Colorful flowers

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\textbf{Abstract}

For a set $A$ let $[A]^k$ denote the family of all $k$-element subsets of $A$. A function $f : [A]^k \rightarrow C$ is a local coloring if it maps disjoint sets of $A$ into different elements of $C$. A family $\mathcal{F} \subseteq [A]^k$ is called a flower if there exists $E \in [A]^{k-1}$ so that $|E \cap F| = E$ for all $F, F' \in \mathcal{F}$, $F \neq F'$. A flower is said to be colorful if $f(F) \neq f(F')$ for any two $F, F' \in \mathcal{F}$. In the paper we find the smallest cardinal $\gamma$ such that there exists a local coloring of $[A]^k$ containing no colorful flower of size $\gamma$. As a consequence we answer a question raised by Pelant, Holický and Kalenda. We also discuss a few results and conjectures concerning a generalization of this problem.

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1. Introduction

For a set $A$, let $[A]^k$ denote the family of all $k$-element subsets of $A$. We will also say that $[A]^k$ is a complete $k$-uniform hypergraph with vertex set $A$ and call the $k$-element subsets of $A$ the edges of $[A]^k$. A function $f : [A]^k \rightarrow C$ is a local coloring of $[A]^k$ if disjoint edges of $[A]^k$ are colored with different colors. In the paper we consider extremal properties of local colorings of $[A]^k$, where $\alpha$ is a cardinal, either finite or infinite. Let $\gamma$ be a cardinal. A $\gamma$-flower in $[A]^k$ is a family $\mathcal{F} \subseteq [A]^k, |\mathcal{F}| = \gamma$, such that there exists $E \in [A]^{k-1}$ so that $|E \cap F| = E$ for all $F, F' \in \mathcal{F}$, $F \neq F'$. The set $E$ will be called the eye of the flower $\mathcal{F}$ and each of the singleton set $F \setminus E, F \in \mathcal{F}$, will be called a petal of the flower $\mathcal{F}$. We say that a flower is colorful if $f(F) \neq f(F')$ for any two $F, F' \in \mathcal{F}$. Let $\alpha$ and $\gamma$ be two cardinals and $k$ be a natural number. We write $\varphi(k, \alpha) = \gamma$ if $\gamma$ is the smallest cardinal for which there exists a local coloring of $[A]^k$ with no colorful $\gamma$-flower. In particular, if $n$ is a natural number, we have $\varphi(k, n) = t + 1$, where $t$ is the maximum number such that each local coloring of $[n]^k$ yields a colorful $t$-flower.

2. Colorful flowers in finite and infinite hypergraphs

2.1. Finite hypergraphs

In this section we estimate $\varphi(k, n)$ for finite $n$ and $k$. Let us start with simple observations. If $k > \frac{n}{2}$ there are no two disjoint $k$-sets in $[n]^k$. Hence coloring all the edges of $[n]^k$ with the same color yields a local coloring of $[n]^k$ which does not
have a colorful flower of size more than 1. Thus $\varphi(k, n) \leq 2$. On the other hand, since $[n]^k$ is non-empty, any local coloring of $[n]^k$ must use at least one color so that $\varphi(k, n) \geq 2$. Consequently, when $k > \frac{n}{2}$, $\varphi(k, n) = 2$.

For $n \geq 2k$, first observe an easy upper bound similar to one on the chromatic number of Kneser graphs. A well-known result of Lovász [3] states that the chromatic number of the Kneser graph $KG(k, n)$, $n \geq 2k - 1$, is $n - 2k + 2$. While most of the known proofs of the lower bound rely on Ulam–Borsuk theorem, the upper bound for the chromatic number is straightforward. Indeed, it is enough to consider the coloring

$$\chi : [n]^k \rightarrow [n - 2k + 2],$$

$$v \mapsto \min(\min v, n - 2k + 2)$$

for every vertex $v$ of $KG(k, n)$. It is easy to see that $\chi$ is a proper coloring of the Kneser graph, i.e., it colors disjoint sets of $[n]^k$ with different colors. In other words, $\chi$ is a local coloring of $[n]^k$, and since it uses only $n - 2k + 2$ colors it contains no colorful $t$-flower for $t > n - 2k + 2$. Consequently when $n > 2k - 1$, $\varphi(k, n) \leq n - 2k + 3$. Note that $\chi$ does not give a better bound since, under this coloring, one can find a colorful $(n - 2k + 2)$-flower. The main result of this section states that $\varphi(k, n)$ grows with $n$ roughly as $n/k$.

**Theorem 1.** Let $k$ and $n$ be natural numbers. We have

$$\frac{n + 1}{k} \leq \varphi(k, n) \leq \frac{n}{k} + k.$$

For the proof of the lower bound of Theorem 1 we shall need a result from extremal set theory. Let $S$ be a family of $k$-element sets. We say that $S$ is an intersecting family of $k$-element sets if any two sets of $S$ share at least one element. The shadow of $S$, denoted as $\partial S$, is the family of $(k - 1)$-element sets which are subsets of sets from $S$, i.e.,

$$\partial S = \{ T \in [n]^{k-1} : T \subseteq S \text{ for some } S \in S \}.$$

The following fact is a special case of a more general theorem of Katona [2].

**Lemma 1.** Suppose that $S$ is an intersecting family of $k$-element sets. Then

$$|S| \leq |\partial S|.$$

**Proof of Theorem 1.** We start with the proof of lower bound for $\varphi(k, n)$. Let $f : [n]^k \rightarrow C$ be a local coloring. For such a coloring we need to show the existence of a colorful $t$-flower, where $t \geq \frac{n + 1}{k} - 1$. Let $C$ be an edge color class of $[n]^k$, i.e., $C = f^{-1}(c)$ for some $c \in C$. Since every two $k$-tuples in $C$ are intersecting, Lemma 1 gives $|C| \leq |\partial C|$ so that

$$\sum_{c} |C| \leq \sum_{c} |\partial C|,$$

where the sum is taken over all color classes. Given a $(k - 1)$-set $M$ of vertices of $[n]^k$, let the degree of $M$, $\deg(M)$, denote the number of all colors which appear among the $k$ sets containing $M$. Let us count the pairs $(M, c)$, where $M \in [n]^{k-1}$, and $c$ is a color such that one of the $k$-element sets $K$ containing $M$ is colored with $c$. Then, from (1), we get

$$\sum_{M} \deg(M) = \sum_{c} |\partial C| \geq \sum_{c} |C| = \binom{n}{k}.$$

Since the number of terms in the first sum is $\binom{n}{k-1}$, there is a $(k - 1)$-element set $M_0$ such that

$$\deg(M_0) \geq \binom{k}{k-1} - 1 = \frac{n - k + 1}{k},$$

i.e., $M_0$ is the eye of a colorful $t$-flower with $t \geq (n - k + 1)/k = (n + 1)/k - 1$. This completes the proof of the lower bound for $\varphi(k, n)$.

In order to show the upper bound for $\varphi(k, n)$ let us define an auxiliary function $I : [n]^k \rightarrow [k]$ setting for $x_0, x_1, \ldots, x_{k-1} \in [n]$

$$I(x_0, x_1, \ldots, x_{k-1}) = \sum_{i=0}^{k-1} x_i \pmod{k}.$$

Furthermore, for $x_0 < x_1 < \cdots < x_{k-1}$, let $f$ be defined by

$$f(x_0, x_1, \ldots, x_{k-1}) = \lambda((x_0, x_1, \ldots, x_{k-1})) = \lambda(v),$$

where $v = (x_0, x_1, \ldots, x_{k-1})$. 

