Classification of Graded Lie Algebras with Two Generators

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This article considers infinite-dimensional Lie algebras over a field of characteristic 0 with basis \( e_1, e_2, \ldots \) which satisfy the condition
\[
[e_i, e_j] = c_{ij} e_{i+j}.
\]
A complete description is given of such algebras with two generators. In particular, it follows from the proposed classification that if the number of independent relations between the generators of a Lie algebra of this type is finite, then it is equal to 2.

In this paper we classify the graded Lie algebras \( \mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i \) over a field \( K \) of characteristic 0, for which \( \dim \mathfrak{g}_i = 1 \), with minimum possible number of generators. Obviously this number is equal to 2.

There are three well-known Lie algebras of the above type: \( L_1, n_1, n_2 \). The Lie algebra \( L_1 \) consists of vector fields on the real line with polynomial coefficients which vanish together with their first derivative at the coordinate origin [1]. The algebras \( n_1 \) and \( n_2 \) are maximal nilpotent subalgebras in Kac-Moody algebras \( A_1^{(1)} \) and \( A_2^{(2)} \), respectively [2].

The problem of classification of algebras such as \( L_1, n_1, n_2 \) is naturally related to the problem formulated by V. Kac in [3], which involves the classification of all simple graded Lie algebras \( L = \bigoplus_{i \in \mathbb{Z}} L_i \), where \( \dim L_i = 1 \).

In the algebra \( \mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i \) we choose a basis of homogeneous elements \( e_i \in \mathfrak{g}_i \). The generators of \( \mathfrak{g} \) are \( e_1 \) and \( e_2 \). Note that \( [e_1, e_2] \neq 0 \) and \( [e_1, [e_1, e_2]] \neq 0 \). We specify the explicit form of the commutator in the algebras \( L_1, n_1, n_2 \) as follows.

\[ L_1 : [e_i, e_j] = (j - i) e_{i+j}, \]
\[ n_1 : [e_i, e_j] = a_{ij} e_{i+j}, \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if} \ j - i \equiv 1 \mod 3 \\ 0 & \text{if} \ j - i \equiv 0 \mod 3 \\ -1 & \text{if} \ j - i \equiv -1 \mod 3 \end{cases}, \]
\[ n_2 : [e_i, e_j] = b_{ij} e_{i+j}, \quad \text{where the numbers} \quad b_{ij} \quad \text{depend only on the residue obtained when dividing} \ i \quad \text{and} \ j \quad \text{by} \ 8 \quad \text{according to the rule} \ b_{ij} + b_{ij'} = 0 \quad \text{if the numbers} \ i + i' \quad \text{and} \ j + j' \quad \text{are divisible by} \ 8. \]
The accompanying table gives the numbers \( b_{ij} \) (the remaining \( b_{ij} \) are determined from the relations \( b_{ij} = -b_{ji} \) and \( b_{ij} + b_{8-i,8-j} = 0 \)).

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In addition to these three algebras, we will need two particular algebras and also a special family of algebras. These are:

$m_1$: The algebra in which $[e_1, e_i] = e_{i+1}$ for $i > 1$ and $[e_i, e_j] = 0$ for $i, j > 1$.

$m_2$: The algebra in which the commutator is set up as follows: $[e_i, e_j] = 0$ for $i, j > 2$, while $[e_1, e_j] = e_{j+1}$ for $j ≥ 2$ and $[e_2, e_j] = e_{j+2}$ for $j > 2$.

$g(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$: A family of Lie algebras with countably many parameters $\lambda_{4k} \in \mathbb{K}P^1$. The commutator is defined as follows: $[e_1, e_4] = 0$, $[e_3, e_4] = 0$, $[e_i, e_j] = 0$ if $i$ is even but not 2 and $j$ is any positive integer. Furthermore, $[e_1, e_{4k-1}] = \alpha_{4k} e_{4k}$, and $[e_2, e_{4k-2}] = \beta_{4k} e_{4k}$, $k = 2, 3, 4, \ldots$.

where the $\alpha_{4k}$ and $\beta_{4k}$ are the homogeneous coordinates of the point $\lambda_{4k} \in \mathbb{K}P^1$. The remaining commutators can be uniquely reconstructed from the above formulas. Their structural constants are homogeneous polynomials of $\alpha_{4k}$ and $\beta_{4k}$. See the Appendix for some explicit formulas for the commutators.

