Cohomology of Nilpotent Subalgebras of Affine Lie Algebras

A. Fialowski


Stable URL: 
http://links.jstor.org/sici?sici=0002-9939%28199409%29122%3A1%3C71%3ACONSOA%3E2.0.CO%3B2-F

*Proceedings of the American Mathematical Society* is currently published by American Mathematical Society.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at 
http://www.jstor.org/about/terms. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained 
prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in 
the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at 
http://www.jstor.org/journals/ams.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed 
page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic 
journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, 
and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take 
advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.
COHOMOLOGY OF NILPOTENT SUBALGEBRAS
OF AFFINE LIE ALGEBRAS

A. FIALOWSKI

(Communicated by Roe Goodman)

ABSTRACT. We compute the cohomology of the maximal nilpotent Lie algebra of an affine Lie algebra \( \hat{g} \) with coefficients in modules of functions on the circle with values in a representation space of \( g \). These modules are not highest weight modules and are somewhat similar to the adjoint representation.

INTRODUCTION

Let \( g \) be a finite-dimensional semisimple Lie algebra, \( b \) a Borel subalgebra of \( g \), and \( n_+ \subset b \) the maximal nilpotent ideal of \( b \). The Bott-Kostant Theorem for Lie algebra cohomology is the following.

Theorem [K]. Let \( V \) be an irreducible representation of \( g \) with dominant highest weight and \( n \) a maximal nilpotent subalgebra of \( g \). Then \( \dim H^i(n; V) \) is equal to the number of elements of length \( i \) in the Weyl group of \( g \).

Consider the affine infinite-dimensional graded Lie algebra \( \hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \) corresponding to \( g \), with \( \hat{g}_i = g \otimes t^i \). There are at least two analogues of the above Theorem for affine algebras. The most direct analogue is the following: if \( V \) is an irreducible representation of the current algebra \( \hat{g} \) with dominant highest weight and \( \hat{n}_+ \) is a maximal nilpotent subalgebra of \( \hat{g} \), that is,

\[
\hat{n}_+ = (n_+ \otimes 1) \oplus (g \otimes t) \oplus (g \otimes t^2) \oplus \cdots,
\]

then \( \dim H^i(\hat{n}_+; V) \) is equal to the number of elements of length \( i \) in the Weyl group. This Theorem was proved by Garland in 1975 [G] and Garland and Lepowsky in 1976 (see [GL]). The proof is similar to that of the finite-dimensional case.

In this paper we present the proof of a different analogue of the Bott-Kostant Theorem obtained jointly with Feigin and announced in [FF]. Namely, we compute the cohomology of \( \hat{n}_+ \) with coefficients in modules of functions on the
circle $S^1$ with values in a representation space of $g$. These modules are not highest weight modules and are somewhat similar to the adjoint representation.

**RESULTS**

Let $V$ be a representation of $g$, $A$ a $\mathbb{C}$-algebra, and $\varphi : \mathbb{C}[t, t^{-1}] \to A$ a homomorphism. Let us define a representation of $\hat{g}$ on $V \otimes A$ by

$$(x \otimes f)(v \otimes a) = x(v) \otimes \varphi(f)a,$$

where $x \in g$, $v \in V$, $f \in \mathbb{C}[t, t^{-1}]$, and $a \in A$.

Consider two special cases for $A$ and $\varphi$:

(a) $A = \mathbb{C}[t, t^{-1}]$ and $\varphi = \text{id}$. In this case denote the module $V \otimes A$ by $\hat{V}$. It consists of rational functions $\mathbb{C} \to V$ that are regular outside the origin.

(b) $A = \mathbb{C}$ and $\varphi(f) = f(1)$. In this case denote the module $V \otimes A$ by $V_1$.

Note that the map assigning to a function $\mathbb{C} \to V$ its value at 1 defines a homomorphism $\hat{V} \to V_1$. The space $\hat{V}$ is endowed with an obvious module structure over the algebra $\mathbb{C}[t, t^{-1}]$, and it is easy to see that multiplication by an element of $\mathbb{C}[t, t^{-1}]$ is a $g$-endomorphism of the $\hat{g}$-module $\hat{V}$. Finally, note that $\hat{V}$ is a graded $\hat{g}$-module, that is, $\hat{V} = \bigoplus_{i \in \mathbb{Z}} \hat{V}_i$, with $\hat{V}_i = V \otimes t^i$.

