TWO APPLICATIONS OF THE DEFICIENCY-ONE ALGORITHM

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**Notations**

- \( \mathcal{X} = \{X_1, X_2, \ldots, X_n\} \) - set of species
- \( \mathcal{C} \) - set of complexes
- \( Y \in \mathbb{R}^{n \times |\mathcal{C}|}_{\geq 0} \) - matrix of complexes
- \( \mathcal{R} \) - set of reactions
- \( (\mathcal{X}, \mathcal{C}, \mathcal{R}) \) - chemical reaction network
- \( I \in \{-1, 0, 1\}^{|\mathcal{C}| \times |\mathcal{R}|} \) - incidence matrix of \((\mathcal{C}, \mathcal{R})\)
- \( \ell \) - number of linkage classes
- \( t \) - number of terminal strong linkage classes
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- \( \kappa = (\kappa_{ij})_{(i,j) \in R} \in \mathbb{R}^{|R|}_+ \) - rate coefficients

- \((\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)\) - mass action system

\[
R_{\kappa}(x) = \begin{bmatrix}
\vdots \\
\kappa_{ij} \prod_{s=1}^{n} x_s^{y_{si}} \\
\vdots \\
\end{bmatrix}_{(i,j) \in R}
\]

\[
\dot{x}(\tau) = Y \cdot I \cdot R_{\kappa}(x(\tau)), \text{ state space: } \mathbb{R}^n_+
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$\delta, E_+^\kappa, \text{AND } S$

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**Definition (Deficiency)**

$$\delta = \dim(\ker Y \cap \text{ran } I)$$

- $E_+^\kappa = \{ x \in \mathbb{R}^n_+ | Y \cdot I \cdot R_\kappa(x) = 0 \}$ - positive steady states
- $S = \text{span}\{ Y_j - Y_i | (i, j) \in \mathcal{R} \} \leq \mathbb{R}^n$ - stoichiometric subspace
- $(p + S) \cap \mathbb{R}^n_+$ for $p \in \mathbb{R}^n_+$ - positive stoichiometric classes (recall that the state space is $\mathbb{R}^n_+$)
\[ \dot{x}(\tau) = Y \cdot I \cdot R_\kappa(x(\tau)) \]

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**Deficiency-One Theorem**

**Theorem (Martin Feinberg, 1979, 1987, 1995)**

Assume

(I) \( \delta_r = 0 \) or \( 1 \) (\( \forall \ r \in \{1, \ell\} \)),

(II) \( \delta_1 + \delta_2 + \cdots + \delta_\ell = \delta \), and

(III) \( \ell = t \).

Then the following two implications hold.

(\( C, R \)) is weakly reversible

\[ E^\kappa_+ \neq \emptyset \]

\[ \left| (p + S) \cap E^\kappa_+ \right| = 1 \ (\forall \ p \in \mathbb{R}^n_+) \]
DEFICIENCY-ONE THEOREM

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What can be said about the possibility of multiple positive steady states for networks with $\delta = 1$ and $\delta_1 = \delta_2 = \cdots = \delta_\ell = 0$?

| $X_1 \leftrightarrow 0 \leftrightarrow X_2$ | for all $\kappa$ we have $|E_+^\kappa| = 1$ |
|----------------------------------------|---------------------------------------------|
| $X_1 + X_2 \leftrightarrow 2X_1$       |                                             |

| $X_1 \leftrightarrow 0 \leftrightarrow X_2$ | there exists $\kappa$ such that $|E_+^\kappa| \geq 2$ |
|----------------------------------------|--------------------------------------------------|
| $2X_1 + X_2 \leftrightarrow 3X_1$       |                                                  |

| $0 \leftrightarrow X_1 \leftrightarrow X_2$ | for all $\kappa$ we have $|E_+^\kappa| = 1$ |
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For a directed graph \((C, R)\) with \(\ell\) weak components, the vertices \(i\) and \(j\) are called a \textit{cut pair} if the number of weak components of the directed graph \((C, R \setminus \{(i, j), (j, i)\})\) is \(\ell + 1\).
Denote by $C'$ the union of the complex sets of the terminal strong linkage classes.

**Definition (Regular Networks)**

A chemical reaction network $(X, C, R)$ is said to be *regular* if

1. the set $\{Y_j - Y_i \mid (i, j) \in R\}$ is positively dependent, \( (R.1) \)
2. $\ell = t$, and \( (R.2) \)
3. for all $i, j \in C'$ with $(i, j) \in R$, the complexes $i$ and $j$ form a cut pair. \( (R.3) \)
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**Regular networks**

**Proposition (WR case)**

Let \((X, C, R)\) be a weakly reversible reaction network. Then \((X, C, R)\) is regular if and only if

\[
\begin{cases}
(C, R) \text{ is reversible and} \\
(C, R) \text{ is “forest-like”}.
\end{cases}
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**Proposition (non-WR case)**

Let \((X, C, R)\) be a reaction network, which satisfies \(\ell = t\), but is not weakly reversible. Then \((R.3)\) is equivalent to

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\text{for all } i \in C \setminus C', \text{ all directed paths from } i \text{ to } C' \text{ must enter } C' \text{ across the exact same element of } C'.
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Confluence vectors

Recall the ODE:

\[ \dot{x}(\tau) = Y \cdot l \cdot R_\kappa(x(\tau)) \]

**Definition (Confluence vector)**

For a reaction network \((X, C, R)\), a vector \(h \in \mathbb{R}^{|C|}\) is called a confluence vector if

- \(h \in \ker Y\),
- \(h \in \text{ran } l\), and
- \(\sum_{i \in \tilde{C}} h_i > 0\) for all \(\tilde{C} \subseteq C\) with \(\varrho^{\text{out}}(\tilde{C}) = \emptyset\) and \(\varrho^{\text{in}}(\tilde{C}) \neq \emptyset\).
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Recall

- the ODE: \( \dot{x}(\tau) = Y \cdot I \cdot R_\kappa(x(\tau)) \) and
- the set of positive steady states: \( E^\kappa_+ = \{ x \in \mathbb{R}^n_+ \mid Y \cdot I \cdot R_\kappa(x) = 0 \} \).

**Proposition (Confluence Vectors and Positive Steady States)**

Let \((\mathcal{X}, \mathcal{C}, \mathcal{R}, \kappa)\) be a mass action system and let \(x \in \mathbb{R}^n_+\). Then the following are equivalent.

(A) The vector \(x\) is in \(E^\kappa_+\), i.e., \(x\) is a positive steady state.

(B) The vector \(I \cdot R_\kappa(x)\) is a confluence vector for \((\mathcal{X}, \mathcal{C}, \mathcal{R})\).

**Proposition (Existence of Confluence Vectors)**

Let \((\mathcal{X}, \mathcal{C}, \mathcal{R})\) be a reaction network. Then the following are equivalent.

(A) Condition \((R.1)\) holds.

(B) There exists a confluence vector.
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(A) The vector \( x \) is in \( E_+^\kappa \), i.e., \( x \) is a positive steady state.

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Let \( (\mathcal{X}, \mathcal{C}, \mathcal{R}) \) be a reaction network. Then the following are equivalent.

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**Proposition (Confluence Vectors and Positive Steady States)**

Let $(\mathcal{X}, C, R, \kappa)$ be a mass action system and let $x \in \mathbb{R}^n_+$. Then the following are equivalent.

(A) The vector $x$ is in $E^\kappa_+$, i.e., $x$ is a positive steady state.

(B) The vector $I \cdot R_\kappa(x)$ is a confluence vector for $(\mathcal{X}, C, R)$.

