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A Trade-Off for Covering the Off-Diagonal Elements of Matrices

by

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ABSTRACT

We would like to cover all the off-diagonal elements of an $n \times n$ matrix by non-necessarily contiguous rectangular submatrices; the diagonal elements cannot be covered. It is not difficult to give a cover with $2\lceil \log n \rceil$ rectangles, where some off-diagonal elements are covered as many as $\lceil \log n \rceil$-times, or another cover, using n rectangles and any off-diagonal elements of the matrix is covered only once. We show that one cannot attain *both* low covering multiplicity and a small number of covering rectangles at the same time: We prove a trade-off between these two numbers.

1 Introduction

We are interested in the following problem: cover all the off-diagonal elements of the $n \times n$ matrix $M = \{m_{ij}\}$ by non-necessarily contiguous rectangular submatrices (called rectangles); the diagonal elements cannot be covered. It is not difficult to give a cover with $2\lceil \log n \rceil$ rectangles, where some off-diagonal elements are covered as many as $\lceil \log n \rceil$-times: Indeed, let us consider submatrices

$$U_t = \{m_{ij} : i_t = 0, j_t = 1\}; \quad V_t = \{m_{ij} : i_t = 1, j_t = 0\}$$

where i_t and j_t denotes the t^{th} bit in the binary form of i and j, respectively. Clearly, both U_t and V_t are rectangles, their combined number is $2\lceil \log n \rceil$, and they cover all the off-diagonal elements of M. It is also obvious, that entry m_{ij} is covered $h(i,j)$-times, where $h(i,j)$ stands for the Hamming-distance of the binary forms of i and j. Consequently, there are entries in M which are covered $\lceil \log n \rceil$-times.

Another cover, where all the off-diagonal entries of M are covered once can be given as follows: For $t = 1, 2, \ldots, n$ we define

$$W_t = \{m_{t,j} : j \in \{1, 2, \ldots, n\} - \{t\}\}.$$

Certainly, the n pairwise disjoint W_t's cover all the off-diagonal elements of M exactly once.

So we have got a cover with few rectangles but with a large multiplicity, and another one with a lot of rectangles and multiplicity 1. The question is whether can we cover the off-diagonal elements with few rectangles and with a low multiplicity? The answer is no, as it is implied by the following theorem:

Theorem 1 *(i) Suppose, that the off-diagonal elements of the $n \times n$ matrix M are covered by s rectangles while none of the diagonal elements are covered, and every entry of M are covered by at most d rectangles. Then*

$$\binom{s}{1} + \cdots + \binom{s}{d} \geq n. \tag{1}$$

(ii) Suppose, that the entries under the diagonal of the $n \times n$ matrix M are covered by s rectangles, and none of the diagonal or above-the-diagonal elements are covered, and every entry of M are covered by at most d rectangles. Then

$$\binom{s}{1} + \cdots + \binom{s}{d} \geq n - 1. \tag{2}$$

Using graph-theoretical language, we immediately get the following re-formulation from part (i):

Corollary 2 *Suppose we would like to cover the edges of the $2n$-vertex complete bipartite graph $K_{n,n}$ with color classes $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ with the edges of complete bipartite graphs, without covering any of the edges $u_i v_i, i = 1, 2, \ldots, n$. Suppose that we have s complete bipartite graphs in the cover, and the maximum multiplicity of covering an edge is d. Then (1) is satisfied.*

An easy corollary of part (ii) is the following:

Corollary 3 *Suppose we would like to cover the edges of the n-vertex complete graph with the edges of complete bipartite graphs. Assume that we have s complete bipartite graphs in the cover, and the maximum multiplicity of covering an edge is d. Then (2) is satisfied.*

Note, that the rectangles in the cover of Corollary 3 can be partitioned into pairs, and the members of these pairs are symmetric to the diagonal, while in Corollary 2 and in Theorem 1 part (i) this property does not necessarily hold.

