PH. D. THESIS
A CONSTRUCTIVE APPROACH TO MATCHING AND ITS GENERALIZATIONS

By

GYULA PAP

ADVISOR: ANDRAS FRANK

(Defended DÁTUM)

This paper is a summary of results presented in the author's Ph.D. thesis "A constructive approach to matching and its generalizations", Eötvös University, Budapest, 2007, written under the supervision of András Frank.

1. Matching in graphs

In the first chapter of the thesis we discuss some matching problems in graphs, including non-bipartite matching, square-free 2-factors, path-matching, and even factors. Our approach is slightly different from Edmonds' [5] well-known method of alternating forests and blossoms. The point is that our approach is easier to generalize to those more general problems, thus we obtain simpler proofs and algorithms than known before, and we also obtain some new results.

1.1. Restricted b-matching in bipartite graphs

Consider a simple bipartite graph $G = (A, B; E)$ and let $b \in \mathbb{N}^{A \cup B}$. A subset $M \subseteq E$ of edges is called a *b-matching* if it satisfies $\delta_M(v) \leq b(v)$ for all nodes $v \in A \cup B$. (Here $\delta_M(v)$ denotes the number of edges in M incident with a node v.) Furthermore, we are given a family $\mathcal{K}$ of some complete bipartite subgraphs of G, which will be considered as "forbidden". A b-matching is called $\mathcal{K}$*-free* if it does not contain every edge of any member of $\mathcal{K}$. Suppose that for every $K \in \mathcal{K}$ we have $|A \cap V(K)| \geq 2$, $|B \cap V(K)| \geq$

≥ 2, $b(v) = |B \cap V(K)|$ for all $v \in A \cap V(K)$, and $b(v) = |A \cap V(K)|$ for all $v \in B \cap V(K)$.

THEOREM 1.1. (Pap, [26], [27]) *If $G, b, \mathcal{K}$ satisfies the above assumptions, then the maximum cardinality of a $\mathcal{K}$-free b-matching is equal to*

$$\min_{Z \subseteq A \cup B} b(A \cup B - Z) + |E[Z]| - c_{\mathcal{K}}(G[Z]), \tag{1}$$

where $c_{\mathcal{K}}(G[Z])$ denotes the number of those components of induced subgraph $G[Z]$ which are members of $\mathcal{K}$. Moreover, a maximum $\mathcal{K}$-free b-matching can be constructed in polynomial time.

The formula generalizes a result of Frank [6] on K_{tt}-free t-matching, and results of Hartvigsen [10] and Z. Király [13] on square-free 2-factor. The algorithm generalizes Hartvigsen's algorithm, and is conceptually simpler than that.

1.2. Even factors

Consider a directed graph $D = (V, A)$. A *path/cycle* is the arcset of a unclosed/closed walk without repetition of nodes. An arc of a digraph is called *symmetric* if its reverse is in the arcset of the digraph, too. A cycle is called symmetric if all of its arcs are symmetric. D is called *odd-cycle-symmetric* if all of its odd cycles are symmetric. An *even factor* is the node-disjoint union of paths and even cycles. A *source-component* of a digraph is a strongly connected component of in-degree zero.

THEOREM 1.2. (Pap, Szegő, [19]) *If $D = (V, A)$ is odd-cycle-symmetric, then the maximum cardinality of an even factor is equal to*

$$\min_{Z \subseteq V} |V| + |\Gamma_D^+(Z)| - \sigma_{odd}(D[Z]), \tag{2}$$

where $\sigma_{odd}(D[Z])$ denotes the number of source-components of $D[Z]$ on an odd number of nodes.

Our original proof is non-constructive. In the dissertation (and in [21], [25]), we provide a simpler proof, which also implies a polynomial time algorithm that is conceptually simpler than that of Cunningham, Geelen [4]. Our result is sloghtly more general, since Cunningham and Geelen only claimed the result for so-called weakly-symmetric digraphs, a subclass of odd-cycle-symmetric digraphs. The even factor algorithm also implies a new algorithm for path-matching. The method of proof can be extended to hypo-matching in digraphs [20], [22], a common generalization of even factors and hypo-matching (see Cornuéjols, Hartvigsen [3]).

