On the multiplicativity of the linear combination of additive representation functions

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Abstract

Let A be a set of positive integers. For a fixed $k \geqslant 1$ and a positive integer n let $R_{A,k}(n)$ denote the number of representations of n as the sum of k terms from the set A. In this paper we give a necessary and sufficient condition to the multiplicativity of the function $c_1R_{A,1}(n) + c_2R_{A,2}(n)$, where c_1 and c_2 are integers and $c_2 \neq 0$.

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1 Introduction

Let $\mathbb{Z}^+$ denote the set of positive integers. Let $A = \{a_1, a_2, \dots\}$, $1 \leqslant a_1 < a_2 < \dots$, be an infinite sequence of positive integers. For $k \in \mathbb{Z}^+$, let $R_{A,k}(n)$ denote the number of solutions of the equation

$$a_{i_1} + \dots + a_{i_k} = n, \quad a_{i_1}, \dots, a_{i_k} \in A,$$

where $n \in \mathbb{Z}^+$. In a series of papers P. Erdős, A. Sárközy and V. T. Sós studied additive representation functions like $R_{A,k}$. In [2], [3], [4] they investigated how regular can be the behaviour of the additive representation function $R_{A,k}$, while [5] and [6] focused on the monotonicity of the additive representation functions. Grekos, Haddad, Helou and Pihko [7] proved that the representation function cannot be periodic. One can find some other results in surveys [11] and [12]. We say that an arithmetic function $f(n)$ is multiplicative if $f(ab) = f(a)f(b)$ for every $a, b \in \mathbb{Z}^+$ which are coprime. Obviously, if $f(n)$ is a multiplicative arithmetic function, then $f(1) = 1$, while $R_{A,k}(1) \neq 1$ if $k \geqslant 2$. This implies that the additive representation function $R_{A,k}(n)$ cannot be multiplicative for $k \geqslant 2$. Define the function $g(n)$ by

$$g(n) = c_1R_{A,1}(n) + c_2R_{A,2}(n) + \dots + c_kR_{A,k}(n),$$

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where $c_1, \dots, c_k$ are integers and $c_k \neq 0$. In this paper we focus on the multiplicativity of $g(n)$. As far as we know this question has not been investigated yet. In the case when $k = 2$, i.e., for the representation function corresponding to the two terms sums we give a necessary and sufficient condition for the multiplicativity of $g(n)$. In particular, we prove the following theorem.

Theorem 1. *If $A \subset \mathbb{Z}^+$ and $c_1, c_2 \in \mathbb{Z}$, $c_2 \neq 0$, then the function $g(n) = c_1 R_{A,1}(n) + c_2 R_{A,2}(n)$ is multiplicative if and only if one of the following conditions holds: $A = \{1\}$ and $c_1 = 1$ or $A = \mathbb{Z}^+$ and $c_1 = c_2 = 1$ or $A = \{n^2 : n \in \mathbb{Z}^+\}$ and $c_1 = c_2 = 1$.*

Unfortunately we can not settle the case $k \geqslant 3$; thus this problem remains open.

Problem 1. *When $k \geqslant 3$ what conditions on the set A and on the coefficients c_i are needed to ensure that the function $g(n)$ is multiplicative?*

Let $h(z)$ denote the generating function of the set A, i.e.,

$$h(z) = \sum_{a \in A} z^a,$$

where $z = re^{2i\pi\alpha}$ and $r < 1$, thus this infinite series is absolutely convergent. Consider the polynomial $p(z) = c_1 z + c_2 z^2$. It is easy to see that $g(m)$ is the coefficient of z^m in $p(h(z))$. In view of this observation Theorem 1. asserts that for an infinitely set A the function $g(n)$ is multiplicative if and only if $p(z) = z + z^2$ and set A is the set of positive integers or the set of positive square numbers . However, our proof of Theorem 1. is elementary, the analytic approach may help to handle the general case when $k \geqslant 3$. On the other hand when $A = \mathbb{Z}^+$ we give a full description of the multiplicativity. Let $S(n, k)$ denote the number of partitions of a set of n elements into k nonepty subsets. By convention, we write $S(0, 0) = 1$. The sequence of $S(n, k)$ is called Stirling numbers of the second kind [13].

Theorem 2. *If $A = \mathbb{Z}^+$ and $p(z) = \sum_{i=1}^{k} c_i z^i$ is a polynomial of degree k, then the function $g(n)$ is multiplicative if and only if $c_i = (i-1)! \cdot S(k, i-1) \sum_{j=i-1}^{k} \binom{k-1}{j}$.*

2 Proof of Theorem 1.

First we prove the sufficiency. We denote the cardinality of a set A by $|A|$. If $A = \{1\}$ and $c_1 = 1$, then it follows that $g(1) = 1$, $g(2) = c_2$, and $g(n) = 0$ if $n \geqslant 3$, thus $g(n)$ is multiplicative. In the next step when $A = \mathbb{Z}^+$ and $c_1 = c_2 = 1$, we get that

$$g(n) = R_{A,1}(n) + R_{A,2}(n) = 1 + (n-1) = n,$$

which is obviously multiplicative. In the last case when $A = \{n^2 : n \geqslant 1, n \in \mathbb{N}\}$ and $c_1 = c_2 = 1$ we use the formula for the number of representations of a positive integer as the sum of two squares [9]. This asserts that if the prime decomposition of n is $n = 2^\alpha \cdot p_1^{\alpha_1} \cdots p_s^{\alpha_s} \cdot q_1^{\beta_1} \cdots q_t^{\beta_t}$, where $p_i \equiv 1 \bmod 4$ and $p_j \equiv 3 \bmod 4$, for every $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$, then

$$|\{(x, y) : x, y \in \mathbb{Z}, x^2 + y^2 = n\}| = \begin{cases} 4 \prod_{i=1}^{s} (\alpha_i + 1), \text{ if } t = 0 \\ 0, \text{ if } t > 0 \end{cases}.$$

This implies that

$$|\{(x,y): x,y \in \mathbb{Z}, x^2+y^2=n\}| = 4|\{(x,y): x,y \in \mathbb{Z}^+, x^2+y^2=n\}|$$

$$+4|\{x: x\in \mathbb{Z}^+, x^2=n\}| = 4R_{A,2}(n)+4R_{A,1}(n),$$

thus we have

$$g(n) = R_{A,1}(n)+R_{A,2}(n) = \begin{cases} \prod_{i=1}^{s}(\alpha_i+1), \text{ if } t=0 \\ 0, \text{ if } t>0 \end{cases}.$$

Then it follows immediately that $g(n)$ is multiplicative. This proves the sufficient conditions.

