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AUGMENTING GRAPHS TO MEET EDGE-CONNECTIVITY REQUIREMENTS*

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Abstract. What is the minimum number γ of edges to be added to a given graph G so that in the resulting graph the edge-connectivity between every pair $\{u, v\}$ of its nodes is at least a prescribed value $r(u, v)$?

Generalizing earlier results of S. Sridhar and R. Chandrasekaran [*Integer Programming and Combinatorial Optimization*, R. Kannan and W. Pulleyblank, eds., Proceedings of a conference held at the University of Waterloo, University of Waterloo Press, Waterloo, Ontario, Canada, 1990, pp. 467–484] (when G is the empty graph), of K. P. Eswaran and R. E. Tarjan [*SIAM Journal on Computing*, 5 (1976), pp. 653–665] (when $r(u, v) = 2$), and of G.-R. Cai and Y.-G. Sun [*Networks*, 19 (1989), pp. 151–172] (when $r(u, v) = k \geqq 2$), we derive a min-max formula for γ and describe a polynomial time algorithm to compute γ. The directed counterpart of the problem is solved in the same sense for the case when $r(u, v) = k \geqq 1$ and is shown to be NP-complete if $r(u, v) = 1$ for $u, v \in T$, and $r(u, v) = 0$ otherwise where T is a specified subset of nodes.

A fundamental tool in the proof is the splitting theorems of W. Mader [*Annals of Discrete Mathematics*, 3 (1978), pp. 145–164] and L. Lovász [lecture, Prague, 1974; North-Holland, Amsterdam, 1979]. We also rely extensively on techniques concerning submodular functions. The method makes it possible to solve a degree-constrained version of the problem. The minimum-cost augmentation problem can also be solved in polynomial time provided that the edge-costs arise from node-costs, while the problem for arbitrary edge-costs was known to be NP-complete even for $r(u, v) = 2$.

Key words. network synthesis, augmenting edge-connectivity of graphs, polymatroids, network design, connectivity, combinatorial optimization

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1. Introduction. A typical problem in combinatorial optimization is to find a minimum number, or more generally, a minimum cost, of edges to be added to a graph so that the resulting graph satisfies some prescribed properties. In this paper we are concerned with edge-connectivity properties.

Main problem. What is the minimum number γ (respectively, a minimum cost) of edges to be added to a given directed or undirected graph G so that in the resulting graph the edge-connectivity $\lambda(u, v)$ between every pair $\{u, v\}$ of nodes is at least a prescribed value $r(u, v)$?

Here, the *edge-connectivity* $\lambda(u, v)$ of u and v means the maximum number of pairwise edge-disjoint (directed) paths from u to v. Note that $\lambda(u, v)$ can be interpreted as the maximum flow value between u and v if the capacities of the edges of G are defined to be 1.

To distinguish between the two versions of the main problem, we will sometimes refer to them as the "cardinality case" and "the min-cost case."

A capacitated version of the main problem is as follows.

Max-flow version. Suppose that $g(u, v)$ is a nonnegative capacity function on the pairs of nodes u, v $(u, v \in V)$, and let $r(u, v)$ be another nonnegative function on the pairs of nodes that serves as a maximum flow requirement. The problem is to increase the existing capacities so that in the resulting network the maximum flow value between u and v is at least $r(u, v)$ for each pair $\{u, v\}$ of nodes, such that the sum of capacity increments is minimum.

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Note that if $g(u, v)$ and $r(u, v)$ are integer-valued and the capacity increments are required to be integer-valued, then the main problem is equivalent to the flow version in the sense that a min-max result for one problem easily transforms to a min-max result for the other. Namely, the main problem can be formulated as a max-flow version by letting $g(u, v) = 1$ when (u, v) is an edge of G, and $g(u, v) = 0$, otherwise. Conversely, if g is integer-valued, we can define a graph having $g(u, v)$ parallel edges between each pair of nodes u and v; then a solution to the main problem on G yields a solution to the integer-valued max-flow problem.

This equivalence, however, does not mean algorithmic equivalence. We are going to develop strongly polynomial time algorithms for the more difficult max-flow augmentation problem.

(A polynomial time algorithm is called *strongly polynomial* if it uses, besides ordinary data manipulation, only the basic operations like comparing, adding, subtracting, multiplying, and dividing numbers, and if the number of these operations is independent of the numbers occurring in the input.)

Another useful observation is that in the flow version we may get better results if fractional increments are allowed. For example, let $V := \{a, b, c\}$, $g \equiv 0$, and $r \equiv 1$. If only integers are allowed for the increments, then the value of the best solution is 2: increase the capacity of edges ab and bc by 1. If we use fractional increments, then the value of the best solution is 1.5: increase the capacity of each edge by 0.5. On the other hand, we will see that, apart from some marginal cases, the integer-valued optimum is at most one half larger than the fractional optimum.

It is natural to consider node-connectivity augmentation problems as well. That is, given a prescribed value $r(u, v)$ for each pair of nodes u, v, what is the minimum number (respectively, a minimum cost) of edges to be added to a given directed or undirected graph G so that in the resulting graph there are $r(u, v)$ openly disjoint paths between every pair of nodes u, v? Two paths connecting u and v are called *openly disjoint* if they are node-disjoint, except for the end-nodes. No new contribution to this problem is given here and we mention it merely for the sake of completeness.

We briefly summarize some known special cases for which the above augmentation problems have been solved. Let us start with undirected graphs.

Gomory and Hu [19] algorithmically solved the fractional case of the max-flow version of the augmentation problem where $g \equiv 0$. See also [8]. The minimum cost version of the same problem was solved by Bland, Goldfarb, and Todd [1] via the ellipsoid method. Sridhar and Chandrasekaran [30] solved the main problem when the starting graph is the empty graph. (Actually, they solved the integer-valued max-flow version when $g \equiv 0$.) Frank and Chou [9] solved the same problem under the additional requirement that no parallel edges are allowed to be added. (Note that Frank and the present author are different.)

Suppose now that $r(u, v)$ is identically k. When $k = 1$, the min-cost case of the main problem transforms into a minimum cost spanning tree problem, that is nicely solvable. For $k = 2$, however, the min-cost problem turns out to be NP-complete, as the Hamiltonian circuit problem can easily be formulated in this form (see Eswaran and Tarjan [7]).

Eswaran and Tarjan described a polynomial time algorithm to find a minimum number of new edges the addition of which makes a graph 2-edge-connected. Generalizing this for arbitrary $k \geq 2$, Watanabe and Nakamura [31] described a polynomial time algorithm to find a minimum number of new edges γ to be added to make a graph k-edge-connected. The same problem was also solved by Cai and Sun [2] who, in addition, provided a nice min-max formula for the minimum. Both solutions are rather complicated. Recently, Naor, Gusfield, and Martel [28] developed an efficient algorithm for this edge-connectivity augmentation problem.

As for node-connectivity augmentation problems, we know much less. The problem was solved by Harary [21] when $r(u, v)$ is identically k and the starting graph is the empty graph. For arbitrary starting graphs the case $k = 2$ was settled by Eswaran and Tarjan [7] and the case $k = 3$ by Watanabe and Nakamura [32].

The following results concern directed graphs.

Suppose we are given a digraph G with a source s and a target t. Let $r(u, v) = k$ if $u = s$, $v = t$, and $r(u, v) = 0$ otherwise. In this case the main problem requires adding a minimum cost of edges so that in the resulting digraph there are k edge-disjoint paths from s to t. This problem can easily be reduced to a minimum cost flow problem in the union graph of the new and the original edges where the costs of the original edges are defined to be zero.

If we are interested in openly disjoint paths, rather than edge-disjoint, from a source-node s to a target-node t, then the problem can easily be reduced to the edge-disjoint case by using a simple node-duplicating device mentioned in [8, § I/11]. Unfortunately, in more general cases, the node-duplicating technique does not seem to help in reducing a node-connectivity augmentation problem to the corresponding edge-connectivity augmentation problem. This is the case in the following situation.

Improve a digraph by adding a minimum cost of new edges so as to have k edge-disjoint paths from a specified source-node to each other node. (That is, in the main problem $r(u, v) = k$ if $u = s$, and $r(u, v) = 0$ otherwise.) This problem can be reduced to a weighted matroid intersection problem where the first matroid is k times the circuit-matroid of the underlying undirected graph (that is, a subset of edges is independent if it is the union of k forests) while the second matroid is a partition matroid where a subset of edges is independent if it contains no more than k edges entering the same node. Since there are good algorithms for the matroid intersection problem [5], this problem is also solvable in polynomial time.

We consider the openly disjoint counterpart of the preceding problem; that is, we improve a digraph by adding a minimum cost of new edges so as to have k openly disjoint paths from a specified source-node to each other node. The problem was solved in [14] with the help of submodular flows (as we were unable to reduce the problem to the edge-disjoint case by using the node-duplicating technique).

Eswaran and Tarjan showed how to make a directed graph strongly connected by adding a minimum number of new edges. They also showed that the minimum cost version of the problem is NP-complete, as the directed Hamiltonian circuit problem can be formulated this way. Note, however, that the problem is solvable in strongly polynomial time if we are allowed to add a new edge (u, v) only if (v, u) is an original edge of the digraph (see Lucchesi and Younger [25] and Frank [10]).

The main problem for directed graphs when $r(u, v) \equiv k$ was solved by Fulkerson and Shapley [16] when the starting graph is $G = (V, \emptyset)$, and by Kajitani and Ueno [22] when the starting graph is a directed tree.

Finally, we mention a paper of Gusfield [20] in which a linear time algorithm is described to make a mixed graph strongly connected by adding a minimum number of new directed edges. (A *mixed graph* is one with possibly directed and undirected edges. It is called *strongly connected* if, for every pair of nodes u, v, there is a path from u to v that consists of directed edges in the right direction and arbitrary undirected edges.)

The main purposes of the present paper are as follows. For undirected graphs we completely solve the cardinality case of the main problem. Along the way we provide a short proof of the theorem of Cai and Sun. For directed graphs, we solve the cardinality case if $r(i, j) \equiv k$. We also consider degree-constrained and minimum-cost augmentations for both directed and undirected graphs. The proofs give rise to algorithms that are strongly polynomial even in the max-flow version.

One basic tool in the proof is the so-called splitting technique. We are going to use three theorems. One is due to Lovász, while the other two are due to Mader. To make the paper as self-contained as possible, we prove Lovász's theorem [24] here, as well as one of the two Mader theorems [26], [27]. A relatively simple proof of Mader's other splitting-off theorem is given in a separate paper [12].

We consider whether there is a solution to the augmentation problem for directed graphs when $r(u, v)$ is arbitrary. Unlike the undirected case this problem turns out to be NP-complete even for the following simple demand function. Let T be a subset of nodes and s a specified node not in T. Define $r(s, v) \equiv 1$ if $v \in T$, and $r(s, v) \equiv 0$ otherwise. (In other words, the problem of finding a minimum number of edges to be added to G so that in the augmented digraph every element of T is reachable from s is NP-complete.)

For both the directed and the undirected case, the method makes it possible to solve a degree-constrained version when, in addition, upper and lower bounds are imposed at every node for the number of newly added edges incident to that node. We will show that in the above cases the minimum-cost augmentation problem can also be solved in polynomial time, provided that the edge-costs arise from node-costs. (As we mentioned above, the problem for arbitrary edge-costs is NP-complete even for $r(u, v) \equiv 2$.)

Another basic technique comes from the theory of submodular functions. (For a survey, see [13].) We describe, however, the main results and proofs so that they can be understood without any prior knowledge in this area. It will be clarified in a separate section how some basic features of submodular functions and polymatroids are in the background of our approach.

The structure of the paper is as follows. Section 2 comprises the necessary notations and notions from graph theory. Section 3 describes the results on directed graphs. Furthermore, a new proof of Mader's directed splitting-off theorem is given. In § 3 we provide a simple proof of a theorem of Cai and Sun, along with some degree-constrained versions. A new proof of Lovász's splitting-off theorem is also presented. Section 5 contains the general result for undirected graphs. Section 6 lists some notions and theorems from polymatroid theory, while §§ 7 and 8 explain the relationship between polymatroids and augmentation problems. In § 9 we consider the fractional augmentation problem and algorithmic aspects.