**Proposition (Existence of Confluence Vectors)**

Let $(\mathcal{X}, C, R)$ be a reaction network. Then the following are equivalent.

(A) Condition (R.1) holds.

(B) There exists a confluence vector.
Let \((C, R)\) be a directed graph with weak components \((C^1, R^1), (C^2, R^2), \ldots, (C^\ell, R^\ell)\).

Denote by \(I \in \mathbb{R}^{|C| \times |R|}\) the incidence matrix of \((C, R)\). Then

\[
\text{ran } I = \text{span}(e^1, e^2, \ldots, e^\ell),
\]

where \(e^1, e^2, \ldots, e^\ell \in \{0, 1\}^{|C|}\) are the characteristic vectors of the sets \(C^1, C^2, \ldots, C^\ell\), respectively, i.e.,

\[
e^r_i = \begin{cases} 
1, & \text{if } i \in C^r, \\
0, & \text{if } i \in C \setminus C^r
\end{cases}
\]

for \(r \in \{1, 2, \ldots, \ell\}\) and \(i \in C\).
THE RANGE OF THE INCIDENCE MATRIX

**Proposition**

Let \((\mathcal{C}, \mathcal{R})\) be a directed graph with weak components 

\[ (\mathcal{C}^1, \mathcal{R}^1), (\mathcal{C}^2, \mathcal{R}^2), \ldots, (\mathcal{C}^\ell, \mathcal{R}^\ell). \]

Denote by \(I(\in \mathbb{R}^{\mid\mathcal{C}\mid \times \mid\mathcal{R}\mid})\) the incidence matrix of \((\mathcal{C}, \mathcal{R})\). Then

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for \(r \in \{1, 2, \ldots, \ell\}\) and \(i \in \mathcal{C}\).
The intersection \( \ker Y \cap \text{ran } I \)

Consider \( Y \) in the block form \( Y = [Y^1, Y^2, \ldots, Y^\ell] \), where the columns of \( Y^r \) correspond to the complexes in the \( r \)th linkage class \( (r \in \{1, 2, \ldots, \ell\}) \).

**Corollary**

Let \((\mathcal{X}, \mathcal{C}, \mathcal{R})\) be a reaction network and denote by \( I \) the incidence matrix of the directed graph \((\mathcal{C}, \mathcal{R})\). Then

\[
\ker Y \cap \text{ran } I = \ker \hat{Y},
\]

where

\[
\hat{Y} = \begin{bmatrix}
Y^1 & Y^2 & \cdots & Y^\ell \\
1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\
\end{bmatrix} \in \mathbb{R}^{(n+\sum_{r=1}^{\ell} 1) \times (\sum_{r=1}^{\ell} |C_r|)}.
\]
**The Intersection** \( \ker Y \cap \text{ran } I \)

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Let \((\mathcal{X}, \mathcal{C}, \mathcal{R})\) be a reaction network and denote by \( I \) the incidence matrix of the directed graph \((\mathcal{C}, \mathcal{R})\). Then

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0 \cdots 0 & 1 \cdots 1 & \cdots & 0 \cdots 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 \cdots 0 & 0 \cdots 0 & \cdots & 1 \cdots 1
\end{bmatrix}
\in \mathbb{R}^{(n+\sum_{r=1}^\ell 1) \times (\sum_{r=1}^\ell |C^r|)}.
\]
THE SET OF CONFLUENCE VECTORS FOR DFC-1 NETWORKS

Recall that $\delta = \dim(\ker Y \cap \text{ran } I) = \dim \ker \hat{Y}$. Assuming $\delta = 1$, let $0 \neq h \in \ker \hat{Y}$.

**Proposition (WR case)**

Let $(\mathcal{X}, \mathcal{C}, \mathcal{R})$ be a weakly reversible deficiency-one reaction network. Then the set of confluence vectors is $\{\alpha h \mid \alpha \in \mathbb{R}\}$.

**Proposition (non-WR case)**

Let $(\mathcal{X}, \mathcal{C}, \mathcal{R})$ be a deficiency-one reaction network, which is not weakly reversible. Assume that (R.1) holds. Then the set of confluence vectors is

$$\begin{cases} 
\{\alpha h \mid \alpha > 0\}, & \text{if } h(C') > 0, \\
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$$
A complex $i$ is said to be reactive if $\varrho^{\text{out}}(\{i\}) \neq \emptyset$.

Denote by $C(1), C(2), \ldots, C(t)$ the complex sets of the terminal strong linkage classes. Thus, we have $C' = \bigcup_{k=1}^{t} C(k)$.

### Definition (Upper-Middle-Lower Partition)

An upper-middle-lower partition for a reaction network is a partition of its reactive complexes into three parts, $U$, $M$, and $L$, called the upper, middle, and lower parts, respectively, such that

- $C \setminus C' \subset M$ and
- for all $k \in \{1, 2, \ldots, t\}$ with $|C(k)| \geq 2$, either $C(k) \subseteq U$ or $C(k) \subseteq M$ or $C(k) \subseteq L$.

A terminal strong linkage class $C(k)$ is nontrivial if $|C(k)| \geq 2$.

The number of upper-middle-lower partitions is $3^{t'}$, where $t'$ is the number of nontrivial terminal strong linkage classes.
A complex $i$ is said to be *reactive* if $\varrho^{\text{out}}(\{i\}) \neq \emptyset$.

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A terminal strong linkage class $C(k)$ is *nontrivial* if $|C(k)| \geq 2$.

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UPPER-MIDDLE-LOWER PARTITIONS OF THE REACTIVE COMPLEXES

- A complex $i$ is said to be reactive if $\varrho^\text{out}({i}) \neq \emptyset$.
- Denote by $\mathcal{C}(1), \mathcal{C}(2), \ldots, \mathcal{C}(t)$ the complex sets of the terminal strong linkage classes. Thus, we have $\mathcal{C}' = \bigcup_{k=1}^{t} \mathcal{C}(k)$.

**DEFINITION (UPPER-MIDDLE-LOWER PARTITION)**

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THE INEQUALITY SYSTEM INDUCED BY A CONFLUENCE VECTOR AND AN UPPER-MIDDLE-LOWER PARTITION

- Fix a reaction network \((\mathcal{X}, \mathcal{C}, \mathcal{R})\) and an upper-middle-lower partition \(\{U, M, L\}\).

- Let

  \[
  \mathcal{M}^1 = \{\mu \in \mathbb{R}^n | \langle Y_j - Y_i, \mu \rangle = 0 \text{ for all } i, j \in M\}.
  \]

- Let

  \[
  \mathcal{M}^2 = \mathcal{M}^2_{MU} \cap \mathcal{M}^2_{LU} \cap \mathcal{M}^2_{LM},
  \]

  where

  \[
  \mathcal{M}^2_{MU} = \{\mu \in \mathbb{R}^n | \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in M, j \in U\},
  \]

  \[
  \mathcal{M}^2_{LU} = \{\mu \in \mathbb{R}^n | \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in L, j \in U\}, \text{ and}
  \]

  \[
  \mathcal{M}^2_{LM} = \{\mu \in \mathbb{R}^n | \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in L, j \in M\}.
  \]
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  \]
  where
  \[
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  \]
  \[
  \mathcal{M}^2_{LU} = \{\mu \in \mathbb{R}^n \mid \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in L, j \in U\}, \quad \text{and}
  \]
  \[
  \mathcal{M}^2_{LM} = \{\mu \in \mathbb{R}^n \mid \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in L, j \in M\}.
  \]
THE INEQUALITY SYSTEM INDUCED BY A CONFLUENCE VECTOR AND AN UPPER-MIDDLE-LOWER PARTITION

- Fix a reaction network \((X, C, R)\) and an upper-middle-lower partition \(\{U, M, L\}\).
- Let
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  \[ \mathcal{M}^2 = \mathcal{M}_{MU}^2 \cap \mathcal{M}_{LU}^2 \cap \mathcal{M}_{LM}^2, \]
  where
  \[ \mathcal{M}_{MU}^2 = \{ \mu \in \mathbb{R}^n \mid \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in M, j \in U \}, \]
  \[ \mathcal{M}_{LU}^2 = \{ \mu \in \mathbb{R}^n \mid \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in L, j \in U \}, \] and
  \[ \mathcal{M}_{LM}^2 = \{ \mu \in \mathbb{R}^n \mid \langle Y_j - Y_i, \mu \rangle > 0 \text{ for all } i \in L, j \in M \}. \]
THE INEQUALITY SYSTEM INDUCED BY A CONFLUENCE VECTOR AND AN UPPER-MIDDLE-LOWER PARTITION

Assume further that (R.1) and (R.3) hold and let $h$ be a confluence vector.