First we will compute the cohomology of $\hat{n}_+$ with coefficients in $\hat{V}$. The Lie algebra $\hat{n}_+$ is a graded subalgebra of $\hat{g}$, and $\hat{V}$ is a graded $\hat{g}$-module and $\hat{n}_+$-module. Denote by $C^*(\hat{n}_+; \hat{V})$ the cochain complex of $\hat{n}_+$ with coefficients in the $\hat{n}_+$-module $\hat{V}$. The complex $C^*(\hat{n}_+; \hat{V})$ and the cohomology $H^*(\hat{n}_+; \hat{V})$ are graded by weights. To state this, we introduce the notation

$$C^*_m(\hat{n}_+; \hat{V}) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}((\wedge^q \hat{n}_+)_r, \hat{V}_{r+m}),$$

where

$$(\wedge^q \hat{n}_+)_r = \wedge^q \hat{n}_+ \cap \left( \bigotimes_{i=1}^q \hat{n}_+ \right)_r$$

and

$$\left( \bigotimes_{i=1}^q \hat{n}_+ \right)_r = \bigoplus_{r_1 + \cdots + r_q = r} ((\hat{n}_+)^{r_1} \otimes \cdots \otimes (\hat{n}_+)^{r_q}).$$

In this notation the grading is

$$C^*\left(\hat{n}_+ ; \hat{V}\right) = \bigoplus_{m \in \mathbb{Z}} C^*_m(\hat{n}_+ ; \hat{V}),$$

and, similarly,

$$H^*\left(\hat{n}_+ ; \hat{V}\right) = \bigoplus_{m \in \mathbb{Z}} H^*_m(\hat{n}_+ ; \hat{V}).$$

**Lemma 1.** $H^*_m(\hat{n}_+ ; \hat{V}) \cong H^*(\hat{n}_+ ; V_1)$ for all $m \in \mathbb{Z}$.

**Proof.** The composition of mappings

$$C^*_m(\hat{n}_+ ; \hat{V}) \xrightarrow{i} C^*\left(\hat{n}_+ ; \hat{V}\right) \xrightarrow{g} C^*\left(\hat{n}_+ ; V_1\right),$$

where
where \( i \) is the embedding and \( s \) is induced by the homomorphism \( \hat{V} \to V_1 \), is obviously a complex isomorphism.

Moreover, the isomorphisms

\[
\hat{V}_i = V \otimes t^i \to V \otimes t^{i+1} = \hat{V}_{i+1}
\]

define a \( \hat{g} \)-isomorphism \( t : \hat{V} \to \hat{V} \) of degree 1, which generates an action of \( \mathbb{C}[t, t^{-1}] \) in \( \hat{V} \) and in \( H^*(\hat{n}_+; \hat{V}) \). Evidently, \( t \) maps \( H^*_{(m)}(\hat{n}_+; \hat{V}) \) isomorphically onto \( H^*_{(m+1)}(\hat{n}_+; \hat{V}) \). Hence we have

**Lemma 1'.** \( H^*(\hat{n}_+; \hat{V}) \cong \mathbb{C}[t, t^{-1}] \otimes \mathbb{C} H^*(\hat{V}; \hat{V}) \).

Now let us compute the cohomology \( H^*(\hat{n}_+; \hat{V}) \). Introduce the subalgebra \( g[t] = g \otimes \mathbb{C}[t] \) of \( \hat{g} \). In the following we will identify \( g \) with \( g \otimes 1 \subset g[t] \). The Lie algebra \( \hat{n}_+ \) is embedded into \( g[t] \), and \( V_1 \) is naturally endowed with a \( g[t] \)-module structure.

**Theorem 1.** We have the following isomorphism of cohomology spaces:

\[
H^i(\hat{n}_+; V_1) \cong \bigoplus_{p+q=i} H^p(\hat{n}_+; \mathbb{C}) \otimes H^q(g[t], g; V_1).
\]

*Proof.* We begin the proof, which will take most of this paper, by introducing two subalgebras of \( \hat{g} \). The first is

\[
\tilde{g} = (t - 1)g \oplus (t - 1)^2g \oplus \ldots
\]

consisting of loops \( \varphi(t) \) which vanish at \( 1 \), and the second is

\[
\bar{g} = \hat{n}_+ \cap \tilde{g}.
\]

Note that \( g[t] = \hat{n}_+ + \tilde{g} \). Let \( G \) be the compact connected, simply connected Lie group, corresponding to the compact real form of \( g \). Next we need the following theorem.

