Note

Ramsey-Type Results for Metric Spaces

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If $X$, a metric space, is either finite, or the one consisting of the points \{${x_i}$; $i \leq \omega$\} with $\text{dist}(x_i, x_j) = 2^{-i}$ for $i < j < \omega$, $\kappa$ is a cardinal, then there is a metric space $Y$ such that $\kappa$ coloring the points of $Y$ there always exists a monocolored isometric copy of $X$.

INTRODUCTION

Recently several partition theorems have been proved for topological spaces, see [4]. Here we are interested in the question when a metric space is split and we have to guarantee an isometric copy of a given metric space in one of the classes. One is tempted to think that a full Ramsey result holds here, but the following remarks show that this is not the case. First, every metric space can obviously be split into the union of countably many bounded sets. Second, under the axiom of constructibility every topological space can be split into two parts, none containing a homeomorphic copy of the Cantor continuum, as was shown by Weiss [4]. These remarks show that positive results can only be proved if the target space is small.

As is usual in partition calculus, we use the coloring terminology, i.e., $X, Y$ are metric spaces, $\kappa$ is a cardinal, and we have to ensure that whenever $Y$ is $\kappa$-colored, there always exists a monocolored isometric copy of $X$.

In Section 1 we prove the result if $X$ is a finite metric space, and $\kappa$ is an arbitrary cardinal. Note that if $\kappa$ is finite, the existence of $Y$ easily follows from the following deep result of Erdős and Hajnal [1]: for every finite $k \geq 2$, $s \geq 3$, $g \geq 3$ there is an $s$-chromatic system of $k$-element sets with no circuits of length at most $g$. As this no longer holds if $s$ is uncountable and $g \geq 4$, we adopt a trickier approach for the general case.
In the rest of the article we do one more step: we treat the case when $X = \{x_i: i \leq \omega\}$ is a space with $\varepsilon_i = \text{dist}(x_i, x_j) = \text{dist}(x_i, x_{\omega})$ for $i < j < \omega$, and $\varepsilon_i$ strictly decreasingly converging to 0. Note that $X$ is ultrametric and in fact our space $Y$ will be ultrametric, as well.

1. THE TARGET SPACE IS FINITE

**Theorem 1.** If $(X, d)$ is a finite metric space, $\kappa$ a cardinal, then there exists a metric space $Y$ with the property that for every $\kappa$-coloring of (the points of) $Y$ there is a monocolored isometric copy of $X$.

**Proof.** Put $k = |X| - 1$, and $g = \lfloor d_{\text{max}}/d_{\text{min}} \rfloor$, where $d_{\text{max}}$, $d_{\text{min}}$ denote the largest and the smallest distances in $X$. We assume that $\kappa$ is infinite. Put $\lambda = \exp_\kappa(\kappa)^{+}$, where $\exp_0(\tau) = \tau$, $\exp_{i+1}(\tau) = 2^{\exp_i(\tau)}$. Enumerate $X$ as $\{x(i): 0 \leq i \leq k\}$, and let $Y = \left[\lambda\right]^{g_k}$, the system of $g_k$-element subsets of a fixed ordered ground set of size $\lambda$, itself denoted by $\lambda$.

If $a(0) < a(1) < \cdots < a(gk)$ are elements of $\lambda$, and

$$A = \{a(i): 0 \leq i \leq gk\}, \quad B = A - \{a(gk)\}, \quad C = A - \{a(gj)\},$$

with $0 < i < j < k$, we put

$$w(B, C) = d(x(i), x(j)).$$

For $A, B \in [\lambda]^{g_k}$, $A \neq B$ put

$$\rho(A, B) = \inf_{B_0 \subseteq A, B_1 \supseteq B} \sum_{i=1}^{t} w(B_{i-1}, B_{i}), \quad (1)$$

where the infimum is taken to all chains $A = B_0, B_1, \ldots, B_t = B$ with each $w(B_{i-1}, B_i)$ defined. As $X$ is finite, $\rho$ will be well defined and positive. If $A = B$, we of course take $\rho(A, B) = 0$. This $\rho$ will obviously be a metric.

If $Y$ is colored by $\kappa$ colors, then by the Erdős–Rado theorem [2, Theorem 39, p. 467], there is a set $T \in [\lambda]^{g_k + 1}$ with $[T]^{g_k}$ monocolored. If $T = \{t(i): 0 \leq i \leq gk\}$ is the monotonic enumeration, the mapping

$$x(i) \mapsto T - \{t(gi)\}$$

is easily seen to be an isometric embedding of $(X, d)$ into $(Y, w)$, we only have to show that $\rho(B, C) = w(B, C)$ whenever this latter quantity is defined.
In other words, we have to show that if \( B_1, \ldots, B_t \) is a \( w \)-circuit (i.e., \( w(B_1, B_2), \ldots, w(B_{t-1}, B_t), w(B_t, B_1) \) are all defined), then
\[
w(B_1, B_t) \leq \sum_{i=1}^{t-1} w(B_i, B_{i+1})
\]
holds. This is obvious from the choice of \( g \) if \( t > g \), as then the r.h.s. is at least \( gd_{\min} \geq d_{\max} \).

