem becomes on the tangent line.

$$(c-a)(d-b) = (d-c)(b-a) + (d-a)(c-b)$$



This identity is easy to prove. The right hand side can be expanded to give

$$db - da - cb + ca + dc - db - ac + ab$$

and some terms cancel to give $dc - da - cb + ab$ which factors as $(c-a)(d-b)$.

The distance between two points on the circle given above when used for all six distances reduces to the identity just proven since the same square roots occur for each product and the absolute values are removed by the order of the slopes.

There is another use of the representation of the interesting point of intersection of the line and the circle. Since this point is on the unit circle, it satisfies $x^2 + y^2 = 1$ where (x, y) is the point $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ in question. Take the values of x and y given above in terms of t and let $t = p/q$. A little algebra gives

$$(q^2 - p^2)^2 + (2pq)^2 = (q^2 + p^2)^2.$$

When $q = 2$, $p = 1$, this gives $3^2 + 4^2 = 5^2$; $q = 3$, $p = 2$ gives $5^2 + 12^2 = 13^2$, etc.

Of course there is a lot more. There is an extension of Ptolemy's theorem to a general quadrilateral. Most of the references to extensions are weak extensions; the equality becomes an inequality when the quadrilateral is not cyclic. However, there is an identity with a new term.

One nice way to derive it is to copy Ptolemy's proof as far as you can, and then complete the proof with the cosine rule as used in the first proof above. See Hobson [5] for details.

Here is a nice problem which is easy with Ptolemy's theorem. Inscribe an equilateral triangle in a circle. Take another point on the circle and connect it to the three vertices of the triangle. Show that the sum of the two shorter line segments is equal to the length of the longest segment.

There are other comments on Ptolemy's theorem [2].

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References

[1] H. T. COLEBROOKE: *Algebra: With Arithmetic and Mensuration, From the Sanskrit,* London, 1817, reprinted, Kessinger

[2] T. CRILLY and C. FLETCHER: *Ptolemy's Theorem, Its Parent and Offspring,* in [6], 42–49

[3] C. V. DURELL and A. ROBSON: *Advanced Trigonometry,* G. Bell, London, 1930, reprinted Dover, 2003

[4] A. GIVENTAL: *Kiselev's Geometry: Book 1, Planimetry,* Sumizdat, 2006

[5] E. W. HOBSON: *A Treatise on Plane Trigonometry,* Cambridge, 1891, 7th edition reprinted by Dover in 2005 as A Treatise on Plane and Advanced Trigonometry

[6] C. PRITCHARD: *The Changing Shape of Geometry,* Cambridge, 2003

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Eötvös Loránd University, November 28, 2008

Péter Maga: Generalized number theoretic functions

We define the number theoretic functions over an arbitrary unique factorization domain and study some important functions. We show that the number theoretic functions form a ring that is isomorphic to a certain power series ring. Finally, we prove that unique factorization holds in the ring of number theoretic functions.

Roland Paulin: Stability constants

Dániel Soukup: Planar topologies using the idea of the Sorgenfrey line

My primary goal was to generalize the convergence idea of the Sorgenfrey-line to the plane and investigate these new topologies. For every euclidean closed $S \subseteq S^1$ we define an $\mathbb{R}^2_S$ topology. In our spaces a sequence (x_n) converges to x iff (x_n) converges to x from directions which are in S. The paper deals with the characterization of the properties of $\mathbb{R}^2_S$ with the defining $S \subseteq S^1$ subsets. We examine such properties as countability, separability, compact subspaces and connectedness. Several interesting questions remained unanswered, such as: how many different topologies we get this way?

Péter Maga: Covers and dimension in infinite profinite groups

Answering a question of Miklós Abért we prove that an infinite profinite group cannot be the union of less than continuum many translates of a compact subset of box dimension less than 1. Furthermore, we show that in any infinite profinite group there exists a compact subset C of Hausdorff dimension 0 such that it is consistent with the axioms of set theory that less than continuum many translates of C cover the group.

Dániel Dobos: Sylow p-subgroups never intersect in a chainlike way

Call a finite group G *p-chainlike* if $\left|Syl_p(G)\right| \geq 3$ and the following two conditions hold for its Sylow p-subgroups $P_1, \ldots, P_\ell$:

$$\left|P_i \cap P_j\right| > 1 \Leftrightarrow i \equiv j \pm 1 \pmod{\ell} \quad \text{and} \quad O_p(G) = \bigcap_{i=1}^{\ell} P_i = e$$

(a suitable labelling is chosen).
We prove that actually p-chainlike groups *do not exist*.

György Hermann: Small lattice simplices in dimension d

Tamás Hubai: Competitive rectangle filling

Two players take alternating turns filling an $n \times m$ rectangular board with unit squares. Each square has to be aligned parallel to the board edges, but may otherwise be arbitrary. In particular, they are not forced to have integer coordinates. Squares may not overlap and the game ends when there is no space for the next one.

The result of the game is the area filled, or equivalently, the number of turns in the game. The *constructor* aims to maximize this quantity while the *destructor* wants to minimize it. We would like to determine this value, at least asymptotically, provided that both players use their optimal strategy.

For the case $n = 1$, which corresponds to the one dimensional variant of the problem, we show that about $\frac{3}{4}$ of the interval can be filled, which is exact if it evaluates to an integer. With a different and more complicated approach we are also able to determine the exact value for the case $n = 2$ (and large m), where we obtain that $\frac{9}{16}$ of the area gets covered. This result coincides with our conjecture for the general case when n and m are even or tend to infinity.

We also prove tight bounds for arbitrary n and m by specifying actual strategies for both players. Finally we look at some multi-dimensional and discrete variants, making some observations that lead to a conjecture for the product of such filling games.