ON THE EXISTENCE OF POSITIVE STEADY STATES OF DEFICIENCY-ONE MASS ACTION SYSTEMS

Balázs Boros

Institute of Mathematics
Eötvös Loránd University
Budapest, Hungary

CRNT Portsmouth
June 23-25, 2014
**Notations**

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ - set of species
- $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ - set of complexes
- $Y \in \mathbb{R}^{n \times c}$ - matrix of complexes
- $\mathcal{R}$ - set of reactions
- $\kappa = (\kappa_{ij})_{(i,j) \in \mathcal{R}}$ - rate coefficients
- $(\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)$ - mass action system
- $\ell$ - number of linkage classes
- $t$ - number of terminal strong linkage classes
NOTATIONS

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ - set of species
- $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ - set of complexes
- $Y \in \mathbb{R}^{n \times c}$ - matrix of complexes
- $\mathcal{R}$ - set of reactions
- $\kappa = (\kappa_{ij})_{(i,j) \in \mathcal{R}}$ - rate coefficients
- $(\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)$ - mass action system
- $\ell$ - number of linkage classes
- $t$ - number of terminal strong linkage classes
NOTATIONS

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ - set of species
- $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ - set of complexes
- $Y \in \mathbb{R}^{n \times c}$ - matrix of complexes
- $\mathcal{R}$ - set of reactions
- $\kappa = (\kappa_{ij})_{(i,j)\in \mathcal{R}}$ - rate coefficients
- $(\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)$ - mass action system
- $\ell$ - number of linkage classes
- $t$ - number of terminal strong linkage classes
**Notations**

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ - set of species
- $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ - set of complexes
- $Y \in \mathbb{R}^{n \times c}$ - matrix of complexes
- $\mathcal{R}$ - set of reactions
- $\kappa = (\kappa_{ij})_{(i,j) \in \mathcal{R}}$ - rate coefficients
- $(\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)$ - mass action system
- $\ell$ - number of linkage classes
- $t$ - number of terminal strong linkage classes
**Notations**

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ - set of species
- $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ - set of complexes
- $Y \in \mathbb{R}^{n \times c}$ - matrix of complexes
- $R$ - set of reactions
- $\kappa = (\kappa_{ij})_{(i,j) \in \mathcal{R}}$ - rate coefficients
- $(\mathcal{X}, \mathcal{C}, R, \kappa)$ - mass action system
- $\ell$ - number of linkage classes
- $t$ - number of terminal strong linkage classes
NOTATIONS

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ - set of species
- $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ - set of complexes
- $Y \in \mathbb{R}^{n \times c}$ - matrix of complexes
- $\mathcal{R}$ - set of reactions
- $\kappa = (\kappa_{ij})_{(i,j) \in \mathcal{R}}$ - rate coefficients
- $(\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)$ - mass action system
- $\ell$ - number of linkage classes
- $t$ - number of terminal strong linkage classes
Notations

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ - set of species
- $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ - set of complexes
- $Y \in \mathbb{R}^{n \times c}$ - matrix of complexes
- $\mathcal{R}$ - set of reactions
- $\kappa = (\kappa_{ij})_{(i,j) \in \mathcal{R}}$ - rate coefficients
- $(\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)$ - mass action system
- $\ell$ - number of linkage classes
- $t$ - number of terminal strong linkage classes
NOTATIONS

- \( \mathcal{X} = \{X_1, X_2, \ldots, X_n\} \) - set of species

- \( \mathcal{C} = \{C_1, C_2, \ldots, C_c\} \) - set of complexes

- \( Y \in \mathbb{R}^{n \times c} \) - matrix of complexes

- \( \mathcal{R} \) - set of reactions

- \( \kappa = (\kappa_{ij})_{(i,j) \in \mathcal{R}} \) - rate coefficients

- \( (\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa) \) - mass action system

- \( \ell \) - number of linkage classes

- \( t \) - number of terminal strong linkage classes
\section*{\psi, A_\kappa, \text{ AND THE ODE}}

\[ \psi(x) = \begin{bmatrix}
\prod_{s=1}^{n} x_{s_1}^{y_{s_1}} \\
\prod_{s=1}^{n} x_{s_2}^{y_{s_2}} \\
\vdots \\
\prod_{s=1}^{n} x_{s_c}^{y_{s_c}}
\end{bmatrix} \]

\[ A_\kappa = \begin{bmatrix}
-\sum_{i=2}^{c} \kappa_{1i} \\
\kappa_{12} & -\left(\kappa_{21} + \sum_{i=3}^{c} \kappa_{2i}\right) & \cdots & \kappa_{c1} \\
\vdots & \vdots & \ddots & \vdots \\
\kappa_{1c} & \kappa_{2c} & \cdots & -\sum_{i=1}^{c-1} \kappa_{ci}
\end{bmatrix} \]

\[ \dot{x}(\tau) = Y \cdot A_\kappa \cdot \psi(x(\tau)), \text{ state space: } \mathbb{R}_{\geq 0}^{n} \]
ψ, $A_κ$, AND THE ODE