2. Notation and basic concepts. Typically, we work with a finite ground set V. We will not distinguish between a one-element set $\{x\}$ and its element x. The union of a set X and an element y is denoted by $X + y$. For $s, t \in V$ a subset X of V is called an $t\bar{s}$-*set* is $t \in X$ and $s \notin X$. For two sets X, Y, $X - Y$ denotes the set of elements in X, but not in Y. $X \subset Y$ denotes that X is a subset of Y and $X \neq Y$. Two subsets X, Y of V are called *intersecting* if none of $X \cap Y$, $X - Y$, $Y - X$ is empty. If, in addition, $V - (X \cup Y)$ is nonempty, then X, Y are called *crossing*. A family of subsets is called *laminar* if it includes no intersecting sets.

By a *partition* $\{X_1, X_2, \cdots, X_t\}$ of a set X, we mean a family of disjoint subsets of X whose union is X. By a *subpartition* of V, we mean a partition of a subset X of V.

Let $\mathcal{F}$ be the family of subsets of V possessing a certain property p. We say that a member $X \in \mathcal{F}$ is *maximal* (with respect to p) if no member of $\mathcal{F}$ includes X as a proper subset. For example, a maximal critical set means a critical set not included in any other critical set (whatever critical means).

Throughout, we use the term "graph" for an undirected graph and "digraph" for a directed graph. Let $G = (V, E)$ be a graph with node set V and edge set E. We denote an edge e connecting nodes u and v by uv or vu. This is not quite precise since there may be parallel edges between u and v. This ambiguity, however, will not cause any trouble. Both parallel edges and loops are allowed.

For a digraph $G = (V, E)$ a directed edge $e = uv$ is meant to be an edge from u to v. In this case, vu means the oppositely directed edge.

For a graph or digraph $G = (V, E)$, $E(X)$ denotes the set of edges with both end-nodes in X and is called the set of edges *induced* by X. For $X, Y \subseteq V$, $d(X, Y)$ denotes the number of edges between $X - Y$ and $Y - X$ (in any direction) and $\bar{d}(X, Y) := d(V - X, Y)$. We denote $d(X) := d(X, V - X)$. If $F \subseteq E$ and G is undirected, $d_F(X)$ stands for the number of edges in F entering X. The number of edges incident to a node v is called the *degree* of v. The contribution of a loop vv to the degree of v is, by definition, two.

Deleting an edge e means that we leave out e from E while the node set V is unchanged. For the resulting graph, we use the notation $G - e$. *Deleting* a subset C of nodes means that we leave out the elements of C and all the edges incident to some elements of C. The resulting graph is denoted by $G - C$. *Splitting off* a pair uv, vz of edges with $u \neq z$ means that we replace the two edges uv, vz by a new edge $e = uz$. Note that if $u = z$, e is a loop.

In a digraph $G = (V, E)$ the *in-degree* $\rho(X)$ (*out-degree* $\delta(X)$) is the number of edges entering (leaving) X. If $F \subseteq E$ $\rho_F(X)$ stands for the number of edges from F entering X. The contribution of a directed loop vv to the in-degree of v and to the out-degree of v is, by definition, one.

We call an edge e (node v) of a graph $G = (V, E)$ a *cut edge* (*cut node*) if $G - e$ ($G - v$) has more components than G.

A graph is called *k-edge-connected* if $d(X) \geqq k$ for every $\emptyset \subset X \subset V$. A digraph is called *k-edge-connected* if $\rho(X) \geqq k$ for every $\emptyset \subset X \subset V$.

One fundamental result from graph theory is as follows.

MENGER'S THEOREM 2.1 (Edge-version in [8]). *In a directed (respectively, undirected) graph $G = (V, E)$ there are k edge-disjoint paths from s to t if and only if $\rho(X) \geqq k$ (respectively, $d(X) \geqq k$) for every $t\bar{s}$-set $X \subset V$.*

In a graph or digraph $\lambda(u, v)$ denotes the maximum number of edge-disjoint paths from u to v. $\lambda(u, v)$ is called the *edge-connectivity* from u to v. (In undirected graphs, obviously $\lambda(u, v) = \lambda(v, u)$.) The following identities are often used throughout the paper.

PROPOSITION 2.2. *Let G be an arbitrary graph $G = (V, E)$ and $X, Y \subseteq V$. Then*

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y), \tag{2.1}$$

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\bar{d}(X, Y). \tag{2.2}$$

PROPOSITION 2.3. *Let G be an arbitrary digraph $G = (V, E)$ and $X, Y \subseteq V$. Then*

$$\delta(X) + \delta(Y) = \delta(X \cap Y) + \delta(X \cup Y) + d(X, Y), \tag{2.3}$$

$$\rho(X) + \rho(Y) = \rho(X \cap Y) + \rho(X \cup Y) + d(X, Y). \tag{2.4}$$

If, in addition, $\delta(X \cap Y) = \rho(X \cap Y)$, then

$$\delta(X) + \delta(Y) = \delta(X - Y) + \delta(Y - X) + \bar{d}(X, Y), \tag{2.5}$$

$$\rho(X) + \rho(Y) = \rho(X - Y) + \rho(Y - X) + \bar{d}(X, Y). \tag{2.6}$$

Each formula can easily be proved by taking into consideration the contribution of each type of edges to the two sides.

Let S be a finite ground-set and $b : 2^S \to R \cup \{\infty\}$ a set function. We call b *fully submodular* or, briefly, *submodular* if $b(X) + b(Y) \geqq b(X \cap Y) + b(X \cup Y)$ holds for every $X, Y \subseteq S$. If the inequality is required only for intersecting sets X, Y, then b is called *intersecting submodular*.

A set function p is called *supermodular* if $-p$ is submodular. A finite-valued set function m is called *modular* if $m(X) + m(Y) = m(X \cap Y) + m(X \cup Y)$ holds for every $X, Y \subseteq S$. A modular function m with $m(\emptyset) = 0$ is determined by its value on the singletons, namely $m(X) = \Sigma(m(s) : s \in X)$.

Suppose that $m : V \to R \cup \{\infty\}$ is a function and $X \subseteq V$. Generally, we will use the notation $m(X) := \Sigma(m(s) : s \in X)$ with the following exceptions: $d(X)$ (see above) is *not* $\Sigma(d(v) : v \in X)$, and the case is the same with ρ and δ.

3. Directed graphs. Fulkerson and Shapley [16] described a method to construct a k-edge-connected digraph on n nodes ($k \leqq n$) with a minimum number of edges. Kajitani and Ueno [22] solved the problem of optimally augmenting a directed tree in order to get a k-edge-connected digraph. Here we solve this problem for arbitrary directed graphs.

THEOREM 3.1. *Given a directed graph $G = (V, E)$ and a positive integer k, G can be made k-edge-connected by adding at most γ new edges if and only if*

$$\Sigma(k - \rho(X_i)) \leqq \gamma \quad \text{and} \tag{3.1}$$

$$\Sigma(k - \delta(X_i)) \leqq \gamma \tag{3.2}$$

hold for every subpartition $\{X_1, X_2, \cdots, X_t\}$ of V.

Proof. Necessity. Suppose $G' = (V, E \cup F)$ is a k-edge-connected supergraph of G, where F denotes the set of new edges. Then every subset X_i of V has at least $k - \rho(X_i)$ new entering edges. Therefore, the number of new edges in G' is at least $\Sigma(k - \rho(X_i))$ and (3.1) follows. (3.2) is analogous. □

Let s be a node not in V, and $V' := V + s$. Let $G' = (V', E')$ be a digraph with in- and out-degree functions ρ' and δ', respectively.

PROPOSITION 3.2. *Suppose for $A, B \subseteq V'$ that $\rho'(A) = \rho'(B) = k \leqq \min(\rho'(A \cap B), \rho'(A \cup B))$. Then $\rho'(A \cap B) = \rho'(A \cup B) = k$ and $d'(A, B) = 0$.*

Proof. Applying (2.4) we obtain $k + k = \rho'(A) + \rho'(B) = \rho'(A \cap B) + \rho'(A \cup B) + d'(A, B) \geqq k + k + d'(A, B)$, from which $k = \rho'(A \cap B) = \rho'(A \cup B)$ and $d'(A, B) = 0$ follows. □

We prove the sufficiency in two steps.

LEMMA 3.3. *G can be extended to a digraph $G' = (V + s, E')$ by adding a new node s, γ new edges entering s, and γ new edges leaving s in such a way that for every subset $\emptyset \neq X \subset V$*

$$\rho'(X) \geqq k \quad \text{and} \tag{3.3a}$$

$$\delta'(X) \geqq k \tag{3.3b}$$

hold where ρ' and δ' denote the in-degree and out-degree function of G', respectively.

Note that by Menger's theorem (3.3) is equivalent to saying that the edge-connectivity in G' between every pair of original nodes is at least k.

Proof. We are going to prove that it is possible to add γ edges leaving s so that (3.3a) is satisfied. This implies (by reorienting every edge) that it is possible to add γ edges entering s so that (3.3b) is satisfied.

First, we add a sufficiently large number of edges leaving s so as to satisfy (3.3a). (It certainly will do if we add k edges from s to v for every $v \in V$.)

Second, discard new edges, one by one, as long as possible without violating (3.3a). Let G' denote the final extended digraph. The following claim implies the lemma.

CLAIM. $\delta'(s) \leqq \gamma$.

Proof. Call a subset $\emptyset \subset X \subset V$ *in-critical* if $\rho'(X) = k$. Let $S := \{v \in V, sv$ is an edge in $G'\}$. An edge sv cannot be left out from G' without violating (3.3a) precisely if sv enters an in-critical set. Therefore, by the minimality of G', there is a family $\mathcal{F} = \{X_1, X_2, \cdots, X_t\}$ of in-critical subsets of V covering S, and we can assume that t is minimal.

Case 1. $\mathcal{F}$ consists of disjoint sets. Then we have $kt = \Sigma(\rho'(X_i): i = 1, \cdots, t) = \delta'(s) + \Sigma(\rho(X_i): i = 1, \cdots, t)$ and, hence, by (3.1), $\delta'(s) = \Sigma(k - \rho(X_i): i = 1, \cdots, t) \leqq \gamma$.

Case 2. There are two intersecting members A, B of $\mathcal{F}$. If $A \cup B \neq V$, then $A \cup B$ is in-critical by Proposition 3.2, and then replacing A and B in $\mathcal{F}$ by $A \cup B$ we are in contradiction with the minimal choice of t. Therefore, $A \cup B = V$.

Let $Y_1 := V - A$ and $Y_2 := V - B$. Then $\delta(Y_1) = \rho(A)$, and $\delta(Y_2) = \rho(B)$. By (3.2) we have $\gamma \geqq k - \delta(Y_1) + k - \delta(Y_2) = k - \rho(A) + k - \rho(B) \geqq k - \rho'(A) + k - \rho'(B) + \delta'(s) = \delta'(s)$.

Therefore, the proof of Lemma 3.3 is complete. □

The theorem immediately follows by γ repeated applications of the following theorem of Mader [27].

THEOREM 3.4 ([27]). *Suppose that for a node s of a digraph* $G' = (V + s, E')$ $\delta'(s) = \rho'(s)$ *and the edge-connectivity between any two nodes distinct from s is at least k (that is,* (3.3) *holds). Then for any edge st there is an edge vs such that vs and st can be split off without violating* (3.3).

Call a pair of edges vs, st *splittable* if they can be split off without violating (3.3). Here we provide a (new) proof of Mader's theorem that will be useful in § 9 to improve the complexity of an algorithm arising from the naive implementation of Mader's theorem.

Proof. We need the following proposition.

PROPOSITION 3.5. *Suppose that* $\delta'(s) = \rho'(s)$ *for a node s of a digraph G' and the edge-connectivity between any two nodes distinct from s is at least k. If X, Y are intersecting subsets of nodes for which* $\{s\} = X \cap Y$ *and* $\delta'(X) = \delta'(Y) = k$, *then* $\delta'(X - Y) = \delta'(Y - X) = k$, *and* $\bar{d}'(X, Y) = 0$.

Proof. Applying (2.5) we obtain $k + k = \delta'(X) + \delta'(Y) = \delta'(X - Y) + \delta'(Y - X) + \bar{d}'(X, Y) \geqq k + k + \bar{d}'(X, Y)$ from which $\delta'(X - Y) = \delta'(Y - X) = k$ and $\bar{d}'(X, Y) = 0$ follows. □

Call a subset $\emptyset \subset X \subset V$ *in-critical* if $\rho'(X) = k$ and *out-critical* if $\delta'(X) = k$. X is called *critical* if it is either out- or in-critical. (Note that V is never critical.)