In the next step we prove the other direction. Assume that the function $g(n)$ is multiplicative. As $A \subset \mathbb{Z}^+$, thus we have $R_{A,2}(1)=0$, which implies that $g(1) = c_1R_{A,1}(1)+c_2R_{A,2}(1) = c_1R_{A,1}(1)$. For a multiplicative function $g(n)$ we know that $g(1)=1$. Thus we have $c_1=1$ and $R_{A,1}(1)=1$, which implies that $1\in A$. Let $A=\{a_1,a_2,\dots\}$, where $1=a_1<a_2<\dots$. Assuming that $k\neq l$, we have

$$g(a_k+a_l) = R_{A,1}(a_k+a_l)+c_2R_{A,2}(a_k+a_l)$$

$$= |\{i: a_i=a_k+a_l\}|+c_2|\{(i,j): a_i+a_j=a_k+a_l\}|.$$

In the above formula we have $|\{i: a_i=a_k+a_l\}|=0$ or $|\{i: a_i=a_k+a_l\}|=1$ and $|\{(i,j): a_i+a_j=a_k+a_l\}|\geqslant 2$, thus we have

$$g(a_k+a_l)\neq 0, \tag{1}$$

if $k\neq l$. In view of the fact that $a_1=1$, for $i>1$ we have

$$g(a_i+1)\neq 0. \tag{2}$$

Proposition 1. *If $A\neq\{1\}$, then $|A|=\infty$.*

Proof. We this prove by contradiction. Assume that $A\neq\{1\}$, but $|A|<\infty$. Let a^* be the maximal element of the set A. If $a^*=2$, then $A=\{1,2\}$, thus we have $g(1)=1$, $g(2)=1+c_2$, $g(3)=2c_2$, $g(4)=c_2$. It follows that $g(12)=g(3)\cdot g(4)=2c_2^2$, but

$$g(12) = |\{i: a_i=12, a_i\in A\}|+c_2|\{(i,j): a_i+a_j=12, a_i,a_j\in A\}| = 0$$

which is absurd. If $a^*>2$ we have two cases.
Case 1. $\frac{a^*}{2}\notin A$. Then we have

$$g(a^*) = |\{i: a_i=a^*, a_i\in A\}|+c_2|\{(i,j): a_i+a_j=a^*, a_i,a_j\in A\}|,$$

which implies that $2\nmid g(a^*)$, thus $g(a^*)\neq 0$ and we get from (2) that $g(a^*+1)\neq 0$. It follows that $g(a^*(a^*+1)) = g(a^*)\cdot g(a^*+1)\neq 0$. Since for every element of A we have

$$a_i \leqslant a^* < \frac{a^*(a^*+1)}{2},$$

it follows that

$$g(a^*(a^*+1)) = |\{i : a_i = a^*(a^*+1), a_i \in A\}|+2c_2|\{(i,j) : a_i+a_j = a^*(a^*+1), a_i, a_j \in A\}| = 0,$$

a contradiction.

Case 2. Assume that $\frac{a^*}{2} \in A$. As $a^* > 2$ and in view of (2) it follows that $g(\frac{a^*}{2}+1) \neq 0$ and $g(a^*+1) \neq 0$. Since $2\left(\frac{a^*}{2}+1\right) - (a^*+1) = 1$ it follows that $\frac{a^*}{2}+1$ and a^*+1 are obviusly coprime. Thus we have

$$g((\frac{a^*}{2}+1)(a^*+1)) = g(\frac{a^*}{2}+1) \cdot g(a^*+1)) \neq 0.$$

On the other hand

$$g((\frac{a^*}{2}+1)(a^*+1)) = |\{i : a_i = (\frac{a^*}{2}+1)(a^*+1), a_i \in A\}|+|\{(i,j) : a_i+a_j = (\frac{a^*}{2}+1)(a^*+1), a_i, a_j \in A\}|.$$

Since

$$a_i \leqslant a^* < a^*+1 < \frac{\frac{a^*}{2}+1}{2}(a^*+1),$$

which implies that $g((\frac{a^*}{2}+1)(a^*+1)) = 0$ a contradiction. □

Let p be a positive prime and $M_p = 2^p - 1$ denote a Mersenne prime. Let $F_n = 2^{2^n}+1$ a Fermat number. In the next proposition we compute the possible values of a_2.

Proposition 2. *If $A \neq \{1\}$, then $a_2 = M_p$ or $a_2 = 8$ or $a_2 = F_m - 1$.*

Proof. In the first step we prove that a_2 is a power of a prime, that is $a_2 = p^\alpha$, $\alpha \geqslant 1$. Assume that $a_2 > 2$ and $a_2 = u \cdot v$, where u, v are coprime positive integers and $u, v > 1$. It is clear that $u, v < a_2$ and $g(a_2) = 1$. On the other hand $g(a_2) = g(uv) = g(u) \cdot g(v)$, thus $g(u) \neq 0$ and $g(v) \neq 0$, but conditions $1 < w < a_2$ and $g(w) \neq 0$ imply $w = 2$, contradiction. The same argument shows that $a_2 + 1$ is also a power of a prime. As a_2 and $a_2 + 1$ have different parity, one of them is a power of two. If $a_2 = 2^n$ and $a_2 + 1$ is a prime number, then $n = 2^m$, that is $a_2 = F_m - 1$. If $a_2 + 1 = p^\alpha$, $\alpha > 1$, then it is well known [8] that the Catalan - equation $2^n + 1 = p^\alpha$ has the only solution $n = 3$, $p = 3$, $\alpha = 2$. This implies that $a_2 = 8$. In the second case when $a_2 + 1 = 2^n$, then $a_2 = 2^n - 1$, but $2^n - 1 \neq p^\alpha$ when $\alpha > 1$, thus $2^n - 1$ is a Mersenne prime. □

In the next proposition we study a_3.

Proposition 3. *Let $a_2 = M_p$. Then $a_3 = 8$ or $a_3 = F_m - 1$.*

Proof. In the first step we prove that a_3 is a power of a prime. Assume that $a_3 = u \cdot v$, where u, v are coprime positive integers and $u, v > 1$. If n is a positive integer such that $g(n) \neq 0$ and $n < a_3$, then $n = 2$ or $n = M_p$ or $n = M_p + 1$ or $n = 2M_p$. This implies that $a_3 = 2M_p$ or $a_3 = M_p(M_p + 1)$. We distinguish three cases.