\[ \psi(x) = \begin{bmatrix} \prod_{s=1}^{n} x_s^{y_{s1}} \\ \prod_{s=1}^{n} x_s^{y_{s2}} \\ \vdots \\ \prod_{s=1}^{n} x_s^{y_{sc}} \end{bmatrix} \]

\[ A_κ = \begin{bmatrix} -\sum_{i=2}^{c} κ_{1i} & κ_{21} & \cdots & κ_{c1} \\ κ_{12} & - (κ_{21} + \sum_{i=3}^{c} κ_{2i}) & \cdots & κ_{c2} \\ \vdots & \vdots & \ddots & \vdots \\ κ_{1c} & κ_{2c} & \cdots & - \sum_{i=1}^{c-1} κ_{ci} \end{bmatrix} \]

\[ \dot{x}(τ) = Y \cdot A_κ \cdot ψ(x(τ)), \text{ state space: } \mathbb{R}_{≥0}^n \]
$\Psi$, $A_\kappa$, AND THE ODE

\[ \Psi(x) = \begin{bmatrix} \prod_{s=1}^{n} x_{s}^{y_{s1}} \\ \prod_{s=1}^{n} x_{s}^{y_{s2}} \\ \vdots \\ \prod_{s=1}^{n} x_{s}^{y_{sc}} \end{bmatrix} \]

\[ A_\kappa = \begin{bmatrix} -\sum_{i=2}^{c} \kappa_{1i} & \kappa_{21} & \cdots & \kappa_{c1} \\ \kappa_{12} & -\left(\kappa_{21} + \sum_{i=3}^{c} \kappa_{2i}\right) & \cdots & \kappa_{c2} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{1c} & \kappa_{2c} & \cdots & -\sum_{i=1}^{c-1} \kappa_{ci} \end{bmatrix} \]

\[ \dot{x}(\tau) = Y \cdot A_\kappa \cdot \Psi(x(\tau)), \text{ state space: } \mathbb{R}_{\geq 0}^{n} \]
\[ \dot{x}(\tau) = Y \cdot A_{\kappa} \cdot \psi(x(\tau)) \]

**Definition (Deficiency)**

\[ \delta = \dim(\ker Y \cap \text{ran} A_{\kappa}), \text{ provided that } \ell = t \]

- \( E_+^{\kappa} = \{ x \in \mathbb{R}_+^n \mid Y \cdot A_{\kappa} \cdot \psi(x) = 0 \} \) - positive steady states
- \( S = \text{span}\{ Y.j - Y.i \mid (i, j) \in \mathcal{R} \} \leq \mathbb{R}^n \) - stoichiometric subspace
- \( (p + S) \cap \mathbb{R}_+^n \) for \( p \in \mathbb{R}_+^n \) - positive stoichiometric class
\[ \dot{x}(\tau) = Y \cdot A_\kappa \cdot \psi(x(\tau)) \]

**Definition (Deficiency)**

\[ \delta = \dim(\ker Y \cap \text{ran } A_\kappa), \text{ provided that } \ell = t \]

- \( E_+^\kappa = \{ x \in \mathbb{R}^n_+ \mid Y \cdot A_\kappa \cdot \psi(x) = 0 \} \) - positive steady states
- \( S = \text{span}\{ Y_j - Y_i \mid (i, j) \in \mathcal{R} \} \leq \mathbb{R}^n \) - stoichiometric subspace
- \( (p + S) \cap \mathbb{R}^n_{\geq 0} \) for \( p \in \mathbb{R}^n_+ \) - positive stoichiometric class
\[
\dot{x}(\tau) = Y \cdot A_{\kappa} \cdot \psi(x(\tau))
\]

**Definition (Deficiency)**
\[
\delta = \dim(\ker Y \cap \text{ran } A_{\kappa}), \text{ provided that } \ell = t
\]

- \( E_{+}^{\kappa} = \{ x \in \mathbb{R}_{+}^{n} \mid Y \cdot A_{\kappa} \cdot \psi(x) = 0 \} \) - positive steady states
- \( S = \text{span}\{ Y_{j} - Y_{i} \mid (i, j) \in \mathcal{R} \} \leq \mathbb{R}^{n} \) - stoichiometric subspace
- \( (p + S) \cap \mathbb{R}_{\geq 0}^{n} \text{ for } p \in \mathbb{R}_{+}^{n} \) - positive stoichiometric class
DEFICIENCY-ONE THEOREM

THEOREM (MARTIN FEINBERG, 1979, 1987, 1995)

Assume

(I) \( \delta_r = 0 \) or \( 1 \) (\( \forall \ r \in 1, \ell \)),

(II) \( \delta_1 + \cdots + \delta_\ell = \delta \), and

(III) \( \ell = t \).

Then the following two implications hold.