PROPOSITION 3.6. *Let A and B be two intersecting critical sets. Then either* (i) $A \cup B$ *is critical or* (ii) $B - A$ *is critical and* $\bar{d}'(A, B) = 0$.

Proof. If both A and B are in-critical and $A \cup B \subset V$, then Proposition 3.2 implies alternative (i). If $A \cup B = V$, then Proposition 3.5, when applied to $X := V + s - A$, $Y := V + s - B$, implies (ii). The situation is analogous if both A and B are out-critical. Finally, let A be in-critical and B out-critical. Proposition 3.2, when applied to A and $V + s - B$, implies (ii).

A pair $\{vs, st\}$ of edges is not splittable precisely if there is a critical set containing both v and t. Therefore, if there is no critical set containing t, any pair $\{vs, st\}$ is splittable.

For two intersecting critical sets A, B containing t, only alternative (i) may hold in Proposition 3.6 since the existence of edge st implies $\bar{d}'(A, B) > 0$. Therefore, the union M of all critical sets containing t is critical again.

We claim that there is an edge vs with $v \in V - M$. We indirectly suppose that no such an edge exists. If M is in-critical, then $\delta'(V-M) < \rho'(M) = k$, contradicting (3.3b). If M is out-critical, then $\delta'(s) = \rho'(s)$ implies that $\rho'(V-M) = \delta'(M+s) < \delta'(M) = k$, contradicting (3.3a).

By the choice of M, no critical set contains both v and t; therefore, vs and st are splittable. □

Our next problem is to find an augmentation of minimum cardinality if upper and lower bounds are imposed both on the in-degrees and the out-degrees of the digraph of newly added edges. Let $f_{\text{in}} \leqq g_{\text{in}}$ and $f_{\text{out}} \leqq g_{\text{out}}$ be four nonnegative integer-valued functions on V (infinite values are allowed for g_{in} and g_{out}). (Recall that $g_{\text{in}}(X)$ denotes $\Sigma(g_{\text{in}}(v) : v \in X)$.)

THEOREM 3.7. *Given a directed graph $G = (V, E)$ and a positive integer k, G can be made k-edge-connected by adding a set F of precisely γ new edges so that*

$$f_{\text{in}}(v) \leqq \rho_F(v) \leqq g_{\text{in}}(v) \quad \textit{and} \tag{3.4a}$$

$$f_{\text{out}}(v) \leqq \delta_F(v) \leqq g_{\text{out}}(v) \tag{3.4b}$$

hold for every node v of G if and only if

$$k - \rho(X) \leqq g_{\text{in}}(X) \quad \textit{and} \tag{3.5a}$$

$$k - \delta(X) \leqq g_{\text{out}}(X) \tag{3.5b}$$

hold for every subset $\emptyset \subset X \subset V$, and

$$\Sigma(k - \rho(X_i) : i = 1, \cdots, t) + f_{\text{in}}(X_o) \leqq \gamma, \tag{3.6a}$$

$$\Sigma(k - \delta(X_i) : i = 1, \cdots, t) + f_{\text{out}}(X_o) \leqq \gamma \tag{3.6b}$$

hold for every partition $\{X_o, X_1, X_2, \cdots, X_t\}$ of V where X_o may be empty.

(At this point we emphasize that both in this theorem and later theorems, degree-constrained augmentations loops are allowed to be added to G. It may be interesting to consider the problem when loops are not allowed.)

Proof of the necessity. Suppose that there is a required set F of new edges. Then $k \leqq \rho(X) + \rho_F(X) \leqq \rho(X) + \Sigma(\rho_F(v): v \in X) \leqq \rho(X) + g_{\text{in}}(X)$, and (3.5a) follows. The proof of (3.5b) is analogous. Similarly, $\rho_F(X_i) \geqq k - \rho(X_i)$ and $\rho_F(v) \geqq f_{\text{in}}(v)$ and hence $|F| \geqq \Sigma(\rho_F(X_i): i = 0, 1, \cdots, t) \geqq \Sigma(k - \rho(X_i): i = 1, \cdots, t) + f_{\text{in}}(X_o)$, and (3.6a) follows. (3.6b) is analogous again. □

To prove the sufficiency, we can apply the method of the proof of Theorem 3.1. Here, we only outline this, and a formal proof is postponed to § 8, where the use of polymatroids make clear why such a proof works. The sketch below also indicates an algorithm to find a desired augmentation.

Sketch of the proof of sufficiency. We can assume that g_{in} and g_{out} is finite, since if $g_{\text{in}}(v)$, say, is infinite, then $g_{\text{in}}(v)$ can be revised to be max $(k, \gamma, f_{\text{in}}(v))$. This modification does not destroy the necessary conditions.

Extend G by a new node s. For each node $v \in V$, add $g_{\text{in}}(v)$ parallel edges from s to v, and $g_{\text{out}}(v)$ parallel edges from v to s. (3.5) ensures that (3.3) holds for the extended digraph G'. Since $\gamma \leqq \min(g_{\text{in}}(V), g_{\text{out}}(V))$, we have $\rho'(s) \geqq \gamma$ and $\delta'(s) \geqq \gamma$. Now delete new edges, one by one, so that (3.3) continues to hold and each node v has at least $f_{\text{in}}(v)$ newly entering and $f_{\text{out}}(v)$ newly leaving edges. This deletion procedure stops when the current in-degree and out-degree of s is γ. If we can reach such a situation, then Mader's splitting-off theorem can be applied, and we are done.

The only trouble may arise if $\delta'(s) > \gamma$ and no new edge leaving s can be left out, or if $\rho'(s) > \gamma$ and no new edge entering s can be left out. Suppose that the first case occurs (the second is analogous). Now a new edge sv may not be left out because sv either enters an in-critical set or $\rho'(v) = f_{in}(v)$. That is, the set $\{v: \rho'(v) > f_{in}(v)\}$ can be covered by a family $\mathcal{F} = \{X_1, \cdots, X_t\}$ of in-critical sets. Suppose that t is as small as possible. If $\mathcal{F}$ consists of disjoint sets, then $\{X_o, X_1, \cdots, X_t\}$ violates (3.6a), where $X_o := V - U(X_i: i = 1, \cdots, t)$. If $\mathcal{F}$ includes two intersecting sets A and B, then $A \cup B = V$ and the partition $\{Y_0, Y_1, Y_2\}$ of V, where $Y_o := A \cap B$, $Y_1 := V - A$, and $Y_2 := V - B$, violates (3.6b).

We are interested in degree-constrained augmentation when there is no requirement for the number of new edges; see the following theorem.

THEOREM 3.8. *Given a directed graph $G = (V, E)$ and a positive integer k, G can be made k-edge-connected by adding a set F of new edges satisfying* (3.4) *if and only if* (3.5) *holds, and*

$$\Sigma(k-\rho(X_i): i=1, \cdots, t)+f_{in}(X_o) \leqq \alpha, \tag{3.7a}$$

$$\Sigma(k-\delta(X_i): i=1, \cdots, t)+f_{out}(X_o) \leqq \alpha \tag{3.7b}$$

hold for every partition $\{X_o, X_1, X_2, \cdots, X_t\}$ of V where X_o may be empty and $\alpha :=$ min $(g_{out}(V), g_{in}(V))$.

Proof. (3.5) is clearly necessary. To see the necessity of (3.7), let F be a set of new edges satisfying the requirements. Then $\Sigma(k - \rho(X_i): i = 1, \cdots, t) + f_{in}(X_o) \leqq |F| \leqq \alpha$, and (3.7a) follows. (3.7b) is analogous.

To see the sufficiency, observe that by choosing $\gamma := \alpha$ if α is finite and $\gamma := k|V| + f_{in}(V) + f_{out}(V))$ if $\alpha = \infty$, condition (3.6) follows from (3.7), and then Theorem 3.7 applies. □

Let us consider the minimum cost k-edge-connected augmentations. As we mentioned in the Introduction, if costs are assigned to the edges, the problem is NP-complete even if $k = 1$. Suppose now that $c_{in} : V \rightarrow R_+$ and $c_{out} : V \rightarrow R_+$ are two nonnegative cost functions on the node-set V of G. Our object is to find a k-edge-connected augmentation of G for which $\Sigma\rho_F(v)c_{in}(v) + \Sigma\delta_F(v)c_{out}(v)$ is minimum, where F is the newly added edges. The algorithm is a version of the proof of Lemma 3.3, in which the selection of new edges to be discarded is governed by the cost of the end-nodes in a greedy fashion.

ALGORITHM TO FIND A MINIMUM NODE-COST k-EDGE-CONNECTED AUGMENTATION OF A DIGRAPH

Add a new node s to V.

PART 1. Add k new parallel edges from s to v for every $v \in V$. (For the resulting digraph G', (3.3a) holds.) Assume that the new edges $f_1, f_2, \cdots$ are ordered according to the decreasing order of the c_{in} costs of their end-node u_i. (The order of parallel edges from s to u_i does not matter.) Go through the new edges in the given order and discard an f_i if this can be done without destroying (3.3a). Let γ_1 be the number of remaining new edges.

PART 2. Add k new parallel edges from v to s for every $v \in V$. (For the resulting digraph G', (3.3b) holds.) Assume that the new edges $f_1, f_2, \cdots$ are ordered according to the decreasing order of the c_{out} costs of their tail-node u_i. (The order of parallel edges from u_i to s does not matter.) Go through the new edges in the given order and discard an f_i if this can be done without destroying (3.3b). Let γ_2 be the number of remaining new edges.

Let $\gamma := \max(\gamma_1, \gamma_2)$. If $\gamma_2 < \gamma_1$, add $\gamma_1 - \gamma_2$ parallel edges from u to s, where u is a node and $c_{out}(u)$ is minimum. If $\gamma_1 < \gamma_2$, add $\gamma_2 - \gamma_1$ parallel edges from s to u, where u is a node and $c_{in}(u)$ is minimum.

PART 3. Let G' denote the final digraph. In G', $\delta'(s) = \rho'(s) = \gamma$, and (3.3) holds. Apply γ times Mader's Theorem 3.4 to G'. Let $G_1 = (V, E \cup F)$ denote the resulting digraph.

THEOREM 3.9. *The graph G_1, constructed above, is a minimum cost k-edge-connected augmentation of G.*

The proof of this theorem is postponed until § 7, where the necessary tools from polymatroid theory are already available. In § 8 we will make comments on algorithmic aspects of the procedure above including its extension to the capacitated case.

Perhaps it is worth mentioning that, by the above algorithm, a minimum cost augmentation is automatically a minimum cardinality augmentation.

We close this section by pointing out that the following two versions of the augmentation problem answered by Theorem 3.1 are NP-complete.

Problem A. Let $G = (V, E)$ be a directed graph, s a specified node of G, $T \subset V$ a specified subset of nodes, and γ a positive integer. Decide if it is possible to add at most γ new edges to G so as to have a path from s to every element of T.

Problem B. Let $G' = (V, E')$ be a directed graph, $R \subset V$ a specified subset of nodes, and γ a positive integer. Decide if it is possible to add at most γ new edges so as to have a path from every node of R to any other node of R.

THEOREM 3.10. *Both problems* A *and* B *are* NP-*complete*.

Proof. The following set covering problem is known to be NP-complete [17]: Given k sets $X_1, \cdots, X_k$ and an integer γ, decide if there is a set X with cardinality at most γ that intersects all X_i's.

First we show that set covering can be solved in polynomial time if Problem A can be solved in polynomial time. Let $S := X_1 \cup \cdots \cup X_k$. For each X_i let t_i be a new element and $T := \{t_1, \cdots, t_k\}$. Let s be an element not in $S \cup T$, and $V := S \cup T \cup \{s\}$. Let $G = (V, E)$ be a directed graph, where $E := \{vt_i : \text{if } v \in X_i\}$.

It is easily seen that if Problem A has a solution, it has a solution in which every new edge is of the form sv where $v \in S$. Then there is a solution to set covering; namely, the heads of new edges from a subset X of at most γ elements intersecting all X_i's. Conversely, if X is a solution to set covering, then $\{sv : v \in X\}$ as the set of new edges forms a solution to Problem A. Therefore, Problem A is NP-complete.