Case 1. When $a_3 = 2M_p$ and $a_2 > 3$, then it follows from (1) that $g(a_3 + a_2) \neq 0$ and since $g(3) = 0$ we have $g(a_3 + a_2) = g(3M_p) = g(3)g(M_p) = 0$ which is absurd.

Case 2. If $a_2 = 3$ and $a_3 = 2a_2 = 6$, then $g(2) = c_2$, $g(3) = 1$, $g(6) \geqslant 1 + c_2$ and $g(6) = g(2)g(3) = c_2$, a contradiction.

Case 3. When $a_3 = M_p(M_p+1)$, then $g(a_3+a_2) = g(M_p(M_p+1)+M_p) = g(M_p(M_p+2)) = g(M_p)g(M_p+2)$. It is clear that if $n < a_3$ then $g(n) \neq 0$ when $n = 1, 2, M_p, M_p+1, 2M_p$. Thus we have $g(a_3+a_2) = 0$ which contradicts (1).

In the next step we prove that a_3+1 is a power of a prime similarly as above. We prove this by contradiction. It follows that $a_3+1 = 2M_p$ or $a_3+1 = M_p(M_p+1)$. If $a_3+1 = 2M_p$ and $p = 2$ then $M_2 = 3$ and $a_3 = 5$. Thus $g(2) = c_2$ and $g(3) = 1$, which implies that $g(6) = g(2)g(3) = c_2$. As $g(6) = R_{A,1}(6) + c_2R_{A,2}(6) = R_{A,1}(6) + 3c_2 = c_2$ so that $R_{A,1}(6) = -2c_2$ a contradiction. If $a_2 = M_p > 3$, then $a_2 \geqslant 7$ and $a_3 = 2M_p - 1$. It follows from (1) that $g(a_3+a_2) \neq 0$. As $M_p \equiv 7 \bmod 8$ it follows that $a_3+a_2 = 3M_p-1 \equiv 4 \bmod 8$. Thus we have

$$0 \neq g(a_3+a_2) = g\Big(4\Big(\frac{3M_p-1}{4}\Big)\Big) = g(4)g\Big(\frac{3M_p-1}{4}\Big).$$

Since $g(4) = 0$, this is a contradiction. In the second case when $a_3 = M_p(M_p+1) - 1$, then

$$g(a_3+a_2) = g(M_p^2 + 2M_p - 1) = g\Big(2\Big(M_p + \frac{M_p^2-1}{2}\Big)\Big) = g(2)g\Big(M_p + \frac{M_p^2-1}{2}\Big).$$

It follows from (1) that

$$g\Big(M_p + \frac{M_p^2-1}{2}\Big) \neq 0.$$

On the other hand if $n < M_p^2 + M_p - 1 = a_3$ and $g(n) \neq 0$ then $n = 1, 2, M_p, M_p+1, 2M_p$ a contradiction. A similar argument to the end of the proof of Proposition 2 gives that a_3 must be $M_{p'}$ or 8 or $F_m - 1$. If $a_3 = M_{p'}$, then $0 \neq g(a_2+a_3) = g(2^p + 2^{p'} - 2) = g(2)g(2^{p-1} + 2^{p'-1} - 1)$, but conditions $1 < w < a_3$, $g(w) \neq 0$ and w odd imply that $w = 2^p - 1$, a contradiction. □

We may assume that $|A| = \infty$. We distiguish three cases.

Case 1. $c_2 < 0$. Then $g(n) \leqslant 1$. We have two subcases.

Case 1a. For every p prime we have $g(p^\alpha) \in \{1, -1, 0\}$.

In this case $g(n) \in \{1, -1, 0\}$. It is clear that if $k \neq l$, then $a_k + a_l \in A$ since otherwise

$$g(a_k+a_l) = R_{A,1}(a_k+a_l) + c_2R_{A,2}(a_k+a_l) = c_2|\{(i,j) : a_i + a_j = a_k + a_l\}| \leqslant -2$$

a contradiction. Thus we have $a_1 + a_2 \in A$, $(a_1+a_2) + a_1 \in A$. Since $2a_1 + 2a_2 = (a_1+a_2) + (a_1+a_2) = (2a_1+a_2) + a_2$ we get

$$g(2a_1+2a_2) = R_{A,1}(2a_1+2a_2) + c_2R_{A,2}(2a_1+2a_2) \leqslant 1 - |\{(i,j) : a_i + a_j = 2a_1+2a_2\}| \leqslant -2,$$

a contradiction.

Case 1b. There exists a prime q such that $g(q^\beta) < -1$ for some $\beta \geqslant 1$. If $q \neq p$ prime and $g(p^\alpha) \leqslant -1$, then $g(p^\alpha q^\beta) = g(p^\alpha)g(q^\beta) \geqslant 2$ a contradiction. This implies that $g(p^\alpha) \in \{0, 1\}$. We proved that if $k \neq l$, then $g(a_k + a_l) \neq 0$. We denote by $p^\alpha || n$ if $p^\alpha | n$ but $p^{\alpha+1} \nmid n$. It follows that if p is a prime such that $p^\alpha || a_k + a_l$, $k \neq l$, then $g(p^\alpha) \neq 0$. We need the following lemma of Erdős and Turán [10].

Lemma 1. *If $1 \leqslant a_1 < a_2 < \ldots < a_{12}$ are integers, then the numbers $a_k + a_l$ have at least four prime divisors, that is there exist p_1, p_2, p_3 different primes, $p_i \neq q$ such that $p_1^{\alpha_1} || a_r + a_s$, $p_2^{\alpha_2} || a_t + a_u$, $p_3^{\alpha_3} || a_v + a_w$, where $1 \leqslant r, s, t, u, v, w \leqslant 6$, $r \neq s$, $t \neq u$, $v \neq w$.*

Then we have $g(p^{\alpha}) \in \{0,1\}$. As $g(p_i^{\alpha_i}) \neq 0$, thus we have $g(p_i^{\alpha_i}) = 1$, which implies that $p_i^{\alpha_i} \in A$. It follows from (1) that $g(p_i^{\alpha_i} + p_j^{\alpha_j}) \leqslant -1$. Since $p_i \neq p_j$ and if $q \nmid n$, then $g(n) \geqslant 0$, it follows that $q | p_1^{\alpha_1} + p_2^{\alpha_2}$, $q | p_1^{\alpha_1} + p_3^{\alpha_3}$, $q | p_2^{\alpha_2} + p_3^{\alpha_3}$. Thus we have

$$q | (p_1^{\alpha_1} + p_2^{\alpha_2}) + (p_1^{\alpha_1} + p_3^{\alpha_3}) - (p_2^{\alpha_2} + p_3^{\alpha_3}) = 2p_1^{\alpha_1}.$$

It follows that $q|2$, thus $q = 2$ and $g(2m+1) \in \{0,1\}$. If a_i is even then it follows from (2) that $g(a_i + 1) \leqslant -1$ a contradiction, which implies that a_i is odd. Then it follows from Proposition 2. that $a_2 = M_p = 2^p - 1$ and $a_3 = 8$ or $a_3 = F_m - 1$ by Proposition 3, a contradiction. The proof of Case 1. is completed.