For $i, j \in C'$ with $(i, j) \in R$, there exists a partition of their linkage class into $W(i)$ and $W(j)$ such that $i \in W(i)$, $j \in W(j)$, and the only links between $W(i)$ and $W(j)$ are $(i, j)$ and $(j, i)$.

Let

$$M^3 = M^3_U \cap M^3_L,$$

where

$$M^3_U = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = \text{sgn}(h(W(j))) \right\} \quad \text{and}$$

$$M^3_L = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = -\text{sgn}(h(W(j))) \right\}.$$ 

Define the *induced polyhedron* by

$$M = M^1 \cap M^2 \cap M^3.$$
THE INEQUALITY SYSTEM INDUCED BY A CONFLUENCE VECTOR AND AN UPPER-MIDDLE-LOWER PARTITION

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Let

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where

$$M^3_U = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = \text{sgn}(h(W(j))) \text{ for all } i, j \in C' \cap U \text{ with } (i, j) \in R \right\}$$

and

$$M^3_L = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = -\text{sgn}(h(W(j))) \text{ for all } i, j \in C' \cap L \text{ with } (i, j) \in R \right\}.$$

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- Let

$$M^3 = M_U^3 \cap M_L^3,$$

where

$$M_U^3 = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = \text{sgn}(h(W(j))) \text{ for all } i, j \in C' \cap U \text{ with } (i, j) \in R \right\}$$

and

$$M_L^3 = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = -\text{sgn}(h(W(j))) \text{ for all } i, j \in C' \cap L \text{ with } (i, j) \in R \right\}.$$

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$$M = M^1 \cap M^2 \cap M^3.$$
Assume further that (R.1) and (R.3) hold and let \( h \) be a confluence vector.

For \( i, j \in C' \) with \( (i, j) \in \mathcal{R} \), there exists a partition of their linkage class into \( W(i) \) and \( W(j) \) such that \( i \in W(i), j \in W(j) \), and the only links between \( W(i) \) and \( W(j) \) are \( (i, j) \) and \( (j, i) \).

Let

\[
\mathcal{M}^3 = \mathcal{M}^3_U \cap \mathcal{M}^3_L,
\]

where

\[
\mathcal{M}^3_U = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = \text{sgn}(h(W(j))) \text{ for all } i, j \in C' \cap U \text{ with } (i, j) \in \mathcal{R} \right\}
\]

and

\[
\mathcal{M}^3_L = \left\{ \mu \in \mathbb{R}^n \mid \text{sgn}(\langle Y_j - Y_i, \mu \rangle) = -\text{sgn}(h(W(j))) \text{ for all } i, j \in C' \cap L \text{ with } (i, j) \in \mathcal{R} \right\}.
\]

Define the \textit{induced polyhedron} by

\[
\mathcal{M} = \mathcal{M}^1 \cap \mathcal{M}^2 \cap \mathcal{M}^3.
\]
THE DEFICIENCY-ONE ALGORITHM

A vector $\mu \in \mathbb{R}^n$ is said to be sign compatible with the stoichiometric subspace if there exists a $\sigma \in \text{ran } S$ such that $\text{sgn}(\mu_s) = \text{sgn}(\sigma_s)$ for all $s \in \{1, 2, \ldots, n\}$.

THEOREM (MARTIN FEINBERG, 1988, 1995)

Let $(\mathcal{X}, \mathcal{C}, \mathcal{R})$ be a regular reaction network with $\delta = 1$. Then the following are equivalent.

(A) There exists a $\kappa : \mathcal{R} \to \mathbb{R}_+$ and there exists a positive stoichiometric class $\mathcal{P}$ such that $|E_+^\kappa \cap \mathcal{P}| \geq 2$.

(B) There exists a confluence vector $h \in \mathbb{R}^m$ and an upper-middle-lower partition $\{U, M, L\}$ such that there exists a nonzero vector in the polyhedron induced by $h$ and $\{U, M, L\}$, which is sign compatible with the stoichiometric subspace.
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Let \((\mathcal{X}, \mathcal{C}, \mathcal{R})\) be a regular reaction network. Let \(h\) be a confluence vector and \(\{U, M, L\}\) be an upper-middle-lower partition. Let \(\tilde{h} = h, \tilde{U} = L, \tilde{M} = M, \text{ and } \tilde{L} = U\). Denote by \(\mathcal{M}\) and \(\widetilde{\mathcal{M}}\) the polyhedron induced by \(h\) and \(\{U, M, L\}\) and the polyhedron induced by \(\tilde{h}\) and \(\{\tilde{U}, \tilde{M}, \tilde{L}\}\), respectively. Then the following are equivalent.

(A) There exists a nonzero \(\mu \in \mathcal{M}\) that is sign compatible with the stoichiometric subspace.

(B) There exists a nonzero \(\mu \in \widetilde{\mathcal{M}}\) that is sign compatible with the stoichiometric subspace.
A FEW SHORTCUTS

PROPOSITION (MARTIN FEINBERG, 1995)

Let \((\mathcal{X}, \mathcal{C}, \mathcal{R})\) be a regular reaction network with no trivial terminal strong linkage classes (i.e., \(t' = t\)). Let \(h\) be a confluence vector and \(\{U, M, L\}\) be an upper-middle-lower partition, and assume that there exists a nonzero element of the induced polyhedron that is sign compatible with the stoichiometric subspace. Then \(U \neq \emptyset\) and \(L \neq \emptyset\).

PROPOSITION (MARTIN FEINBERG, 1995)

Let \((\mathcal{X}, \mathcal{C}, \mathcal{R})\) be a regular reaction network with exactly one trivial terminal strong linkage class (i.e., \(t' = t - 1\)). Let \(h\) be a confluence vector, \(\{U, M, L\}\) be an upper-middle-lower partition, and assume that there exists a nonzero element of the induced polyhedron that is sign compatible with the stoichiometric subspace. Then at least one of \(U\) and \(L\) is nonempty.
A FEW SHORTCUTS

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**THE NUMBER OF UPPER-MIDDLE-LOWER PARTITIONS THAT ARE SUFFICIENT TO EXAMINE (BASED ON THE SHORTCUTS)**

<table>
<thead>
<tr>
<th></th>
<th>$\ell = t = 1$</th>
<th>$\ell = t = 2$</th>
<th>$\ell = t = 3$</th>
<th>$\ell = t = 4$</th>
<th>$\ell = t = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t' = t$</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>25</td>
<td>90</td>
</tr>
<tr>
<td>$t' = t - 1$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>13</td>
<td>40</td>
</tr>
<tr>
<td>$t' = t - 2$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>$t' = t - 3$</td>
<td></td>
<td>1</td>
<td>2</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>$t' = t - 4$</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>$t' = t - 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
The network $aX_1 \rightleftharpoons bX_1$, $cX_1 \rightleftharpoons dX_1$

- Assume $a$, $b$, $c$, and $d$ are four distinct nonnegative numbers.