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i.e., $f$ chooses from $\{x_0, x_2, \ldots, x_{k-1}\}$ that $x_j$ for which $j \equiv \sum_{i=0}^{k-1} x_i \pmod{k}$. Since the function $f : [n]^k \to [n]$ satisfies $f(K) \in K$ for every $K \in [n]^k$, it is a local coloring. Fix a $(k - 1)$-set $\{c_1, c_2, \ldots, c_{k-1}\}$, $c_1 < c_2 \cdots < c_{k-1}$, and consider a colorful flower $\mathcal{F}$ with eye $(c_1, c_2, \ldots, c_{k-1})$ and a maximal set of petals $(p_1, p_2, \ldots, p_\ell)$ such that for every $1 \leq i \leq \ell$,

$$f((c_1, c_2, \ldots, c_{k-1}, p_i)) = p_i.$$  \hfill (2)

We will show that $\ell \leq \lceil (n - k + 1)/k \rceil$ and, consequently, the largest colorful t-flower satisfies $t \leq \lceil (n - k + 1)/k \rceil + k - 1$. Define the following intervals: $L_1 = \{1, 2, \ldots, c_1 - 1\}$, $L_j = \{c_{j-1} + 1, c_{j-1} + 2, \ldots, c_j - 1\}$ for $2 \leq j \leq k - 1$ and $L_k = \{c_{k-1} + 1, c_{k-1} + 2, \ldots, n\}$. If $p \in L_j$ is a petal of the flower $\mathcal{F}$, then, by (2), we have

$$\sum_{i=0}^{k-1} c_i + p = j \pmod{k}.\tag{3}$$

For $1 \leq j \leq k$ let $L'_j$ be the interval $L_j$ shifted to the left by $j - 1$. Thus, for example, $L'_1 = L_1$ and $L'_2 = \{c_1, c_1 + 1, \ldots, c_2 - 2\}$. The number of solutions of the equation $\sum_{i=1}^{k-1} c_i + p = j \pmod{k}$ satisfying $p \in L_j$ is equal to the number of solutions of the equation $\sum_{i=1}^{k-1} c_i + p = 0 \pmod{k}$ satisfying $p \in L'_j$. Hence the number of petals of $\mathcal{F}$ is given by the number of solutions of the equation $\sum_{i=1}^{k-1} c_i + p = 0 \pmod{k}$ satisfying $p \in \bigcup_{j=1}^k L'_j = \{1, 2, \ldots, n - (k - 1)\}$. Consequently $|\mathcal{F}| \leq \lceil (n - k + 1)/k \rceil$, and hence the size of a largest colorful flower $\mathcal{F}_{\text{max}}$ satisfies $|\mathcal{F}_{\text{max}}| \leq |\mathcal{F}| + k - 1 \leq \lceil (n - k + 1)/k \rceil + k - 1 \leq n/k + k - 1$, concluding the proof. \hfill $\square$

2.2. Infinite hypergraphs

For an infinite cardinal $\alpha$ the value $\varphi(k, \alpha)$ can be much smaller than $\alpha$. We write $\gamma^+$ for the successor of the cardinal $\gamma$ and denote by $\gamma^{1+\gamma} = ((\gamma^+)^+ \cdots)^+$, the $k$th iterated successor of $\gamma$.

**Proposition 1.** Let $\alpha$ and $\beta$ be two cardinals, $k \geq 1$, and let $C$ be any set. If $\beta^{++(k-1)} \leq \alpha$, then any local coloring $f : [\alpha]^k \to C$ has a colorful $\beta$-flower.

**Proof.** The proof was given by Pelant and Rödl in [5]. For completeness we include it here. Let $\alpha$ and $\beta$ be two cardinals such that $\beta^{++(k-1)} \leq \alpha$, for a natural number $k$. We shall show by induction on $k$ that any local coloring $f : [\alpha]^k \to C$ has a colorful $\beta$-flower. Note that since $|[\alpha]^k| = \alpha$, we can assume without loss of generality that $C = \alpha$. If $k = 1$, then a local coloring colors the singletons, each with a different color, and thus the statement follows. Assuming the inequality holds for $k - 1 \geq 1$ we prove it for $k$. Let $f : [\alpha]^k \to C$ be a local coloring. Take a subset $X$ of $\alpha$ such that $\beta^{++(k-2)} \leq |X| < \alpha$, and let $Y = \alpha \setminus X$. For each $x \in [X]^{k-1}$ denote by $Y_x$ the set of all $y \in Y$ for which there exists $x' \in [X]^{k-1}$, $x' \cap x = \emptyset$, such that $f(x \cup \{y\}) = f(x' \cup \{y\})$. First observe that for each $x \in [X]^{k-1}$ the set $F_x = \{x \cup \{y\}, y \in Y_x\}$ is a colorful flower. Indeed, for each $x \in [X]^{k-1}$ and any $y, y' \in Y_x$ there exists $x' \in [X]^{k-1}$, $x' \cap x = \emptyset$ such that

$$f(x \cup \{y\}) = f(x' \cup \{y\}) \neq f(x \cup \{y'\}).$$

Thus, if for some $x \in [X]^{k-1}$ we have $|Y_x| > \beta$, we are done. Consequently, we can assume that $|Y_x| < \beta$ for every $x \in [X]^{k-1}$. Since $|Y| = \alpha$ while $\bigcup_{x \in [X]^{k-1}} Y_x < \alpha$, there exists $y_0 \in Y \setminus \bigcup_{x \in [X]^{k-1}} Y_x$. Then, for every disjoint $(k - 1)$-element subsets $x, x'$ of $X$, we have $f(x \cup \{y_0\}) \neq f(x' \cup \{y_0\})$, so the function $g : [X]^{k-1} \to X$ defined as $g(x) = f(x \cup \{y_0\})$ is a local coloring. By induction hypothesis, there exists a colorful $\beta$-flower $\mathcal{F}$ for $g$ which can be extended to be a colorful $\beta$-flower for $f$ by adding $y_0$ to the eye of $\mathcal{F}$, completing the proof. \hfill $\square$

The next result states that Proposition 1 is sharp.

**Proposition 2.** Let $\alpha$ and $\beta$ be two cardinals, $k \geq 1$, and let $C$ be any set. If $\alpha < \beta^{++(k-1)}$, then there exists a local coloring $f : [\alpha]^k \to C$ with no colorful $\beta$-flower.

**Proposition 2** follows from a generalization which will be stated and proved in the next section. Combining the last two propositions, we obtain:

**Theorem 2.** For every integer $k \geq 1$ and every infinite cardinal $\alpha$ we have $\varphi(k, \alpha) = \min(\beta; \beta^{++(k-1)} > \alpha)$.

3. Generalized colorful flowers in infinite and finite hypergraphs

The notion of a $\gamma$-flower can be generalized in a natural way. Let $\ell < k$ be natural numbers and let $\gamma$ be a cardinal. A $(k, \ell, \gamma)$-flower is a family $\mathcal{F} \subseteq [A]^k$, $|\mathcal{F}| = \gamma$, such that there exists $E \in [A]^{k-\ell}$ so that $E \subseteq F$ for all $F \in \mathcal{F}$. The set $E$ is
the eye of the flower $\mathcal{F}$ and each of the set $F \setminus E$, $F \in \mathcal{F}$, we call a petal of the flower $\mathcal{F}$. Note that for two petals $F \neq F'$, we may have $(F \setminus E) \cap (F' \setminus E) \neq \emptyset$. We say that a flower is colorful if $f(F) \neq f(F')$ for any two $F, F' \in \mathcal{F}$. As before for integers $k, \ell$, and cardinals $\gamma, \alpha$ we write $\varphi(k, \ell, \alpha) = \gamma$ if $\gamma$ is the smallest cardinal for which there exists a local coloring of $[\alpha]^k$ without $(k, \ell, \gamma)$-flower. In this section we will determine the value of $\varphi(k, \ell, \alpha)$ for any $k > \ell \geq 1$ and $\alpha$ any infinite cardinal. We will only provide partial results when $\alpha$ is a finite cardinal.