\((C, R)\) is weakly reversible

\[ E_+^\kappa \neq \emptyset \]

\[ |(p + S) \cap E_+^\kappa| = 1 \ (\forall \ p \in \mathbb{R}_+^n) \]
DEFICIENCY-ONE THEOREM

Theorem (Martin Feinberg, 1979, 1987, 1995)

Assume

(I) $\delta_r = 0 \text{ or } 1 \ (\forall \ r \in \overline{1, \ell})$,

(II) $\delta_1 + \cdots + \delta_\ell = \delta$, and

(III) $\ell = t$.

Then the following two implications hold.

$(\mathcal{C}, \mathcal{R})$ is weakly reversible

$$E_+^\kappa \neq \emptyset$$

$$| (p + S) \cap E_+^\kappa | = 1 \ (\forall \ p \in \mathbb{R}_+^n)$$
Supplement to the Deficiency-One Theorem

Assume \( \ell = t = 1 \) and \((C, R)\) is not weakly reversible. Let

\[
C = C' \cup^* C'',
\]

where \(C'\) is the complex set of the only terminal strong linkage class. Then

\[
x \in E_+^{\kappa} \iff \begin{bmatrix} Y' & Y'' \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} A'_{\kappa} & \ast \end{bmatrix} & \begin{bmatrix} \ast & 0 \\ \ast & A''_{\kappa} \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} \Psi'(x) \\ \Psi''(x) \end{bmatrix} = 0.
\]

Theorem (BB, 2010, 2012)

Beyond the above, assume that \( \delta = 1 \) and let \( 0 \neq h \in \ker Y \cap \text{ran} A_{\kappa} \) be such that \( \sum_{i \in C''} h_i \leq 0 \). Then

\[
E_+^{\kappa} \neq \emptyset \iff (A''_{\kappa})^{-1} h'' \gg 0.
\]
Supplement to the Deficiency-One Theorem

Assume $\ell = t = 1$ and $(C, \mathcal{R})$ is not weakly reversible. Let

$$C = C' \cup^* C'' ,$$

where $C'$ is the complex set of the only terminal strong linkage class. Then

$$x \in E^\kappa_+ \iff \begin{bmatrix} Y' & Y'' \end{bmatrix} \cdot \begin{bmatrix} A' \kappa & * \\ 0 & A'' \kappa \end{bmatrix} \cdot \begin{bmatrix} \Psi'(x) \\ \psi''(x) \end{bmatrix} = 0 .$$

Theorem (BB, 2010, 2012)

Beyond the above, assume that $\delta = 1$ and let $0 \neq h \in \ker Y \cap \text{ran} A_\kappa$ be such that $\sum_{i \in C''} h_i \leq 0$. Then

$$E^\kappa_+ \neq \emptyset \iff (A''_\kappa)^{-1} h'' \gg 0 .$$
**Special Case:** \((\mathcal{C}, \mathcal{R})\) is a Chain

**Theorem (BB, 2013)**

Assume that \((\mathcal{C}, \mathcal{R})\) is of the form

\[
\begin{align*}
\mathcal{C}_c & \xleftarrow{\kappa_{c-1,c}} \cdots \xleftarrow{\kappa_{23}} \mathcal{C}_2 \xrightarrow{\kappa_{21}} \mathcal{C}_1.
\end{align*}
\]

Assume that \(\delta = 1\). Let \(0 \neq h \in \ker Y \cap \text{ran} A_{\kappa}\) be such that \(\sum_{i=2}^{c} h_i \leq 0\). Let us define \(\vartheta_2, \ldots, \vartheta_c\) by the recursion

\[
\begin{align*}
\vartheta_2 &= -\frac{1}{\kappa_{21}} \sum_{i=2}^{c} h_i \text{ and} \\
\vartheta_j &= \frac{\kappa_{j-1,j}}{\kappa_{j,j-1}} \vartheta_{j-1} - \frac{1}{\kappa_{j,j-1}} \sum_{i=j}^{c} h_i, \quad j = 3, \ldots, c.
\end{align*}
\]

Then \(E_+^\kappa \neq \emptyset \iff \vartheta_j > 0 \quad (\forall j \in \{2, 3, \ldots, c\})\).
**Special Case: \((C, R)\) is a Chain**

**Theorem (BB, 2013)**

Assume that \((C, R)\) is of the form

\[
\begin{align*}
C_c & \xleftarrow{\kappa_{c,c-1}} \cdots \xleftarrow{\kappa_{c-1,c}} C_2 \xrightarrow{\kappa_{21}} C_1. \\
& \xleftarrow{\kappa_{c-1,c}} \xrightarrow{\kappa_{32}} C_2
\end{align*}
\]

Assume that \(\delta = 1\). Let \(0 \neq h \in \ker Y \cap \text{ran } A_{\kappa}\) be such that \(\sum_{i=2}^c h_i \leq 0\). Let us define \(\vartheta_2, \ldots, \vartheta_c\) by the recursion

\[
\begin{align*}
\vartheta_2 &= -\frac{1}{\kappa_{21}} \sum_{i=2}^c h_i \quad \text{and} \\
\vartheta_j &= \frac{\kappa_{j-1,j}}{\kappa_{j,j-1}} \vartheta_{j-1} - \frac{1}{\kappa_{j,j-1}} \sum_{i=j}^c h_i, \quad j = 3, \ldots, c.
\end{align*}
\]

