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k-wise Set-Intersections and k-wise Hamming-Distances

by

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ABSTRACT

We prove a version of the Ray-Chaudhuri–Wilson and Frankl-Wilson theorems for k-wise intersections and also generalize a classical code-theoretic result of Delsarte for k-wise Hamming distances. A set of code-words $a^1, a^2, \ldots, a^k$ of length n have k-wise Hamming-distance ℓ, if there are exactly ℓ such coordinates, where not all of their coordinates coincide (alternatively, exactly $n - \ell$ of their coordinates are the same). We show a Delsarte-like upper bound: codes with few k-wise Hamming-distances must contain few code-words.

1 Introduction

In this paper we give bounds on the size of set-systems and codes, satisfying some k-wise intersection-size or Hamming-distance properties. For $k = 2$, these theorems were proven by Ray-Chaudhuri and Wilson [12], Frankl and Wilson [9], and Delsarte [6], [5]. The $k > 2$ case was asked (partially) by T. Sós [13], and Füredi [10] proved, that for uniform set-systems with small sets, the order of magnitude of the largest set-system satisfying k-wise or just pair-wise intersection constraints are the same (his constant was huge). Grolmusz [11] proved a k-wise intersection analog of the Deza-Frankl-Singhi theorem [7], and gave direct applications for explicit coloring of k-uniform hypergraphs without large monochromatic sets.

Here we first strengthen the result of [11], giving at the same time a much shorter proof, and then prove a k-wise version of the Delsarte-bounds [6], [5] for codes. In the last section we present a construction which shows that some of our bounds are asymptotically tight.

2 Set systems

In this section we present results on set-systems with restricted k-wise intersections. We begin with the following extension of results from [12].

Theorem 1 *Let L be a subset of non-negative integers of size s. Let $k \geq 2$ be an integer and let $\mathcal{H}$ be a family of subset of n-element set such that $|H_1 \cap \ldots \cap H_k| \in L$ for any collection of k distinct sets from $\mathcal{H}$. Then*

$$|\mathcal{H}| \leq (k-1) \sum_{i=0}^{s} \binom{n}{i}.$$

If in addition the size of every member of $\mathcal{H}$ belongs to the set $\{k_1, \ldots, k_t\}$ and $k_i > s-t$ for every i, then

$$|\mathcal{H}| \leq (k-1) \sum_{i=s-t+1}^{s} \binom{n}{i}.$$

This theorem has the following modular version, which generalize the theorem of Frankl and Wilson [9] and strengthen the result from [11].

Theorem 2 *Let p be a prime and L be a subset of $\{0, 1, \ldots, p-1\}$ of size s. Let $k \geq 2$ be an integer and let $\mathcal{H}$ be a family of subsets of n-element set such that $|H| \pmod p \notin L$ for every $H \in \mathcal{H}$ but $|H_1 \cap \ldots \cap H_k| \pmod p \in L$ for any collection of k distinct sets from $\mathcal{H}$. Then*

$$|\mathcal{H}| \leq (k-1) \sum_{i=0}^{s} \binom{n}{i}.$$

If in addition there exist $t \leq s$ integers $k_1, \ldots, k_t \in \{0, 1, \ldots, p-1\}$ so that $k_i > s-t$ for each i and $|H| \ (mod\ p) \in \{k_1, \ldots, k_t\}$ for every $H \in \mathcal{H}$, then

$$|\mathcal{H}| \leq (k-1) \sum_{i=s-t+1}^{s} \binom{n}{i}.$$

We start with the proof of Theorem 2 and then we show how to modify it to get Theorem 1. Our proof combines an approach introduced in [1] with some additional ideas.