3.1. Infinite hypergraphs

Concerning infinite hypergraphs we can prove the following result generalizing Proposition 1.

**Proposition 3.** If $q \ell \leq k \leq (q + 1)\ell$ and $\alpha \geq \beta^{+q}$, then every local coloring of $[\alpha]^k$ contains a colorful $(k, \ell, \beta)$-flower.

**Proof.** Let $k, \ell, \alpha$ and $\beta$ be as in the statement and let $f : [\alpha]^k \rightarrow \alpha$ be a local coloring. Define $1 \leq s \leq \ell$ to be the integer such that $k = q\ell + s$. Let $\alpha = A \cup B$ be a partition of $\alpha$ into two sets $A$ and $B$ satisfying $|A| = \beta^q$ and $|B| = \beta^{q+1}$. We further partition $A = \bigcup_{k \in \beta^q} A_\theta$ into finite sets $A_\theta$, each of cardinality $s$, and $B = \bigcup_{k \in \beta^{q+1}} B_\theta$ into finite sets $B_\theta$, each of cardinality $\ell$. Define $A = \{A_\theta ; \theta \in \beta^q\}$ and $B = \{B_\theta ; \theta \in \beta^{q+1}\}$. For each $Q \in [B]^{\ell}$, denote by $A_Q$ the set of all $A_\theta \in A$ for which there exists $Q' \in [B]^q$, $Q \cap Q' = \emptyset$, such that $f(\bigcup Q \cup A_\theta) = f(\bigcup Q' \cup A_\theta)$. Observe that for each $Q \in [B]^{\ell}$, the set $\mathcal{F}_Q = \bigcup Q \cup A ; A \in A_Q$ is a colorful $(k, \ell, |\mathcal{F}_Q|)$-flower.

Indeed, for each $Q \in [B]^{\ell}$ and any $A, A' \in A_Q$ there exists $Q' \in [B]^{\ell}$, $Q \cap Q' = \emptyset$, such that

$$f(\bigcup Q \cup A) = f(\bigcup Q' \cup A') \neq f(\bigcup Q \cup A').$$

Thus, if for some $Q \in [B]^{\ell}$ we have $|A_Q| \geq \beta$, we are done. Consequently, we can assume that $|A_Q| < \beta$ for every $Q \in [B]^{\ell}$. Since $|A| = \beta^q$ and $|Q_\in \beta| = \beta^{q+1}$, the set $A \setminus \bigcup_{Q \in [B]} A_Q$ is non-empty. Fix $A_\beta \in A \setminus \bigcup_{Q \in [B]} A_Q$. Then, for every disjoint $q$-element subsets $Q, Q'$ of $B$, we have $f(\bigcup Q \cup A_\beta) \neq f(\bigcup Q' \cup A_\beta)$, so the function $g : [B]^{\ell} \rightarrow \beta$ defined as $g(Q) = f(\bigcup Q \cup A_\beta)$ is a local coloring. Since $|B| = \beta^{q+1}$ we can apply Proposition 1 of the paper to conclude that there exists a colorful $(q, 1, \beta)$-flower $\mathcal{F}$ for the coloring $g$. The flower $\mathcal{F} = (\bigcup F \cup A_\beta ; F \in \mathcal{F})$ is a colorful $(\ell q + s, \ell, \beta)$-flower for $f$, concluding the proof. $\square$

The next result shows that Proposition 3 is sharp. It also implies Proposition 2 stated earlier (with $\ell = 1$ and $q = k - 1$).

**Proposition 4.** If $k > q\ell$ and $\alpha \leq \beta^{q+1}$ then there is a local coloring of $[\alpha]^k$ with no colorful $(k, \ell, \beta)$-flowers.

We start the proof with two claims.

**Claim 1.** If $|V| \leq \beta^{q+1}$ then there is a function $f : [V]^q \rightarrow [V]^\beta$ such that for every finite $X \subseteq V$, $|X| > q$ there is a $P \in [X]^q$ with $X \setminus P \subseteq f(P)$.

**Proof.** We prove this by induction on $q$. If $q = 1$, enumerate $V$ as $V = \{v_\gamma ; \gamma < \beta\}$, and set $f(v_\gamma) = \{v_\theta ; \theta < \gamma\}$. If now $X \subseteq V$ is finite, $X = \{v_{\gamma_1}, \ldots, v_{\gamma_n}\}$ with $\gamma_1 < \cdots < \gamma_n$ then $X \setminus \{v_{\gamma_n}\} \subseteq f(v_{\gamma_n})$.

Assume now that we have the statement for $q$ and we want to prove it for $q + 1$. Assume that $|V| = \beta^{q+1}$. Enumerate $V$ as $V = \{v_\gamma ; \gamma < \beta^{q+1}\}$. Let $f_\beta$ be a function as in the statement for $\{v_\theta ; \theta < \gamma\}$. If $\gamma_1 < \cdots < \gamma_{q+1} < \beta^{q+1}$ then set

$$f((v_{\gamma_1}, \ldots, v_{\gamma_{q+1}})) = f_\beta((v_{\gamma_1}, \ldots, v_{\gamma_q})).$$

We show that this function is as required. Assume that $X \subseteq V$ is finite, $|X| \geq q + 2$. Let $v_{\gamma} \in X$ be the element with largest index. By the inductive assumption there is a set $Q \subseteq X \setminus \{v_{\gamma}\}$, $|Q| = q$, such that $X \setminus Q \setminus \{v_{\gamma}\} \subseteq f_\beta(Q)$. Then, $P = Q \cup \{v_{\gamma}\}$ has $X \setminus P \subseteq f(P)$.

**Claim 2.** Let $f$ be the function in Claim 1. There is a function $F : [\alpha]^k \rightarrow \alpha$ with $F(A) \in A$ such that the following holds. If $s \subseteq A - \{F(A)\}$, $|s| = k - \ell$, then there is a $t \subseteq s$, $|t| = q$ with $F(A) \in f(t)$.

**Proof.** Given $A \in [\alpha]^k$ we determine $F(A)$ as follows. Choose by induction the sets $B_1, \ldots, B_{\ell}$, $|B_i| = q$, as follows. If $B_1, \ldots, B_i$ are already selected, let $B_{i+1} \subseteq A - (B_1 \cup \cdots \cup B_i)$ be such that $A - (B_1 \cup \cdots \cup B_{i+1}) \subseteq f(B_{i+1})$ holds. This is possible by Claim 1. Finally let $F(A)$ be an arbitrary element of $A - (B_1 \cup \cdots \cup B_\ell)$ (a non-empty set, as $k > q\ell$).