Then \(E^\kappa_+ \neq \emptyset \iff \vartheta_j > 0 \ (\forall j \in \{2, 3, \ldots, c\})\).
Special Case: \((\mathcal{C}, \mathcal{R})\) is a Chain

**Theorem (BB, 2013)**

**Assume that** \((\mathcal{C}, \mathcal{R})\) **is of the form**

\[
\begin{align*}
\mathcal{C}_c & \xleftarrow[\kappa_{c-1,c}]{\kappa_{c,c-1}} \cdots \xleftarrow[\kappa_{23}]{\kappa_{32}} \mathcal{C}_2 \xrightarrow[\kappa_{21}]{\kappa_{21}} \mathcal{C}_1.
\end{align*}
\]

**Assume that** \(\delta = 1\). **Let** \(0 \neq h \in \ker Y \cap \text{ran} A_{\kappa} \) **be such that** \(\sum_{i=2}^{c} h_i \leq 0\). **Let us define** \(\vartheta_2, \ldots, \vartheta_c\) **by the recursion**

\[
\begin{align*}
\vartheta_2 &= -\frac{1}{\kappa_{21}} \sum_{i=2}^{c} h_i \quad \text{and} \\
\vartheta_j &= \frac{\kappa_{j-1,j}}{\kappa_{j,j-1}} \vartheta_{j-1} - \frac{1}{\kappa_{j,j-1}} \sum_{i=j}^{c} h_i, \quad j = 3, \ldots, c.
\end{align*}
\]

Then \(E_+^{\kappa} \neq \emptyset \iff \vartheta_j > 0 \quad (\forall j \in \{2, 3, \ldots, c\})\).
**Special Case:** $(C, R)$ is a chain $(\exists \kappa$ and $\forall \kappa)\nabla

**Corollary (BB, 2013)**

Assume that $(C, R)$ is of the form

\[ C_c \iff \cdots \iff C_2 \rightarrow C_1. \]

Assume that $\delta = 1$. Let $0 \neq h \in \ker Y \cap \text{ran} A_{\kappa}$ be such that

\[ \sum_{i=2}^{c} h_i \leq 0. \]

Then

(A) $\exists \kappa : R \rightarrow R_+ \text{ such that } E_{\kappa}^+ \neq \emptyset \text{ if and only if } \sum_{i=2}^{c} h_i < 0 \text{ and }$

(B) $\forall \kappa : R \rightarrow R_+ \text{ we have } E_{\kappa}^+ \neq \emptyset \text{ if and only if }$

\[ \sum_{i=2}^{c} h_i < 0 \text{ and } \sum_{i=j}^{c} h_i \leq 0 \text{ (} \forall j \in \{3, \ldots, c\} \text{).} \]
**Special Case:** \((C, R)\) is a chain \((\exists \kappa \text{ and } \forall \kappa)\)

**Corollary (BB, 2013)**

Assume that \((C, R)\) is of the form

\[
C_c \leftrightarrow \cdots \leftrightarrow C_2 \rightarrow C_1.
\]

Assume that \(\delta = 1\). Let \(0 \neq h \in \ker Y \cap \text{ran } A_\kappa\) be such that \(\sum_{i=2}^{c} h_i \leq 0\). Then

(A) \(\exists \kappa : R \rightarrow \mathbb{R}_+\) such that \(E_+^\kappa \neq \emptyset\) if and only if \(\sum_{i=2}^{c} h_i < 0\) and

(B) \(\forall \kappa : R \rightarrow \mathbb{R}_+\) we have \(E_+^\kappa \neq \emptyset\) if and only if

\[
\sum_{i=2}^{c} h_i < 0 \text{ and } \sum_{i=j}^{c} h_i \leq 0 \quad (\forall j \in \{3, \ldots, c\}).
\]
**Special Case: \((C, R)\) is a chain (\(\exists \kappa\) and \(\forall \kappa\))**

**Corollary (BB, 2013)**

Assume that \((C, R)\) is of the form

\[
C_c \leftrightarrow \cdots \leftrightarrow C_2 \rightarrow C_1.
\]

Assume that \(\delta = 1\). Let \(0 \neq h \in \ker Y \cap \text{ran} A_\kappa\) be such that

\[
\sum_{i=2}^{c} h_i \leq 0.
\]

Then

(A) \(\exists \kappa : R \rightarrow \mathbb{R}_+\) such that \(E_+^\kappa \neq \emptyset\) if and only if \(\sum_{i=2}^{c} h_i < 0\) and

(B) \(\forall \kappa : R \rightarrow \mathbb{R}_+\) we have \(E_+^\kappa \neq \emptyset\) if and only if

\[
\sum_{i=2}^{c} h_i < 0 \quad \text{and} \quad \sum_{i=j}^{c} h_i \leq 0 \quad (\forall \ j \in \{3, \ldots, c\}).
\]
Theorem (BB, 2013)