Proof: Let $L = \{l_1, \ldots, l_s\}$ and let $\mathcal{H}$ be a set system satisfying assertion of the theorem. We repeat the following procedure until $\mathcal{H}$ is empty. At round i if $\mathcal{H} \neq \emptyset$ we choose a maximal collection $H_1, \ldots, H_d$ from $\mathcal{H}$ such that $|\cap_{j=1}^{d} H_j| \ (mod\ p) \notin L$ but for any additional set $H' \in \mathcal{H}$ we have that $|\cap_{j=1}^{d} H_j \cap H'| \ (mod\ p) \in L$. Clearly by definition such family always exists and $1 \leq d \leq k-1$. Denote $A_i = H_1$, $B_i = \cap_{j=1}^{d} H_j$ and remove all sets $H_1, \ldots, H_d$ from $\mathcal{H}$. Note that as the result of this process we obtain at least $m \geq |\mathcal{H}|/(k-1)$ pairs of sets A_i, B_i. By definition, $|A_i \cap B_i| = |B_i| \ (mod\ p) \notin L$ but $|A_r \cap B_i| \ (mod\ p) \in L$ for any $r > i$. With each of the sets A_i, B_i we associate its characteristic vector which we denote a_i, b_i respectively.

Let $\mathbf{Q}$ denote the set of rational numbers. For $x, y \in \mathbf{Q}^n$, let $x \cdot y$ denote their standard scalar product. Clearly $a_r \cdot b_i = |A_r \cap B_i|$. For $i = 1, \ldots, m$ let us define the multilinear polynomial f_i in n variables as

$$f_i(x) = \prod_{j=1}^{s} (x \cdot b_i - l_j),$$

where for each monomial, we reduce the exponent of each occurring variable to 1. Clearly

$$f_i(a_i) = \prod_{j=1}^{s} \Big(|A_i \cap B_i| - l_j\Big) = \prod_{j=1}^{s} \Big(|B_i| - l_j\Big) \neq 0 \ (mod\ p) \text{ for all } 1 \leq i \leq m,$$

but

$$f_i(a_r) = \prod_{j=1}^{s} \Big(|A_r \cap B_i| - l_j\Big) = 0 \ (mod\ p) \text{ for } 1 \leq i < r \leq m.$$

We claim that the polynomials $f_1, \ldots, f_m$ are linearly independent as a functions over $\mathbf{F}_p$, the finite field of order p . Indeed, assume that $\sum \alpha_i f_i(x) = 0$ is a nontrivial linear relation, where $\alpha_i \in \mathbf{F}_p$. Let i_0 be the largest index such that $\alpha_{i_0} \neq 0$. Substitute a_{i_0} for x in this relation. Clearly all terms but the one with index i_0 vanish, with the consequence $\alpha_{i_0} = 0$, contradiction. On the other hand, each f_i belongs to the space of multilinear polynomials of degree at most s. The dimension of this space is $\sum_{j=1}^{s} \binom{n}{i}$, implying the desired bound on m and thus on $|\mathcal{H}|$.

We now extend the idea above to prove the second part of the theorem. This extension uses a technique employed by Blokhuis [4] (see also [1]). For a subset $I \subseteq$

$\{1,\ldots,n\}=[n]$ denote by v_I its characteristic vector and by $x_I=\prod_{i\in I}x_i$. In particular $x_\emptyset=1$ and it is easy to see that for any $J\subseteq[n]$, $x_I(v_J)=1$ if and only if $I\subseteq J$ and zero otherwise. In what follows we use the notation introduced in the first part of the proof.

In addition to polynomials f_i we define a new set of multilinear polynomials

$$g_I(x)=x_I\cdot\prod_{j=1}^{t}\Big(\sum_{i=1}^{n}x_i-k_j\Big)\quad\text{for } I\subseteq[n].$$

Here again we reduce the exponent of each occurring variable to 1 to make g_I multilinear. We claim that the functions g_I are linearly independent over $\mathbf{F}_p$ for $|I|\le s-t$. Denote by $h(x)=\prod_{j=1}^{t}(\sum_{i=1}^{n}x_i-k_j)$. Since $k_i>s-t$ for all i, note that $h(v_I)\neq 0$ for all $|I|\le s-t$. Let us arrange all the subsets of $\{1,2,\ldots,n\}$ in a linear order, denoted by $\prec$, such that $J\prec I$ implies that $|J|\le|I|$. Clearly if $|I|,|J|\le s-t$ by definition, $g_I(v_J)=x_I(v_J)h(v_J)$ is equal to $h(v_J)\neq 0$ if $I=J$ and zero if $J\prec I$. Now the linear independence of $g_I(x)$ follows easily. Indeed, if $\sum_{|I|\le s-t}\beta_I g_I(x)=0$ is a nontrivial relation, let I_0 to be a minimal index (with respect to $\prec$), such that $\beta_{I_0}\neq 0$. By substituting $x=v_{I_0}$ we immediately obtain a contradiction.