We have to show that $F(A)$ satisfies the claim. If $s \subseteq A - \{F(A)\}$, $|s| = k - \ell$, then there is some $i$ with $B_i \subseteq s$. Now $t = B_i$ is as required.$\square$

**Proof of Proposition 4.** The function in Claim 2 gives a coloring with no colorful $(k, \ell, \beta)$-flower. Indeed, assume that $|s| = k - \ell$, and $s \subseteq A_{\gamma}$ with $F(A_{\gamma})$ distinct $\gamma < \beta$. By Claim 2, $F(A_{\gamma}) \in \bigcup f(t) ; \gamma \in [s]^q$ for every $\gamma < \beta$, and the right-hand side is the union of finitely many sets of cardinality less than $\beta$, so itself is of cardinality less than $\beta$. $\square$
Recall that for integers $k$, $\ell$, and cardinals $\gamma$, $\alpha$ we write $\psi(k, \ell, \alpha) = \gamma$ if $\gamma$ is the smallest cardinal for which there exists a local coloring of $[\alpha]^k$ without a $(k, \ell, \gamma)$-flower. Hence under appropriate assumptions on $q$, Proposition 3 states that $\psi(k, \ell, \alpha) \geq \gamma^{+q}$ for any $\alpha \geq \gamma^{+q}$ while Proposition 4 yields $\psi(k, \ell, \alpha) \leq \beta$ for any $\alpha \leq \beta^{+(q-1)}$. This can be summarized as:

**Theorem 3.** For every integers $k > \ell \geq 1$ and every infinite cardinal $\alpha$ we have $\psi(k, \ell, \alpha) = \min\{\beta; \beta^{+q} > \alpha\}$, where $q = \lceil \frac{k-1}{\ell} \rceil$.

**Proof.** Let $k > \ell \geq 1$ be integers, $q = \lceil \frac{k-1}{\ell} \rceil$ and $\alpha$ any infinite cardinal. Notice that $\lceil \frac{k-1}{\ell} \rceil < \frac{k}{\ell} \leq \lceil \frac{k-1}{\ell} \rceil + 1$ so that with our definition of $q$ we have $q(\ell+1) < k \leq (q+1)\ell$, as needed to apply Propositions 3 and 4.

Let $\beta$ be a cardinal such that $\beta^{+q} > \alpha$, or equivalently $\beta^{+(q-1)} > \alpha$. Proposition 4 implies $\psi(k, \ell, \alpha) - \beta$. Since $\psi(k, \ell, \alpha) \leq \beta$ for every such $\beta$ we obtain $\psi(k, \ell, \alpha) = \min\{\beta; \beta^{+q} > \alpha\}$.

Now let $\gamma$ be smaller than $\min\{\beta; \beta^{+q} > \alpha\}$. Then we have $\gamma^{+q} < \alpha$ and by Proposition 3 this implies $\psi(k, \ell, \alpha) > \gamma$, concluding the proof. \(\square\)

3.2. Finite hypergraphs

We now turn to finite hypergraphs. As we have already mentioned, the fact that the chromatic number of the Kneser graph $KG(n, k)$ is $n - 2k + 2$ implies that for any set $C$ of cardinality $n - 2k + 2$, there exists a local coloring $f : [n]^k \to C$. Such a map $f$ has no $(k, \ell, \gamma)$-flower for $t > n - 2k + 2$. Thus, for each finite $n$ and every $k$ and $\ell$, $1 \leq \ell < k \leq n$, we have $\psi(k, \ell, n) \leq n - 2k + 3$. In the first section we improved this crude upper bound in the special case $\ell = 1$, proving $\psi(k, 1, n) \leq n/k + k$. We also showed that $\psi(k, 1, n) \geq (n+1)/k$ and so, if $n$ tends to infinity, we have $\psi(k, 1, n) = n/k + O(1)$. Here we concentrate on the ‘opposite’ case, when $\ell = k - 1$. It turns out that the asymptotic behavior of $\psi(k, k - 1, n)$ is strongly related to the well-known Caccetta–Häggkvist Conjecture so let us first recall some basic facts concerning oriented cycles in digraphs. A digraph is simple if it contains no double arcs, i.e., if for every pair of vertices $v, w$, the digraph contains at most one of the arcs $(v, w)$ and $(w, v)$. Let $dCH(n, k)$ denote the smallest natural number $j$ such that each simple digraph on $n$ vertices with the minimum out-degree at least $j$ contains a directed cycle of length at most $k$.

**Claim 1.** $dCH(n, k) \geq \lceil n/k \rceil$.

**Proof.** Let $m = \lceil n/k \rceil - 1 < n/k$. Consider a digraph $D$ with the vertex set $\{0, 1, \ldots, n-1\}$ and the edges $(i, i + j \pmod n)$, $i = 0, 1, \ldots, n - 1$, $j = 1, 2, \ldots, m$. Clearly, $D$ is $m$ out-regular digraph without directed cycles shorter than $k + 1$. \(\square\)

**Caccetta–Häggkvist Conjecture.** $dCH(n, n) = \lceil n/k \rceil$ for $n \geq k \geq 3$.

Before we proceed further, let us make the following simple observation on dense families of subsets of $[n]$.

**Claim 2.** For every integers $r, s, r \geq s \geq 1$, and a constant $a > 0$, there exists a constant $b = b(r, s, a) > 0$, such that the following holds. For every family $\mathcal{F} \subseteq [n]^s$ with at least $an^s$ elements, at least $bn^r$-sets from $[n]^s$ can be represented as a union of sets from $\mathcal{F}$.

**Proof.** We use induction on $s$. For $s = 1$ the assertion clearly holds. Let us assume that $s \geq 2$ and that the assertion is true for $s - 1$. Let $\mathcal{F}_s = \{F \setminus \{x\}; x \in F \in \mathcal{F}\}$, and $S = \{x; |\mathcal{F}_s| \geq \frac{san^{s-1}}{2}\}$. It is clear that $\bigcup_{x \in S} |\mathcal{F}_x| \leq \binom{n}{s-1}$ and that $\bigcup_{x \in S} |\mathcal{F}_x| \leq \frac{san^{s-1}}{2}n$. We obtain

$$\sum_{x \in [n]} |\mathcal{F}_x| = san^s \leq |S| \binom{n}{s-1} + \frac{san^{s-1}}{2}n$$

and thus

$$\frac{san^s}{2} \leq |S| \binom{n}{s-1}.$$ 

Hence $\frac{san^s}{2} \leq |S|$. The inducational hypothesis implies that for a given $x \in S$ at least $b(r - 1, s - 1 - \frac{s^2a}{2})n^{r-1}$ $(r - 1)$-sets from $[n] \setminus \{x\}$ are unions of sets from $\mathcal{F}_x$. Thus at least

$$b(r - 1, s - 1 - \frac{s^2a}{2})n^{r-1} \frac{|S|}{r} \geq b(r - 1, s - 1 - \frac{s^2a}{2}) \frac{s^2a}{2r}n^r$$

$r$-sets from $[n]$ are unions of sets from $\mathcal{F}$ and the assertion follows. \(\square\)

Our argument will rely on the following lemma which implies that a digraph on $n$ vertices with out-degree slightly larger than $dCH(k, n)$ contains a lot of short cycles.
Lemma 2. For every $k \geq 3$, $\eta \geq 1/k$, and $\epsilon > 0$, there exist a constant $b > 0$ and natural numbers $m_0$, $n_0$, such that the following holds. If $d_{CH}(k, m_0) \leq \eta m_0$, then in every digraph $D$ on $n$ vertices, $n \geq n_0$, with out-degree larger than $(\eta + \epsilon)n$, there are at least $bn^k$ different $k$-sets $K$, such that in a subgraph induced in $D$ by $K$ each vertex belongs to an oriented cycle.