Assume that \( \ell = t = 1 \) and \( \delta = 1 \). Assume also that \((C, R)\) is not weakly reversible and let

\[
C = C' \cup^* C''
\]

where \(C'\) is the complex set of the only terminal strong linkage class. Let

\[
0 \neq h \in \ker Y \cap \text{ran } A_\kappa
\]

be such that

\[
\sum_{i \in C''} h_i \leq 0.
\]

Then there exists \(\kappa : R \to \mathbb{R}_+\) such that \(E_\kappa^+ \neq \emptyset\) if and only if

\[
\sum_{i \in \tilde{C}} h_i < 0 \quad \text{for all } \emptyset \neq \tilde{C} \subset C \text{ with } \varrho^\text{in}(\tilde{C}) = \emptyset,
\]

where \(\varrho^\text{in}(\tilde{C})\) is the set of reactions that enter \(\tilde{C}\).
\((\mathcal{C}, \mathcal{R})\) IS OF GENERAL FORM \((\exists \kappa)\)

**Theorem (BB, 2013)**

Assume that \(\ell = t = 1\) and \(\delta = 1\). Assume also that \((\mathcal{C}, \mathcal{R})\) is not weakly reversible and let

\[
\mathcal{C} = \mathcal{C}' \cup^* \mathcal{C}'',
\]

where \(\mathcal{C}'\) is the complex set of the only terminal strong linkage class. Let

\[
0 \neq h \in \ker Y \cap \text{ran} \ A_\kappa \text{ be such that } \sum_{i \in \mathcal{C}''} h_i \leq 0.
\]

Then there exists \(\kappa : \mathcal{R} \to \mathbb{R}_+\) such that \(E_+^\kappa \neq \emptyset\) if and only if

\[
\sum_{i \in \mathcal{C}} h_i < 0 \text{ for all } \emptyset \neq \overset{\sim}{\mathcal{C}} \subsetneq \mathcal{C} \text{ with } \varrho^\text{in}(\overset{\sim}{\mathcal{C}}) = \emptyset,
\]

where \(\varrho^\text{in}(\overset{\sim}{\mathcal{C}})\) is the set of reactions that enter \(\overset{\sim}{\mathcal{C}}\).
Theorem

Assume that \((V,A)\) is a weakly connected directed graph and \(h: V \rightarrow \mathbb{R}\) is such that \(\sum_{i \in V} h_i = 0\). Then there exists \(z: A \rightarrow \mathbb{R}_+\) with excess \(z = h\) if and only if

\[
\sum_{i \in U} h_i < 0 \quad \text{for all } \emptyset \neq U \subset V \text{ with } \varrho^\text{in}(U) = \emptyset,
\]

where \(\varrho^\text{in}(U)\) is the set of arcs that enter \(U\).
**Theorem**

Assume that \((V, A)\) is a weakly connected directed graph and \(h : V \rightarrow \mathbb{R}\) is such that \(\sum_{i \in V} h_i = 0\). Then there exists \(z : A \rightarrow \mathbb{R}_+\) with \(\text{excess}_z = h\) if and only if

\[
\sum_{i \in U} h_i < 0 \text{ for all } \emptyset \neq U \subset V \text{ with } \varrho^{\text{in}}(U) = \emptyset,
\]

where \(\varrho^{\text{in}}(U)\) is the set of arcs that enter \(U\).
$(C, \mathcal{R})$ is of general form ($\forall \kappa$)

- $U(i) = \{ k \in C'' \mid \text{each } k \leadsto C' \text{ path crosses } i \}$,

- $C''(j) = \{ k \in C'' \mid \text{there exist both } k \leadsto j \text{ and } j \leadsto k \text{ path} \}$

- $U(C''(j)) = \{ k \in C'' \mid \text{each } k \leadsto C' \text{ path crosses } C''(j) \}$

- $W(j) = \left\{ k \in C'' \mid \text{there exist paths } k \leadsto j \text{ and } k \leadsto C' \text{ that intersect each other only in } k \right\}$

- let $\mathcal{J} \subseteq C''$ be such that

  - for all $j \in C''$ we have $|C''(j) \cap \mathcal{J}| = 1$ and

  - for all $j \in \mathcal{J}$ we have $\varrho^{\text{out}}(j) \cap \varrho^{\text{out}}(C''(j)) \neq \emptyset$. 
$(\mathcal{C}, \mathcal{R})$ is of general form $(\forall \kappa)$

- $U(i) = \{ k \in \mathcal{C}'' \mid \text{each } k \leadsto \mathcal{C}' \text{ path crosses } i \}$,

- $\mathcal{C}''(j) = \{ k \in \mathcal{C}'' \mid \text{there exist both } k \leadsto j \text{ and } j \leadsto k \text{ path} \}$

- $U(\mathcal{C}''(j)) = \{ k \in \mathcal{C}'' \mid \text{each } k \leadsto \mathcal{C}' \text{ path crosses } \mathcal{C}''(j) \}$

- $W(j) = \left\{ k \in \mathcal{C}'' \mid \begin{array}{l}
\text{there exist paths } k \leadsto j \text{ and } k \leadsto \mathcal{C}' \\
\text{that intersect each other only in } k
\end{array} \right\}$

- let $\mathcal{J} \subseteq \mathcal{C}''$ be such that

  - for all $j \in \mathcal{C}''$ we have $|\mathcal{C}''(j) \cap \mathcal{J}| = 1$ and
  - for all $j \in \mathcal{J}$ we have $\varrho^{\text{out}}(j) \cap \varrho^{\text{out}}(\mathcal{C}''(j)) \neq \emptyset$.  