Case 2. $c_2 \geqslant 2$. In view of Proposition 2. we have three possibilities for a_2. As $a_1 = 1$, if $a_2 = F_m - 1$ we have

$$g(F_m) = g(a_2 + 1) \geqslant c_2 |\{(i,j) : a_i + a_j = a_2 + 1\}| \geqslant 2c_2,$$

and

$$g(2) \geqslant c_2 |\{(i,j) : a_i + a_j = 2\}| \geqslant c_2.$$

Thus we have $g(2F_m) = g(2)g(F_m) \geqslant 2c_2^2$. The quantity $2F_m$ has the following three possible representations as the sum of two terms from the sequence A.

$$2F_m = 1 + (2F_m - 1) = (F_m - 1) + (F_m + 1) = F_m + F_m.$$

We prove that $F_m + 1$ cannot be contained in A. We prove this by contradiction. Assume that $F_m + 1 \in A$. If $m > 0$ then we have

$$g(F_m + 1) = g(2^{2^m} + 2) = g(2(2^{2^m - 1} + 1)) = g(2)g(2^{2^m - 1} + 1).$$

On the other hand $2 < 2^{2^m - 1} + 1 \leqslant 2^{2^m} - 1 < F_m$, thus we have $g(2^{2^m - 1} + 1) \neq 0$ a contradiction. If $m = 0$, then we have $a_1 = 1$, $a_2 = 2$, $a_3 \leqslant 4$, which implies that $g(2) = c_2 + 1$, $g(3) \geqslant 2c_2$ so that $g(6) = g(2)g(3) \geqslant (c_2 + 1)2c_2$. Clearly $6 = 1 + 5 = 2 + 4 = 3 + 3$, thus we have

$$g(6) \leqslant 1 + 2c_2 + 2c_2 + c_2 = 5c_2 + 1 < (c_2 + 1)2c_2 \leqslant g(6)$$

a contradiction. It follows that the quantity $2F_m$ has the following two possible representations as the sum of two terms from the sequence A.

$$2F_m = 1 + (2F_m - 1) = F_m + F_m.$$

Thus we have

$$2c_2^2 \leqslant g(2F_m) \leqslant 1 + 2c_2 + c_2 = 3c_2 + 1,$$

which is a contradiction if $c_2 \geqslant 2$.

In the second case we assume that $a_2 = 8$. As $9 = 8 + 1 = a_2 + a_1$ we have $g(9) \geqslant 2c_2$. It follows that $g(18) = g(9)g(2) \geqslant 2c_2c_2 = 2c_2^2$. It is clear that 18 has the following possible representations as the sum of two terms from A: $1 + 17 = 8 + 10 = 9 + 9$. As $g(10) = g(2)g(5)$, if $10 \in A$, then $g(10) \neq 0$ and so $g(5) \neq 0$ a contradiction. Hence $g(18) \leqslant 1 + 2c_2 + c_2 = 1 + 3c_2 < 2c_2^2 \leqslant g(18)$, a contradiction.

Thus $a_2 = M_p$. It follows from Proposition 3. that $a_3 = 8$ or $a_3 = F_m - 1$. Assume that $a_3 = 8$. Then $a_2 = 3$ or $a_2 = 7$. If $a_2 = 3$, then $g(3) = 1$ and $g(4) = 2c_2$, thus we have $g(12) = g(3)g(4) = 2c_2$. It is clear that 12 has the following possible representations as the sum of two terms from A: $1 + 11 = 3 + 9$. If $9 \in A$ then clearly $a_4 = 9$. It follows that $g(10) = g(2)g(5) = g(a_4 + a_1) \neq 0$, thus we have $g(5) \neq 0$ a contradiction. If $11 \in A$, then $g(14) = g(2)g(7) = g(3 + 11) \neq 0$, thus we have $g(7) \neq 0$ a contradiction. Thus $g(12) \leqslant 1$ which is a contradiction. If $a_2 = 7$, then $g(a_2 + a_3) = g(15) = g(3)g(5) \neq 0$, which implies that $g(5) \neq 0$ a contradiction.

In the next case assume that $a_3 = F_m - 1$, $m \geqslant 1$. Then we have $g(2) = c_2$. We show that $g(F_m) \geqslant 2c_2$. Clearly

$$g(F_m) = |\{i : a_i = F_m\}| + c_2|\{(i, j) : a_i + a_j = F_m\}|.$$

Applying the fact that $F_m = 1 + (F_m - 1) = a_1 + a_3$, we obtain $g(F_m) \geqslant 2c_2$ and therefore $g(2F_m) = g(2)g(F_m) \geqslant 2c_2^2$ we have $g(2F_m) \geqslant 2c_2^2$. The quantity $2F_m$ has the following four possible representations as the sum of two terms from the sequence A.

$$2F_m = 1 + (2F_m - 1) = (F_m - 1) + (F_m + 1) = F_m + F_m = M_p + (F_m - M_p).$$

In the next step we prove that $g(2F_m - 1) = 0$. We prove it by contradiction. Assume that $g(2F_m - 1) \neq 0$. Then clearly $3 \mid 2F_m - 1$, i.e., $2F_m - 1 = 3^u v$, where $3 \nmid v$. For $m > 1$ we know from Catalan's equation that $2F_m - 1 = 2^{2^m+1} + 1 \neq 3^u$, thus $v > 1$. It follows that $g(2F_m - 1) = g(3^u)g(v)$. We proved above that if n is odd and $n < a_3$ and $g(n) \neq 0$, then $n = 1$ or $n = M_p$ which is a contradiction. For $m = 1$ we have $a_2 = M_p = 3$, $a_3 = F_m - 1 = F_1 - 1 = 4$. Thus we have $g(2) = c_2$, $g(3) = 1$, so that $g(6) = c_2$. It is clear that 6 has the following possible representations as the sum of two terms from A: $6 = 1 + 5 = 3 + 3$. It follows that $5, 6 \notin A$. Thus we have $g(4) = 2c_2 + 1$, $g(5) = 2c_2$. Thus we have $g(10) = g(2)g(5) = 2c_2^2$. It is clear that 10 has the following possible representations as the sum of two terms from A: $10 = 1 + 9 = 3 + 7$. Since $c_2 \mid g(10)$, thus $10 \notin A$ and $7, 9 \in A$. An easy calculation shows that $g(7) = 2c_2 + 1$. Similarly we get that $g(12) = g(3)g(4) = 2c_2 + 1$, thus $11 \notin A$. Thus we have $g(14) = g(2)g(7) = c_2(2c_2 + 1)$. It is clear that 14 has the following possible representations as the sum of two terms from A: $14 = 1 + 13 = 7 + 7$, which implies that $g(14) \leqslant 1 + 3c_2$ a contradiction.