2. Matroid matching

In the second chapter we discuss matroid matching, in particular we focus on those special cases admitting a good characterization and a polynomial time algorithm. First we propose a matroid intersection algorithm, which is obtained from a generalization of the bipartite matching algorithm in the first chapter. Then we discuss Lovász' [14] linear matroid matching formula, and its algorithmic proof given by Orlin and Vande Vate [29]. Then we propose a new, algorithmic proof of a recent result of Makai and Szabó [17] on polymatroid matching, which implies a polynomial time algorithm for a theorem of Frank, Jordán, Szigeti [7]. Finally we discuss the lesser-known matroid fractional matching problem, results of Vande Vate [31], and Gijswijt [9]. We also provide a new application of matroid fractional matching, namely fractional packing of $\mathcal{A}$-paths.

2.1. Linear matroid matching

Let $E = \{l_1, \ldots, l_n\}$ be a set of *lines*, i.e. a set of 2-dimensional subspaces of a given vectorspace V. A line is assumed given by a pair of spanning vectors. A subset $M \subseteq E$ of lines is called a **matching** if $r(M) = 2|M|$. Let $\nu(E)$ denote the maximum cardinality of a matching. A pair K, π is called a *cover* if K is a subspace, and $\pi = \{A_1, \ldots, A_k\}$ is a partition of E. The *value* of a cover is defined by $val(K,\pi) := r(K) + \sum_{A_i \in \pi} \lfloor \frac{1}{2} r_{V/K}(A_i) \rfloor$.

THEOREM 2.1. (Lovász, [14]) $\nu(E) = \min val(K,\pi)$, *where the minimum is taken over covers* K, π.

In addition to this min-max formula, Lovász also provided a polynomial time algorithm, on the assumption that the lines are given by an explicit representation over a specific field, and arithmetic operations over this field take constant time.

In this section of the thesis, we discuss another matroid matching algorithm provided by Orlin and Vande Vate [29], which in many regards is analogous to those graph matching algorithms in the first chapter. It is based on the observation that, in Lovász' formula, the optimum cover is attained by a special kind of cover, a so-called *very strong cover*. The algorithm maintains a very strong cover of a growing subset of lines, and concludes by exhibiting a maximum matching and a minimum cover of the whole set of lines.

Note that, if the matroid is given by a linear representation over the rationals, then the original statement of Lovász' min-max formula is NOT a good characterization, since some subspace K may only be represented by a basis using very large numbers. Luckily, we show that its minimum attains with a very strong cover, which, using the following theorem, implies that Lovász' formula is a good characterization, and that the algorithm runs in polynomial time.

THEOREM 2.2. *If $K, \pi = \{A_1, \ldots, A_k\}$ is a very strong cover, then there are maximum matchings $M_1, \ldots, M_k$ such that*

$$K = \bigvee_{i=1}^{k} (sp(A_i) \wedge sp(M_i - A_i)).$$

2.2. Matching in ntcdc-free polymatroids

Consider a polymatroid function b on a finite groundset S, and let $\mathcal{P}(b)$ denote its induced polymatroid. A vector is called *even* if all of its entries are even. *Matchings* are the even vectors in $\mathcal{P}(b)$, the size of which is defined by half the sum of its entries. Let $\nu(b)$ denote the maximum size of a matching. A non-trivial compatible double-circuit (*ntcdc*, for short) is an integer vector $w \geq 0$ such that $w \notin \mathcal{P}(b)$, $w - \chi_s \notin \mathcal{P}(b)$ for all $s \in S$, and $supp(w)$ has a partition into at least three components such that $w - \chi_s - \chi_t \in \mathcal{P}(b)$ iff s, t is in distinct components of the partition. A polymatroid-function is called *ntcdc-free* if there is no ntcdc.