**Theorem 2.** We have

\[
H^*(\tilde{g}) \cong H^*(\hat{n}_+ \oplus \tilde{g}) \otimes H^*(\Omega G),
\]

where \( \Omega G \) is the loop space of \( G \).

*Proof.* Since we have the embeddings

\[
\begin{align*}
\hat{n}_+ & \quad \hat{n}_+ \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n} & \quad g[t] \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n}_+ & \quad \hat{n}_+ \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n} & \quad g[t] \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n}_+ & \quad \hat{n}_+ \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n} & \quad g[t] \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n}_+ & \quad \hat{n}_+ \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n} & \quad g[t] \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n}_+ & \quad \hat{n}_+ \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n} & \quad g[t] \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n}_+ & \quad \hat{n}_+ \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n} & \quad g[t] \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n}_+ & \quad \hat{n}_+ \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]

\[
\begin{align*}
\hat{n} & \quad g[t] \\
\tilde{g} & \quad \tilde{g}
\end{align*}
\]
with $\mathfrak{g} \to \mathfrak{g}[t] \to \mathfrak{g}$, we also have the diagram:

$$
\begin{array}{ccc}
C^*(\hat{n}_+) & \xleftarrow{c} & C^*(\mathfrak{g}[t]) \\
\downarrow & & \downarrow \\
C^*(\hat{n}) & \xleftarrow{c} & C^*(\mathfrak{g}) \\
\downarrow & & \downarrow \\
C^*(\hat{\mathfrak{g}})
\end{array}
$$

Consequently,

$$C^*(\hat{n}) = C^*(\hat{n}_+) \otimes_{C^*(\mathfrak{g}[t])} C^*(\hat{\mathfrak{g}}),$$

where the tensor product is taken in the category of differential algebras. In such a situation there exists an Eilenberg-Moore spectral sequence, connecting these four differential algebras. Its second term is

$$E_2 = \text{Tor}_{H^*} (\mathfrak{g}[t]) (H^*(\hat{n}_+), H^*(\hat{\mathfrak{g}})),$$

and its limit term is $H^*(\hat{n})$. We know that $H^*(\mathfrak{g}[t]) \cong H^*(\mathfrak{g})$ (see, for example, [F]). On the other hand, since $H^*(\hat{\mathfrak{g}})$ acts trivially on $H^*(\hat{n})$ and also on $H^*(\hat{n}_+)$, we conclude that the composition $H^*(\hat{\mathfrak{g}}) \to H^*(\hat{n}_+) \to H^*(\hat{n}_+)$ is trivial. So we have

$$E_2 = \text{Tor}_{H^*} (\hat{\mathfrak{g}}) (H^*(\hat{n}_+), H^*(\hat{\mathfrak{g}})) \cong H^*(\hat{n}_+) \otimes H^*(\hat{\mathfrak{g}}) \otimes \text{Tor}_{H^*} (\mathbb{C}, \mathbb{C}).$$

On the other hand,

$$\text{Tor}_{H^*} (\hat{\mathfrak{g}}) (\mathbb{C}, \mathbb{C}) \cong H^*(\Omega G).$$

Indeed, the cohomology algebra of $\mathfrak{g}$ with trivial coefficients coincides with the cohomology algebra of $G$, and by the Hopf Theorem it is commutative and free (see [S]). Using the computation of $\text{Tor}_A (\mathbb{C}, \mathbb{C})$ for the free commutative algebra $A$ ([M, Proposition 7.3] and see also [A]) and the connection between the cohomology of $G$ and $\Omega G$, we get the isomorphism $\text{Tor}_A (\mathbb{C}, \mathbb{C}) \cong H^*(\Omega G)$:

$$H^*(\mathfrak{g}) = \bigwedge^* (e_{\alpha_1}, \ldots, e_{\alpha_k}),$$

where $e_{\alpha_i} \in H^{\alpha_i}$. So with the mapping $G \to \Omega G$ we have

$$\text{Tor}_{\bigwedge^* (e_{\alpha_1}, \ldots, e_{\alpha_k})} (\mathbb{C}, \mathbb{C}) = S^* (c_{\alpha_1-1}, \ldots, c_{\alpha_k-1}),$$

where $\deg c_{\alpha_i-1} = \alpha_i - 1$. The generators of the cohomology are the homotopy groups. To complete the proof of Theorem 2 we will need the next proposition.

**Proposition 1.** The spectral sequence degenerates, namely, its second term $E_2$ coincides with the limit term $E_\infty$.