Equation $g(2F_m - 1) \neq 0$ implies that $2F_m - 1 \notin A$ and $F_m \notin A$ because $F_m - 1 \in A$. We prove that $F_m + 1 \notin A$. We prove by contradiction. Let us suppose that $F_m + 1 \in A$, which implies $g(F_m + 1) \neq 0$. On the other hand

$$g(F_m + 1) = g(2^{2^m} + 2) = g(2(2^{2^m-1} + 1)) = g(2)g(2^{2^m-1} + 1),$$

and $m > 0$, thus we have $g(2^{2^m-1} + 1) \neq 0$. We proved above that if n is odd and $n < a_3 = F_m - 1$ and $g(n) \neq 0$, then $n = 1$ or $n = M_p$. In the latter case we have $2^{2^m-1} + 1 = M_p = 2^p - 1$, which implies that $p = 2$, $m = 1$. In this case $a_1 = 1$, $a_2 = 3$ and $a_3 = 4$. We have already seen this case.

We get that the only possible representations of $2F_m$ from A $2F_m = M_p + (F_m - M_p)$. It follows that $g(2F_m) \leqslant 1 + 2c_2$. As we proved above that $g(2F_m) \geqslant 2c_2^2$ we get a contradiction. The proof of Case 2. is completed.

Case 3. $c_2 = 1$. In this case we have $g(n) = R_{1,A}(n) + R_{2,A}(n)$. We prove that if $a_2 = 2$, then $A = \mathbb{Z}^+$. We distinguish two subcases.

Case 3a. Assume that $a_2 = 2$, $a_3 = 3$. We prove by induction that $a_n = n$ and $g(n) = n$. For $n \leqslant 3$ the statement is obvious. It follows from the well known Bertrand postulate that if $n \geqslant 3$ there exists an odd prime p between $n/2$ and n. Thus we get from the inductive step that $g(2p) = g(2)g(p) = 2p$. On the other hand $g(2p) = |\{i : a_i = 2p\}| + |\{(i,j) : a_i + a_j = 2p\}|$. It is easy to see that $|\{(i,j) : a_i + a_j = 2p\}| \leqslant 2p - 1$ and the equality holds if and only if $a_i = i$ for $i \leqslant 2p - 1$. This implies that if $i \leqslant 2p$ then $a_i = i$. In our situation $2p \geqslant n + 1$, we obtain that $a_{n+1} = n + 1$ and $g(n+1) = n + 1$.

Case 3b. Assume that $a_2 = 2$, $a_3 > 3$. Then we have $g(2) = 2$ and $g(3) = 2$, thus we have $g(6) = g(2)g(3) = 4 = |\{i : a_i = 6\}| + |\{(i,j) : a_i + a_j = 6\}|$. This implies that $a_3 = 4$, $a_4 = 5$ and $6 \notin A$. It follows that $g(4) = 2$, $g(5) = 3$. We get that $g(10) = g(2)g(5) = 6 = |\{i : a_i = 10\}| + |\{(i,j) : a_i + a_j = 10\}|$. This implies that $8, 9, 10 \in A$. It is clear that 12 has the following possible representations as the sum of two terms from A: $12 = 1 + 11 = 2 + 10 = 4 + 8 = 5 + 7$, and $g(12) = g(3)g(4) = 4$, which implies that $7, 11 \notin A$. Thus $g(7) = 2$. It is clear that $g(14) = g(2)g(7) = 4$ and 14 has the following representations as the sum of two terms from A: $4 + 10 = 5 + 9$, which implies that $12, 13, 14 \notin A$. Obviously $g(15) = g(3)g(5) = 6$, but counting the possible representation as the sum of two terms from set A we get that $g(15) \leqslant 5$ which is a contradiction.

In the next step we prove that if $a_2 > 2$ then $a_2 = 4$. We have two subcases.

Case 3c. Assume that $a_2 = 3$. Then we have $g(2) = 1$, $g(3) = 1$. As $g(6) = g(2)g(3) = 1$, thus we have $5, 6 \notin A$. If $4 \notin A$, we get that $g(4) = 2$, $g(5) = 0$, which implies that $g(10) = g(2)g(5) = 0$ thus we have $7, 9, 10 \notin A$. We get that $g(7) = 0$, which implies that $g(14) = g(2)g(7) = 0$. This gives that $11, 13, 14 \notin A$. As $g(12) = g(3)g(4) = 2$ and in view of the above facts we obtain that $g(12) \leqslant 1$ which is a contradiction. Thus we get that $a_3 = 4$, $g(4) = 3$, $g(5) = 2$. Assume that $9 \in A$. Then we have from $10 = 1 + 9$ that $g(10) = g(2)g(5) = 2$, so $7, 10 \notin A$, and $g(7) = 2$. It follows that $g(12) = g(3)g(4) = 3$ and $12 = 3 + 9$, thus we have $8, 11 \notin A$ and $12 \in A$. In view of $g(14) = g(2)g(7) = 2$, thus $13 \in A$ and $14 \notin A$. The representation $21 = 9 + 12$ and $g(21) = g(3)g(7) = 2$ imply that $17, 18, 20, 21 \notin A$. It is clear that 20 has the following possible representations as the sum of two terms from A: $20 = 1 + 19 = 4 + 16$, and $g(20) = g(4)g(5) = 6$, which implies that $g(20) \leqslant 4$ which is a contradiction. We get that $9 \notin A$. Because of $4 \in A$, we have $g(1) = 1$, $g(2) = 1$, $g(3) = 1$, $g(4) = 3$, therefore $g(6) = g(2)g(3) = 1$, thus $5, 6 \notin A$, and $g(5) = 2$. In view of $g(10) = g(2)g(5) = 2$ we obtain that $7 \in A$ and $10 \notin A$. As $g(12) = g(3)g(4) = 3$, thus we have $12 \in A$. It is clear that $g(7) = 3$. Since $g(15) = g(3)g(5) = 2$, we have $8, 11, 14, 15 \notin A$, and therefore $g(12) \leqslant 1$, which is absurd. Thus we have $a_2 \neq 3$.