To see that Problem B is NP-complete, suppose that it is solvable in polynomial time. We then show that Problem A is also solvable in polynomial time. Indeed, a set F of new edges is a solution to Problem A with input $\{G = (V, E), s, T, \gamma\}$ if and only if F is a solution to Problem B with input $\{G' = (V, E'), R, \gamma\}$, where $E' := E \cup \{vs : v \in T\}$ and $R := T + s$. □

4. Undirected graphs. In this section we first provide a simpler proof of a theorem of Cai and Sun [2]. One advantage of this proof is that it can be extended to the degree-constrained case. Another one is that we use Lovász's splitting-off theorem [23], [24] rather than Mader's, which is much more difficult. This way, the proof becomes self-contained as we provide a (new) proof of Lovász's theorem.

THEOREM 4.1 ([2]). *Given an undirected graph $G = (V, E)$ and an integer $k \geqq 2$, G can be made k-edge-connected by adding at most γ new edges if and only if*

$$\Sigma(k - d(X_i)) \leqq 2\gamma \tag{4.1}$$

holds for every subpartition $\{X_1, X_2, \cdots, X_t\}$ of V.

Proof. The proof is analogous to that of Theorem 3.1.

Necessity. Suppose $G' = (V, E \cup F)$ is a k-edge-connected supergraph of G, where F denotes the set of new edges. Then every subset X_i of V has at least $k - d(X_i)$ newly entering edges. Therefore, the number of new edges in G' is at least $\Sigma(k - d(X_i))/2$, and (4.1) follows.

We prove the sufficiency in two steps. Let s be a node not in V, and $V' := V + s$.

LEMMA 4.2. *G can be extended to a graph $G' = (V + s, E')$ by adding a new node s, and 2γ new edges between V and s in such a way that for every subset $\emptyset \neq X \subset V$*

$$d'(X) \geqq k \tag{4.2}$$

holds where d' denotes the degree function of G'.

Proof. First, we add a sufficiently large number of edges leaving s so as to satisfy (4.2). (It certainly will do if we add k edges from s to v for every $v \in V$.)

Second, we discard new edges, one by one, as long as possible without violating (4.2). Let G' denote the final extended graph. The following claim implies the lemma.

CLAIM. $d'(s) \leqq 2\gamma$.

Proof. Call a subset $\emptyset \subset X \subset V$ *critical* if $d'(X) = k$.

PROPOSITION 4.3. *If X and Y are intersecting critical, then both $X - Y$ and $Y - X$ are critical, and $\bar{d}'(X, Y) = 0$.*

Proof. We have $k + k = d'(X) + d'(Y) = d'(X - Y) + d'(Y - X) + 2\bar{d}'(X, Y) \geqq k + k$ from which the proposition follows. □

Let $S := \{u \in V: su \in E'\}$. An edge su cannot be left out without violating (4.2), precisely if there is a critical set containing u. Let M_u denote a minimal critical set containing u ($u \in S$) and let $\mathcal{F} := \{M_u: u \in S\}$. Let $X_1, X_2, \cdots, X_t$ be the maximal members of $\mathcal{F}$.

PROPOSITION 4.4. *Sets X_i ($i = 1, \cdots, t$) are pairwise disjoint.*

Proof. We prove that $\mathcal{F}$ is laminar. If $M_u, M_v \in \mathcal{F}$ are intersecting, then, by Proposition 4.3, $M_v - M_u$ is critical and $\bar{d}'(M_u, M_v) = 0$, therefore $v \in M_v - M_u$ contradicting the minimal choice of M_v.

By (4.2) we have $d'(s) = \Sigma(d'(X_i) - d(X_i): i = 1, \cdots, t)) = \Sigma(k - d(X_i): i = 1, \cdots, t) \leqq 2\gamma$, as required for the claim.

Add one extra edge sv if $d'(s)$ is odd to make it even, and let $\gamma' := d'(s)/2$ ($\leqq \gamma$). Theorem 4.1 follows by γ' repeated applications of the following theorem of Lovász. □

THEOREM 4.5 ([23], [24]). *Suppose that in a graph $G' = (V + s, E')$ $d'(s) > 0$ is even, and for every subset $\emptyset \neq X \subset V$ (4.2) holds. Then for every edge st there is an edge su so that the pair $\{st, su\}$ can be split off without violating* (4.2).

Remark. Lovász announced this theorem in Prague [23] and gave a proof in his problem book [24]. There Lovász broke up the problem into two parts. Problem 6.51 is the above statement (with different notation) formulated for Eulerian graphs while Problem 6.53 in Lovász's book sounds as follows, "*Prove that, provided $k \geqq 2$, the assertion of* 6.51 *holds for non-Eulerian graphs as well.*" However, this formulation is not completely precise since the evenness of the degree of s cannot be dropped, as is shown by the complete graph on four nodes. The proof (which is otherwise correct) given by Lovász [24, p. 287] uses a "tripartite" submodular inequality. Here we provide another proof that avoids this and will also be useful in § 9 to improve the efficiency of an algorithm arising from the naive implementation of Theorem 4.5.

Proof. Call a set $\emptyset \subset X \subset V$ *dangerous* if

$$d'(X) \leqq k + 1. \tag{4.3}$$

A pair $\{st, su\}$ of edges is called *splittable* if they can be split off without violating (4.2). This is the case precisely if there is no dangerous set X with $t, u \in X$. Let $S := \{v \in V: sv \in E'\}$.

CLAIM A. *Let A and B be intersecting dangerous sets with $t \in A \cap B$. Then*
(i) $\bar{d}'(A, B) = 1$ *and*
(ii) $S \not\subseteq A \cup B$ *(in particular, $A \cup B \neq V$).*

Proof. By (2.2) we have $(k+1)+(k+1) \geqq d'(A)+d'(B) = d'(A-B)+d'(B-A)+2\bar{d}'(A,B) \geqq k+k+2$ from which (i) follows.

Suppose that, indirectly, $S \subseteq A \cup B$. Let $\alpha := d'(s, A-B)$ and $\beta := d'(s, B-A)$. By symmetry, we can assume that $\alpha \geqq \beta$. Since $\bar{d}'(A,B) = 1$ we have $k \leqq d'(V-A) = d'(A+s) = d'(A) - \alpha + \beta - 1 \leqq d'(A) - 1 \leqq k$ from which $\alpha \leqq \beta$, and thus $\alpha = \beta$ follows. But this is impossible since, if $S \subseteq X \cup Y$, then $d'(s) = \alpha + \beta + 1 = 2\alpha + 1$, an odd number. □

CLAIM B. *If A and B are intersecting dangerous sets with $t \in A \cap B$, and A is maximal dangerous, then $d'(A) = d'(B) = k+1$ and $d'(A \cap B) = k$.*

Proof. By Claim A, $A \cup B \neq V$, and by the maximality of A, $d'(A \cup B) \geqq k+2$. From (2.1) we have $(k+1)+(k+1) \geqq d'(A)+d'(B) \geqq d'(A \cup B) + d'(A \cap B) \geqq (k+2)+k$, from which the statement follows. □

If there is at most one maximal dangerous set X with $t \in X$, then for any edge sv with $v \notin X$ the pair st, sv is splittable. Such an edge exists since otherwise $d'(V-X) = d'(X+s) = d'(X) - d'(s) \leqq (k+1) - 2 = k-1$, contradicting (4.2).

Suppose that X and Y are two distinct maximal dangerous sets with $t \in X \cap Y$ for which $M := X \cap Y$ is maximal. Then X and Y are intersecting, and Claim A implies that there is an edge sv with $v \notin X \cup Y$.

CLAIM. *The pair sv, st is splittable.*

Proof. Suppose that, indirectly, there is a maximal dangerous set Z with $t, v \in Z$. Applying Claim B to $A := X$ and $B := Y$ we have $d'(M) = k$. Z and M must not be intersecting for otherwise Claim B could be applied to $A := X$ and $B := M$ implying $d'(M) = k+1$. Therefore $M \subseteq Z$ and by the maximal choice of M we have $X \cap Z = Y \cap Z = M$. By Claim A $\bar{d}'(X, Y) = \bar{d}'(Z, Y) = \bar{d}'(Z, X) = 1$, and therefore no other edge than st can leave M, contradicting $k \geqq 2$. □

Theorem 4.1 can be extended to a degree-constrained case when upper and lower bounds are imposed on the degrees of the graph of newly added edges. Let $f \leqq g$ be two nonnegative integer-valued functions on V (infinite values are allowed for g).

THEOREM 4.6. *Given an undirected graph $G = (V, E)$ and an integer $k \geqq 2$, G can be made k-edge-connected by adding a set F of precisely γ new edges so that*

(4.4) $$f(v) \leqq d_F(v) \leqq g(v)$$

holds for every node v of G if and only if $2\gamma \leqq g(V)$ and

(4.5) $$k - d(X) \leqq g(X),$$

holds for every subset $\emptyset \subset X \subset V$ and

(4.6) $$\Sigma(k - d(X_i): i = 1, \cdots, t) + f(X_o) \leqq 2\gamma$$

holds for every partition $\{X_o, X_1, X_2, \cdots, X_t\}$ of V where X_o may be empty.

Remark. This theorem, when applied to $f \equiv 0$, $g \equiv \infty$, immediately implies the theorem of Cai and Sun and, therefore, it would not have been necessary to prove first Theorem 4.1. We did so to exhibit the simplicity of the idea behind the proof. The next proof uses the very same idea along with some technicalities.

Proof. The necessity of the conditions is straightforward. To see the sufficiency, first, add a new node s to V and $\min(g(v), f(v)+k)$ new parallel edges between s and v for every $v \in V$. For the enlarged graph G', the number $d'(v) - d(v)$ of new edges incident to v is at most $g(v)$ for every $v \in V$ and, by (4.5), (4.2) holds. If $d'(s) < 2\gamma$, add $2\gamma - d'(s)$ appropriate edges leaving s in such a way that $d'(v) - d(v)$ is still at most $g(v)$ $(v \in V)$. This is possible because we have assumed that $2\gamma \leqq g(V)$.

Second, discard new edges, one by one, as long as possible without violating (4.2) and the following inequalities: $d'(s) \geqq 2\gamma$, $d'(v) - d(v) \geqq f(v)$ $(v \in V)$. Let G' denote the final extended graph. By Proposition 4.4 the set $A := \{v \in V: d'(v) - d(v) > f(v)\}$ can be covered by disjoint critical sets $X_1, X_2, \cdots, X_t$. Let $X_o := V - \cup(X_i: i = 1, \cdots, t)$. Applying (4.6) we have $2\gamma \leqq d'(s) = \Sigma(d'(X_i) - d(X_i): i = 1, \cdots, t) + f(X_o) = \Sigma(k - d(X_i): i = 1, \cdots, t) + f(X_o) \leqq 2\gamma$. Hence $d'(s) = 2\gamma$, and by γ applications of Theorem 4.5 we obtain the desired augmentation of G. □

We are interested in degree-constrained augmentations where the number of new edges does not matter; see the following theorem.

THEOREM 4.7. *Given an undirected graph $G = (V, E)$ and an integer $k \geqq 2$, G can be made k-edge-connected by adding a set F of new edges so that* (4.4) *holds for every node v of G if and only if* (4.5) *holds for every subset $\emptyset \subset X \subset V$ and* (∗) *there is no partition $\mathcal{F} := \{X_o, X_1, X_2, \cdots, X_t\}$ of V, where only X_o may be empty, with the following properties: $f(X_o) = g(X_o)$, $g(X_i) = k - d(X_i)$, and $g(V)$ is odd.*

Proof. The necessity of (4.5) is clear. To see the necessity of (∗) let $\mathcal{F}$ be a partition with the given properties, and F a set of new edges satisfying the requirements. Then $d_F(v) = g(v)$ for $v \in X_o$, $d_F(X_i) = k - d(X_i) = g(X_i)$ for $i = 1, \cdots, t$, and, furthermore, no X_i induces elements of F. Therefore $g(V) = \Sigma(g(X) : X \in \mathcal{F}) = 2|F|$, an even number.

To see the sufficiency, extend the graph with a new node s and add $\min(g(v), f(v) + k)$ new parallel edges from s to v for every $v \in V$. Let $V' := V + s$ and let $G' = (V', E')$ denote the extended graph. The number of new (parallel) edges between s and v is $d'(v) - d(v)$.

By this construction $f(v) \leqq d'(v) - d(v) \leqq g(v)$, and (4.5) implies that (4.2) holds for every subset $\emptyset \subset X \subset V$. Therefore, if $d'(s)$ is even, then Lovász's Theorem 4.5 implies the theorem.