Proof. We prove Lemma 2 using an elementary probabilistic argument. Let us fix $k \geq 3$, $\eta \geq 1/k$, and $\epsilon > 0$. Assume that $d_{CH}(k, m_0) \leq \eta m_0$, for some $m_0 = m_0(k, \eta, \epsilon)$ large enough (if $\epsilon < 0.01$, then $m_0 = 40k^{-2} \ln(k)\epsilon$ will do). Now take $n$ much larger than $m_0$ and consider a digraph $D$ with vertex set $n$ and minimum out-degree at least $\lceil (\eta + \epsilon)n \rceil$. Choose uniformly at random a subset $W$ of $m_0$ vertices of $D$. Let $X$ denote the number of $k$-sets $K$, contained in $W$, such that the subgraph induced in $D$ by $K$ contains an oriented cycle. We first give a lower bound on $EX$, the expectation of $X$. The number of out-neighbors of a vertex $x \in K$ in $K$ is bounded from below by a random variable $Y$ with a hypergeometric distribution with parameters $\lceil (\eta + \epsilon)n \rceil$, $m_0 - 1$, $n - 1$. Since in our case $m_0$ and $n$ are large enough, a well-known estimate for the tail of the hypergeometric distribution (cf. [1, Theorem 2.10]) shows that the probability that $Y < \eta m_0$ is smaller than $1/(2m_0)$. Thus, the probability that there is a vertex in $K$ which has fewer than $\eta m_0$ out-neighbors in $K$ is smaller than $1/2$. Consequently, with probability at least $1/2$ all vertices of $K$ have at least $\eta m_0$ out-neighbors in $K$, and, since $d_{CH}(k, m_0) \leq \eta m_0$, we have $EX \geq 1/2$. Now let $M$ denote the total number of $k$-sets $K$, contained in $D$, such that the subgraph induced in $D$ by $K$ contains an oriented cycle. The probability that such a $k$-set is contained in $W$ is, clearly,

$$\frac{(\frac{n-k}{m_0})}{(\frac{n}{m_0})} \leq \frac{m_0 - k}{n - k} \leq \frac{2m_0^k}{n^k}.$$ 

Hence,

$$\frac{1}{2} \leq EX \leq M\frac{2m_0^k}{n^k}$$

and so $M \geq (4m_0)^{-k}n^k$. It means however that for some $i$, $3 \leq i \leq k$, the number of oriented cycles of length $i$ in $D$ is at least $(4m_0)^{-k}/k)n_i!$. Now the assertion follows from Claim 2 with $r = k$, $s = i$, and $a = (4m_0)^{-k}/k$. □

Let us note an immediate consequence of Lemma 2.

Corollary. For every $k \geq 3$, the limit

$$\eta(k) = \lim_{n \to \infty} \frac{d_{CH}(k, n)}{n}$$

exists.

The main result of this section gives a simple connection between the asymptotic value $\psi(k, k - 1, n)$ and $\eta(k)$.

Theorem 4. For every $k \geq 3$,

$$\psi(k, k - 1, n) = (1 - \eta(k))n + o(n).$$

In particular, if the Caccetta–Häggkvist Conjecture holds, we have

$$\psi(k, k - 1, n) = n - n/k + o(n).$$

Proof. We first estimate $\psi(k, k - 1, n)$ from above. To this end, let $D$ be a simple digraph with vertex set $[n]$ and minimum out-degree $d_{CH}(k, n) - 1$, which contains no oriented cycle shorter than $k + 1$. Then, in every $k$-set $\{x_1, \ldots, x_k\}$, there exists a vertex which has no in-neighbor in $\{x_1, \ldots, x_k\}$. We call each such vertex a leader. Let us define a coloring $\chi : [n]^k \to [k]$ setting $\chi([x_1, \ldots, x_k]) = y$, where $y \in \{x_1, \ldots, x_k\}$ is the smallest leader of the set of integer $\{x_1, \ldots, x_k\}$. Clearly, $\chi$ is a local coloring. Note that if $z \in [n]$, then no $k$-set containing $z$ is colored by one of the out-neighbors of $z$. Thus, since $D$ is simple, $\chi$ leads to no colorful $(k, k - 1, m)$-flower with

$$m \geq n - 1 - (d_{CH}(k, n) - 1) + 1 = n - d_{CH}(k, n) + 1,$$

so $\psi(k, k - 1, n) \leq n - d_{CH}(k, n) + 1$. Now let us show the lower bound for $\psi(k, k - 1, n)$. Thus, we have to show that for every $\epsilon > 0$ and $n$ large enough, every local coloring $f : [n]^k \to C$ leads to a colorful $(k, k - 1, m)$-flower for some $m \geq n - \eta(k)n - \epsilon n$. Note that in $f$ we have two types of color classes. If, for some $c \in C$, $C = f^{-1}(c)$ is such that $\cap C$ is non-empty, the family $C$ is a flower with eye $\cap C$. On the other hand, we may have color classes $D_i$ which are not flowers. Let $[n]^k = F_1 \cup F_2 \cup \cdots \cup F_t \cup D_1 \cup D_2 \cup \cdots \cup D_s$ be the partition of $[n]^k$ into color classes and where $F_i$ is a flower for $1 \leq i \leq t$, and $\cap D_i = \emptyset$ for $1 \leq i \leq s$. For $1 \leq i \leq s$, define $d_i = |D_i|$ as the number of $k$-sets in $D_i$, and let $v_i = |\cup_{D \in D_i} D|$ be the number of vertices covered by the color class $D_i$. □
Claim 3. There exists a constant $c_k$ depending only on $k$ such that $d_i \leq c_k v_i^{k-2}$.

Proof. Fix a color class $D_1$ and let $D_0 \in \mathcal{D}_1$. Since $\bigcap \mathcal{D}_i = \emptyset$, for every $x \in D_0$ there is a $k$-set $D_x \in \mathcal{D}_i$ with $x \not\in D_x$. Define $K = D_0 \cup \bigcup_{x \in D_0} D_x$. By construction we have $|K| \leq k(k-1)$. Note that each $D \in \mathcal{D}_1$ shares with $K$ at least two elements. Indeed, since $\mathcal{D}_1$ is intersecting, $D \cap D_0 \neq \emptyset$. If $|D \cap D_0| \geq 2$, we are done, so let us assume that $D \cap D_0 = \{y\}$. But then

$$D \cap D_y = D \cap (D_y \setminus \{y\}) \neq \emptyset,$$

so $|D \cap K| \geq 2$. Thus, $|\mathcal{D}_1| \leq \binom{k(k-1)}{2} (v_y^2)$, concluding the proof of Claim 3. □

Let us set $\mathcal{D}_1 = \bigcup \mathcal{D}_i$. The remaining part of the proof of Theorem 4 we split into two cases, according to the size of $\mathcal{D}$.

Case 1. $|\mathcal{D}| \geq n^{k-1/2}$.