Case 3d. Assume that $a_2 > 4$. We know that $a_2 = F_m - 1$ or $a_2 = M_p$ or $a_2 = 8$.

Let us suppose that $a_2 = F_m - 1$, $m \geqslant 2$. Then we have $g(a_2 + a_1) = g(F_m) \geqslant 2$ and $g(2F_m) = g(2)g(F_m) \geqslant 2$. But the possible representations as the sum of two terms of A are $2F_m = (F_m - 1) + (F_m + 1) = F_m + F_m = 1 + (2F_m - 1)$. If $F_m + 1 \in A$, then

$$g(F_m + 1) = g(2^{2^m} + 2) = g(2(2^{2^m - 1} + 1)) = g(2)g(2^{2^m - 1} + 1) \neq 0,$$

thus $g(2^{2^m-1} + 1) \neq 0$, a contradiction, which implies that $F_m + 1 \notin A$. We show that $g(2F_m - 1) = 0$. Suppose that $g(2F_m - 1) > 0$. It is clear that $3 | 2F_m - 1$ and so $2F_m - 1 = 3^\alpha \cdot t$, where $t > 1$ and $3 \nmid t$. It follows that

$$g(2F_m - 1) = g(3^\alpha \cdot t) = g(3^\alpha) \cdot g(t) \neq 0,$$

thus we have $g(3^\alpha) \neq 0$, $g(t) \neq 0$ a contradiction. We obtain that $F_m, 2F_m - 1 \notin A$, because $2F_m - 1 = (F_m - 1) + F_m$. Thus $g(2F_m) \leqslant 1$ which is absurd. This implies that $a_2 \neq F_m - 1$.

Assume that $a_2 = 8$. Then we have $g(9) = g(1+8) \geqslant 2$, therefore $g(18) = g(2)g(9) \geqslant 2$ and we have three possibilities to write integer 18 as the sum of two terms from A as $18 = 1 + 17 = 9 + 9 = 8 + 10$. If $9 \in A$ or $10 \in A$, then $g(10) \neq 0$. Hence we have $g(10) = g(2)g(5) \neq 0$, thus $g(5) \neq 0$ a contradiction. It follows that $17 \in A$. Then we have $g(8 + 17) = g(25) \geqslant 2$, thus $g(50) \geqslant 2$. As $g(3) = g(4) = g(5) = g(7) = 0$, and for every decomposition $50 = k + (50 - k)$, $1 \leqslant k \leqslant 25$ one can find a prime power $p^\alpha \in \{3, 4, 5, 7\}$ and an integer $l \in \{0, 1, 8, 17\}$ such that $p^\alpha || k + l$ or $p^\alpha || 50 - k + l$ we get that either $g(k) = 0$ or $g(50-k) = 0$, therefore either $k \notin A$ or $50-k \notin A$, thus $g(50) \leqslant 1$, contradiction.

It follows that $a_2 = M_p$. By Proposition 3 we have $a_3 = F_m - 1$ or $a_3 = 8$. If $a_3 = F_m - 1$, then $g(2^{2^m} + 1) \geqslant 2$, hence $g(2^{2^m+1} + 2) = g(2)g(2^{2^m} + 1) \geqslant 2$. It follows that the possible representations of $2^{2^m+1} + 2$ are the following

$$2^{2^m+1}+2 = 1+(2^{2^m+1}+1) = 2^p-1+(2^{2^m+1}-2^p+3) = 2^{2^m}+(2^{2^m}+2) = (2^{2^m}+1)+(2^{2^m}+1).$$

It is clear that if $g(2^{2^m+1}+1) \neq 0$, then $2^{2^m+1}+1 = 3^\alpha t$, $3 \nmid t$, $t > 1$ (now $m > 1$), therefore $g(3^\alpha) \neq 0$ and $g(t) \neq 0$, which is absurd, because only for one odd w, $1 < w < a_3$ holds $g(w) \neq 0$. This implies that $2^{2^m} + 1 \notin A$ and $2^{2^m+1} + 1 \notin A$. If $2^{2^m+1} - 2^p + 3 \in A$ then we have $g((2^{2^m+1} - 2^p + 3) + 1) = g(4)g(2^{2^m-1} - 2^{p-2} + 1) \neq 0$, thus $g(4) \neq 0$ which is absurd. In the last case if $2^{2^m} + 2 \in A$ then we have $g(2^{2^m} + 2) = g(2)g(2^{2^m-1} + 1) \neq 0$, thus $g(2^{2^m-1} + 1) \neq 0$ which is impossible. Hence $g(2^{2^m+1} + 2) \leqslant 1$, a contradiction. This implies that $a_2 = 4$.

In the next step we prove that $a_3 = 9$. If $a_3 = 5$, then $g(2) = 1$, $g(3) = 0$ and $2 \leqslant g(6) = g(2)g(3) = 0$ a contradiction. If $a_3 = 6$, then $g(2) = 2$, $g(3) = 0$ and $1 = g(6) = g(2)g(3) = 0$ a contradiction. If $a_3 = 7$, then $g(3) = 0$ and $g(9k + 3) = g(3(3k + 1)) = g(3)g(3k + 1) = 0$, thus $9k + 3 \notin A$. On the other hand $g(9k + 6) = g(3(3k + 2)) = g(3)g(3k + 2) = 0$, thus $9k + 6 \notin A$. It is clear that if $9k + 2 \in A$, then $9k + 3 = (9k + 2) + 1$, thus $g(9k + 3) \geqslant 2$ which is impossible. It is easy to see similarly that $9k - 1$, $9k - 4 \notin A$. This implies that $3k + 2 \notin A$, specially $8 \notin A$. The equality $g(10) = g(2)g(5) = 2$ implies $9 \in A$ and $10 \notin A$. Then $g(14) = g(2)g(7) = 1$, which implies that $13 \notin A$. We know $g(18) = g(2)g(9) = 1$, which implies that $18 \notin A$. Thus we have $g(40) = g(5)g(8) = 2 \cdot 3 = 6$, but $g(40) \leqslant 5$, because the possible representation as the sum of two terms from the set A are $40 = 4 + 36 = 9 + 31$. If $a_3 = 8$, then $g(12) = g(3)g(4) = 0$, but 12=4+8, $4, 8 \in A$, therefore $g(12) > 0$, which is absurd. On the other hand if $a_3 \geqslant 10$, then $1 \geqslant g(10) = g(2)g(5) = 1 \cdot 2 = 2$ which is impossible. It follows that $a_3 = 9$.