Example: For the algebra $g(1, 1, 1, \ldots)$ the commutators are

$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4$

$[e_1, e_{2k+1}] = [e_2, e_{2k}] = e_{2k+2}$ if $k ≥ 2$

$[e_2, e_{2k-1}] = e_{2k+1}$ if $k ≥ 2$

the other commutators are 0.

**Theorem** Let $g = \bigoplus_{i=1}^{\infty} g_i$ be an $\mathbb{N}$-graded Lie algebra, where $\dim g_i = 1$, with basis $e_1, e_2, e_3, \ldots$, generated by $e_1$ and $e_2$. Then $g$ is one of the following.

a) Assume $[e_1, e_4] \neq 0$ and $[e_2, e_3] \neq 0$. If $[e_3, e_4] \neq 0$, then $g \cong L_1$ while if $[e_3, e_4] = 0$, then $g \cong m_2$.

b) Assume $[e_2, e_3] = 0$. If $[e_3, e_4] \neq 0$, then $g \cong n_2$ while if $[e_3, e_4] = 0$, then $g \cong m_1$.

c) Assume $[e_1, e_4] = 0$. If $[e_3, e_4] \neq 0$, then $g \cong n_1$ while if $[e_3, e_4] = 0$, then $g \cong g(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$ for some choice of the $\lambda_8, \lambda_{12}, \lambda_{16}, \ldots$.

**Proof:** (sketch) The Jacobi identity yields a system of linear equations for the usual structural constants $c_{ij}$ ($[e_i, e_j] = c_{ij} e_{i+j}$). The numbers $c_{12}$ and $c_{13}$ can be regarded as arbitrary but non-zero. To be specific, assume that $c_{12} = 1$. Then the coefficients $c_{14}$ and $c_{23}$ can no longer be chosen arbitrarily since they must satisfy a linear equation. We fix some solution of this equation after which $c_{15}$ and $c_{24}$ are determined uniquely. At the next step we obtain two equations for $c_{16}$, $c_{25}$ and $c_{34}$, etc.
As an example we consider the case in which $c_{14} \neq 0$, $c_{23} \neq 0$, and $c_{34} \neq 0$. We can assume that $c_{13} = 2$, $c_{23} = c_{34} = 1$. Solving this system step-by-step we find that if $c_{14}c_{25} \neq 9$, then the system of equations for $c_{ij}$ with $i + j = 16$ does not have a non-zero solution. If, however, $c_{14}c_{25} = 9$, then all the equations can be solved non-trivially and uniquely so we obtain an algebra that is isomorphic to $L_1$.

We can similarly consider the cases in which some of the numbers $c_{14}, c_{23}$, and $c_{34}$ are zero.

Of interest are the relations that link the generators $e_1$ and $e_2$. It is easy to show that these generators should satisfy at least two independent relations of weights 5 and 7 of the form

$$\lambda[e_1, [e_1, [e_1, e_2]]] + \mu[e_2, [e_2, e_1]] = 0,$$

$$\alpha[e_1, [e_1, [e_1, e_2]]] + \beta[e_2, [e_2, e_1]] + \gamma[e_1, [e_1, [e_2, e_1]]] = 0.$$

In the algebras $L_1$, $n_1$, $n_2$, and $m_2$, the relations (*) make up a complete system of defining relations while for the algebras $m_1$ and $g(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$ this system is infinite. Specifically, in the algebra $m_1$ we should add relations of weights 13, 17, 21, ... to (*); the relations between generators in the algebras $g(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$ are not yet computed.

**Corollary 1** If the number of relations between the generators of the Lie algebra $g$ is finite, then $g$ is isomorphic to one of the four algebras $L_1$, $n_1$, $n_2$, and $m_2$, and the number of relations is 2.