THEOREM 2.3. (Makai, Szabó, [17], Makai, Pap, Szabó, [18]) *If b is an ntcdc-free polymatroid-function, then $\nu(b) = \min_{\pi} \sum_{S_i \in \pi} \lfloor \frac{1}{2} b(S_i) \rfloor$, where the minimum is taken over partitions $\pi = \{S_1, \ldots, S_t\}$ of S.*

Makai and Szabó originally provided a non-constructive proof. In the dissertation, we provide a different, constructive proof. Special cases of this result are: A result of T. Király, Szabó [12] on parity constrained orientations of a graph covering a given non-negative submodular function. This generalizes to a great extent the result of Frank, Jordán, Szigeti [7], and also Nebeský's [28] characterization of maximum genus graph embedding.

2.3. Matroid fractional matching

Consider an arbitrary matroid $\mathcal{M} = (S, \mathcal{I})$. Let E be a family of some two-element subsets of S, which are called pairs. For a set $K \subseteq S$, let $d_K \in \{0,1,2\}^E$ denote the vector such that $d_K(e) := r(K \cap sp(e))$ for all $e \in E$. A *fractional matching* is a vector $x \in \mathbb{R}^E$ satisfying $x \geq 0$ and $d_K \cdot x \leq r(K)$ for all $K \subseteq S$. The size of a fractional matching is defined by the sum of its entries. Let $\nu^*(E)$ denote the maximum size of a fractional matching.

THEOREM 2.4 (Vande Vate, [31]) $\nu^*(E) = \min_{K \subseteq S}$

$$r(K) + \frac{1}{2} r_{\mathcal{M}/K}(\{e : r_{V/K}(e) = 2\}).$$

THEOREM 2.5. (Gijswijt, Pap, [9]) *The above description of the matroid fractional matching polytope is totally dual half-integer.*

Vande Vate provided two special cases of matroid fractional matching: graph fractional matching, and matroid intersection. In the dissertation we show that fractional packing of $\mathcal{A}$-paths is also a special case.

3. Packing A-paths

In the third chapter, we consider Mader's [16] path-packing min-max formulae, its generalizations, and polynomial time algorithms. Consider an undirected graph $G = (V, E)$ and a set $A \subseteq V$. A path is called an *A-path* if its two distinct endpoints are in A. Let $\widehat{\nu}(G, A)$ denote the maximum number of pairwise fully node-disjoint A-paths. Gallai [8] proved that determining $\widehat{\nu}(G, A)$ reduces to maximum matching in an auxiliary graph. If, moreover, we are given a partition $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of A, then an A-path is called an *$\mathcal{A}$-path* if it joins two distinct sets A_i. Let $\nu(G, \mathcal{A})$ denote the maximum number of pairwise fully node-disjoint $\mathcal{A}$-paths. Mader provided a min-max formula for $\nu(G, \mathcal{A})$. A polynomial time algorithm follows from Lovász' [15] linear matroid matching algorithm, and Schrijver's [30] linear representation.

3.1. Fractional packing of $\mathcal{A}$-paths

We assume $G, \mathcal{A}$ is given as above, and we are given a vector $b \in \mathbb{N}^V$ of node-capacities. A *fractional b-packing* of $\mathcal{A}$-paths is a vector x : "all $\mathcal{A}$-paths" $\to \mathbb{R}$ satisfying the inequalities (4)–(5). The linear program (3)–(5) is called the maximum fractional b-packing problem.

$$\max \mathbf{1}x \tag{1}$$

$$x(P) \geq 0 \quad \text{for all } \mathcal{A}\text{-paths } P, \tag{2}$$

$$x(\{P : v \in V(P)\}) \leq b(v) \quad \text{for all } v \in V. \tag{3}$$

THEOREM 3.1. *Both (3)–(5), and its dual admit a half-integral optimum solution.*