To complete the argument we show that the functions f_i remain linear independent even together with all the functions g_I for $|I|\le s-t$. For a proof of this claim assume that

$$\sum_i \alpha_i f_i(x)\quad+\sum_{|I|\le s-t}\beta_I g_I(x)=0,$$

for some $\alpha_i,\beta_I\in\mathbf{F}_p$. Substitute $x=a_i$. All terms in the second sum vanish since $|A_i|\ (mod\ p)\in\{k_1,\ldots,k_t\}$ and hence $h(a_i)=0$. In this case we can deduce that all $\alpha_i=0$ as previously. But then we get a relation only among the polynomials g_I and it was already proved that such relation should be trivial.

Therefore we found $m+\sum_{i=0}^{s-t}\binom{n}{i}$ linearly independent functions, all of which belong to space of multilinear polynomials of degree at most s. As we already mentioned, the dimension of this space is $\sum_{j=1}^{s}\binom{n}{i}$. This implies the desired bound on m and thus on $|\mathcal{H}|$. $\square$

An easy modification of above proof establishes Theorem 1.

Sketch of proof of Theorem 1. We repeat the following procedure. At step i, if $|H\cap H'|\in L$ for any two distinct sets in $\mathcal{H}$, then let H_1 be the largest set remaining in $\mathcal{H}$. Denote $A_i=B_i=H_1$ and remove H_1 from $\mathcal{H}$. Otherwise there exist a collection $H_1,\ldots,H_d$ from $\mathcal{H}$ such that $|\cap_{j=1}^{d}H_j|\notin L$ but for any additional set $H'\in\mathcal{H}$ we have that $|\cap_{j=1}^{d}H_j\cap H'|\in L$ and $2\le d\le k-1$. Denote $A_i=H_1$, $B_i=\cap_{j=1}^{d}H_j$ and remove all sets $H_1,\ldots,H_d$ from $\mathcal{H}$. By definition, $|A_i\cap B_i|=|B_i|$ but $|A_r\cap B_i|\in L$ and has size strictly smaller than $|B_i|$ for all $r>i$. With each of the sets A_i,B_i we associate its characteristic vector which we denote a_i,b_i respectively.

We will also need a slightly different definition of polynomials f_i. For $i = 1, \ldots, m$ let us define the multilinear polynomial f_i in n variables as

$$f_i(x) = \prod_{l_j < |B_i|} (x \cdot b_i - l_j).$$

By our construction $f_i(a_i) \neq 0$ but $f_i(a_r) = 0$ for all $r > i$. Now the rest of the proof is identical with that of Theorem 2 and we omit it here. □

3 Codes

Let $A = \{0, 1, 2, \ldots, q-1\}$. The Hamming-distance of two elements of A^n is the number of coordinates in which they differ. A q-ary code of length n is simply a $C \subset A^n$. The following result is a classical inequality of Delsarte [6], [5]:

Theorem 3 (Delsarte) *Let C be a q-ary code of length n. If the set of Hamming distances which occur between distinct codewords of C has cardinality s, then*

$$|C| \leq \sum_{i=0}^{s} (q-1)^i \binom{n}{i}.$$

Frankl [8] proved the modular generalization of this result, and it was further strengthened by Babai, Snevily and Wilson [3].

Our goal here is to give generalizations of this theorem for k-wise Hamming distances.

Definition 4 *Let $a^i \in A^n$, for $i = 1, 2, \ldots, k$. Their k-wise Hamming distance,*

$$d_k\left(a^1, a^2, \ldots, a^k\right)$$

is ℓ, if there exist exactly ℓ coordinates, in which they are not all equal. (Equivalently, their coordinates are all equal on $n - \ell$ positions).