We cannot show that our Theorem 1 is sharp. It is easy to construct rectangle-covers of $m + 2\lceil\log\lceil n/m\rceil\rceil$ rectangles and $1 + \lceil\log\lceil n/m\rceil\rceil$ mutiplicity by unifying the n-cover and the $2\log n$ cover, but we do not think that this construction is optimal.

1.1 Related work

Graham and Pollack [GP72] asked that how many edge-disjoint bipartite graphs can cover the edges of an n-vertex complete graph. They proved that $n - 1$ bipartite graphs are sufficient and necessary. Later, Tverberg gave a very nice proof for this statement [Tve82]. Having relaxed the disjointness-property, Babai and Frankl [BF92] asked that what is the minimum number of bipartite-graphs, which covers every edge of an n-vertex complete graph an odd multiplicity. Babai and Frankl proved that $(n-1)/2$ bipartite graphs are necessary. The optimum upper bound for the odd-cover was proved by Radhakrishnan, Sen and Vishwanathan [RSV00]. Radhakrishnan, Sen and Vishwanathan also gave matching upper bounds for covers, when the off-diagonal elements of matrix M are covered by multiplicity 1 modulo a prime number.

1.2 Motivation

Our questions, leading to Theorem 1 were motivated by examining the constructibility or at least the existence of low-rank matrices, containing 0's in the diagonal and non-zeroes elsewhere. We have given constructions of such matrices in [Gro00], with the help of the BBR-polynomials [BBR94]. Our goal was to find lower rank matrices of this property, and it turned out that the rank depends on the degree of the BBR polynomial. Roughly, the degree of the BBR-polynomial in that construction depends on the covering-multiplicity d, so that is the reason that we were interested in covers with low multiplicity.

2 Proof

Obviously, the intersection of two rectangles are also a (possibly empty) rectangle. Similarly, the intersection of a finite number of rectangles is also a rectangle.

We should also remark, that if a matrix M of rank r can be given over some field as a sum of t all-1 rectangles, then $t \geq r$, simply because then

$$M = \sum_{i=1}^{t} M_i,$$

where (the rank-1) matrix M_i has entries 1, corresponding to the elements of rectangle i, and 0's everywhere else, for $i = 1, 2, \ldots, t$, and using the sub-additivity property of the rank function.

Now, let us consider matrix M, and a cover of s rectangles with multiplicity d. Let us correspond matrix M_i to the rectangle i as in the previous paragraph. Then obviously the $n \times n$ matrix $\sum_{i=1}^{s} M_i$ has 0's in the diagonal and non-0's everywhere else, and the largest entry is at most d.

For a $K \subset \{1, 2, \ldots, s\}$ let M_K denote the rank-1 matrix, corresponding to the intersection of the rectangles i, where $i \in K$. Let J denote the all-1 matrix and let I denote the unit-matrix.

Now we prove part (i) of the Theorem.

From the inclusion-exclusion formula:

$$J - I = \sum_{K \subset \{1,2,\ldots,s\}, |K|=1} M_K - \sum_{K \subset \{1,2,\ldots,s\}, |K|=2} M_K + \cdots + (-1)^{d+1} \sum_{K \subset \{1,2,\ldots,s\}, |K|=d} M_K$$

The right-hand-side of this formula is a sum of $\binom{s}{1} + \cdots + \binom{s}{d}$ rank-1 matrices; the left-hand-side has rank n for $n \geq 2$ over the rationals, so (1) follows.

The proof of part (ii):

Let H denote the $n \times n$ matrix with 0's in the diagonal and above, and 1's under the diagonal. Obviously, this is a matrix of rank $n - 1$. On the other hand:

$$H = \sum_{K \subset \{1,2,\ldots,s\}, |K|=1} M_K - \sum_{K \subset \{1,2,\ldots,s\}, |K|=2} M_K + \cdots + (-1)^{d+1} \sum_{K \subset \{1,2,\ldots,s\}, |K|=d} M_K,$$

so (2) follows again.□

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