- Then $\hat{Y} = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $h = \begin{bmatrix} \alpha \\ -\alpha \\ 1 \\ -1 \end{bmatrix}$ or $h = \begin{bmatrix} -\alpha \\ \alpha \\ -1 \\ 1 \end{bmatrix}$ for some $\alpha \neq 0$, and $\text{ran } S = \mathbb{R}$. (Easy: $h = 0$ cannot provide a solution.)

- Let $U = \{aX_1, bX_1\}$ and $L = \{cX_1, dX_1\}$.

- Then

  $M^1 = \mathbb{R}$,

  $M^2 = \left\{ \mu \in \mathbb{R} \mid (a - c)\mu > 0, (a - d)\mu > 0, (b - c)\mu > 0, (b - d)\mu > 0 \right\}$, and

  $M^3 = \left\{ \mu \in \mathbb{R} \mid \text{sgn}((a - b)\mu) = \text{sgn}(\alpha), (c - d)\mu < 0 \right\}$ or

  $M^3 = \left\{ \mu \in \mathbb{R} \mid \text{sgn}((a - b)\mu) = \text{sgn}(-\alpha), (c - d)\mu > 0 \right\}$.

- Thus, there exists $\kappa$ such that $|E^\kappa_+| \geq 2$ if and only if either $\max(a, b) < \min(c, d)$ or $\max(c, d) < \min(a, b)$. 

Balázs Boros (Eötvös Univ., Budapest)
THE NETWORK \( aX_1 \Leftrightarrow bX_1, \ cX_1 \Leftrightarrow dX_1 \)

- Assume \( a, b, c, \) and \( d \) are four distinct nonnegative numbers.
- Then \( \hat{Y} = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \), \( h = \begin{bmatrix} \alpha \\ -\alpha \\ 1 \\ -1 \end{bmatrix} \) or \( h = \begin{bmatrix} -\alpha \\ \alpha \\ -1 \\ 1 \end{bmatrix} \) for some \( \alpha \neq 0 \), and \( \text{ran } S = \mathbb{R} \). (Easy: \( h = 0 \) cannot provide a solution.)
- Let \( U = \{aX_1, bX_1\} \) and \( L = \{cX_1, dX_1\} \).
- Then

\[
\mathcal{M}^1 = \mathbb{R},
\]
\[
\mathcal{M}^2 = \left\{ \mu \in \mathbb{R} \mathrel{|} (a - c)\mu > 0, (a - d)\mu > 0, (b - c)\mu > 0, (b - d)\mu > 0 \right\}, \ 	ext{and}
\]
\[
\mathcal{M}^3 = \left\{ \mu \in \mathbb{R} \mathrel{|} \text{sgn}((a - b)\mu) = \text{sgn}(\alpha), (c - d)\mu < 0 \right\} \text{ or }
\]
\[
\mathcal{M}^3 = \left\{ \mu \in \mathbb{R} \mathrel{|} \text{sgn}((a - b)\mu) = \text{sgn}(-\alpha), (c - d)\mu > 0 \right\}.
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- Thus, there exists \( \kappa \) such that \( |E^\kappa_+| \geq 2 \) if and only if either \( \max(a, b) < \min(c, d) \) or \( \max(c, d) < \min(a, b) \).
The network $aX_1 \iff bX_1$, $cX_1 \iff dX_1$

- Assume $a$, $b$, $c$, and $d$ are four distinct nonnegative numbers.

- Then $
\hat{Y} = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ h = \begin{bmatrix} \alpha \\ -\alpha \\ 1 \\ -1 \end{bmatrix} \text{ or } h = \begin{bmatrix} -\alpha \\ \alpha \\ -1 \\ 1 \end{bmatrix} \text{ for some } 
\alpha \neq 0, \text{ and } \text{ran } S = \mathbb{R}. \text{ (Easy: } h = 0 \text{ cannot provide a solution.)}

- Let $U = \{aX_1, bX_1\}$ and $L = \{cX_1, dX_1\}$.

- Then

\[ M^1 = \mathbb{R}, \]

\[ M^2 = \left\{ \mu \in \mathbb{R} \mid (a - c)\mu > 0, (a - d)\mu > 0, (b - c)\mu > 0, (b - d)\mu > 0 \right\}, \text{ and} \]

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Then \( \hat{Y} = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ h = \begin{bmatrix} \alpha \\ -\alpha \\ 1 \\ -1 \end{bmatrix} \) or \( h = \begin{bmatrix} -\alpha \\ \alpha \\ -1 \\ 1 \end{bmatrix} \) for some \( \alpha \neq 0 \), and \( \text{ran } S = \mathbb{R} \). (Easy: \( h = 0 \) cannot provide a solution.)

Let \( U = \{aX_1, bX_1\} \) and \( L = \{cX_1, dX_1\} \).

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\[
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Thus, there exists \( \kappa \) such that \( |E_+^\kappa| \geq 2 \) if and only if either \( \max(a, b) < \min(c, d) \) or \( \max(c, d) < \min(a, b) \).
THE NETWORK $aX_1 \Leftrightarrow bX_1$, $cX_1 \Leftrightarrow dX_1$

- Assume $a$, $b$, $c$, and $d$ are four distinct nonnegative numbers.

- Then $\hat{Y} = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $h = \begin{bmatrix} \alpha \\ -\alpha \\ -1 \end{bmatrix}$ or $h = \begin{bmatrix} -\alpha \\ \alpha \\ -1 \end{bmatrix}$ for some $\alpha \neq 0$, and $\text{ran } S = \mathbb{R}$. (Easy: $h = 0$ cannot provide a solution.)

- Let $U = \{aX_1, bX_1\}$ and $L = \{cX_1, dX_1\}$.

- Then

  $M^1 = \mathbb{R}$,

  $M^2 = \left\{ \mu \in \mathbb{R} \bigg| \begin{array}{l} (a - c)\mu > 0, (a - d)\mu > 0, \\ (b - c)\mu > 0, (b - d)\mu > 0 \end{array} \right\}$, and

  $M^3 = \left\{ \mu \in \mathbb{R} \bigg| \text{sgn}((a - b)\mu) = \text{sgn}(\alpha), (c - d)\mu < 0 \right\}$ or

  $M^3 = \left\{ \mu \in \mathbb{R} \bigg| \text{sgn}((a - b)\mu) = \text{sgn}(-\alpha), (c - d)\mu > 0 \right\}$.

- Thus, there exists $\kappa$ such that $|E_+^\kappa| \geq 2$ if and only if either $\max(a, b) < \min(c, d)$ or $\max(c, d) < \min(a, b)$. 
The network $X_1 \rightleftharpoons X_2$, $X_1 + X_2 \rightleftharpoons 2X_2$

- Then ran $S = \left\{ \begin{bmatrix} \beta \\ -\beta \end{bmatrix} \bigg| \beta \in \mathbb{R} \right\}$. 
- Let $U = \{X_1, X_2\}$ and $L = \{X_1 + X_2, 2X_2\}$. 
- Then
  \[
  \mathcal{M}^2 = \{ \mu \in \mathbb{R}^2 \mid \mu_1 > \mu_1 + \mu_2, \mu_2 > \mu_1 + \mu_2, \mu_1 > 2\mu_2, \mu_2 > 2\mu_2 \} = \\
  = \{ \mu \in \mathbb{R}^2 \mid \mu_2 < 0, \mu_1 < 0, \mu_1 > 2\mu_2 \}. 
  \]
- There is no $\mu \in \mathcal{M}^2$ that is sign compatible with the stoichiometric subspace.
- Thus, for all $\kappa$ and for all positive stoichiometric classes $\mathcal{P}$, we have $|E_+^\kappa \cap \mathcal{P}| = 1$. (We know the existence from elsewhere.)
The network $X_1 \rightleftharpoons X_2, X_1 + X_2 \rightleftharpoons 2X_2$

Then ran $S = \left\{ \begin{bmatrix} \beta \\ -\beta \end{bmatrix} | \beta \in \mathbb{R} \right\}$.