We prove the following theorems. The first one generalizes Delsarte's original bound [6], [5] to k-wise Hamming distance:

Theorem 5 *Let C be a q-ary code of length n. If the set of k-wise Hamming distances which occur between k distinct codewords of C has cardinality s, then*

$$|C| \leq (k-1) \sum_{i=0}^{s} (q-1)^i \binom{n}{i}. \tag{1}$$

The second result is the modular version of Theorem 5, it is a k-wise generalization of the modular upper bound of Frankl [8] and also a result of Babai, Snevily and Wilson [3]:

Theorem 6 *Let C be a q-ary code of length n, p be a prime and let L be a subset of $\{1,\ldots,p-1\}$ of size s. If the set of k-wise Hamming distances which occur between k distinct codewords of C lie in L modulo p, then*

$$|C| \leq (k-1)\sum_{i=0}^{s}(q-1)^i\binom{n}{i}.$$

If in addition, there exist $t \leq s$ integers $w_1,\ldots,w_t \in \{0,1,\ldots,p-1\}$, so that $w_i > s-t$ for each i and the weight of any member of C is congruent to some element of $\{w_1,\ldots,w_t\}$ modulo p, then

$$|C| \leq (k-1)\sum_{i=s-t+1}^{s}(q-1)^i\binom{n}{i}.$$

Two definitions are needed for the proof.

Definition 7 *Let a and b be two codewords of length n. Then let $a \sqcap b$ denote a codeword which contains only those coordinates of a and b which are equal. Let $|a \sqcap b|$ denote the length of word $a \sqcap b$.*

For example, if $a = 01134230$, $b = 12134111$, then $a \sqcap b = 134$, and $|a \sqcap b| = 3$.

Definition 8 ([3]) *For a fixed integer $a \in A$, let $\varepsilon(a,x)$ be the polynomial in one variable with rational coefficients such that for every $b \in A$*

$$\varepsilon(a,b) = \begin{cases} 1, & \text{if } b = a, \\ 0, & \text{if } b \neq a. \end{cases}$$

Since k-wise Hamming distances which occur between k distinct codewords are always nonzero, then the proof of Theorem 5 follows from the statement of Theorem 6 if we choose a prime $p > n$. Therefore we present only the proof of Theorem 6.

Proof: We start with the proof of the second part of the theorem. Our approach combines the ideas from [1] and [3].

Let L be the set of k-wise Hamming-distances which occur between the elements of C and let $L' = \{l_1,\ldots,l_s\} = \{(n-l) \ (mod\ p) \ | l \in L\}$. Note that since $0 \notin L$ we have $n \ (mod\ p) \notin L'$. Now repeat the following procedure until C is empty.

At round i if set C is still not empty we choose a maximal subset $a^1,\ldots,a^d$ from C such that $|a^1 \sqcap a^2 \sqcap \ldots \sqcap a^d| \ (mod\ p) \notin L'$, but for any additional word $a' \in C$ we have that $|a^1 \sqcap a^2 \sqcap \ldots \sqcap a^d \sqcap a'| \ (mod\ p) \in L'$. Clearly, by definition, such codeword-set always exists and $1 \leq d \leq k-1$. Next define $c^i = a^1$, $b^i = a^1 \sqcap a^2 \sqcap \ldots \sqcap a^d$ and let $X_i \subseteq [n]$ be the set of indices of the coordinates in which $a^j, 1 \leq j \leq d$ are all equal. Note that $|c^i \sqcap b^i| = |b^i| \ (mod\ p) \notin L'$ but $|c^r \sqcap b^i| \ (mod\ p) \in L'$ for any $r > i$. Finally remove $a^1,\ldots,a^m$ from C and proceed to the next round.