From Claim 3 we have

$$\frac{1}{c_k} d_i^{1/(k-2)} \leq v_i \leq n,$$

so

$$\sum_{i=1}^{s} v_i \geq \frac{1}{c_k} \sum_{i=1}^{s} d_i^{1/(k-2)}. \quad (6)$$

The function $f(x) = x^{1/(k-2)}$ is concave. Thus, the sum $\sum_{i=1}^{s} d_i^{1/(k-2)}$ is minimized when as many $d_i$ are as large as possible. This means that

$$\frac{\sum_{i=1}^{s} d_i}{\max_{1 \leq i \leq s} d_i} \left( \max_{1 \leq i \leq s} d_i \right)^{1/(k-2)} \leq \sum_{i=1}^{s} d_i^{1/(k-2)}. \quad (7)$$

By assumption we have $\sum_{i=1}^{s} d_i = |\mathcal{D}| \geq n^{k-1/2}$ and Claim 3 gives $d_i \leq c_k v_i^{k-2}$. Thus,

$$\frac{n^{k-1/2}}{c_k v_i^{k-2}} \left( c_k v_i^{k-2} \right)^{1/(k-2)} \leq \sum_{i=1}^{s} d_i^{1/(k-2)},$$

and after simplification

$$\frac{n^{k-1/2}}{c_k v_i^{k-2}} \leq \sum_{i=1}^{s} d_i^{1/(k-2)}.$$

Using the fact that $v_i \leq n$ the last inequality becomes

$$n^2 c_k^{1/(k-2)} \leq \sum_{i=1}^{s} d_i^{1/(k-2)}. \quad (8)$$

From (6) and (8) we infer that

$$\sum_{i=1}^{s} v_i \geq c'_k n^{5/2}$$

for $c'_k = c_k^{1/(k-2) - 2}$. Consequently, there is a vertex $x \in [n]$ which is counted at least $c'_k n^{3/2}$ times in the sum on the left-hand side of the above equation, i.e., $x$ belongs to at least $c'_k n^{3/2}$ color classes $\mathcal{D}_i$. The observation that for large enough $n$ we have $c'_k n^{3/2} \geq n - \eta(k)n$, completes the proof of Case 1.

Case 2. $|\mathcal{D}| \leq n^{k-1/2}$.

We build an oriented graph $\vec{G}_T$ on the vertex set $V = [n]$ as follows. For every $1 \leq i \leq t$ we choose a vertex $y_i$ in the eye of the flower $F_i$. Let $\{x_1, x_2, \ldots, x_k\}$ be a $k$-set in $\mathcal{F}_i$. For some index $j_0$, $1 \leq j_0 \leq k$, we have $y_i = x_{j_0}$. We call the vertex $x_{j_0}$ a leader of $\{x_1, x_2, \ldots, x_k\}$ and place in $\vec{G}_T$ all the directed edges of the form $(x_{j_0}, x_j)$ for $1 \leq j \leq k$, $j \neq j_0$. It is easy to see that if some vertex $z \in [n]$ has in-degree $m$, then $\vec{G}_T$ contains a $(k, k - 1, m)$-flower. Thus, let us suppose that the minimum
in-degree of $\tilde{G}_f$ is smaller than $n - \bar{\eta}(k)n - \epsilon n$. We shall show that this assumption leads to a contradiction. We call a pair $(u, v)$ of vertices of $\tilde{G}_f$ an empty pair if neither $(u, v)$, nor $(v, u)$ is an arc of $\tilde{G}_f$. Our first goal is to give an upper bound for the number of empty pairs. Let $(u, v)$ be an empty pair. Denote by $W_{[u,v]}$ the set of vertices $w$ that are never a leader of a $k$-set containing both the vertices $u$ and $v$. Note that if there exists an empty pair $(u, v)$ such that $|W_{[u,v]}| \leq \bar{\eta}(k)n$ then both vertices $u$ and $v$ have in-degree at least $n - \bar{\eta}(k)n$ and we are done. Thus, we may and shall assume that for every empty edge $(u, v)$, we have $|W_{[u,v]}| \geq \bar{\eta}(k)n$. But then the number $M$ of $k$-sets without a leader is bounded from below by

$$M \geq \sum \left( \frac{\bar{\eta}(k)n}{k-2} \right) \binom{k}{2},$$

where the sum is over all empty pairs $(u, v)$. On the other hand,

$$M \leq |D| \leq n^{k-1/2},$$

and consequently the number of empty edges can be bounded by

$$\frac{(k)n^{k-1/2}}{(k-2)} \leq cn^2,$$

for some constant $c$ depending only on $k$. Consider the graph $G$ on the vertex set $[n]$ with an edge between any pair of vertices that is an empty pair for $\tilde{G}_f$. By (11), the graph $G$ has at most $cn^{3/2}$ edges. Hence $G$ has at most $2cn^{3/4}$ vertices of degree larger than $n^{3/4}$ and the subgraph $G'$ of $G$ induced by vertices of $G$ of degree at most $n^{3/4}$ has $m \geq n - 2cn^{3/4}$ vertices. Now let us recall that the minimum in-degree of $\tilde{G}_f$ is smaller than $n - \bar{\eta}(k)n - \epsilon n$. Let $\tilde{G}_f'$ be the oriented subgraph of $\tilde{G}_f$ induced on the vertex set $V(G')$. Then, $\tilde{G}_f'$ has $m \geq n - 2cn^{3/4}$ vertices and the minimum out-degree at least $n/k + \epsilon n - n^{3/4} \geq m/k + \epsilon m/2$. Thus by Lemma 2, $\tilde{G}_f'$ contains at least $\eta^k n^{k-1/2} \geq |D|$ $k$-sets $R$ which are unions of short oriented cycles and consequently have no leaders. However, by the definition of $f$, each $k$-set which does not belong to $D$ must have a leader. Consequently, the assumption that $\varphi(k, k-1, n) \leq n - \bar{\eta}(k)n - \epsilon n$ leads to a contradiction, and the assertion follows.

4. Applications and concluding remarks

In [4], Pelant, Holický and Kalenda, inspired by a certain topological problem, asked the following question.

**Question 1.** Let $\alpha$ be an uncountable cardinal, $k \in \mathbb{N}$ and $f : [\alpha]^k \rightarrow \alpha$ a mapping. Must one of the following conditions hold?

1. There are pairwise disjoint sets $K_n \in [\alpha]^k$ such that $f$ is constant on $\{K_n \mid n \in \mathbb{N}\}$.
2. There is $U \in [\alpha]^{k-1}$ such that $\{f(U \cup \{x\}) \mid x \in \alpha \setminus U\}$ is infinite.

Note that in our notation property 2 can be rephrased as “$f$ has a colorful $(k, 1, \omega_0)$-flower.” It is not hard to see that Proposition 2 can be used to answer Question 1 in the negative. Indeed, let $\alpha = \omega_1$ and $k = 3$. Then $\omega_1 < \omega_2 = \omega_0^{\omega_1-(k-1)}$, so that Proposition 2 guarantees the existence of a local coloring $f : [\omega_1]^3 \rightarrow \omega_1$ with no colorful $\omega_0$-flower. Thus, $f$ does not satisfy property 2, and since it is a local coloring, $f$ does not satisfy property 1 either. However, Theorem 5 below shows that a modified (and generalized) problem of Pelant, Holický and Kalenda has a positive answer. First we state and prove a lemma dealing with the case $k = 2$.

**Lemma 3.** Let $\beta$ be an infinite cardinal. Any map $f : [\beta^+]^2 \rightarrow \beta^+$ satisfies one of the following.

1. There are pairwise disjoint sets $K_0 \in [\beta^+]^2$, $\theta < \beta^+$, such that $f$ is constant on $\{K_0 \mid \theta \in \beta^+\}$.
2. $f$ has a colorful $(2, 1, \beta^r)$-flower.