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$$\mathcal{M}^2 = \{ \mu \in \mathbb{R}^2 | \mu_1 > \mu_1 + \mu_2, \mu_2 > \mu_1 + \mu_2, \mu_1 > 2\mu_2, \mu_2 > 2\mu_2 \} = \{ \mu \in \mathbb{R}^2 | \mu_2 < 0, \mu_1 < 0, \mu_1 > 2\mu_2 \}.$$  

There is no $\mu \in \mathcal{M}^2$ that is sign compatible with the stoichiometric subspace.

Thus, for all $\kappa$ and for all positive stoichiometric classes $\mathcal{P}$, we have $|E_+^{\kappa} \cap \mathcal{P}| = 1$. (We know the existence from elsewhere.)
The network $X_1 \rightleftharpoons X_2, X_1 + X_2 \rightleftharpoons 2X_2$

- Then ran $S = \left\{ \begin{bmatrix} \beta \\ -\beta \end{bmatrix} \right| \beta \in \mathbb{R} \right\}$.
- Let $U = \{X_1, X_2\}$ and $L = \{X_1 + X_2, 2X_2\}$.
- Then
  \[ M^2 = \{ \mu \in \mathbb{R}^2 \mid \mu_1 > \mu_1 + \mu_2, \mu_2 > \mu_1 + \mu_2, \mu_1 > 2\mu_2, \mu_2 > 2\mu_2 \} = \{ \mu \in \mathbb{R}^2 \mid \mu_2 < 0, \mu_1 < 0, \mu_1 > 2\mu_2 \}. \]

- There is no $\mu \in M^2$ that is sign compatible with the stoichiometric subspace.
- Thus, for all $\kappa$ and for all positive stoichiometric classes $\mathcal{P}$, we have $|E_+^\kappa \cap \mathcal{P}| = 1$. (We know the existence from elsewhere.)
The network $X_1 \Leftrightarrow X_2, X_1 + X_2 \Leftrightarrow 2X_2$

- Then ran $S = \begin{bmatrix} \beta \\ -\beta \end{bmatrix} \mid \beta \in \mathbb{R}$.
- Let $U = \{X_1, X_2\}$ and $L = \{X_1 + X_2, 2X_2\}$.
- Then
  \[ M^2 = \{ \mu \in \mathbb{R}^2 \mid \mu_1 > \mu_1 + \mu_2, \mu_2 > \mu_1 + \mu_2, \mu_1 > 2\mu_2, \mu_2 > 2\mu_2 \} = \{ \mu \in \mathbb{R}^2 \mid \mu_2 < 0, \mu_1 < 0, \mu_1 > 2\mu_2 \} \]

- There is no $\mu \in M^2$ that is sign compatible with the stoichiometric subspace.
- Thus, for all $\kappa$ and for all positive stoichiometric classes $\mathcal{P}$, we have $|E_{\kappa}^+ \cap \mathcal{P}| = 1$. (We know the existence from elsewhere.)
The network $X_1 \Leftrightarrow X_2, \ X_1 + X_2 \Leftrightarrow 2X_2$

- Then ran $S = \left\{ \begin{bmatrix} \beta \\ -\beta \end{bmatrix} \mid \beta \in \mathbb{R} \right\}$.
- Let $U = \{X_1, X_2\}$ and $L = \{X_1 + X_2, 2X_2\}$.
- Then

$$M^2 = \{\mu \in \mathbb{R}^2 \mid \mu_1 > \mu_1 + \mu_2, \mu_2 > \mu_1 + \mu_2, \mu_1 > 2\mu_2, \mu_2 > 2\mu_2\} =$$

$$= \{\mu \in \mathbb{R}^2 \mid \mu_2 < 0, \mu_1 < 0, \mu_1 > 2\mu_2\}.$$  

- There is no $\mu \in M^2$ that is sign compatible with the stoichiometric subspace.
- Thus, for all $\kappa$ and for all positive stoichiometric classes $\mathcal{P}$, we have $|E^{\kappa}_+ \cap \mathcal{P}| = 1$. (We know the existence from elsewhere.)
The network $X_1 \Leftrightarrow X_2, X_1 + 2X_2 \Leftrightarrow 3X_2$

Then ran $S = \left\{ \left[ \begin{array}{c} \beta \\ -\beta \end{array} \right] \mid \beta \in \mathbb{R} \right\}$.

Let $U = \{X_1, X_2\}$ and $L = \{X_1 + 2X_2, 3X_2\}$.

Then

$M^1 = \mathbb{R}^2$ and

$M^2 = \{\mu \in \mathbb{R}^2 \mid \mu_1 > \mu_1 + 2\mu_2, \mu_2 > \mu_1 + 2\mu_2, \mu_1 > 3\mu_2, \mu_2 > 3\mu_2\} = \{\mu \in \mathbb{R}^2 \mid \mu_2 < 0, \mu_1 + \mu_2 < 0, \mu_1 > 3\mu_2\}$. 
The network $X_1 \leftrightarrow X_2, X_1 + 2X_2 \leftrightarrow 3X_2$

- Then ran $S = \left\{ \left[ \begin{array}{c} \beta \\ -\beta \end{array} \right] \mid \beta \in \mathbb{R} \right\}$.
- Let $U = \{X_1, X_2\}$ and $L = \{X_1 + 2X_2, 3X_2\}$.
- Then

  $\mathcal{M}^1 = \mathbb{R}^2$ and
  $\mathcal{M}^2 = \{\mu \in \mathbb{R}^2 \mid \mu_1 > \mu_1 + 2\mu_2, \mu_2 > \mu_1 + 2\mu_2, \mu_1 > 3\mu_2, \mu_2 > 3\mu_2\} = \{\mu \in \mathbb{R}^2 \mid \mu_2 < 0, \mu_1 + \mu_2 < 0, \mu_1 > 3\mu_2\}$. 
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Let $U = \{X_1, X_2\}$ and $L = \{X_1 + 2X_2, 3X_2\}$.

Then

$M^1 = \mathbb{R}^2$ and

$M^2 = \{\mu \in \mathbb{R}^2 \mid \mu_1 > \mu_1 + 2\mu_2, \mu_2 > \mu_1 + 2\mu_2, \mu_1 > 3\mu_2, \mu_2 > 3\mu_2\} = \{\mu \in \mathbb{R}^2 \mid \mu_2 < 0, \mu_1 + \mu_2 < 0, \mu_1 > 3\mu_2\}$. 


The network $X_1 \iff X_2, X_1 + 2X_2 \iff 3X_2$

- We have $\hat{Y} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

- For $h = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $h = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, and $h = 0$, we have $M^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 < \mu_2\}$, $M^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 > \mu_2\}$, and $M^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 = \mu_2\}$, respectively.

- In the first and third cases, there is no nonzero element of $M$ that is sign compatible with the stoichiometric subspace.

- However, in the second case, $\mu_1 = 1$ and $\mu_2 = -2$ is a solution.