Let $f_i(x)$ be the following polynomial of n variables $x_1,\ldots,x_n$:

$$f_i(x) = \prod_{u=1}^{s}\Big(\sum_{j\in X_i}\varepsilon(b^i_j, x_j) - l_u\Big),$$

where b^i_j is the value of the coordinate of b^i which corresponds to index $j \in X_i$ and the summation is restricted only to these indices. Note that by our construction, the number of such polynomials is at least $m = |C|/(k-1)$. By definition

$$f_i(c^i) = \prod_{u=1}^{s}\Big(|c^i \sqcap b^i| - l_u\Big) = \prod_{u=1}^{s}\Big(|b^i| - l_u\Big) \neq 0 \ (mod\ p),$$

but for all $r > i$

$$f_i(c^r) = \prod_{u=1}^{s}\Big(|c^r \sqcap b^i| - l_u\Big) = 0 \ (mod\ p).$$

Similarly to the proof of Theorem 2, we next define an additional set of polynomials. Let $\delta(x)$ be the polynomial in one variable with rational coefficients such that $\delta(0) = 0$ and $\delta(i) = 1$ for all $i = 1,\ldots,q-1$. Note that for any vector $x \in A^n$, the value of $\sum_{l=1}^{n}\delta(x_l)$ is equal to the weight of x. For all subsets $I \subset [n], |I| \leq s-t$ and for all vectors $v \in \{1,\ldots,q-1\}^I$, we define a polynomial

$$g_{I,v}(x) = \left(\prod_{i\in I}\varepsilon(x_i, v_i)\right)\ \prod_{j=1}^{t}\Big(\sum_{l=1}^{n}\delta(x_l) - w_j\Big),$$

where v_i are the entries of the vector v. Clearly, the number of such polynomials is equal to $\sum_{i=0}^{s-t}(q-1)^i\binom{n}{i}$, and by definition, the value $g_{I,v}(x)$ is an integer for all $x \in A^n$. In addition for every $x \in A^n$ with weight at most $s-t$, we have $g_{I,v}(x) \neq 0 \ (mod\ p)$ if and only if the vector x, restricted to I, equals to v.

We claim that the polynomials f_i and $g_{I,v}$ are linearly independent over the rationals. For a proof of this claim assume that

$$\sum \alpha_i f_i(x) \quad + \sum_{|I|\leq s-t}\beta_{I,v}g_{I,v}(x) = 0,$$

is a nontrivial relation. Clearly we can make all α_i and $\beta_{I,v}$ to be integers and in addition, since the above relation is nontrivial we can assume that not all of them are divisible by p. Let i_0 be the largest index such that $\alpha_{i_0} \neq 0 \ (mod\ p)$. Then, by substituting $x = c^{i_0}$ we obtain a contradiction. Indeed, $f_{i_0}(c^{i_0}) \neq 0 \ (mod\ p)$ but $f_i(c^{i_0}) = 0 \ (mod\ p)$ for all $i < i_0$ and also $g_{I,v}(c^{i_0}) = 0 \ (mod\ p)$, since the weight of c^{i_0} is equal w_j modulo p for some $1 \leq j \leq t$. Next suppose that all $\alpha_i = 0 \ (mod\ p)$, and let I_0 be the smallest set with the property $\beta_{I_0,v_0} \neq 0 \ (mod\ p)$ for some $v_0 \in \{1,\ldots,q-1\}^{I_0}$. Let $x_0 \in A^n$ be a vector which is equal to v_0 on the coordinates from I_0 and is zero everywhere else. Since all w_j are greater than the weight of x_0, by substituting $x = x_0$ into relation we obtain $g_{I_0,v_0}(x_0) \neq 0 \ (mod\ p)$, but as we explain above,

$g_{I,v}(x_0) = 0 \ (mod\ p)$ for all $|I| \geq |I_0|$ and $v \neq v_0$. This contradiction proves the linear independence of f_i and $g_{I,v}$.

Next note that all our computations are over the domain where $x_i(x_i-1)\dots(x_i-q+1)=0$ for each variable $1 \leq i \leq n$. Thus we can assume that in polynomials f_i and $g_{I,v}$, every variable x_i has exponent at most $q-1$. If not, we simply reduce these polynomials modulo $x_i(x_i-1)\dots(x_i-q+1)$ for all i. Also, in addition, every term of f_i and $g_{I,v}$ is the monomial with at most s variables. The space of such polynomials has dimension $\sum_{i=0}^{s}(q-1)^i\binom{n}{i}$ and we have found $m+\sum_{i=0}^{s-t}(q-1)^i\binom{n}{i}$ independent functions in this space. This immediately implies the desired bound on m and hence on $|C|$.