**Proof.** We shall indicate explicit cocycles of $C^*(\hat{n})$ which represent the generators of $E_2$. For this we apply the continuous cohomology theory. Let $n(0, 1)$ be the Lie algebra of infinitely differentiable functions $f : [0, 1] \to \mathfrak{g}$ such that $f(0) \in n$ and $f(1) = 0$. Denote by $C^*_c(0, 1)$ the complex of cochains of $n(0, 1)$, continuous in the $C^\infty$-topology. Let $\alpha$ be a generator of $H^*(\mathfrak{g})$ and...
\(\bar{\alpha}\) a cochain representing \(\alpha\). For \(p \in [0, 1]\) denote by \(\varphi_p\) the homomorphism \(\bar{n} \to \bar{g}\), "the value at \(p\):

\[
\varphi_p((t - 1)g_1, (t - 1)^2g_2, \ldots) = \sum_m (p - 1)^m g_m.
\]

Let \(\alpha_p = \varphi_p^*\bar{\alpha}\), \(\alpha_p \in C^*_c(0, 1)\). Choose \(\bar{\alpha}\) in such a way that \(\alpha_0 = \alpha_1 = 0\). Let \(p \neq 0, 1\); then we can define the cochain \(\frac{\partial \alpha}{\partial x}(p)\), where \(x\) is the coordinate on \([0, 1]\). It is shown in [F] that \(\frac{\partial \alpha}{\partial x}(p)\) is a coboundary: \(\frac{\partial \alpha}{\partial x}(p) = \delta \omega(p)\), where \(\delta\) is the differential in \(C^*_c(0, 1)\). Indeed, let \(K_p(p \neq 0)\) be the cochain complex of \(\bar{n}\) with support at \(p\). It is proved in the same paper that the cohomology of \(K_p\) is isomorphic to \(H^*(\bar{g})\). The complex \(K_p\) is a \(W_l\)-module, where \(W_l\) is the Lie algebra of formal vector fields at the point \(p\). But \(H^*(\bar{g})\) is finite dimensional and \(W_l\) has no nontrivial finite-dimensional representations. We conclude that if \(\nu \in K_p\) and \(\delta \nu = 0\) then \(\frac{\partial \nu}{\partial x}\) is the differential of some other cocycle \(\bar{\nu} \in K_p\).

This means that

\[
\alpha_p - \alpha_q = \delta \int_q^p \omega(x) \, dx.
\]

In particular, \(\delta \int_0^1 \omega(x) \, dx = 0\). Suppose that \(\alpha' = \int_0^1 \omega(x) \, dx\). The cochain \(\alpha'\) represents a nontrivial cohomology class of \(\bar{n}\).

The Lie algebras \(\bar{n}_+\) and \(\bar{g} = (t - 1)g \oplus (t - 1)^2g \oplus \cdots\) are graded. Similarly, the cochain complexes are also graded. Note that the cochain complex \(K_0\) of \(n(0, 1)\) with support in 0 is isomorphic to \(\bigoplus C^*_c(\bar{n}_+)\) and the cochain complex \(K_1\) with support in 1 is isomorphic to \(\bigoplus C^*_C(\bar{g})\). It follows from this that the cohomologies of \(K_0\) and \(K_1\) are isomorphic to \(H^*(\bar{n}_+)\) and \(H^*(\bar{g})\), respectively.

Recall that \(H^*(\bar{g})\) is isomorphic to the free graded commutative algebra with generators \(\xi_1, \xi_2, \ldots\), with \(\deg \xi_k = 2k + 1\). Using the above construction, let us assign to each \(\xi_i\) a representative cocycle \(\xi'_i\).

**Lemma 2.** The space \(H^*(\bar{n})\) is generated by the cohomology classes of cochains of the form \(u \wedge v \wedge P(\xi'_1, \xi'_2, \ldots)\), where \(u \in K_0\) and \(v \in K_1\) are cocycles corresponding to the elements of \(H^*(\bar{n}_+)\) and \(H^*(\bar{g})\), respectively, and \(P\) is an arbitrary polynomial with generators \(\xi'_1, \xi'_2, \ldots\).

The proof of Lemma 2 follows from the construction above for continuous cohomology (a similar argument in a more difficult situation was used in [FR]). In particular, we have an explicit construction of cochains, representing the generators of \(E_2\) in the proof of Theorem 2, surviving until \(E_\infty\). Thus, Theorem 2 is proved.