- Thus, there exists a $\kappa$ and there exists a positive stoichiometric class $P$ such that $|E_+^\kappa \cap P| \geq 2$. 
The network $X_1 \leftrightarrow X_2$, $X_1 + 2X_2 \leftrightarrow 3X_2$

- We have $\hat{Y} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

- For $h = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $h = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, and $h = 0$, we have $\mathcal{M}^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 < \mu_2\}$, $\mathcal{M}^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 > \mu_2\}$, and $\mathcal{M}^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 = \mu_2\}$, respectively.

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The network $X_1 \Leftrightarrow X_2, X_1 + 2X_2 \Leftrightarrow 3X_2$

- We have $\hat{Y} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

- For $h = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $h = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, and $h = 0$, we have $M^3 = \{ \mu \in \mathbb{R}^2 \mid \mu_1 < \mu_2 \}$, $M^3 = \{ \mu \in \mathbb{R}^2 \mid \mu_1 > \mu_2 \}$, and $M^3 = \{ \mu \in \mathbb{R}^2 \mid \mu_1 = \mu_2 \}$, respectively.

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- Thus, there exists a $\kappa$ and there exists a positive stoichiometric class $\mathcal{P}$ such that $|E^\kappa_+ \cap \mathcal{P}| \geq 2$. 
The network $X_1 \Leftrightarrow X_2, X_1 + 2X_2 \Leftrightarrow 3X_2$

- We have $\hat{Y} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

- For $h = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $h = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, and $h = 0$, we have $\mathcal{M}^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 < \mu_2\}$, $\mathcal{M}^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 > \mu_2\}$, and $\mathcal{M}^3 = \{\mu \in \mathbb{R}^2 \mid \mu_1 = \mu_2\}$, respectively.

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- Thus, there exists a $\kappa$ and there exists a positive stoichiometric class $\mathcal{P}$ such that $|E_+^\kappa \cap \mathcal{P}| \geq 2$. 
The network $X_1 \leftrightarrow X_2, X_1 + 2X_2 \leftrightarrow 3X_2$

- We have $\hat{Y} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

- For $h = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $h = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, and $h = 0$, we have $M^3 = \{ \mu \in \mathbb{R}^2 | \mu_1 < \mu_2 \}$, $M^3 = \{ \mu \in \mathbb{R}^2 | \mu_1 > \mu_2 \}$, and $M^3 = \{ \mu \in \mathbb{R}^2 | \mu_1 = \mu_2 \}$, respectively.

- In the first and third cases, there is no nonzero element of $M$ that is sign compatible with the stoichiometric subspace.

- However, in the second case, $\mu_1 = 1$ and $\mu_2 = -2$ is a solution.

- Thus, there exists a $\kappa$ and there exists a positive stoichiometric class $\mathcal{P}$ such that $|E_+^\kappa \cap \mathcal{P}| \geq 2$. 
THE NETWORK $X_1 \leftrightarrow X_2$, $X_1 + 2X_2 \leftrightarrow 3X_2$
**THE NETWORK** \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s, \)
\( \sum_{s=1}^{n} c_s X_s \rightleftharpoons \sum_{s=1}^{n} d_s X_s \)

**PROPOSITION**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s, \)
\( \sum_{s=1}^{n} c_s X_s \rightleftharpoons \sum_{s=1}^{n} d_s X_s, \) where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}^n \geq 0 \) with \( b - a \) and \( d - c \) being linearly dependent. Then

(I) the network is a regular deficiency-one network and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \to \mathbb{R}_+ \) and there exists a positive stoichiometric class \( \mathcal{P} \) such that \( |E^\kappa_+ \cap \mathcal{P}| \geq 2. \)

(B) There exists an \( s^* \in \{1, 2, \ldots, n\} \) such that either

\[
\min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}) < \min(c_{s^*}, d_{s^*}) < \max(c_{s^*}, d_{s^*})
\]

or

\[
\min(c_{s^*}, d_{s^*}) < \max(c_{s^*}, d_{s^*}) < \min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}).
\]
Consider the reaction network \(\sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s\),
\(\sum_{s=1}^{n} c_s X_s \rightleftharpoons \sum_{s=1}^{n} d_s X_s\), where \(a, b, c,\) and \(d\) are four distinct elements of \(\mathbb{R}^n_{\geq 0}\) with \(b - a\) and \(d - c\) being linearly dependent. Then

(1) the network is a regular deficiency-one network and

(II) the following are equivalent.

(A) There exists a \(\kappa : \mathcal{R} \rightarrow \mathbb{R}_+\) and there exists a positive stoichiometric class \(\mathcal{P}\) such that \(|E_+^\kappa \cap \mathcal{P}| \geq 2\).

(B) There exists an \(s^* \in \{1, 2, \ldots, n\}\) such that either

\[\min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}) < \min(c_{s^*}, d_{s^*}) < \max(c_{s^*}, d_{s^*})\]

or

\[\min(c_{s^*}, d_{s^*}) < \max(c_{s^*}, d_{s^*}) < \min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}).\]
THE NETWORK \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s \),
\( \sum_{s=1}^{n} c_s X_s \rightleftharpoons \sum_{s=1}^{n} d_s X_s \)

**Proposition**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s \),
\( \sum_{s=1}^{n} c_s X_s \rightleftharpoons \sum_{s=1}^{n} d_s X_s \), where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}^n \geq 0 \) with \( b - a \) and \( d - c \) being linearly dependent. Then

(I) the network is a regular deficiency-one network and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) and there exists a positive stoichiometric class \( \mathcal{P} \) such that \( |E^\kappa_+ \cap \mathcal{P}| \geq 2 \).

(B) There exists an \( s^* \in \{1, 2, \ldots, n\} \) such that either

\[
\min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}) < \min(c_{s^*}, d_{s^*}) < \max(c_{s^*}, d_{s^*})
\]

or

\[
\min(c_{s^*}, d_{s^*}) < \max(c_{s^*}, d_{s^*}) < \min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}).
\]
THE NETWORK \[ \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s, \]
\[ \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s, \]

PROPOSITION

Consider the reaction network \[ \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s, \]
\[ \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s, \] where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}^n_{\geq 0} \) with \( b - a \) and \( d - c \) being linearly dependent. Then

(I) the network is a regular deficiency-one network and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) and there exists a positive stoichiometric class \( \mathcal{P} \) such that \( |E^*_+ \cap \mathcal{P}| \geq 2. \)

(B) There exists an \( s^* \in \{1, 2, \ldots, n\} \) such that either

\[ \min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}) < c_{s^*} < d_{s^*} \]

or

\[ d_{s^*} < c_{s^*} < \min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}). \]
**Proposition**

Consider the reaction network \[ \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s, \]
\[ \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s, \]
where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}^n \geq 0 \) with \( b - a \) and \( d - c \) being linearly dependent. Then

(I) the network is a regular deficiency-one network and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) and there exists a positive stoichiometric class \( \mathcal{P} \) such that \( |E_+^\kappa \cap \mathcal{P}| \geq 2. \)

(B) There exists an \( s^* \in \{1, 2, \ldots, n\} \) such that either

\[
\min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}) < c_{s^*} < d_{s^*}
\]

or

\[
d_{s^*} < c_{s^*} < \min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}).
\]
Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s \), 
\( \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s \), where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}_{\geq 0} \) with \( b - a \) and \( d - c \) being linearly dependent. Then

(I) the network is a regular deficiency-one network and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) and there exists a positive stoichiometric class \( \mathcal{P} \) such that \( |E_+^\kappa \cap \mathcal{P}| \geq 2 \).