Suppose that $d'(s)$ is odd. If there is a node $v \in V$ with $d'(v) - d(v) < g(v)$, then by adding one more edge sv to G' we are at the case of $d'(s)$ even.

Therefore, we can assume that $d'(v) - d(v) = g(v)$ for every $v \in V$. If there is an edge $e = su$ for which $f(u) < g(u)$ and su does not enter any critical set, then e can be deleted without destroying (4.4) and $f(v) \leqq d'(v) - d(v)$, and then $d'(s)$ becomes again even. So suppose that there is no such an edge; that is, every edge sv either enters a critical set or has $f(v) = g(v)$.

Then, by Proposition 4.4, there are disjoint critical sets $X_1, X_2, \cdots, X_t$ for which $k = d'(X_i) = d(X_i) + g(X_i)$ so that $\cup X_i$ contains all nodes v with $f(v) < g(v)$. Let $X_0 := \{v \in V: f(v) = g(v), v \notin \cup X_i\}$. We obtain that $f(X_o) = g(X_o)$, $g(X_i) = k - d(X_i)$ for $i = 1, \cdots, t$, and $g(V) = \Sigma(g(X) : X \in \mathcal{F}) = d'(s)$ is odd, contradicting (∗). □

5. Generalization. In this section we exhibit a natural generalization of results from the preceding section. Let $G = (V, E)$ be an undirected graph and $r(u, v)$ $(u, v \in V)$ a nonnegative integer-valued function on the pair of nodes that serves as the demand for edge-connectivity between u and v. When can G be extended by adding γ new edges so that in the extended graph G' the edge-connectivity number $\lambda'(u, v)$ is at least $r(u, v)$ for every pair of nodes u, v? Such an augmentation is called *good* (with respect to the demand $r(u, v)$). It will be convenient to assume (and this can be done without loss of generality) that

(5.0a) $$r(u,v) \geqq \lambda(u,v) \text{ for every } u,v \in V, \quad \text{and}$$

(5.0b) $$r(u,x) \geqq 1 \quad \text{and} \quad r(v,x) \geqq 1 \text{ imply that } r(u,v) \geqq 1.$$

By Menger's Theorem 2.1, G' is a good augmentation of G if

(5.1) $$d'(X) \geqq R(X)$$

holds for every set $\emptyset \subset X \subset V$ where $R(X) := \max(r(u, v): u \in X, v \in V - X)$. (The maximum on the empty set is defined to be 0.)

In order to obtain more general results on optimal augmentations, we need stronger theorems about splitting-off.

In a graph $G' = (V + s, E')$, let $\lambda'(X) := \max(\lambda'(u, v): u \in X, v \in V - X)$. Obviously $d'(X) \geqq \lambda'(X)$. We call a pair $\{su, sv\}$ of edges of G' *splittable* if after splitting off $\{su, sv\}$ the edge-connectivity between every two nodes distinct from s remains the same. Obviously, $\{su, sv\}$ is splittable precisely if there is no subset $X \subseteq V$ with u, $v \in X$ for which $d'(X) \leqq \lambda'(X) + 1$. We call such a set X *dangerous*.

Mader [26] proved the following extremely powerful result.

THEOREM 5.1 ([26]). *Let $G' = (V + s, E')$ be a connected undirected graph with $d'(s) \neq 3$ or 1.*

(a) *If s is not a cut-node (that is, $G' - s$ is connected), then there is a splittable pair of edges $\{su, sv\}$.*

(b) *If s is a cut-node but there is no cut-edge incident to s, then any pair of edges $\{su, sv\}$ is splittable provided that u and v belong to distinct components of $G' - s$.*

Remarks. The original proof of this theorem is rather difficult. In [12] a relatively simple proof is given. It is not necessarily true that, under the above assumptions, for a given edge st there is an edge su so that st and su are splittable. Furthermore, the theorem does not hold in general if $d'(s) = 3$, as is shown by a complete graph on four nodes. Note that Theorem 4.5 is a special case of Mader's theorem.

COROLLARY 5.2. *Suppose that in an undirected graph $G' = (V + s, E')$ degree $d'(s)$ is even and there is no cut-edge incident to s. Then the edge incident to s can be paired in such a way that splitting off each pair results in a graph with vertex set V in which the edge-connectivity between every two nodes u, v is equal to the original edge-connectivity $\lambda'(u, v)$.*

Proof. Apply Theorem 5.1 $d'(s)/2$ times and observe that after a splitting no edge incident to s becomes a cut-edge. □

Let us turn back to the augmentation problem. We introduce the following notation: $q(A) := R(A) - d(A)$. That is, $q(A)$ is the deficiency of $A \subseteq V$. The following condition is clearly necessary for the existence of a good augmentation using at most γ new edges:

(5.2) $$\Sigma q(X_i) \leqq 2\gamma$$

for every subpartition $\{X_1, \cdots, X_t\}$ of V. Theorem 4.1 of Cai and Sun asserted that, in the special case when $r(u, v) \equiv k \geqq 2$, (5.2) is sufficient as well. However, it is not sufficient, in general, as is shown by the empty graph on four nodes with $r(u, v) \equiv 1$.

Let $C\ (\neq V)$ be a component of G. We call C a *marginal* component (with respect to the demand function $r(u, v)$) if $q(X) \leqq 0$ for every $X \subset C$ and $q(C) \leqq 1$. This is equivalent to saying that $r(u, v) \leqq \lambda(u, v)$ for $u, v \in C$ and $r(u, v) \leqq \lambda(u, v) + 1$ for $u \in C, v \in V - C$.

Our solution to the problem of finding a good augmentation consists of two steps. First, we show that marginal components can be easily eliminated; second, we prove that if there are no marginal components, then (5.2) is sufficient.

Let $\gamma(G, r)$ denote the minimum number of edges the addition of which to G results in a graph G' satisfying (5.1). Let C be a marginal component of G, and $G_1 := G - C$. Let r_1 denote the function r restricted on the pairs of nodes of $V - C$.

THEOREM 5.3. *For a marginal component C of G $\gamma(G, r) = \gamma(G_1, r_1) + q(C)$.*

Proof. Let $\gamma := \gamma(G, r)$ and $\gamma_1 := \gamma(G_1, r_1)$. First, we show that $\gamma \leqq \gamma_1 + q(C)$. Let G'_1 denote a minimal augmentation of G_1. If $q(C) = 0$, then clearly G'_1, along with C, yields a good augmentation of G and hence $\gamma \leqq \gamma_1 = \gamma_1 + q(C)$. If $q(C) = 1$, then there is a pair a, b with $a \in C$ and $b \in V - C$ for which $r(a, b) = 1$. We claim that adding C and an edge ab to G'_1 yields a good augmentation G' of G. Indeed, if this were not true, there would be a pair s, t of nodes of G for which $r(s, t) > \lambda'(s, t)$. Then precisely one of s and t, say s, is in C and t in $V - C$ (because C is marginal and G'_1 is good with respect to r_1). Since C is marginal, $r(s, t) = 1$ and then $\lambda'(s, t) = 0$. Hence in G'_1 there is no path between t and b, and therefore $r_1(t, b) = r(t, b) = 0$. (5.0b) shows that a and s must be distinct. Then, by (5.0a) $r(s, a) \geqq 1$. Applying (5.0b) twice, we obtain that $r(a, t) \geqq 1$ and $r(b, t) \geqq 1$, a contradiction. Therefore $\gamma \leqq \gamma_1 + q(C)$.

To see the other direction let $G_o = (V_o, E_o)$ be a graph obtained from G by replacing C with a new node v_c. Define $r_o(u, v) := r(u, v)$ if $u, v \in V - C$ and $r_o(v_c, v) := q(C)$. Let $\gamma_o := \gamma(G_o, r_o)$. Obviously, $\gamma_o \leqq \gamma$.

Let $G'_o = (V_o, E_o \cup F)$ be a minimal augmentation of G_o good with respect to r_o such that the number t of elements in F incident to v_c is as small as possible. If $t = 0$, then $q(C) = 0$ and the elements of F are induced by $G - C$. Hence $\gamma_1 \leqq |F| = \gamma_o \leqq \gamma = \gamma - q(C)$, as required. If $t = 1$, then, by the minimality of F, $q(C) = 1$. Let $f \in F$ be the edge incident to v_c. Adding $F - f$ to G_1, we obtain a good augmentation of G_1 and then $\gamma_1 \leqq |F| - 1 = \gamma_o - 1 \leqq \gamma - q(C)$, as required.

Finally, suppose that $t \geqq 2$ and let $v_cu_1, \cdots, v_cu_t$ be the t edges in F incident to v_c. Let F' be obtained from F by replacing v_cu_i by u_1u_i $(i = 2, 3, \cdots, t)$. It is not hard to see that $G_o + F'$ is also a good (and minimal) augmentation of G_o, contradicting the minimal choice of t. □

By Theorem 5.3 we can easily reduce the augmentation problem to a case when there is no marginal component. Namely, proceed as follows. Let $C_1, C_2, \cdots, C_t$ be components of G such that C_i is a marginal component of $G - (C_1 \cup \cdots \cup C_{i-1})$ $(i = 1, \cdots, t)$ and $G - (C_1 \cup \cdots \cup C_t)$ has no marginal components. Leave out each C_i and find a minimal augmentation of the remaining graph (as to be described below). Take back the components C_i, and for each component C_i add $q(C_i)$ (which is 0 or 1) new edges, as described in the first part of the proof of Theorem 5.3.

Before formulating the main result of this section we prove the following.

PROPOSITION 5.4. *For arbitrary $X, Y \subseteq V$ at least one of the following inequalities holds*:

$$R(X)+R(Y)\leqq R(X\cap Y)+R(X\cup Y), \tag{5.3a}$$

$$R(X)+R(Y)\leqq R(X-Y)+R(Y-X). \tag{5.3b}$$

Proof. Suppose that $R(X) = r(x, x')$ and $R(Y) = r(y, y')$, where X separates x and x', and Y separates y and y'. Assume first that one of the two pairs, say x, x', is separated by both X and Y. By taking the complement of Y, if necessary, we can assume that $x \in X - Y$ and $x' \in Y - X$. (If Y is replaced by its complement, then (5.3a) and (5.3b) transform into each other.) If y, y' are separated by X, then $R(Y) = R(X) \leqq \min(R(X - Y), R(Y - X))$, and (5.3b) follows.

If y, y' are not separated by X, then either one of y and y', say y, is in $X \cap Y$ and $y' \in X - Y$; or else one of y and y', say y, is in $Y - X$ and $y' \in V - (X \cup Y)$. In the first case, $R(X - Y) \geqq R(Y)$ and $R(Y - X) \geqq R(X)$, and (5.3b) follows. In the second case, $R(Y - X) \geqq R(Y)$ and $R(X - Y) \geqq R(X)$, and (5.3b) follows again.

Finally, assume that neither x, x' are separated by Y nor are y, y' separated by X. Again, we can assume that $x \notin Y$. Then $x' \in V - (X \cup Y)$. Now either one of y and y', say y, is in $X \cap Y$ and $y' \in X - Y$; or else one of y and y', say y, is in $Y - X$ and $y' \in V - (X \cup Y)$. In the first case, $R(X) \leqq R(X \cup Y)$ and $R(Y) \leqq R(X \cap Y)$, from which (5.3a) follows. In the second case, $R(X) \leqq R(X - Y)$ and $R(Y) \leqq R(Y - X)$, and (5.3b) follows. □

The main result of this section is as follows.

THEOREM 5.5. *If G has no marginal components, there is a good augmentation using at most γ new edges if and only if (5.2) holds for every subpartition $\{X_1, \cdots, X_t\}$ of V.*

Proof. The following lemma and Corollary 5.2 imply the theorem.

LEMMA 5.6. *G can be extended to a graph $G' = (V + s, E')$ by adding a new node s, and 2γ new edges between V and s so that none of the new edges is a cut-edge of G' and for every subset $\varnothing \subseteq X \subseteq V$*

$$d'(X) \geqq R(X) \tag{5.4}$$

holds where d' denotes the degree function of G'.

Proof. First, add a sufficiently large number of edges leaving s so as to satisfy (5.4). Second, discard new edges, one by one, as long as possible without violating (5.4). Let G' denote the final extended graph.

Claim. $d'(s) \leqq 2\gamma$.

Proof. We call a set $\varnothing \subset X \subseteq V$ *critical* if $d'(X) = R(X)$.