**Proof.** Let us assume that the assertion does not hold, i.e., that there exists a map $f : [\beta^+]^2 \rightarrow \beta^+$ which satisfies neither property 1 nor property 2. We will use the fact that property 1 fails to construct a subset $X = \bigcup_{\theta < \beta^r} X_\theta \subset \beta^+$ with the following property that for any two pairs $p, p' \in [X]^2$, $f(p) = f(p')$ implies $\min p = \min p'$. More precisely we will use transfinite induction to construct two sequences $(X_\gamma)_{\gamma < \beta^r}$, $(Y_\gamma)_{\gamma < \beta^r}$ of subsets of $\beta^+$, such that for every $\gamma < \beta^+$ the following holds:

(i) $X_\gamma = \{x_\theta : \theta < \gamma\}$, where $x_\theta = \min Y_\theta$;
(ii) the sequence $(Y_\gamma)_{\gamma < \beta^r}$ is decreasing, i.e., $Y_{\gamma'} \subset Y_\gamma \subset \beta^+$ for $\gamma < \gamma' < \beta^+$;
(iii) $X_\gamma \cap Y_\gamma = \emptyset$;
(iv) \(|\beta^+ \setminus Y_{\gamma} < \beta^+|\);  
(v) if for two pairs \(p, p' \in [X_{\gamma} \cup Y_{\gamma}]^2\) we have \(f(p) = f(p')\), then either \(\min p = \min p' \in X_{\gamma}\), or \(p \cup p' \subseteq Y_{\gamma}\).

Let us set \(x_0 = 0, X_0 = \emptyset, Y_0 = \beta^+\). Let \(C_0\) be the set of all colors appearing among the edges of the form \([x_0, \theta], \theta < \beta^+\). We have assumed that there are no colorful \((2,1,\beta)\)-flowers, so \(|C_0| < \beta^+\). For each \(c \in C_0\) set
\[
M_c = \{\gamma < \beta^+: \exists \gamma' > \gamma \, f((\gamma, \gamma')) = c\}.
\]
Observe that \(|M_c| < \beta^+\) for otherwise there would be \(\beta^+\) disjoint pairs colored by \(c\). Set \(Y_1 = \beta^+ \setminus \bigcup_{c \in C_0} M_c, x_1 = \min Y_1,\) and \(X_1 = \{x_0\}\. Since \(|C_0| < \beta^+\) and \(|M_c| < \beta^+\) for every \(c \in C_0\), we have \(|\beta^+ \setminus Y_1| < \beta^+.\) Observe that \(X_1 \cap Y_1 = \emptyset,\) and, because of our construction, \((\nu)\) holds for \(\nu = 1\). Now, let \(\gamma < \beta^+\) and assume that for all \(\theta < \gamma\) we have already constructed sets \(X_{\theta}, Y_{\theta}\) for which properties (i)-(v) hold. In order to find \(X_{\gamma}\) and \(Y_{\gamma}\) we distinguish two cases, when \(\gamma\) is a successor ordinal and when \(\gamma\) is a limit ordinal. Let \(\gamma\) be a successor ordinal, and let \(\gamma^-\) denote its predecessor (i.e., \((\gamma^-)^+ = \gamma\)). Let \(C_{\gamma^-}\) be the set of all colors which occur among the edges of the form \([x_{\gamma^-}, \theta], x_{\gamma^-} < \theta < \beta^+.\) Since we have assumed that there are no colorful \((2,1,\beta)\)-flowers, \(|C_{\gamma^-}| < \beta^+.\) Set \(X_{\gamma} = X_{\gamma^-} \cup [x_{\gamma^-}]\) and \(Y_{\gamma} = Y_{\gamma^-} \setminus \bigcup_{c \in C_{\gamma^-}} M_c.\) Since \(x_{\gamma^-} \notin Y_{\gamma}\), and by the induction hypothesis \(X_{\gamma^-} \cap Y_{\gamma^-} = \emptyset,\) we also have \(X_{\gamma} \cap Y_{\gamma} = \emptyset,\) i.e., (iii) holds. Similarly as before, since \(|M_c| < \beta^+\) for every color \(c\), we infer that \(|Y_{\gamma^-} \setminus Y_{\gamma}| < \beta^+\) and consequently also \(|\beta^+ \setminus Y_{\gamma}| < \beta^+.\) We now verify (v). Since it holds for \(X_{\theta}\) and \(Y_{\theta}\) for every \(\theta < \gamma,\) it is enough to check it for two pairs \(p, p' \in [X_{\gamma} \cup Y_{\gamma}]^2\) with \(x_{\gamma^-} = \min p \leq \min p'.\) Then \(p = (x_{\gamma^-}, \theta)\) and assume that \(p' \in [Y_{\gamma}]^2\) (i.e., \(\min p' > \min p\)). Then, \(\min p' \notin \bigcup_{c \in C_{\gamma^-}} M_c\) and hence \(f(p) \neq f(p')\) establishing (v). Now let \(\gamma\) be a limit ordinal. We set \(Y_{\gamma} = \bigcap_{\theta < \gamma} X_{\theta}\). Since for every \(\theta < \gamma\)
\[
\emptyset = X_0 \cap Y_0 \supseteq X_0 \cap Y_{\gamma},
\]
we have
\[
X_{\gamma} \cap Y_{\gamma} = \bigcup_{\theta < \gamma} (X_0 \cap Y_{\gamma}) = \emptyset.
\]
Moreover, for every \(\theta < \beta^+\) we have \(|\beta^+ \setminus Y_0| < \beta^+\), and so \(|\beta^+ \setminus Y_{\gamma}| < \beta^+.\) Finally, for every \(\theta < \gamma,\) and each \(c \in C_0\) the set of all pairs colored \(c\) have the form \([x_{\theta}, \lambda]\) for \(\lambda > x_{\theta}\), so (v) holds. Now, having constructed the sequences \((X_{\gamma})_{\gamma < \beta^+}\) \((Y_{\gamma})_{\gamma < \beta^+}\) with properties (i)-(v), let us define \(X = \bigcup_{\gamma < \beta^+} X_{\gamma}\). Then, for all \(p, p' \in [X]^2\) for which \(f(p) = f(p')\), we have \(\min p = \min p'.\) Since clearly \(|X| = \beta^+,\) there exists an element \(z_0 \in X\) such that \(|\{x \in X: x < z_0\}| = \beta.\) The family \(F = \{x, z_0\}: x \in X\) and \(x < z_0\) is then a colorful \((2,1,\beta)\)-flower, contradicting the assumption on \(f\) and thus concluding the proof of Theorem 3. \(\square\)

More in general, the following is true.

**Theorem 5.** Let \(k \geq 2\) be an integer, and let \(\alpha\) and \(\beta\) be two cardinals such that \(\alpha \geq \beta^{+(k-1)}\). Then any map \(f: [\alpha]^k \rightarrow \alpha\) satisfies one of the following.

1. There are pairwise disjoint sets \(K_{\theta} \in [\alpha]^k, \theta < \beta^{+(k-1)}\), such that \(f\) is constant on \(K_{\theta}: \theta < \beta^{+(k-1)}\).

2. \(f\) has a colorful \((k, 1, \beta)\)-flower.

**Proof.** By induction on \(k.\) The case \(k = 2\) is handled in Lemma 3. Assume we have the statement for \(k\) and we want to prove it for \(k + 1.\) Set \(\alpha = \beta^{+(k-1)}\) and assume that we are given a coloring \(F: [\alpha]^k+1 \rightarrow \alpha\) with no colorful \((k + 1, 1, \beta)\)-flower. \(\square\)

**Case 1.** There is an \(x \in [\alpha]^k\) such that for every \(\theta < \alpha\) there exist \(\xi < \alpha,\) \(\eta \in [\alpha]^k\) with \(\theta < \xi,\) \(\theta < \min(\eta)\) and \(F(x \cup \{\xi\}) = F(y \cup \{\eta\})\) for every \(\xi < \alpha,\) \(\eta \in [\alpha]^k\) such that \(sup(\{\xi, \max(\eta, p); p < \theta\}) < \xi,\) \(\min(\eta)\) and \(F(x \cup \{\xi\}) = F(y \cup \{\eta\})\) for every \(\xi < \alpha,\) \(\eta \in [\alpha]^k\) such that \(\sup(\{\xi, \max(\eta, p); p < \theta\}) < \xi,\) \(\min(\eta)\) and \(F(x \cup \{\xi\}) = F(y \cup \{\eta\})\).