We now return to complete the proof of Theorem 1. We want to prove the isomorphism

\[
H^*(\bar{g}[t], \bar{g}; V_1) \otimes H^*(\bar{n}_+; \mathbb{C}) \cong H^*(\bar{n}_+; V_1).
\]

Consider the Serre-Hochschild spectral sequence associated with the algebra \(\bar{n}_+\), its ideal \(\bar{n}\), and the module \(V_1\). The Lie algebra \(\bar{n}\) acts on \(V_1\) trivially. The second term of this spectral sequence is

\[
E_2^{ij} = H^j(\bar{n}_+/\bar{n}; H^i(\bar{n}; V_1)) = H^j(\bar{n}_+/\bar{n}; H^i(\bar{n}; \mathbb{C}) \otimes V_1).
\]
But by Theorem 2, this is the same as

$$H^i\left( \hat{n}_+ / \hat{\mathfrak{g}}, \bigoplus_{p+q+r=j} H^p(\hat{n}_+) \otimes H^q(\hat{\mathfrak{g}}) \otimes H^r(\Omega G) \otimes V_1 \right).$$

Since $\hat{n}_+ / \hat{\mathfrak{g}} \cong \mathfrak{g}$, we get that $E_2$ is then isomorphic to

$$H^*(\mathfrak{g}, \mathbb{C}) \otimes [H^*(\hat{n}_+) \otimes H^*(\hat{\mathfrak{g}}) \otimes H^*(\Omega G) \otimes V_1]^\mathfrak{g},$$

where $[ \ ]^\mathfrak{g}$ denotes the invariant space.

Let us note the following facts:

(a) $\mathfrak{g}$ acts on $H^*(\hat{n}_+)$ trivially (this action is extended by the projection $\hat{n}_+ \to \mathfrak{g}$ to the canonical action of $\hat{n}_+$, and a Lie algebra acts trivially on the cohomology of itself).

(b) $\mathfrak{g}$ acts trivially on $H^*(\Omega G) = \text{Tor}_{H^*(\mathfrak{g})}(\mathbb{C}, \mathbb{C})$.

These imply that

$$E_2^{ij} = H^i(\mathfrak{g}) \otimes \left( \bigoplus_{p+q+r=j} (H^p(\hat{n}_+) \otimes H^q(\Omega G) \otimes [H^r(\hat{\mathfrak{g}}) \otimes V_1]^\mathfrak{g}) \right)$$

and

$$[H^*(\hat{\mathfrak{g}}) \otimes V_1]^\mathfrak{g} = H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1).$$

The differentials act in the following way.

(i) Differential on $H^*(\hat{n}_+) \otimes H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1)$ are zero. We have the map

$$H^*(\hat{n}_+) \otimes H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1) \to H^*(\hat{n}_+ ; V_1).$$

Thus, elements of the left side survive in $E_\infty$.

(ii) Differential on

$$H^*(\mathfrak{g}) \otimes H^*(\Omega G) = \bigwedge^* \{ e_{\alpha_i} \} \otimes S^* \{ c_{\alpha_i-1} \}$$

map the generators of the algebra $H^*(\Omega G)$ into the generators of $H^*(\mathfrak{g})$ (the differential maps $c_{\alpha_i-1} \mapsto e_{\alpha_i}$).

Consider the Serre path fibration $E_G \to G$. Since the paths are contractible, $H^0(E_G) = \mathbb{C}$ and $H^i(E_G) = 0$ for $i > 0$. In addition, we have $H^*(\mathfrak{g}) = H^*(G)$, so

$$H^*(G) \otimes H^*(\Omega G) \text{ converges to } H^*(E_G) \cong \mathbb{C}.$$
and $\text{Hom}_g(g; H^i(tg[t])) = 0$ if $i \neq 1$ and is equal to $\mathbb{C}$ if $i = 1$ (see [L]). Since $H^i(g[t], g; V) \cong \text{Hom}_g(V; H^i(tg[t]))$, this gives us that $H^i(g[t], g; g) = 0$ for $i \neq 1$ and is equal to $\mathbb{C}$ for $i = 1$. After this, it is enough to apply Theorem 1 to find that $H^i(\hat{\mathfrak{n}}_+; V_1) = H^{i-1}(\hat{\mathfrak{n}}_+)$.

ACKNOWLEDGMENT

The authors would like to thank the referee for the useful comments.

REFERENCES


Department of Mathematics, University of California, Davis, California 95616-8633

E-mail address: fialowsk@math.ucdavis.edu