According to the following result, in case of $b \equiv \mathbf{1}$, the fractional b-packing problem not only admits a half-integral optimum, but the half-integral optimum attains at a special kind of half-integral solution. A half-integral packing is called an odd-A-cycle, if it arises in the following way. Suppose C is a cycle and $P_1, \ldots, P_{2m+1}$ are pairwise node-disjoint paths such that one of its endpoints s_i is in A, and the other endpoint is in $V(C)$, in this cyclic order. A half-integral packing is given by assigning $\frac{1}{2}$ with those unique s_i–s_{i+1} paths. Half-integral packings of this kind are called *odd-A-cycles*.

THEOREM 3.2. *There is a maximum half-integral $\mathbf{1}$-packing x which decomposes into the fully node-disjoint union of odd-A-cycles and some paths P with $x(P) = 1$.*

This theorem is attractive by itself, but it also proved useful in the proof of the following result, claiming that fractional $\mathbf{1}$-packing reduces to a special case of matroid fractional matching. Let $\mathcal{E} = \mathcal{E}(G, \mathcal{A})$ denote the set of those lines of a euclidian vectorspace constructed by Schrijver [30], which implies $\nu(G, \mathcal{A}) = \nu(\mathcal{E}) - |V - A|$. The following theorem claims that applying matroid fractional matching for the same set of lines implies a solution of the problem of fractional packing.

THEOREM 3.3 *The maximum size of a fractional $\mathbf{1}$-packing is equal to $\nu^*(\mathcal{E}) - |V - A|$.*

3.2. Mader matroids are gammoids

The problem of packing $\mathcal{A}$-paths induces a matroid on the set A of terminals, which is called the Mader matroid. A subset of A is defined independent if there is a maximum packing covering each of those nodes in the subset. It is quite easy to show that this gives a matroid. Schrijver [30] published as an open question whether all Mader matroids are gammoids. (Gammoids are the minors of transversal matroids.) In the dissertation we show that the answer is positive.

THEOREM 3.4. *Every Mader matroid is a gammoid.*

3.3. Packing non-returning A-paths

We propose a model of packing A-paths which generalizes the problem of packing $\mathcal{A}$-paths. The problem is to pack a maximum number of non-returning A-paths in a permutation-labeled graph, for which an extension of Mader's formula is proven, and a polynomial time algorithm is constructed.

Consider an undirected graph $G = (V, E)$ and a given subset $A \subseteq V$ of nodes. Let Ω be an arbitrary set disjoint from the graph. We assign an element $\omega(v) \in \Omega$ with every node $v \in A$. We assign with every edge $ab = e \in E$ a reference-orientation, and a permutation $\pi(e)$ of S. We define $\pi(e,a) := = \pi(e)$ and $\pi(e,b) := \pi(e)^{-1}$, where we take the reference-orientation into consideration. An A-path $(v_1, \dots, v_m)$ is called *non-returning* if $\pi(v_1v_2, v_1) \circ \cdots \pi(v_{m-1}v_m, v_{m-1})(\omega(v_1)) \neq \omega(v_m)$ holds.

The model of $\mathcal{A}$-paths easily reduces to this setting, and thus we obtain a generalization of Mader's formula below. This theorem is, in fact, a generalization of Chudnovsky et al.'s [2] result on packing non-zero A-paths in a group-labeled graph.

THEOREM 3.5. ([23], [24]) *The maximum number of pairwise node-disjoint non-returning A-paths is equal to* $\min \widehat{v}(G-X, A \cup V(F)-X)$*, where the minimum is taken over pairs $X \subseteq V, F \subseteq E$ such that $G - X - F$ does not contain any non-returning A-paths. The maximum may be determined in running time polynomial in $|V| + |E| + |\Omega|$.*

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Gyula Pap
Address: (Fill in, please)

`email: Fill in, please`