By recursion on \(\theta < \alpha\) we can choose the values \(\xi, \eta \in [\alpha]^k\) such that \(sup(\{\xi, \max(\eta, p); p < \theta\}) < \xi,\) \(\min(\eta)\) and \(F(x \cup \{\xi\}) = F(y \cup \{\eta\})\). The sets \(\{\xi\} \cup \{\eta\}; \theta < \alpha\) therefore will be pairwise disjoint. As there is no colorful \((k + 1, 1, \beta)\)-flower, there are only less than \(\beta\) different values of the form \(F(x \cup \{\xi\})\), so for some \(|X| = \alpha^+\) we have that \(F(x \cup \{\xi\}) = i\) holds for \(\theta \in X\). But then the sets \(\{\xi\} \cup \{\eta\}; \theta < \alpha\) are disjoint and \(F\) assumes the same value \(i\) on each of them.

**Case 2.** For every \(x \in [\alpha]^k\) there is a bound \(\theta(x) < \alpha\) such that there are no \(\xi < \alpha,\) \(\eta \in [\alpha]^k\) such that \(\theta(x) < \xi,\) \(\theta(x) < \min(\eta)\) and \(F(x \cup \{\xi\}) = F(y \cup \{\eta\})\) holds.

Let \(\delta_{\gamma}: \gamma < \alpha\) be an increasing sequence of ordinals such that for each \(\gamma\) we have \(\delta_{\gamma} < \alpha\), if \(x \in [\delta_{\gamma}]^k\) then \(\theta(x) < \delta_{\gamma},\) and finally \(\delta_{\gamma} = sup(\delta_{\gamma}; \gamma < \epsilon)\) holds for every limit \(\epsilon < \alpha.\) Let also \(\delta\) be such that \(\delta_{\gamma} < \delta < \alpha\) for every \(\gamma < \alpha.\)

We construct a derived coloring \(f\) as follows. The underlying set is \(V = [\delta_{\gamma}; \gamma < \alpha]\). If \(x \in [V]^k\), then set \(f(x) = F(x \cup \{\delta\})).\)
By the inductive hypothesis the theorem applies to $V$, $F'$. But if $\{x_t: \tau < \beta\}$ is a colorful $(k, 1, \beta)$-flower for $F'$, then we obtain a colorful $(k + 1, 1, \beta)$-flower for $F$ by adding $\delta$ to the eye of the flower.

Assume finally that there are pairwise disjoint sets $\{x_i: i < \alpha\}$ in $[V]^k$ on which $F'$ obtains the same value. As they are disjoint, there can be only less than $\alpha$ of them containing an element of $\{\delta_y: \delta_y \leq \max(x_0)\}$. There is, therefore, an $i < \alpha$ such that $\max(x_0) < \min(x_i)$. If now $\gamma < \alpha$ is some value with $\max(x_0) < \min(x_i)$ then by noticing that $F(x_0 \cup \{\delta\}) = F(x_i \cup \{\delta\})$ we get a contradiction as $\theta(x_0) < \theta(x_i) < \theta(x_0)$, $\delta$ forbids a configuration like that. □

We conclude the paper with some open problems and conjectures concerning colorful flowers on finite cardinal. Note first that, in view of Theorems 1 and 4, it would be tempting to conjecture that for the finite case we have $\varphi(k, \ell, n) = n\ell/k + o(n)$.

**Question 2.** Is it true that for every integers $k, \ell, \ell < k$, we have $\varphi(k, \ell, n) = n\ell/k + o(n)$?

At this moment we cannot even show (nor disprove) that for some function $\theta(k) = \alpha(k)$ we have $\varphi(k, 2, n) \leq \theta(k)n$. We can show that $n/3 \leq \varphi(4, 2, n) \leq 5n/8$ but do not include a rather technical proof here. At first sight it seems that the relation $\varphi(k - 1, \ell - 1, n) \leq \varphi(k, \ell, n)$ holds for every $k \geq 2, k \geq \ell$ and $n \in \mathbb{N}$. However this is not true in general. For instance, since the constant function on $[4]^k$ is a local coloring, clearly $\varphi(3, 2, 4) = 2$. On the other hand, each local coloring of $[4]^2$ must use at least two colors, so $\varphi(2, 1, 4) \geq 3$ (in fact we have $\varphi(2, 1, 4) = 3$). However, we conjecture that the relation $\varphi(k - 1, \ell - 1, n) \leq \varphi(k, \ell, n)$ holds whenever $n$ is larger than some constant depending only on $k$ and $\ell$. By Theorem 4, $\varphi(2, 1, n) = n/2 + o(n)$ and $\varphi(3, 2, n) = 2n/3 + o(n)$, so that the conjecture is true for $k = 3$ and $\ell = 2$. An internal coloring is a map $f : [n]^k \to n$ with $f(A) \in A$. Restricting to internal colorings, the inequality become true. We make this precise with the following definition. We write $\varphi'(k, \ell, n) = t$ if $t$ is the smallest integer for which there exists an internal coloring $f : [n]^k \to n$ without $(k, \ell, t)$-flowers.

**Theorem 6.** For every integers $k \geq 2, k \geq \ell$ and $n$ integers we have $\varphi'(k - 1, \ell - 1, n) \leq \varphi'(k, \ell, n)$.

**Proof.** Let $f : [n]^k \to n$ be an internal map with no colorful $(k, \ell, t)$-flower for $t \geq \varphi'(k, \ell, n)$. The map related to Kneser graph given at the beginning of part 2 implies $\varphi'(k, 1, t) \leq n - 2k + 3 < n - k$, since $k \geq 2$. This implies that for every $\{x_1, x_2, \ldots, x_{k-1}\} \in [n]^{k-1}$ the set $\{p \in n \setminus \{x_1, x_2, \ldots, x_{k-1}\}: f(\{x_1, x_2, \ldots, x_{k-1}, p\}) = p\}$ has fewer than $n - k$ elements. Consequently for every choice of $\{x_1, x_2, \ldots, x_{k-1}\} \in [n]^{k-1}$ there exists $p_{\{x_1, x_2, \ldots, x_{k-1}\}} \in n \setminus \{x_1, x_2, \ldots, x_{k-1}\}$ such that

$$f(\{x_1, x_2, \ldots, x_{k-1}, p_{\{x_1, x_2, \ldots, x_{k-1}\}}\}) = x_i.$$

Now define an internal map $g : [n]^{k-1} \to n$ setting $g(\{x_1, x_2, \ldots, x_{k-1}\}) = x_i$. By the definition of $\varphi'(k - 1, \ell - 1, n)$, there exists a colorful $(k - 1, \ell, t_0)$-flower $\mathcal{F}$ for the map $g$ with $t_0 = \varphi'(k - 1, \ell - 1, n) - 1$. Consider the family

$$\mathcal{F}' = \{\{x_1, x_2, \ldots, x_{k-1}, p_{\{x_1, x_2, \ldots, x_{k-1}\}}\}: \{x_1, x_2, \ldots, x_{k-1}\} \in \mathcal{F}\}.$$

It is easy to see that $\mathcal{F}'$ is a colorful $(k, \ell, t_0)$-flower for $f$. But we chose $f$ in such a way that $t_0 < \varphi(k, \ell, n)$, i.e., $\varphi(k - 1, \ell - 1, n) \leq \varphi(k, \ell, n)$. □

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**References**


**Further reading**