PROPOSITION 5.7. *If X and Y are critical sets, then at least one of the following statements holds:*

$$\text{both } X \cap Y \text{ and } X \cup Y \text{ are critical;} \tag{5.5a}$$

$$\text{both } X - Y \text{ and } Y - X \text{ are critical and } \bar{d}'(X, Y) = 0. \tag{5.5b}$$

Proof. If (5.3a) holds, then $R(X) + R(Y) = d'(X) + d'(Y) \geqq d'(X \cap Y) + d'(X \cup Y) \geqq R(X \cap Y) + R(X \cup Y) \geqq R(X) + R(Y)$ and (5.5a) follows.

If (5.3b) holds, then $R(X) + R(Y) = d'(X) + d'(Y) = d'(X - Y) + d'(Y - X) + 2\bar{d}'(X, Y) \geqq R(X - Y) + R(Y - X) + 2\bar{d}'(X, Y) \geqq R(X) + R(Y) + 2\bar{d}'(X, Y)$ and (5.5b) follows.

Let $S := \{u \in V : su \in E'\}$. An edge su cannot be left out without violating (5.4) precisely if there is a critical set containing u. Let $\mathcal{F} := \{X_1, X_2, \cdots, X_t\}$ be a family of critical sets that cover S so that t is minimal and, given this minimal t, $\Sigma |X_i|$ is minimal. We claim that the sets X_i's are disjoint.

Indeed, for $X, Y \in \mathcal{F}$ their union $X \cup Y$ cannot be critical by the minimality of t. Therefore (5.5b) must apply. Hence $X - Y$ and $Y - X$ are both critical and $\bar{d}'(X, Y) = 0$, from which $S \cap (X \cap Y) = \varnothing$. This means that if we replace X and Y by $X - Y$ and $Y - X$, then we obtain another family of t critical sets covering S. By the minimal choice of $\Sigma |X_i|$ we have that $|X| = |X - Y|$ and $|Y| = |Y - X|$; that is, X and Y are disjoint.

By (5.2) we have $d'(s) = \Sigma(d'(X_i) - d(X_i) : i = 1, \cdots, t) = \Sigma(q(X_i) : i = 1, \cdots, t) \leqq 2\gamma$, which proves the claim.

By adding one extra edge (parallel to an existing edge su), if $d'(s)$ is odd, we can assume that $d'(s) = 2\gamma'$ ($\leqq 2\gamma$). We claim that no edge incident to s is a cut edge of G'. Indeed, if $e = sv$ were a cut-edge, then let C be the component of $G' - e$ containing v but not s. There is precisely one edge in G' leaving C and therefore C must be a marginal component of G contradicting the assumption.

This way the proof of Lemma 5.6 is complete and so is the proof of the theorem. □

We mention two degree-constrained versions of Theorem 5.5. Their proofs are analogous, respectively, to those of Theorems 4.6 and 4.7, and we do not repeat them here. However, a proof will be provided in § 7 relying on a relationship between good augmentations and polymatroids.

THEOREM 5.8. *Suppose that $f(C) \geqq 2$ for every marginal component C of G. There is a good augmentation of G using a set F of precisely γ new edges so that* (4.4) *holds if and only if $2\gamma \leqq g(V)$ and*

$$q(X) \leqq g(X), \tag{5.6}$$

holds for every subset $\emptyset \subset X \subset V$ and

$$\Sigma(q(X_i): i=1, \cdots, t) + f(X_o) \leqq 2\gamma \tag{5.7}$$

holds for every partition $\{X_o, X_1, X_2, \cdots, X_t\}$ of V where X_o may be empty.

THEOREM 5.9. *Suppose that $f(C) \geqq 2$ for every marginal component C of G. There is a good augmentation of G using a set F of new edges so that* (4.4) *holds if and only if* (5.6) *holds for every subset $\emptyset \subset X \subset V$ and* (*) *there is no partition $\mathscr{F} := \{X_o, X_1, X_2, \cdots, X_t\}$ of V, where only X_o may be empty, with the following properties: $f(X_o) = g(X_o)$, $g(X_i) = q(X_i)$, and $g(V)$ is odd.*

To close this section we consider minimum cost augmentations. As we mentioned in the Introduction, if costs are assigned to the edges, the problem is NP-complete even if $r \equiv 2$. Suppose now that $c: V \rightarrow R_+$ is a nonnegative cost function on the node-set V of G. Our object is to find a good augmentation of G for which $\Sigma d_F(v)c(v)$ is minimum, where F is the set of newly added edges.

We are concerned only with the case when G has no marginal components. If G does have marginal components, a reduction analogous to the one described in Theorem 5.3 can be applied.

Assume that the elements of V are ordered in such a way that $c_1 \geqq c_2 \geqq \cdots \geqq c_n$ where $c_i := c(v_i)$. Let $k := \max(q(X): X \subseteq V)$.

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First, add a new node s to V, and k new parallel edges between s and v for every $v \in V$. For the resulting graph G', (5.1) holds; that is, $d'(X) \geqq R(X)$ for every $X \subseteq V$.

Assume that the new edges $f_1, f_2, \cdots$ are ordered according to the decreasing order of their end-node u_i. That is, first come the parallel edges from s to u_1, then the parallel edges from s to u_2, and so on. (The order of parallel edges between s and u_i does not matter.)

Next, go through the new edges in the given order and discard an f_i if this can be done without destroying (5.1). If at the end of the procedure there is an odd number of edges incident to s, add one further edge between s and v_n and let $G' = (V+s, E')$ denote the final graph. Note that the newly added edge is not a cut-edge of G', as we assumed that G has no marginal components.

Therefore we can apply Corollary 5.2 to G'. Let $G_1 = (V, E \cup F)$ denote the resulting graph.

THEOREM 5.10. *The graph G_1 constructed above is a minimum-cost good augmentation of G.*

This theorem will be proved in § 7, where the necessary tools from polymatroid theory are already available.

6. Generalized polymatroids. Before going into the details, let us indicate how polymatroid theory became involved in this research. The starting point was the paper of Cai and Sun [2]. We were trying to find a simple proof of their theorem when we realized that the degree vectors of good augmentations span a (generalized) polymatroid. This observation led to the desired proof and almost immediately to a proof of the directed version. The nice properties of polymatroids gave rise to the solution of the degree-constrained and the minimum node-cost augmentation problems.

Since there may be those who are interested in the augmentation problem but have no prior knowledge in polymatroid theory, some of the original proofs are converted here to avoid polymatroids. These proofs are included in the preceding sections. However, we do not want to hide this relationship, so this and the next two sections have been inserted.

First, we collect some results from polyhedral combinatorics. This environment helps us understand the background behind the results occurring in earlier sections. It also gives rise to possible generalizations and a proof of the two algorithms described at the end of §§ 3 and 5.

Generalizing the concept of matroid polyhedra, Edmonds [4] defined a polymatroid to be a polyhedron $P(b) := \{x \in R^V : x \geqq 0, x(A) \leqq b(A) \text{ for every } A \subseteq V\}$, where b is a submodular, monotone-increasing, finite-valued set-function with $b(\emptyset) = 0$. There are other classes of polyhedra of similar type. For example, Shapley [29] introduced, as we call it here, contrapolymatroids. Submodular polyhedra and basis polyhedra have been defined and investigated by Fujishige. For a general account, see [15]. Generalized polymatroids (in short, g-polymatroids) [11] serve as a common framework for all of these polyhedra.

Throughout this section we assume that any function in question is integer-valued (allowing $\pm\infty$).

Let $p : 2^V \to Z \cup \{-\infty\}$ be a supermodular function, and $b : 2^V \to Z \cup \{\infty\}$ a submodular function with $p(\emptyset) = b(\emptyset) = 0$ for which

(6.1) $$b(X) - p(Y) \geqq b(X-Y) - p(Y-X)$$

holds for every $X, Y \subseteq V$.

A pair (p, b) with the above properties is called a *strong pair*. A polyhedron $Q(p, b) := \{x \in R^V: p(A) \leqq x(A) \leqq b(A) \text{ for every } A \subseteq V\}$ is called a *g-polymatroid*. For technical convenience, we consider the empty set a g-polymatroid. For a detailed account on properties of g-polymatroids, see [13]. Here we cite some without proof.

PROPOSITION 6.1. *A g-polymatroid* $Q = Q(p, b)$ *is nonempty and is spanned by its integral points.* Q *uniquely determines its defining strong pair, namely,* $p(A) = \min(x(A): x \in Q)$ *and* $b(A) = \max(x(A): x \in Q)$.

A pair (p', b') is called a *weak pair* if p' (respectively, b') is supermodular (respectively, submodular) only on intersecting sets, and (6.1) is required only for intersecting X and Y.

PROPOSITION 6.2. *For a weak pair* (p', b') *the polyhedron* $Q = Q(p', b')$ *is a g-polymatroid.* Q *is nonempty if and only if*

(6.2a) $$\Sigma p'(Z_i) \leqq b'(\cup Z_i) \quad \text{and}$$

(6.2b) $$\Sigma b'(Z_i) \geqq p'(\cup Z_i)$$

hold for every subpartition $\{Z_1, \cdots, Z_t\}$ *of* V. *If* Q *is nonempty, it contains an integer point.*

Let $Q = Q(p, b)$ be a g-polymatroid defined by a strong pair (p, b). Let $f: V \to Z \cup \{-\infty\}$ and $g: V \to Z \cup \{\infty\}$ be two functions with $f \leqq g$ and let $-\infty \leqq \alpha \leqq \beta \leqq \infty$ be two integers.

PROPOSITION 6.3. *The intersection Q_o of a plank $\{x \in R^V: \alpha \leqq x(V) \leqq \beta\}$ and a g-polymatroid $Q(p, b)$ is a g-polymatroid. If Q_o is nonempty, its unique strong pair (p_o, b_o) is given by*

(6.3a) $$p_o(X) = \max(p(X), \alpha - b(V-X)),$$

(6.3b) $$b_o(X) = \min(b(X), \beta - p(V-X)).$$

PROPOSITION 6.4. *The intersection Q_1 of a box $\{x \in R^V: f \leqq x \leqq g\}$ and a g-polymatroid $Q(p_o, b_o)$ is a g-polymatroid. If Q_1 is nonempty, its unique strong pair (p_1, b_1) is given by*

(6.4a) $$p_1(X) = \max(p_o(Y) + f(X-Y) - g(Y-X): Y \subseteq V),$$

(6.4b) $$b_1(X) = \min(b_o(Y) + g(X-Y) - f(Y-X): Y \subseteq V).$$

PROPOSITION 6.5. *For a g-polymatroid $Q = Q(p, b)$ if $x \leqq y \leqq z$ are vectors so that $x, z \in Q$, then $y \in Q$.*

The greedy algorithm can also be extended to work on g-polymatroids Q. We need it only in the special case when the objective function $c: V \to R_+$ is nonnegative (c need not be integer-valued) and Q is bounded from below. (By Proposition 6.1 this is equivalent to requiring p to be finite.) The objective is to minimize cx over Q.

Suppose that the elements of V are ordered so that $c_1 \geqq c_2 \geqq \cdots \geqq c_n$.

PROPOSITION 6.6. *If Q is given by a strong pair (p, b), then* $\min(cx: x \in Q)$ *is attained by a vector z where $z_t = p(v_1, \cdots, v_t) - p(v_1, \cdots, v_{t-1})$ $(t = 1, \cdots, n)$.*

This proposition has a useful corollary, which shows that the greedy algorithm may be applied even if Q is not given by its strong pair.

COROLLARY 6.7. *Let Q be a g-polymatroid bounded from below. Define iteratively the components of a vector z as follows. Suppose $z_1, z_2, \cdots, z_{t-1}$ have already been defined. Let z_t be the smallest number for which $(z_1, \cdots, z_t, x_{t+1}, \cdots, x_n)$ belongs to Q for some appropriate $x_{t+1}, \cdots, x_n$. Then z is an integer-valued solution to* $\min(cx: x \in Q)$.

Note that the procedure in the corollary has nothing to do with the form in which Q is given. Therefore, it becomes a usable algorithm only if there is a way to compute the current z_t.

Actually, we will use the properties listed above mainly for the special case of contrapolymatroids. Let p be a supermodular, monotone-increasing (integer-valued) set-function with $p(\emptyset) = 0$. A polyhedron $C(p) := \{x \in R : x(A) \geqq p(A)$ for every $A \subseteq V\}$ is called a *contrapolymatroid*. A contrapolymatroid is a g-polymatroid, since $C(p) = Q(p, b)$ where $b :\equiv \infty$ (except $b(\emptyset) = 0$), and this (p, b) is a strong pair.