Finally we remark that the first part of this theorem follows already from independence of the polynomials f_i. This completes the proof. □

4 Concluding remarks

- It is natural to ask how tight are the results of Theorems 1, 2, 5 and 6. In particular do we need to have a multiplicative factor $(k-1)$ in all upper bounds? The following construction shows that in Theorem 2 this factor is indeed needed when p is fixed and n tends to infinity. We do not have analogous constructions for other theorems.

  Let p be a fixed prime, $s < p$ and suppose $2^{t-1} < k-1 \leq 2^t$ for some integer $t = o(n)$. Note that in this example we do not fix the value of k and it can be as big as $2^{o(n)}$. Let X be an n-element set and let $Y_1, \dots, Y_t$ be disjoint subsets of X, each of size p. Denote by $Y = X - \cup_i Y_i$. By definition $|Y| = n' = n - \lceil \log_2(k-1) \rceil p = (1+o(1))n$. Since the number of subsets of $\{1, \dots, t\}$ is $2^t \geq k-1$, let $I_1, \dots, I_{k-1}$ be any $k-1$ of these distinct subsets of $\{1, \dots, t\}$. Finally, the family $\mathcal{H}$ consists of all subsets of X of the form $A \cup (\cup_{i \in I_j} Y_i)$ for all subsets A of Y of size s and all $1 \leq j \leq k-1$. Clearly the number of sets in the family $\mathcal{H}$ equals to

$$(k-1)\binom{n'}{s} = (1+o(1))(k-1)\binom{n}{s},$$

  and it is easy to see that every set $H \in \mathcal{H}$ has size equal to s modulo p and every collections of k distinct sets from $\mathcal{H}$ satisfies that $|H_1 \cap \dots \cap H_k| = r \ (mod\ p)$ for some integer $0 \leq r \leq s-1$. Note, that the pairwise intersections of the sets of $\mathcal{H}$ do not satisfy the assumptions of the Frankl-Wilson theorem [9], since their sizes are not separated from the size of the sets itself; however, the k-wise intersection-sizes are already separated from s modulo p.

- An interesting open question is extension of the results of Theorems 2 and 6 to composite moduli. In this case the polynomial upper bound is no longer valid in general. In particular for any $k \geq 2$, $q = 6$ and $L = \{1, \dots, 5\}$ there exist a family of subset of n-element set of super polynomial size which satisfies the assertion of Theorem 2, see [11] for details. On the other hand for the special case of prime power moduli q and $s = q - 1$ one can still get a polynomial upper bounds.

It is not difficult to see, that our proofs of Theorems 2 and 6 together with the tools of Babai, Snevily and Wilson ([3], Theorem 6) and Babai and Frankl ([2], Theorem 5.30) give the following two results, whose proof will be left to the reader.

Theorem 9 *Let $k \geq 2$ and r be integers and p^α be a prime power. If $\mathcal{H}$ is a family of subset of n-element set such that $|H| = r \pmod{p^\alpha}$ for every $H \in \mathcal{H}$ but $|H_1 \cap \ldots \cap H_k| \neq r \pmod{p^\alpha}$ for all collections of k distinct sets from $\mathcal{H}$, then*

$$|\mathcal{H}| \leq (k-1) \sum_{i=0}^{p^\alpha - 1} \binom{n}{i}. \qquad \square$$

Theorem 10 *Let C be a q-ary code of length n and p^α be a prime power. If the set of k-wise Hamming distances which occur between k distinct codewords of C are never divisible by p^α, then*

$$|C| \leq (k-1) \sum_{i=0}^{p^\alpha - 1} (q-1)^i \binom{n}{i}. \qquad \square$$

- It is easy to see that when $k = 2$, one can deduce Theorem 2 from the Theorem 6. But for $k \geq 3$ these two statements do not seem to be related and need different proofs.

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