Balázs Boros (Eötvös Univ., Budapest)  
Existence of positive steady states  
Portsmouth, June 23-25, 2014  
11 / 22
\((C, R)\) IS OF GENERAL FORM \((\forall \kappa)\)

- \(U(i) = \{ k \in C'' \mid \text{each } k \rightsquigarrow C' \text{ path crosses } i \}\),
- \(C''(j) = \{ k \in C'' \mid \text{there exist both } k \rightsquigarrow j \text{ and } j \rightsquigarrow k \text{ path} \}\)
- \(U(C''(j)) = \{ k \in C'' \mid \text{each } k \rightsquigarrow C' \text{ path crosses } C''(j) \}\)
- \(W(j) = \left\{ k \in C'' \mid \text{there exist paths } k \rightsquigarrow j \text{ and } k \rightsquigarrow C' \text{ that intersect each other only in } k \right\}\)
- let \(\mathcal{J} \subseteq C''\) be such that
  - for all \(j \in C''\) we have \(|C''(j) \cap \mathcal{J}| = 1\) and
  - for all \(j \in \mathcal{J}\) we have \(\rho^{\text{out}}(j) \cap \rho^{\text{out}}(C''(j)) \neq \emptyset\).
$(C, R)$ is of general form ($\forall \kappa$)

- $U(i) = \{ k \in C'' \mid \text{each } k \rightsquigarrow C' \text{ path crosses } i \}$,

- $C''(j) = \{ k \in C'' \mid \text{there exist both } k \rightsquigarrow j \text{ and } j \rightsquigarrow k \text{ path} \}$

- $U(C''(j)) = \{ k \in C'' \mid \text{each } k \rightsquigarrow C' \text{ path crosses } C''(j) \}$

- $W(j) = \left\{ k \in C'' \mid \begin{array}{l} \text{there exist paths } k \rightsquigarrow j \text{ and } k \rightsquigarrow C' \text{ that intersect each other only in } k \end{array} \right\}$

- let $J \subseteq C''$ be such that
  - for all $j \in C''$ we have $|C''(j) \cap J| = 1$ and
  - for all $j \in J$ we have $\rho^\text{out}(j) \cap \rho^\text{out}(C''(j)) \neq \emptyset$. 

Balázs Boros (Eötvös Univ., Budapest)  Existence of positive steady states

Portsmouth, June 23-25, 2014
\((C, \mathcal{R})\) is of general form \((\forall \kappa)\)

- \(U(i) = \{ k \in C'' \mid \text{each } k \rightsquigarrow C' \text{ path crosses } i \}\),

- \(C''(j) = \{ k \in C'' \mid \text{there exist both } k \rightsquigarrow j \text{ and } j \rightsquigarrow k \text{ path} \}\)

- \(U(C''(j)) = \{ k \in C'' \mid \text{each } k \rightsquigarrow C' \text{ path crosses } C''(j) \}\)

- \(W(j) = \left\{ k \in C'' \middle| \begin{align*}
\text{there exist paths } k \rightsquigarrow j \text{ and } k \rightsquigarrow C' \\
\text{that intersect each other only in } k
\end{align*} \right\}\)

- Let \(J \subseteq C''\) be such that
  - for all \(j \in C''\) we have \(|C''(j) \cap J| = 1\) and
  - for all \(j \in J\) we have \(\varrho^\text{out}(j) \cap \varrho^\text{out}(C''(j)) \neq \emptyset\).
\((\mathcal{C}, \mathcal{R})\) is of general form \((\forall \kappa)\)

**Theorem (BB, 2013)**

Assume that \(\ell = t = 1\) and \(\delta = 1\). Assume also that \((\mathcal{C}, \mathcal{R})\) is not weakly reversible and let

\[ \mathcal{C} = \mathcal{C}' \cup^* \mathcal{C}'' , \]

where \(\mathcal{C}'\) is the set of complexes in the only terminal strong linkage class. Let

\[ 0 \neq h \in \ker Y \cap \text{ran} A_\kappa \text{ be such that } \sum_{k \in \mathcal{C}''} h_k \leq 0. \]

Then we have \(E^\kappa_+ \neq \emptyset\) for all \(\kappa : \mathcal{R} \to \mathbb{R}_+\) if and only if