(B) There exists an \( s^* \in \{1, 2, \ldots, n\} \) such that either

\[
\min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}) < c_{s^*} < d_{s^*}
\]

or

\[
d_{s^*} < c_{s^*} < \min(a_{s^*}, b_{s^*}) < \max(a_{s^*}, b_{s^*}).
\]
Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s \),
\( \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s \), where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}_{\geq 0}^n \) with
\[
d - c = \alpha \cdot (a - b) \text{ for some } \alpha \neq 0.
\]
Then
(I) the network is of deficiency-one and
(II) the network is regular if and only if \( \alpha > 0 \).
Consider the reaction network
\[ \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, \]
\[ \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s, \]
where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}^n_{\geq 0} \) with
\[ d - c = \alpha \cdot (a - b) \quad \text{for some} \quad \alpha \neq 0. \]

Then

(I) the network is of deficiency-one and

(II) the network is regular if and only if \( \alpha > 0 \).
Consider the reaction network 
\[ \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, \]
\[ \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s, \]
where \(a, b, c,\) and \(d\) are four distinct elements of \(\mathbb{R}^n\) with
\[ d - c = \alpha \cdot (a - b) \text{ for some } \alpha \neq 0. \]

Then
(1) the network is of deficiency-one and
(2) the network is regular if and only if \(\alpha > 0.\)
Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, \)
\( \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s, \)
where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}_{\geq 0}^n \) with \( b - a \) and \( d - c \) being linearly dependent.
Assume that the network is regular. Then the following are equivalent.

(A) There exists a \( \kappa : R \rightarrow \mathbb{R}_+ \) and there exists a positive stoichiometric class \( \mathcal{P} \) such that \( |E_+^\kappa \cap \mathcal{P}| \geq 2. \)
Consider the reaction network
\[ \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, \]
\[ \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s, \]
where \( a, b, c, \) and \( d \) are four distinct elements of \( \mathbb{R}_{\geq 0}^n \) with \( b - a \) and \( d - c \) being linearly dependent.
Assume that the network is regular. Then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) and there exists a positive stoichiometric class \( \mathcal{P} \) such that \( |E^\kappa_\pm \cap \mathcal{P}| \geq 2. \)
The network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, \)
\( \sum_{s=1}^{n} c_s X_s \rightarrow \sum_{s=1}^{n} d_s X_s \)

**Proposition (continued from the previous slide)**

**(B) Either**

- there exists a partition \( \mathcal{X} = \mathcal{X}_1 \cup^* \mathcal{X}_2 \) such that
  
  \( a_s = b_s \) for all \( s \in \mathcal{X}_1 \) and \( a_s = c_s \) for all \( s \in \mathcal{X}_2 \) or

- there exist \( s^*, s^{**} \in \{1, 2, \ldots, n\} \) such that either
  
  \[ b_{s^*} > a_{s^*} < c_{s^*} > d_{s^*} \quad \text{and} \quad b_{s^{**}} < a_{s^{**}} < c_{s^{**}} < d_{s^{**}}, \]
  
  \[ b_{s^*} > a_{s^*} < c_{s^*} > d_{s^*} \quad \text{and} \quad b_{s^{**}} > a_{s^{**}} > c_{s^{**}} > d_{s^{**}}, \]
  
  \[ d_{s^*} > c_{s^*} < a_{s^*} > b_{s^*} \quad \text{and} \quad d_{s^{**}} < c_{s^{**}} < a_{s^{**}} < b_{s^{**}}, \quad \text{or} \]
  
  \[ d_{s^*} > c_{s^*} < a_{s^*} > b_{s^*} \quad \text{and} \quad d_{s^{**}} > c_{s^{**}} > a_{s^{**}} > b_{s^{**}}. \]
The network \( \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftrightarrow 0 \)
\((1 \leq s \leq n)\)

From now on, assume that \( a \) and \( b \) are two distinct elements of \( \mathbb{R}^n_{\geq 0} \) with \( a, b, e_1, e_2, \ldots, e_n, \) and 0 being \( n + 3 \) distinct elements of \( \mathbb{R}^n_{\geq 0} \).

**Proposition (Badal Joshi, 2013)**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftrightarrow 0 \)
\((1 \leq s \leq n)\). Then

(I) the network is regular and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathbb{R} \rightarrow \mathbb{R}^+ \) such that \( |E_+^{\kappa}| \geq 2 \).

(B) At least one of the sums

\[
\sum_{s \in \{1, \ldots, n\}} a_s \quad \text{and} \quad \sum_{s \in \{1, \ldots, n\}} b_s
\]

\( \quad a_s < b_s \quad \text{and} \quad a_s > b_s \)

is strictly greater than 1.
The network \( \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftrightarrow 0 \) 
\((1 \leq s \leq n)\)

From now on, assume that \( a \) and \( b \) are two distinct elements of \( \mathbb{R}^n_{\geq 0} \) with \( a, b, e_1, e_2, \ldots, e_n, \) and 0 being \( n + 3 \) distinct elements of \( \mathbb{R}^n_{\geq 0} \).

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Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftrightarrow 0 \) 
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(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E_+^\kappa| \geq 2 \).

(B) At least one of the sums

\[
\sum_{s \in \{1, \ldots, n\} \atop a_s < b_s} a_s \quad \text{and} \quad \sum_{s \in \{1, \ldots, n\} \atop a_s > b_s} b_s
\]

is strictly greater than 1.
THE NETWORK $\sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftarrow 0$

$(1 \leq s \leq n)$

From now on, assume that $a$ and $b$ are two distinct elements of $\mathbb{R}^{n}_{\geq 0}$ with $a$, $b$, $e_1$, $e_2$, $\ldots$, $e_n$, and $0$ being $n + 3$ distinct elements of $\mathbb{R}^{n}_{\geq 0}$.

**Proposition (Badal Joshi, 2013)**

Consider the reaction network $\sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftarrow 0$

$(1 \leq s \leq n)$. Then

(I) the network is regular and

(II) the following are equivalent.

(A) There exists a $\kappa : \mathcal{R} \rightarrow \mathbb{R}_{+}$ such that $|E_{+}^{\kappa}| \geq 2$.

(B) At least one of the sums

$$\sum_{s \in \{1, \ldots, n\} \atop a_s < b_s} a_s$$

and

$$\sum_{s \in \{1, \ldots, n\} \atop a_s > b_s} b_s$$

is strictly greater than 1.
Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftrightarrow \) 0 \((1 \leq s \leq n)\).

Then

(1) the network is regular and

(2) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E^\kappa_+| \geq 2 \).

(B) The sum

\[
\sum_{s \in \{1, \ldots, n\} \atop a_s < b_s} a_s
\]

is strictly greater than 1.
The network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftrightarrow 0 \) 
(\( 1 \leq s \leq n \))

**Proposition (Badal Joshi, 2013)**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \leftrightarrow 0 \) 
(\( 1 \leq s \leq n \)). Then

(I) the network is regular and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E_+^{\kappa}| \geq 2 \).

(B) The sum

\[
\sum_{s \in \{1, \ldots, n\}, \ a_s < b_s} a_s
\]

is strictly greater than 1.
Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, \ X_s \leftrightarrow 0 \) (\( 1 \leq s \leq n \)). Then

(I) the network is regular and

(II) the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E^\kappa_+| \geq 2 \).

(B) The sum

\[
\sum_{s \in \{1, \ldots, n\}: \ a_s < b_s} a_s
\]

is strictly greater than 1.
**Proposition**

Consider the reaction network
\[ \sum_{s=1}^{n} a_s X_s \iff \sum_{s=1}^{n} b_s X_s, \ X_s \to 0 \]
\((1 \leq s \leq n)\). Then

(I) the network is regular if and only if either

\[ a_s < b_s \text{ for all } 1 \leq s \leq n \text{ or } a_s > b_s \text{ for all } 1 \leq s \leq n \]

and

(II) if the network is regular then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \to \mathbb{R}_+ \) such that \( |E^\kappa| \geq 2 \).