In the next step we will prove that $a_4 = 16$. Assume that the first three elements of A are $1, 4, 9 \in A$. For the fourth element of A we have seven possibilities. If $10 \in A$, then $3 = g(10) = g(2)g(5) = 2$ which is absurd. If $11 \in A$, then $2 \leqslant g(12) = g(3)g(4) = 0$ which is absurd. If $12 \in A$, then $1 \leqslant g(12) = g(3)g(4) = 0$ which is absurd. If $13 \in A$, then $2 \leqslant g(14) = g(2)g(7) = 0$ which is absurd. If $14 \in A$, then $1 = g(14) = g(2)g(7) = 0$ which is absurd. If $15 \in A$, then $1 = g(15) = g(3)g(5) = 0$ which is absurd. On the other hand if $a_4 \geqslant 17$, then $g(20) = g(4)g(5) = 2$, then $19 \in A$, thus $2 \leqslant g(28) = g(4)g(7) = 0$ which is impossible. It follows that $a_4 = 16$.

In the next step we will prove that $a_5 = 25$. Assume that the first four elements of A are $1, 4, 9, 16 \in A$. For the fifth element of A we have nine possibilities. If $17 \in A$, then $2 \leqslant g(21) = g(3)g(7) = 0$ which is absurd. If $18 \in A$, then $2 \leqslant g(22) = g(2)g(11) = 0$ which is absurd. If $19 \in A$, then $4 \leqslant g(20) = g(4)g(5) = 2$ which is absurd. If $20 \in A$, then $3 = g(20) = g(4)g(5) = 2$ which is absurd. If $21 \in A$, then $1 = g(21) = g(3)g(7) = 0$ which is absurd. If $22 \in A$, then $1 = g(22) = g(2)g(11) = 0$ which is absurd. If $23 \in A$, then $2 \leqslant g(24) = g(3)g(8) = 0$ which is absurd. If $24 \in A$, then $1 = g(24) = g(3)g(8) = 0$ which is absurd. On the other hand if $a_5 \geqslant 26$, then $1 \geqslant g(26) = g(2)g(13) = 2$ which is impossible. It follows that $a_5 = 25$.

We will prove that the nth element of A is $a_n = n^2$ for every n.

Assume that the first five elements of A are $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16$ and $a_5 = 25$. It is clear that If $26 \in A$, then $3 = g(26) = g(2)g(13) = 2$ which is absurd. If $27 \in A$, then $2 \leqslant g(28) = g(4)g(7) = 0$ which is absurd. If $28 \in A$, then $1 = g(28) = g(4)g(7) = 0$ which is absurd. If $29 \in A$, then $2 \leqslant g(30) = g(3)g(10) = 0$ which is absurd. If $30 \in A$, then $1 = g(30) = g(3)g(10) = 0$ which is absurd. If $31 \in A$, then $2 \leqslant g(35) = g(5)g(7) = 0$ which is absurd.

Let us suppose that $a_1 = 1, a_2 = 4, \ldots, a_n = n^2$ for $n \geqslant 5$ and $a_{n+1} \geqslant 32$. We prove that $a_{n+1} = (n+1)^2$. We have two cases.

Case 1. $a_{n+1} < (n+1)^2$. We prove the following proposition.

Proposition 4. *If a_{n+1} and $a_{n+1} + 1$ are both prime power and a_{n+1} is not a square, $a_{n+1} \geqslant 32$, then $(n+1)^2 - a_{n+1} > 16$.*

Proof. It is clear that one of a_{n+1} and $a_{n+1} + 1$ is even, thus we have two possibilities. If $a_{n+1} = 2^\alpha$, so that α is odd, thus $3 | a_{n+1} + 1 = 2^\alpha + 1$. It follows that $2^\alpha + 1 = 3^\beta$. This implies that $\alpha = 3$ and $a_{n+1} = 8$ a contradiction. If $a_{n+1} + 1 = 2^\alpha$, but a_{n+1} is a power of a prime, thus $2^\alpha - 1 = p^\gamma$. It follows from [8] that a_{n+1} is a prime, then we have $a_{n+1} = M_p$, where $p \geqslant 7$. It is easy to see that $2^p + 1 \neq (n+1)^2$, $2^p + 4 \neq (n+1)^2$ and $2^p + 9 \neq (n+1)^2$, thus we have $2^p + 16 \leqslant (n+1)^2$, therefore $(n+1)^2 - a_{n+1} \geqslant 17$. □

In the next step we show that both a_{n+1} and $a_{n+1} + 1$ must be a power of a prime. Assume on the contrary that $a_{n+1} = u \cdot v$, where u and v are coprime positive integers and $u, v > 1$. Let $G(k)$ denote that values of $g(k)$ which correspond to the set of squares, i.e., when A is the set of squares define $G(k)$ by

$$G(k) = c_1 R_{A,1}(n) + c_2 R_{A,2}(n) + \ldots + c_k R_{A,k}(n).$$

Then we have

$$g(a_{n+1}) = g(u)g(v) = G(u)G(v) = G(a_{n+1}).$$

On the other hand

$$g(a_{n+1}) = 1 + |\{(i,j) : a_i + a_j = a_{n+1}, i, j \leqslant n\}| = 1 + |\{(i,j) : i^2 + j^2 = a_{n+1}\}| = 1 + G(a_{n+1}),$$

which is a contradiction. Assume that $a_{n+1} + 1 = u \cdot v$, where u and v are coprime positive integers and $u, v > 1$. Then we have

$$g(a_{n+1} + 1) = g(uv) = g(u)g(v) = G(u)G(v) = G(a_{n+1} + 1).$$

On the other hand

$$g(a_{n+1}+1) \geqslant 2 + |\{(i,j) : i^2+j^2 = a_{n+1}+1\}|,$$

and

$$G(a_{n+1}+1) \leqslant 1 + |\{(i,j) : i^2+j^2 = a_{n+1}+1, i,j \leqslant n\}|,$$

which is absurd. We have already shown that if both a_{n+1} and $a_{n+1}+1$ are a power of a prime and $a_n \geqslant 32$, then $a_{n+1} < (n+1)^2 - 16$. It is clear that there are exactly two even numbers among a_{n+1}, $a_{n+1}+1$, $a_{n+1}+4$, $a_{n+1}+9$ and neither of them are a power of 2, i.e., $a_{n+1}+l^2 = 2^\alpha v$, where $v > 1$ odd and $0 \leqslant l \leqslant 3$. Then we have

$$g(a_{n+1}+l^2) = g(2^\alpha v) = g(2^\alpha)g(v) = G(2^\alpha)G(v) = G(a_{n+1}+l^2).$$

On the other hand

$$g(a_{n+1}+l^2) \geqslant 1+|\{(i,j) : i^2+j^2 = a_{n+1}+l^2\}| > |\{(i,j) : i^2+j^2 = a_{n+1}+l^2\}| = G(a_{n+1}+l^2),$$

which is impossible. The proof of Case 1. is completed.