- for all \(i \in \mathcal{C}''\) we have \(\sum_{k \in U(i)} h_k \leq 0\) and

- for all \(j \in \mathcal{J}\) with \(W(j) \subseteq \mathcal{C}''(j)\), we have \(\sum_{k \in U(\mathcal{C}''(j))} h_k < 0\).
\((C, \mathcal{R})\) is of general form \((\forall \kappa)\)

**Theorem (BB, 2013)**

Assume that \(\ell = t = 1\) and \(\delta = 1\). Assume also that \((C, \mathcal{R})\) is not weakly reversible and let

\[ C = C' \cup^* C'', \]

where \(C'\) is the set of complexes in the only terminal strong linkage class. Let

\[ 0 \neq h \in \ker Y \cap \text{ran } A_{\kappa} \text{ be such that } \sum_{k \in C''} h_k \leq 0. \]

Then we have \(E_+^{\kappa} \neq \emptyset\) for all \(\kappa : \mathcal{R} \to \mathbb{R}_+\) if and only if

- for all \(i \in C''\) we have \(\sum_{k \in U(i)} h_k \leq 0\) and
- for all \(j \in \mathcal{J}\) with \(W(j) \subseteq C''(j)\), we have \(\sum_{k \in U(C''(j))} h_k < 0\).
Theorem (William Thomas Tutte, 1948)

The signed determinant corresponding to the $ij$th element of $A_\kappa$ is

$$(-1)^{c-1} \cdot \sum_{\tilde{A} \in T(j)} \left( \prod_{a \in \tilde{A}} \kappa_a \right).$$

Theorem

Let $Q \subseteq \{1, 2, \ldots, c\}$ and $i, j \in \{1, 2, \ldots, c\} \setminus Q$. Then the determinant of the matrix obtained from $A_\kappa$ by deleting the rows corresponding to $Q \cup \{i\}$ and the columns corresponding to $Q \cup \{j\}$ is

$$(-1)^{i+j} \cdot (-1)^{c-|Q|-1} \cdot \sum_{\tilde{A} \in T^{ij}(Q \cup \{j\})} \left( \prod_{a \in \tilde{A}} \kappa_a \right).$$
To the proof: Matrix-Tree Theorem

**Theorem (William Thomas Tutte, 1948)**

The signed determinant corresponding to the $ij$th element of $A_\kappa$ is

$$(-1)^{c-1} \cdot \sum_{\tilde{A} \in T(j)} \left( \prod_{a \in \tilde{A}} \kappa_a \right).$$

**Theorem**

Let $Q \subseteq \{1, 2, \ldots, c\}$ and $i, j \in \{1, 2, \ldots, c\} \setminus Q$. Then the determinant of the matrix obtained from $A_\kappa$ by deleting the rows corresponding to $Q \cup \{i\}$ and the columns corresponding to $Q \cup \{j\}$ is

$$(-1)^{i+j} \cdot (-1)^{c-|Q|-1} \cdot \sum_{\tilde{A} \in T_{ij}(Q \cup \{j\})} \left( \prod_{a \in \tilde{A}} \kappa_a \right).$$
**Theorem (BB, 2013)**

Assume that

(I) $\delta = 1$ and

(II) $(\mathcal{C}, \mathcal{R})$ is weakly reversible.

Then

$$|(p + S) \cap E^\kappa_+| \geq 1 \ (\forall \ p \in \mathbb{R}^n).$$

Moreover, if $\ell \leq 2$ then

$$|(p + S) \cap E^\kappa_+| < \infty \ (\forall \ p \in \mathbb{R}^n).$$
**Further mass action systems with $\delta = 1$**

**Theorem (BB, 2013)**

Assume that

(I) $\delta = 1$ and

(II) $(C, R)$ is weakly reversible.

Then

$$|(p + S) \cap E^\kappa_+| \geq 1 \quad (\forall \ p \in \mathbb{R}^n_+).$$

Moreover, if $\ell \leq 2$ then

$$|(p + S) \cap E^\kappa_+| < \infty \quad (\forall \ p \in \mathbb{R}^n_+).$$
To the proof: a power function

Assume that $l = t = 1$ and let $\bar{y} \in \mathbb{R}^c_{\geq 0}$, $y^* \in \mathbb{R}^c_{\geq 0}$, and $h \in \mathbb{R}^c \setminus \{0\}$ be such that $A_\kappa \bar{y} = 0$, $A_\kappa y^* = h$,

for all $j \in C$ we have $j \in C'$ if and only is $\bar{y}_j > 0$,

for all $j \in C \setminus C'$ we have $y^*_j > 0$, and

there exists $j \in C'$ such that $y^*_j = 0$.