(B) Both of the inequalities

\[ \sum_{s=1}^{n} a_s > 1 \text{ and } \sum_{s=1}^{n} b_s > 1 \]

hold.
The network \( \sum_{s=1}^{n} a_s X_s \iff \sum_{s=1}^{n} b_s X_s, X_s \to 0 \) 
(\(1 \leq s \leq n\))

**Proposition**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \iff \sum_{s=1}^{n} b_s X_s, X_s \to 0 \) 
(\(1 \leq s \leq n\)). Then

(I) the network is regular if and only if either

\[ a_s < b_s \text{ for all } 1 \leq s \leq n \text{ or } a_s > b_s \text{ for all } 1 \leq s \leq n \]

and

(II) if the network is regular then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \to \mathbb{R}_+ \) such that \( |E^\kappa_+| \geq 2 \).

(B) Both of the inequalities

\[ \sum_{s=1}^{n} a_s > 1 \text{ and } \sum_{s=1}^{n} b_s > 1 \]

hold.
**The network** \[ \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, \ X_s \rightarrow 0 \]

(1 \leq s \leq n)

**Proposition**

Consider the reaction network \[ \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s, \ X_s \rightarrow 0 \]

(1 \leq s \leq n). Then

(I) the network is regular if and only if either

\[ a_s < b_s \text{ for all } 1 \leq s \leq n \text{ or } a_s > b_s \text{ for all } 1 \leq s \leq n \]

and

(II) if the network is regular then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E_{++}^{\kappa}| \geq 2 \).

(B) Both of the inequalities

\[ \sum_{s=1}^{n} a_s > 1 \text{ and } \sum_{s=1}^{n} b_s > 1 \]

hold.
THE NETWORK \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \rightarrow 0 \) 
\( 1 \leq s \leq n \)

**PROPOSITION**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \rightarrow 0 \) 
\( 1 \leq s \leq n \). Then

(I) the network is regular if and only if

\[ a_s < b_s \text{ for all } 1 \leq s \leq n \]

holds and

(II) if the network is regular then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E_+^\kappa| \geq 2 \).

(B) The equality

\[ \sum_{s=1}^{n} a_s = 1 \]

holds.
THE NETWORK $\sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \rightarrow 0$
($1 \leq s \leq n$)

**Proposition**

Consider the reaction network $\sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \rightarrow 0$
($1 \leq s \leq n$). Then

(I) the network is regular if and only if

$$a_s < b_s \text{ for all } 1 \leq s \leq n$$

holds and

(II) if the network is regular then the following are equivalent.

(A) There exists a $\kappa : \mathcal{R} \rightarrow \mathbb{R}_+$ such that $|E_+^\kappa| \geq 2$.

(B) The equality

$$\sum_{s=1}^{n} a_s = 1$$

holds.
The network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \rightarrow 0 \) 
(1 \( \leq s \leq n \))

**Proposition**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s, X_s \rightarrow 0 \) 
(1 \( \leq s \leq n \)). Then

(I) the network is regular if and only if

\[ a_s < b_s \text{ for all } 1 \leq s \leq n \]

holds and

(II) if the network is regular then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E^\kappa_+| \geq 2 \).

(B) The equality

\[ \sum_{s=1}^{n} a_s = 1 \]

holds.
0 → \( X_n \)

**Proposition**

Assume we have either

\[
\sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s \quad \text{or} \quad \sum_{s=1}^{n} a_s X_s \rightarrow \sum_{s=1}^{n} b_s X_s.
\]

Assume further that we have the reactions

\[
X_s \rightarrow 0 \text{ for } 1 \leq s \leq k,
\]

\[
X_s \rightleftharpoons 0 \text{ for } k + 1 \leq s \leq n - 1,
\]

\[
0 \rightarrow X_n
\]

for some \( 0 \leq k \leq n - 1 \). Then for all \( \kappa \) we have \( |E_{+}^{\kappa}| \leq 1 \).
Proposition

Assume we have either

\[ \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s \text{ or } \sum_{s=1}^{n} a_s X_s \to \sum_{s=1}^{n} b_s X_s. \]

Assume further that we have the reactions

\[ X_s \to 0 \text{ for } 1 \leq s \leq k, \]
\[ X_s \leftrightarrow 0 \text{ for } k + 1 \leq s \leq n - 1, \]
\[ 0 \to X_n \]

for some \(0 \leq k \leq n - 1\). Then for all \(\kappa\) we have \(|E_+^\kappa| \leq 1\).
Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \leftrightarrow \sum_{s=1}^{n} b_s X_s \) and
\[
X_s \to 0 \text{ for } 1 \leq s \leq k, \\
X_s \leftrightarrow 0 \text{ for } k + 1 \leq s \leq n
\]
for some \( 1 \leq k \leq n - 1 \). Then

(1) the network is regular if and only if either
\[
a_s < b_s \text{ for all } 1 \leq s \leq k \text{ or } a_s > b_s \text{ for all } 1 \leq s \leq k
\]
and

(II) if the network is regular with \( a_s < b_s \) for all \( 1 \leq s \leq k \) then the following are equivalent.

(A) There exists a \( \kappa: \mathbb{R} \to \mathbb{R}_+ \) such that \( |E^\kappa_+| \geq 2 \).
(B) The inequality \( \sum_{s:a_s < b_s} a_s > 1 \) holds.
Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s \) and
\[
X_s \rightarrow 0 \text{ for } 1 \leq s \leq k, \\
X_s \leftrightarrow 0 \text{ for } k + 1 \leq s \leq n
\]
for some \( 1 \leq k \leq n - 1 \). Then

(I) the network is regular if and only if either
\[
a_s < b_s \text{ for all } 1 \leq s \leq k \text{ or } a_s > b_s \text{ for all } 1 \leq s \leq k
\]

and

(II) if the network is regular with \( a_s < b_s \) for all \( 1 \leq s \leq k \) then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E_{+}^\kappa| \geq 2 \).
(B) The inequality \( \sum_{s:a_s < b_s} a_s > 1 \) holds.
The network \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s, \ X_s \rightarrow 0 \) (1 ≤ s ≤ k), \( X_s \rightleftharpoons 0 \) (k + 1 ≤ s ≤ n)

**Proposition**

Consider the reaction network \( \sum_{s=1}^{n} a_s X_s \rightleftharpoons \sum_{s=1}^{n} b_s X_s \) and

\[
X_s \rightarrow 0 \text{ for } 1 \leq s \leq k, \\
X_s \rightleftharpoons 0 \text{ for } k + 1 \leq s \leq n
\]

for some 1 ≤ k ≤ n − 1. Then

(I) the network is regular if and only if either

\[ a_s < b_s \text{ for all } 1 \leq s \leq k \text{ or } a_s > b_s \text{ for all } 1 \leq s \leq k \]

and

(II) if the network is regular with \( a_s < b_s \) for all 1 ≤ s ≤ k then the following are equivalent.

(A) There exists a \( \kappa : \mathcal{R} \rightarrow \mathbb{R}_+ \) such that \( |E^\kappa_+| \geq 2 \).

(B) The inequality \( \sum_{s:a_s<b_s} a_s > 1 \) holds.
REFERENCES


- B. Boros. Revisiting the Deficiency-One Algorithm. *In preparation.*
These slides are available at

www.cs.elte.hu/~bboros