In applications we will encounter contrapolymatroids that are not given by their unique monotone supermodular function. Let $q: 2^V \to Z_+$ be a nonnegative integer-valued function. Suppose that $Q := \{x \in R^V: x(A) \geqq q(A)$ for every $A \subseteq V\}$ is a contrapolymatroid. Let $k := \max(q(X): X \subseteq V)$. Then, obviously, $(k, k, \cdots, k) \in Q$. For Q the greedy algorithm can be formulated as follows.

COROLLARY 6.7′. *Define iteratively the components of a vector y as follows. Suppose $y_1, y_2, \cdots, y_{t-1}$ have already been defined. Let y_t be the smallest number for which $(y_1, \cdots, y_t, k, \cdots, k)$ belongs to Q. Then y is an integer-valued solution to* $\min(cx: x \in Q)$.

Proof. We show that $z = y$ where z is the vector constructed in Corollary 6.7. If this were not true, then there is a smallest subscript t for which $z_t \neq y_t$. Obviously, $z_t < y_t$. Let $V_t := \{v_1, \cdots, v_t\}$. By the definition of y_t, there is a set A that prevented y_t from being smaller. This means that $v_t \in A$ and $q(A) = y(V_t \cap A) + k|A - V_t|$. Now if $A \subseteq V_t$, then $q(A) = y(V_t \cap A) > z(V_t \cap A) = z(A)$, contradicting that $z \in Q$. If $A \not\subseteq V_t$, then $q(A) \leqq y_t + k \leqq q(A)$ (by the definition of k). Hence $0 \leqq z_t < y_t = 0$, a contradiction. □

Let Q be the same as before. We will have to be able to solve the following optimization problem:

$$(*) \qquad \min\,(cx : x \in Q, x(V) \text{ is even}).$$

By applying the greedy algorithm, compute an integer vector $z' \in Q$ that minimizes cx over Q. If $z'(V)$ is even, let $z := z'$. If $z'(V)$ is odd, revise z' by increasing $z'(v_n)$ by 1. Let z denote the resulting vector.

PROPOSITION 6.8. *Vector z is an optimal solution to* $(*)$.

Proof. Suppose that $Q = C(p)$, where p is the unique supermodular function defining Q. If $p(V)$ is odd, modify p by increasing $p(V)$ by 1. The resulting p_o is fully supermodular and monotone increasing. The proposition follows by observing that the greedy algorithm described in Corollary 6.7′, when applied to $C(p_o)$, outputs vector z constructed above. □

Let $Q := C(p)$ be a contrapolymatroid, let $f: V \to Z$ and $g : V \to Z \cup \{\infty\}$ be two functions with $f \leqq g$, and let $0 \leqq \alpha \leqq \beta \leqq \infty$ be two integers. Let $Q_1 := \{x \in Q: f \leqq x \leqq g, \alpha \leqq x(V) \leqq \beta\}$.

PROPOSITION 6.9. *Q_1 is a g-polymatroid. Q_1 is nonempty if and only if*

$$(6.5) \qquad \alpha \leqq g(V) \text{ and}$$

$$(6.6) \qquad p(X) + f(V - X) \leqq \min\,(\beta, g(V)),$$

$$(6.7) \qquad p(X) \leqq g(X)$$

hold for every subset $X \subseteq V$.

Proof. By Propositions 6.3 and 6.4, it follows that Q_1 is a g-polymatroid.

Let $b(X) := \infty$ if $X \neq \emptyset$ and $b(\emptyset) = 0$. By applying Proposition 6.3 to this p and b, we obtain that $Q_o := \{x \in Q: \alpha \leqq x(V) \leqq \beta\}$ is a g-polymatroid with strong pair (p_o, b_o), where $p_o(X) := p(X)$ if $X \subset V$, $p_o(V) := \max\,(p(V), \alpha)$, and $b_o(X) := \beta - p(V - X)$ if $X \neq \emptyset$.

Define p' and b' as follows. $p'(X) := \max\,(p_1(X), f(X))$, $b'(X) := \min\,(b_1(X), g(X))$. By Proposition 6.4 (p', b') is a weak pair, and $Q_1 = Q(p', b')$. Therefore, Proposition 6.2 applies, and (6.2) in this case is equivalent to (6.5)–(6.7). □

Let (p_1, b_1) denote the strong pair defining Q_1. From (6.3) and (6.4) we can read off that

$$(6.8\text{a}) \qquad p_1(V) = \max\,[\alpha, \max\,(p(Y) + f(V - Y): Y \subseteq V)] \quad \text{and}$$

$$(6.8\text{b}) \qquad b_1(V) = \min\,(\beta, g(V)).$$

PROPOSITION 6.10. *Suppose that Q_1 is nonempty. Q_1 contains no integer vector y with $y(V)$ even if and only if $p_1(V) = b_1(V)$ is odd.*

Proof. Obviously, if $m := p_1(V) = b_1(V)$, then $x(V) = m$ for every $x \in Q_1$; therefore, if m is odd, $x(V)$ is odd as well. Conversely, if $p_1 \neq b_1$, then, by Proposition 6.1, there is an integer vector $x \in Q_1$ with $x(V) = p_1(V)$, and there is an integer vector $z \in Q_1$ with $z(V) = b_1(V)$. By applying Proposition 6.5, we obtain that there is an integer vector $y \in Q$ with $y(V)$ even. If $p_1(V) = b_1(V)$ is even, then any integer vector of Q_1 will do. □

Finally, we mention one more result from [13].

PROPOSITION 6.11. *The intersection Q of two g-polymatroids $Q(p_1, b_1)$ and $Q(p_2, b_2)$ defined by strong pairs is nonempty if and only if $p_1 \leqq b_2$ and $p_2 \leqq b_1$. Moreover, Q is spanned by its integer points.*

7. Undirected augmentations and G-polymatroids. In this section we reveal a relationship between augmentations and g-polymatroids. First, let us consider the augmentation problem analysed in § 5 and recall the definition of a good augmentation. In Lemma 5.6 we showed how to extend G by a new node s and some new edges incident to s so as to satisfy (5.4). Given such an extension, let $z(v)$ denote the number of parallel edges between $v \in V$ and s. (5.4) is clearly equivalent to

$$z(A) \geqq q(A) \text{ for every } A \subseteq V. \tag{7.1}$$

Note that $q(A) := R(A) - d(A)$ denotes the deficiency of $A \subseteq V$. Also note that q is not intersecting supermodular, in general. Still, the following theorem asserts that q defines a contrapolymatroid.

THEOREM 7.1. *$Q := \{z \in R^V: z \geqq 0$ and z satisfies (7.1)$\}$ is a contrapolymatroid $C(p)$, where the unique supermodular function defining Q is*

$$p(A) := \max\,(\Sigma q(A_i): \{A_1, A_2, \cdots, A_t\} \textit{ a subpartition of } A, A_i \neq \emptyset). \tag{7.2}$$

Proof. First, we show that $C(p) = Q$. Indeed, $C(p) \subseteq Q$ since $\{A\}$ is a subpartition of A. On the other hand, let $z \in Q$ and assume that $p(A) = \Sigma_i q(A_i)$ for some subpartition $\{A_1, A_2, \cdots, A_t\}$ of A $(A_i \neq \emptyset)$. Since z satisfies (7.1) and is nonnegative, we have $z(A) = \Sigma_i z(A_i) + z(A - \cup A_i) \geqq \Sigma_i q(A_i) = p(A)$, and therefore $z \in C(p)$.

As p is clearly monotone increasing, all we have to show is that p is supermodular. Let A and B be two arbitrary subsets of V. Assume that $p(A) = \Sigma q(A_i)$ for some subpartition $\{A_1, A_2, \cdots, A_k\}$ of A, and let $p(B) = \Sigma q(B_i)$ for some subpartition $\{B_1, \cdots, B_h\}$ of B.

Let $\mathcal{F} := \{A_1, \cdots, A_k, B_1, \cdots, B_h\}$. Then $\mathcal{F}$ satisfies the following:

(7.3) every $v \in A \cap B$ is covered at most twice,
every $v \in (A - B) \cup (B - A)$ is covered at most once by $\mathcal{F}$.

By Propositions 2.2 and 5.4, q satisfies at least one of the following inequalities for every two subsets X, Y of V:

$$q(X) + q(Y) \leqq q(X \cap Y) + q(X \cup Y), \tag{7.4a}$$

$$q(X) + q(Y) \leqq q(X - Y) + q(Y - X). \tag{7.4b}$$

Denote $q(\mathcal{F}) := \Sigma(q(X): X \in \mathcal{F})$. Assume that there are two intersecting sets A_i and B_j in $\mathcal{F}$. If $X := A_i$ and $Y := B_j$ satisfy (7.4a) (respectively, (7.4b)), revise $\mathcal{F}$ by replacing A_i and B_j by $X \cap Y$ and $X \cup Y$ (respectively, $X - Y$ and $Y - X$). Then the new family $\mathcal{F}_1$ satisfies (7.3), and by (7.4), $q(\mathcal{F}_1) \geqq q(\mathcal{F})$.

Apply this "uncrossing" operation as long as there are intersecting sets. Since in every step $\Sigma(|X|^2 - 2|V||X| : X \in \mathcal{F})$ strictly increases, after a finite number of steps we obtain an $\mathcal{F}_o$ satisfying (7.3), for which $q(\mathcal{F}_o) \geqq q(\mathcal{F})$ and $\mathcal{F}_o$ is laminar. Let $\mathcal{P}_1$ consist of the minimal members of $\mathcal{F}_o$ that are subsets of $A \cap B$, and $\mathcal{P}_2 := \mathcal{F}_o - \mathcal{P}_1$. Then $\mathcal{P}_1$ is a subpartition of $A \cap B$, and $\mathcal{P}_2$ is a subpartition of $A \cup B$. By definition, $p(A \cap B) \geqq q(\mathcal{P}_1)$ and $p(A \cup B) \geqq q(\mathcal{P}_2)$ so we have $p(A) + p(B) = q(\mathcal{F}) \leqq q(\mathcal{F}_o) = q(\mathcal{P}_1) + q(\mathcal{P}_2) \leqq p(A \cap B) + p(A \cup B)$, as required. □

Theorem 7.1, with the help of Theorem 5.2, provides the following relation between good augmentations of G and integer vectors in $C(p)$.

COROLLARY 7.2. *Let F be a set of new edges and define a vector $z_F \in Z^V$ by $z_F(v) := d_F(v)$ (for $v \in V$). If $(V, E \cup F)$ is a good augmentation, then $z_F \in C(p)$ and $z(V)$ is even. If $z \in C(p)$ is an integer-valued vector with $z(V)$ even and*

$$(7.5) \qquad z(C) \neq 1 \text{ for every component } C \text{ of } G,$$

then there is a set F of new edges for which $z(v) = d_F(v)$ (for $v \in V$) and for which $(V, E \cup F)$ is a good augmentation.

Proof. The first part is clear from the definitions. To see the second part, let $z \in C(p)$ be a vector having the required properties. Extend G by a new node s and by $z(sv)$ new parallel edges between s and v ($v \in V$). By the hypotheses, the extended graph G' satisfies the hypotheses of Corollary 5.2,′ and therefore the required augmentation exists. □

Corollary 7.2 ensures that the results of § 6 concerning g-polymatroids can be utilised for augmentations. Let us first prove Theorems 5.8 and 5.9.

Proof of Theorem 5.8. The necessity of the conditions is clear, and we concentrate only on their sufficiency. Let p be the set-function defined in (7.2), and let $\alpha := \beta := 2\gamma$. We claim that the hypotheses of Theorem 5.8 imply (6.5)–(6.7). Indeed, $2\gamma \leqq g(V)$ implies (6.5). Since g is modular, (5.6) and (7.2) imply (6.7). Similarly, (5.7) and (7.2) imply $p(X) + f(V - X) \leqq \beta$, and by $f \leqq g$ we also have $p(X) + f(V - X) \leqq g(V)$; that is, (6.6) follows.