Case 2. $a_{n+1} > (n+1)^2$. It is clear that there are exactly one even numbers among $(n+1)^2$, $(n+1)^2+1$. We have two possibilities. If $(n+1)^2$ is even, but not a power of 2, then $(n+1)^2 = u \cdot v$, where u, $v > 1$ coprime positive integers. Then we have

$$g((n+1)^2) = g(uv) = g(u)g(v) = G(u)G(v) = G((n+1)^2).$$

On the other hand

$$G((n+1)^2) = 1+|\{(i,j) : i^2+j^2 = (n+1)^2, i,j > 0\}| > |\{(i,j) : i^2+j^2 = (n+1)^2, i,j > 0\}| = g((n+1)^2),$$

which is impossible. If $(n+1)^2+1$ is even, then $n+1$ is odd, thus $(n+1)^2$ congruent to 1 modulo 4, which implies that $(n+1)^2+1 = 2v$, where $2 \nmid v$.

$$g((n+1)^2+1) = g(2v) = g(2)g(v) = G(2)G(v) = G((n+1)^2+1).$$

On the other hand

$$G((n+1)^2+1) = |\{(i,j) : i^2+j^2 = (n+1)^2+1, i,j > 0\}| =$$
$$2 + |\{(i,j) : i^2+j^2 = (n+1)^2+1, 1 \leqslant i,j \leqslant n+1\}|$$
$$> 1 + |\{(i,j) : i^2+j^2 = (n+1)^2+1, 1 \leqslant i,j \leqslant n\}| \geqslant g((n+1)^2+1)$$
$$> 2 + |\{(i,j) : i^2+j^2 = (n+1)^2+1, 1 \leqslant i,j \leqslant n+1\}|$$
$$> 1 + |\{(i,j) : i^2+j^2 = (n+1)^2+1, 1 \leqslant i,j \leqslant n+1\}| \geqslant g((n+1)^2+1),$$

which is impossible.

If $(n+1)^2 = 2^{2m}$, then we have

$$G(2^{2m}+4) = G(4(1+2^{2m-2})) = G(4)G(1+2^{2m-2}) = g(4)g(1+2^{2m-2}) = g(2^{2m}+4).$$

$$G(2^{2m}+4) = 2 + |\{(i,j) : i^2+j^2 = 2^{2m}+4, 0 \leqslant i,j < 2^m\}| = g(2^{2m}+4),$$

therefore $2^{2m}+3 \in A$. Thus we have

$$g(2^{2m}+12) = g(4(3+2^{2m-2})) = g(4)g(3+2^{2m-2}) = G(4)G(3+2^{2m-2}) = G(2^{2m}+12).$$

On the other hand

$$G(2^{2m}+12) = |\{(i,j) : i^2+j^2 = 2^{2m}+12, 0 < i,j < 2^m\}|$$

but

$$g(2^{2m}+12) = 2 + |\{(i,j) : i^2+j^2 = 2^{2m}+12, 0 < i,j < 2^m\}|,$$

which is impossible. Thus the proof of Case 2. and that of Theorem 1. are completed.

3 Proof of Theorem 2.

It is easy to see that

$$p(g(z)) = p\Big(\frac{z}{1-z}\Big) = c_1\frac{z}{1-z} + c_2\Big(\frac{z}{1-z}\Big)^2 + \ldots + c_k\Big(\frac{z}{1-z}\Big)^k$$

$$= c_1\sum_{j=1}^{\infty}\binom{j-1}{0}z^j + c_2\sum_{j=2}^{\infty}\binom{j-1}{1}z^j + c_3\sum_{j=3}^{\infty}\binom{j-1}{2}z^j + \ldots + c_k\sum_{j=k}^{\infty}\binom{j-1}{k-1}z^j.$$

As $g(m) = \sum_{s=1}^{k} c_s\binom{m-1}{s-1}$ if $c_k < 0$, then $g(m) < 0$, for $m \geqslant m_0$, then for $p,q > m_0$, where $(p,q) = 1$, then we have $g(p) < 0$, $g(q) < 0$ which implies that $0 > g(pq) = g(p)g(q) > 0$ a contradiction. It follows that $c_k > 0$. This implies that there exists m_0 such that for $m > m_0$ we have $g(m) > 0$. In fact, for every $m \in \mathbb{Z}^+$, $g(m) > 0$, because for prime number $p > m_0$ and $p > m$ we have $g(kp) = g(p)g(k) > 0$, and $g(p) > 0$, which implies that $g(k) > 0$. It is clear that there exists $m \geqslant m_1$ such that $g(m)$ is monotonous increasing. As $g(m)$ is multiplicative, then $\log g(m)$ is additive and $\liminf(\log g(m+1) - \log g(m)) \geqslant 0$, then it follows form a well known theorem of Erdős [1] that $\log g(m) = c\log m$, where c is a positive constant. Thus we have $g(m) = m^{k-1}$. On the other hand it is well known (see [13] , p. 83.) that

$$(n-1)^{k-1} = \sum_{j=1}^{k}\binom{n-1}{j-1}(j-1)!\cdot S(k-1,j-1).$$

Thus we have

$$n^{k-1} = (n-1+1)^{k-1} = \sum_{l=0}^{k-1}\binom{k-1}{l}(n-1)^j$$

$$= \sum_{l=0}^{k-1}\binom{k-1}{j}\sum_{j=1}^{l+1}\binom{n-1}{j-1}(j-1)!\cdot S(k-1,j-1)$$

$$= \sum_{j=1}^{k}\binom{n-1}{j-1}(j-1)!\cdot S(k-1,j-1)\sum_{l=j-1}^{k-1}\binom{k-1}{l} = \sum_{j=1}^{k}\binom{n-1}{j-1}c_j,$$

where $c_j = (j-1)!\cdot S(k-1,j-1)\sum_{l=j-1}^{k-1}\binom{k-1}{l}$. The proof of Theorem 2. is completed.

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