Let us consider a Lie algebra with generators $e_1$ and $e_2$ and relations (*). It turns out that for most points $(\lambda, \mu, \alpha, \beta, \gamma) \in \mathbb{C}^5$ this algebra is finite-dimensional. One can compute the following. If $6\lambda = \mu$, $120\alpha = 3\beta + 20\gamma$, then the Lie algebra is isomorphic to $L_1$; if $\mu = 0$, $\lambda = 0$, then it is isomorphic to $n_1$; if $\lambda = 0$, $\gamma = 0$, it is isomorphic to $n_2$; and if $\lambda = \mu$, $\alpha = \beta + \gamma$, it is isomorphic to $m_2$. If, however, $\lambda = 0$, $\alpha = 0$ or $\mu = 0$, $\beta = 0$, then the dimension of the space of weight $i$ in the Lie algebra increases exponentially with $i$, and adding additional relations converts this algebra into one of the algebras of the family $g(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$.

In conclusion, we should note that the cohomology of the Lie algebras $L_1$, $n_1$ and $n_2$ with trivial coefficients are known [4], [5]. In all cases, for $i > 0$ the $i$-th cohomology space is two-dimensional. It would be interesting to calculate the cohomology of the other algebras considered in this article.

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**References**


Appendix

(added June 1997)

Here are some details for the structural constants of the algebras \( g(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots) \). We compute \([e_i, e_j]\) for all levels \( i + j \leq 17 \). For these levels, we list only the (generically) non-zero brackets. Using a computer one can extend these much further.

\[
\begin{align*}
i + j &= 3 : & [e_1, e_2] &= e_3 \\
i + j &= 4 : & [e_1, e_3] &= e_4 \\
i + j &= 5 : & [e_2, e_3] &= e_5 \\
i + j &= 6 : & [e_1, e_5] &= [e_2, e_4] = e_6 \\
i + j &= 7 : & [e_2, e_5] &= e_7 \\
i + j &= 8 : & [e_1, e_7] &= \alpha_8 e_8, & [e_2, e_6] &= \beta_8 e_8, & [e_3, e_5] &= (\alpha_8 - \beta_8)e_8 \\
i + j &= 9 : & [e_2, e_7] &= e_9 \\
i + j &= 10 : & [e_2, e_9] &= e_{10}, & [e_1, e_9] &= (2\alpha_8 - \beta_8)e_{10}, & [e_3, e_7] &= (\alpha_8 - \beta_8)e_{10} \\
i + j &= 11 : & [e_2, e_9] &= e_{11} \\
i + j &= 12 : & [e_1, e_{11}] &= \alpha_{12} e_{12}, & [e_2, e_{10}] &= \beta_{12} e_{12} \\
& [e_3, e_9] &= (\alpha_{12} + (\beta_8 - 2\alpha_8)\beta_{12})e_{12}, \\
& [e_5, e_7] &= ((3\alpha_8 - 2\beta_8)\beta_{12} - \alpha_{12})e_{12} \\
i + j &= 13 : & [e_2, e_{11}] &= e_{13} \\
i + j &= 14 : & [e_2, e_{12}] &= e_{14}, \\
& [e_1, e_{13}] &= (3\alpha_{12} + (3\beta_8 - 5\alpha_8)\beta_{12})e_{14}. \\
& [e_3, e_{11}] &= (2\alpha_{12} + (3\beta_8 - 5\alpha_8)\beta_{12})e_{14}. \\
& [e_5, e_9] &= ((3\alpha_8 - 2\beta_8)\beta_{12} - \alpha_{12})e_{14} \\
i + j &= 15 : & [e_2, e_{13}] &= e_{15} \\
i + j &= 16 : & [e_1, e_{15}] &= \alpha_{16} e_{16}, & [e_2, e_{14}] &= \beta_{16} e_{16} \\
& [e_3, e_{13}] &= (\alpha_{16} - 3\beta_{16}\alpha_{12} - \beta_{16}\beta_{12}(3\beta_8 - 5\alpha_8))e_{16} \\
& [e_5, e_{11}] &= (-\alpha_{16} + 5\beta_{16}\alpha_{12} + 2\beta_{16}\beta_{12}(3\beta_8 - 5\alpha_8))e_{16} \\
& [e_7, e_9] &= (\alpha_{16} - 6\beta_{16}\alpha_{12} + \beta_{16}\beta_{12}(13\alpha_8 - 8\beta_8))e_{16} \\
i + j &= 17 : & [e_2, e_{15}] &= e_{17}
\end{align*}
\]