By Propositions 6.1 and 6.9, Q_1 contains an integer point z, the hypotheses of Corollary 7.2 hold, and therefore the required augmentation exists. □

Proof of Theorem 5.9. Again we are concerned only with the sufficiency. Let p be the set-function defined in (7.2) and let $\alpha := 0$ and $\beta := \infty$. Now (6.5) holds. (5.6) and (7.2) imply (6.7) and (6.6). Therefore Proposition 6.9 applies, and Q_1 is non-empty. We claim that Q_1 contains an integer vector z with $z(V)$ even. If this were not the case, then by Proposition 6.10, $b_1(V) = g(V)$ is odd, and $b_1(V) = p_1(V)$. That is, by (7.2) and (6.8a) we would have $g(V) = \Sigma q(X_i) + f(V - \cup X_i)$ for some subpartition $\{X_1, X_2, \cdots, X_t\}$ of V, contradicting (∗) in Theorem 5.9.

We finish by applying Corollary 7.2 to this vector z. Note that (7.5) is satisfied because $f(C) \geqq 2$ for every marginal component C of G. □

Using the same technique, a good characterization can be derived from Propositions 6.9 and 6.10 for the existence of a set F of new edges for which $(V, E \cup F)$ is a good augmentation, $f(v) \leqq d_F(v) \leqq g(v)$ for every $v \in V$ and $\varphi \leqq |F| \leqq \gamma$.

Proof of Theorem 5.10. The theorem follows if we put together Corollaries 7.2 and 6.7′ and Proposition 6.8. □

8. Directed augmentations and G-polymatroids. Let $G = (V, E)$ be a digraph. Suppose that G can be extended by γ new edges to a k-edge-connected digraph; that is, (3.1) and (3.2) hold.

THEOREM 8.1. *$Q_{in} := \{z \in R^V: z \geqq 0, z(V) \geqq \gamma, z(X) \geqq k - \rho(X)$ for every $\emptyset \subset X \subset V\}$ is a contrapolymatroid $C(p_{in})$, where*

$$(8.1) \qquad p_{in}(A) = \max\,(\Sigma(k - \rho(A_i)) : \{A_1, \cdots, A_t\} \text{ a subpartition of } A)$$

if $A \subset V$ and $p_{in}(V) := \gamma$.

Proof. In the proof we will abbreviate p_{in} by p, and Q_{in} by Q. First, we show that $C(p) = Q$. Indeed, $C(p) \subseteq Q$ since $\{A\}$ is a subpartition of A. On the other hand, let $z \in Q$ and assume that $p(A) = \Sigma_i q(A_i)$ for some subpartition $\{A_1, A_2, \cdots, A_t\}$ of

A ($A_i \neq \emptyset$). Since $z(X) \geqq q(X)$ for $X \subseteq V$ and z is nonnegative, we have $z(A) = \Sigma_i z(A_i) + z(A - \cup A_i) \geqq \Sigma_i q(A_i) = p(A)$, and therefore $z \in C(p)$.

By the definition of γ we have $\gamma \geqq \max(\Sigma(k - \rho(X_i))\colon \{X_1, \cdots, X_t\}$ a subpartition of V), and hence p is monotone increasing. We are going to show that p is fully supermodular. Let $q(X) := k - \rho(X)$ if $\emptyset \subset X \subset V$, and $q(\emptyset) := q(V) := 0$. Then q is supermodular on crossing sets.

Let A and B be two arbitrary subsets of V. Suppose that $p(A) = \Sigma q(A_i)$ for some subpartition $\{A_1, A_2, \cdots, A_k\}$ of A, and let $p(B) = \Sigma q(B_i)$ for some subpartition $\{B_1, \cdots, B_h\}$ of B.

Let $\mathscr{F} := \{A_1, \cdots, A_k, B_1, \cdots, B_h\}$. Then $\mathscr{F}$ satisfies the following:

(8.2) every $v \in A \cap B$ is covered at most twice,
every $v \in (A - B) \cup (B - A)$ is covered at most once by $\mathscr{F}$.

Denote $q(\mathscr{F}) := \Sigma(q(X)\colon X \in \mathscr{F})$. If there are two crossing sets A_i and B_j in $\mathscr{F}$, revise $\mathscr{F}$ by replacing A_i and B_j by $A_i \cap B_j$ and $A_i \cup B_j$. The new family $\mathscr{F}_1$ satisfies (8.2) and, since q is supermodular on crossing sets, $q(\mathscr{F}_1) \geqq q(\mathscr{F})$.

Apply this "uncrossing" operation as long as there are crossing sets. Since in every step $\Sigma(|X|^2\colon X \in \mathscr{F})$ strictly increases, after a finite number of steps the procedure stops with an $\mathscr{F}_o$ satisfying (8.2) for which $q(\mathscr{F}_o) \geqq q(\mathscr{F})$.

Assume first that $\mathscr{F}_o$ includes no intersecting sets; that is, $\mathscr{F}_o$ is laminar. Let $\mathscr{P}_1$ consist of the minimal members of $\mathscr{F}_o$ that are subsets of $A \cap B$, and $\mathscr{P}_2 := \mathscr{F}_o - \mathscr{P}_1$. Then $\mathscr{P}_1$ is a subpartition of $A \cap B$, and $\mathscr{P}_2$ is a subpartition of $A \cup B$. By definition, $p(A \cap B) \geqq q(\mathscr{P}_1)$ and $p(A \cup B) \geqq q(\mathscr{P}_2)$, so we have $p(A) + p(B) = q(\mathscr{F}) \leqq q(\mathscr{F}_o) = q(\mathscr{P}_1) + q(\mathscr{P}_2) \leqq p(A \cap B) + p(A \cup B)$, as required.

Second, assume that $\mathscr{F}_o$ includes two intersecting sets X and Y. They are not crossing, therefore $X \cup Y = V$. By (8.2), the other members of $\mathscr{F}_o$ are pairwise disjoint subsets of $A \cap B$. Therefore $p(A \cap B) \geqq q(\mathscr{F}_o) - q(X) - q(Y)$.

By the assumption, (3.2) holds. Hence $k - \delta(V - X) + k - \delta(V - Y) \leqq \gamma$; that is, $q(X) + q(Y) = k - \rho(X) + k - \rho(Y) \leqq \gamma$. We have $p(A) + p(B) = q(\mathscr{F}) \leqq q(\mathscr{F}_o) \leqq p(A \cap B) + q(X) + q(Y) \leqq p(A \cap B) + \gamma = p(A \cap B) + p(A \cup B)$, as required. □

By interchanging δ and ρ in Theorem 8.1 we obtain the following result.

THEOREM 8.1′. *$Q_{\text{out}} := \{z \in R^V\colon z \geqq 0,\ z(V) \geqq \gamma,\ z(X) \geqq k - \delta(X)$ for every $\emptyset \subset X \subset V\}$ is a contrapolymatroid $C(p_{\text{out}})$, where*

(8.1′) $p_{\text{out}}(A) = \max(\Sigma(k - \delta(A_i))\colon \{A_1, \cdots, A_t\}$ *a subpartition of* A)

if $A \subset V$ and $p_{\text{out}}(V) := \gamma$.

COROLLARY 8.2. *If F is a set of γ new edges and $(V, E \cup F)$ is k-edge-connected, then $z \in C(p_{\text{in}})$ ($z \in C(p_{\text{out}})$) where $z \in Z$ is defined by $z(v) := \rho_F(v)$ ($z(v) := \delta_F(v)$) for every $v \in V$. Conversely, if $z_{\text{in}} \in C(p_{\text{in}})$ and $z_{\text{out}} \in C(p_{\text{out}})$ are integer-valued vectors with $\gamma := z_{\text{in}}(V) = z_{\text{out}}(V)$, then there is a set F of γ new edges for which $(V, E \cup F)$ is a k-edge-connected and $z_{\text{in}}(v) = \rho_F(v)$ and $z_{\text{out}}(v) = \delta_F(v)$ hold for every $v \in V$.*

Proof. The first part is clear from the definitions. To see the second part, let z_{in} and z_{out} be two vectors having the required properties. Extend G by a new node s, by $z_{\text{in}}(v)$ new parallel edges from s to v, and by $z_{\text{out}}(v)$ new parallel edges from v to s ($v \in V$). By the hypotheses the extended graph G' satisfies the hypotheses of Theorem 3.4, and therefore the required augmentation exists. □

By now we are in a position to prove Theorems 3.7 and 3.9.

Proof of Theorem 3.7. By Corollary 8.2 and Theorem 3.4 all we have to prove is that there is an integer vector z_{in} in $C(p_{\text{in}})$ for which $f_{\text{in}} \leqq z_{\text{in}} \leqq g_{\text{in}}$, $z_{\text{in}}(V) = \gamma$, and that there is an integer vector z_{out} in $C(p_{\text{out}})$ for which $f_{\text{out}} \leqq z_{\text{out}} \leqq g_{\text{out}}$, $z_{\text{out}}(V) = \gamma$. Apply

Proposition 6.9 to $C(p_{in})$ and to $C(p_{out})$ (separately) with the choice $\alpha := \beta := \gamma$. The assumption $\gamma \leqq \min(g_{in}(V), g_{out}(V))$ implies (6.5). Equation (6.6) follows from (3.6) and (8.1). Equation (6.7) follows from (3.5). □

Proof of Theorem 3.9. The proof is immediate if we observe that the algorithm in question is nothing but two (separate) applications of the greedy algorithm described in Corollary 6.7′ to Q_{in} and Q_{out}. □

9. Max-flow version and algorithmic aspects. This section is offered to make some comments on the complexity of algorithms implied by the proofs. It is certainly not our purpose to describe a detailed algorithm with data structure and precise time-bound. Instead, we briefly indicate the idea of a strongly polynomial algorithm.

Basically, we encountered two types of problems. Problem A consists of finding an appropriate enlargement of a starting graph or digraph using a new node s. Problem B consists of performing algorithmically the splitting operation.

We will consider these algorithms concerning the max-flow version. Let $G = (V, E)$ be a graph or a digraph, and $g : E \to Z_+$ an integer-valued capacity function. Let $r(u, v)$ be an integer-valued demand function so that there is no marginal component in the undirected case and $r \equiv k$ in the directed one. Recall that the max-flow version of the augmentation problem is as follows. Extend G by adding new edges with suitable capacities so that in the enlarged digraph the maximum flow value from every node u to any other node v is at least $r(u, v)$, and so that the sum of capacities of the newly added edges is minimum. (The algorithms below work with the same time complexity if g and r are not necessarily integer-valued.)

By replacing every edge e by $g(e)$ parallel edges, we can see that this max-flow version is theoretically equivalent to its noncapacitated case analyzed in §§ 3 and 5. We do not formulate the corresponding theorems but only mention a corollary of Theorems 5.5 and 3.1.

COROLLARY 9.1. (a) *Let $G = (V, E)$ be an undirected graph, $r(u, v)$ an integer-valued demand-function such that G has no marginal components, and g an integer-valued capacity function on E. There is an optimal solution to the undirected max-flow augmentation problem that is half integral. Furthermore, an optimal integer-valued solution is either optimal among the real-valued augmentations or its total increment is one half bigger than that of a (real-valued) optimal solution.*

(b) *If G is a directed graph and $r(u, v) \equiv k$, then there is an optimal solution to the directed max-flow augmentation problem that is integer-valued.*

Proof. Let γ and γ^* denote the minimum total increment of an integer-valued and a real-valued augmentation, respectively. By Theorem 5.5, $\gamma = \lceil \frac{1}{2} \max \Sigma q(X_i) \rceil$. Let us consider the augmentation problem concerning capacity function $g' := 2g$ and demand function $r' := 2r$. Let q' $(=2q)$ denote the deficiency function and γ' the minimum total increment of an integral augmentation x'. Clearly, $x'/2$ is a fractional solution to the original augmentation problem. Therefore $\gamma^* \leqq \gamma'/2$. On the other hand, by Theorem 5.5 again, we have $\gamma' = \lceil \frac{1}{2} \max \Sigma q'(X_i) \rceil = \frac{1}{2} \max \Sigma q'(X_i) = \max \Sigma q(X_i) \leqq 2\gamma^* \leqq \gamma'$.

Hence $x'/2$ is an optimal solution to the original augmentation problem and $\gamma = \lceil \gamma^* \rceil$ from which part (a) follows.

Part (b) follows directly from Theorem 3.1. □

Gomory and Hu [19] described a very simple solution method to the undirected max-flow augmentation problem when the starting graph G is the empty graph. Their algorithm provides only a half-integer solution. For the same problem, Sridhar and Chandrasekaran [30] described a polynomial time algorithm that finds an integer-valued optimal augmentation. Bland, Goldfarb, and Todd [1] showed how to apply the ellipsoid