Define $\beta^* \in \mathbb{R}$ and the function $p : (\beta^*, \infty) \rightarrow \mathbb{R}_+$ by

$$
\beta^* = \max \left\{ -\frac{\bar{y}_i}{y^*_i} \mid i \in C \text{ and } y^*_i > 0 \right\}
$$

and

$$
p(\beta) = \prod_{i \in C} (\beta y^*_i + \bar{y}_i)^{h_i} \quad (\beta \in (\beta^*, \infty)),
$$

respectively.
**To the proof: a power function**

Assume that $\ell = t = 1$ and let $\bar{y} \in \mathbb{R}^c_{\geq 0}$, $y^* \in \mathbb{R}^c_{\geq 0}$, and $h \in \mathbb{R}^c \setminus \{0\}$ be such that $A_\kappa \bar{y} = 0$, $A_\kappa y^* = h$,

for all $j \in C$ we have $j \in C'$ if and only is $\bar{y}_j > 0$,

for all $j \in C \setminus C'$ we have $y^*_j > 0$, and

there exists $j \in C'$ such that $y^*_j = 0$.

Define $\beta^* \in \mathbb{R}$ and the function $p : (\beta^*, \infty) \rightarrow \mathbb{R}_+$ by

$$
\beta^* = \max \left\{ -\frac{\bar{y}_i}{y^*_i} \bigg| i \in C \text{ and } y^*_i > 0 \right\} \text{ and }
\quad p(\beta) = \prod_{i \in C} (\beta y^*_i + \bar{y}_i)^{h_i} \quad (\beta \in (\beta^*, \infty)),
$$

respectively.
**To the proof: a power function**

**Lemma**

Let \( p : (\beta^*, \infty) \rightarrow \mathbb{R}_+ \) be as in the previous slide. Then

(A) \( \lim_{\beta \rightarrow \beta^*+0} p(\beta) = \infty, \)

(B) \( \lim_{\beta \rightarrow \infty} p(\beta) = 0, \)

(C) *the derivative of \( p \) is negative on \( (\beta^*, \infty) \), and*

(D) \( p : (\beta^*, \infty) \rightarrow \mathbb{R}_+ \) is a bijection.

**The function \( p \)**

\[
\begin{align*}
\text{The function } p &: (\beta^*, \infty) \rightarrow \mathbb{R}_+ \\
\beta^* &< 0
\end{align*}
\]
To the proof: the Bolzano Theorem in 2 dimensions

Theorem

Assume that \( f : [0, 1]^2 \to \mathbb{R}^2 \) is continuous and

\[
\begin{align*}
f_2 &\leq 0 \\
f_1 &\leq 0 \\
f_2 &\geq 0 \\
f_1 &\geq 0
\end{align*}
\]

Then \( \exists x \in [0, 1]^2 \) with \( f(x) = 0 \in \mathbb{R}^2 \).
To the proof: the Bolzano Theorem in $n$ dimensions

**Theorem**

Assume that $f : [0, 1]^n \rightarrow \mathbb{R}^n$ is continuous and

$$(\forall x \in \partial [0, 1]^n)(\forall i \in \overline{1, n})[x_i = 0 \Rightarrow f_i(x) \geq 0] \text{ and }$$

$$(\forall x \in \partial [0, 1]^n)(\forall i \in \overline{1, n})[x_i = 1 \Rightarrow f_i(x) \leq 0].$$

Then $\exists x \in [0, 1]^n$ with $f(x) = 0 \in \mathbb{R}^n$. 
Can the previous theorem be generalised substantially?

**Conjecture (Jian Deng, Martin Feinberg, Chris Jones, Adrian Nachman)**

Assume that \((\mathcal{C}, \mathcal{R})\) is weakly reversible. Then

\[
1 \leq |(p + S) \cap E_\kappa^\kappa| < \infty \quad (\forall \ p \in \mathbb{R}_+^n).
\]

**Remark**

Once the above conjecture is proved, the set

\[
E_\kappa^\kappa = \{ x \in \partial \mathbb{R}_+^n \mid Y \cdot A_\kappa \cdot \psi(x) = 0 \}
\]

can also be described.
Can the previous theorem be generalised substantially?

**Conjecture (Jian Deng, Martin Feinberg, Chris Jones, Adrian Nachman)**

Assume that \((\mathcal{C}, \mathcal{R})\) is weakly reversible. Then

\[
1 \leq |(p + S) \cap E^\kappa_+| < \infty \quad (\forall \ p \in \mathbb{R}_+^n).
\]

**Remark**

Once the above conjecture is proved, the set

\[
E_0^\kappa = \{ x \in \partial \mathbb{R}_+^n \mid Y \cdot A_\kappa \cdot \psi(x) = 0 \}
\]

can also be described.
An EnvZ-OmpR model in which ATP (resp. ADP) is the cofactor in phospho-OmpR dephosphorylation

**Cofactor: ATP**

**Cofactor: ADP**
For these three systems, we have

- $\delta_1 = \delta_2 = 0 (= \delta_3)$ and $\delta = 1$ and
- $(\mathcal{C}, \mathcal{R})$ is not weakly reversible.
For these three systems, we have

- $\delta_1 = \delta_2 = 0 (= \delta_3)$ and $\delta = 1$ and
- $(C, R)$ is not weakly reversible.
www.cs.elte.hu/~bboros