Assume now that \( t \leq g \). We construct a directed graph \( D \) on \( X \) as the vertex set as follows. If \( B_s, B_{s+1} \) are two successive members of our circuit (\( s = t, s+1 = 1 \) is allowed) and \( \{y\} = B_s - B_{s+1}, \{z\} = B_{s+1} - B_s \), \( y \) and \( z \) are the \( g \)th, \( g \)th element of \( B_s \cup B_{s+1} \), respectively, draw a directed edge of \( D \) from \( x(i) \) to \( x(j) \). Following the circuit \( B_s, B_{s+1}, \ldots, B_t \), the point \( z \) must be removed at some stage, i.e., \( \{z\} = B_r - B_{r+1} \) for some \( r \). The index of \( z \) in \( B_s \cup B_{s+1} \) is strictly between \( g(j-1) \) and \( g(j+1) \) as \( r-s \) is at most \( (g-1) \). It is, therefore, exactly, \( gj \). Pairing re- and disappearances as above, we get that for every vertex in \( D \), the indegree and the outdegree of the vertex are the same.

We now show (2). If the pair \( (B_s, B_t) \) gives rise the edge \( x(i) \rightarrow x(j) \) in \( D \), by the just proved property of \( D \), there is a path \( x(j) = x(j_0) \rightarrow \cdots \rightarrow x(j_u) = x(i) \) in \( D \). We may assume that there are not repeated vertices in it. From this, as \( (X, d) \) is metric, we get
\[
w(B_1, B_t) = d(x(i), x(j)) \leq \sum_{m=1}^{u} d(x(j_{m-1}), x(j_m))
\]
and the r.h.s. is a subsum of the r.h.s. of (2).

2. Ultrametric Spaces

**Definition 1.** A metric space \( (X, \rho) \) is ultrametric, if
\[
\rho(x, y) \leq \max(\rho(x, z), \rho(z, y))
\]
always holds.

**Proposition 1.** If \( (X, \rho) \) is ultrametric, then \( \rho(x, y) = \max(\rho(x, z), \rho(z, y)) \) whenever \( \rho(x, z) \neq \rho(z, y) \).

From now on, all spaces considered will be ultrametric.
PROPOSITION 2. If \((X_\beta, \rho_\beta)\) is an increasing sequence of ultrametric spaces \((\beta < \alpha)\), then

\[
\left( \bigcup_{\beta < \alpha} X_\beta, \bigcup_{\beta < \alpha} \rho_\beta \right)
\]

is ultrametric, as well.

DEFINITION 2. A subset \(W \subseteq X\) with \(\rho(w, w') = r\) for any \(w, w' \in W\), \(w \neq w'\) is called a simplex with diameter \(r\).

PROPOSITION 3. If \((X, \rho)\) is ultrametric, \(W \subseteq X\) is a simplex with diameter \(r\), a point \(t\) can be added to \(X\), such that \(W \cup \{t\}\) is still a simplex with diameter \(r\).

Proof. Define \(\rho'(w, t) = r\) for \(w \in W\), and

\[
\rho'(x, t) = \inf_{w \in W} \max(\rho(x, w), r)
\]

for \(x \in X - W\), \(\rho'(x_1, x_2) = \rho(x_1, x_2)\) for \(x_1, x_2 \in X\). We have to show that \(\rho'\) is an ultrametric. We first show that for \(x_1, x_2 \in X\),

\[
\rho(x_1, x_2) \leq \max(\rho'(x_1, t), \rho'(t, x_2)). \tag{4}
\]

For arbitrary \(\varepsilon > 0\), there are \(w_1, w_2 \in W\) with

\[
\max(\rho(x_i, w_i), r) < \rho'(x_i, t) + \varepsilon
\]

for \(i = 1, 2\). From this,

\[
\rho(x_1, x_2) \leq \max(\rho(x_1, w_1), \rho(w_1, w_2), \rho(w_2, x_2))
\]

\[
\leq \max(\rho(x_1, w_1), r, \rho(w_2, x_2))
\]

\[
< \max(\rho'(x_1, t), \rho'(x_2, t)) + \varepsilon.
\]

As this holds for every \(\varepsilon > 0\), we are done with (4).

Next we show that

\[
\rho'(x_1, t) \leq \max(\rho(x_1, x_2), \rho'(x_2, t)) \tag{5}
\]

for \(x_1, x_2 \in X\). For this, note that for \(w \in W\),

\[
\rho(x_1, w) \leq \max(\rho(x_1, x_2), \rho(x_2, w))
\]

as \((X, \rho)\) is ultrametric. This implies

\[
\max(\rho(x_1, w), r) \leq \max(\rho(x_1, x_2), \rho(x_2, w), r).
\]

Taking infima, we get (5).
**Proposition 4.** If \((X, \rho)\) is ultrametric, \(\langle x_i \rangle_{i=0}^\infty\) is a Cauchy-convergent sequence in \(X\) with no limit point, a point \(t\) can be added to \(X\), such that \(t\) will be the limit of \(\langle x_i \rangle_{i=0}^\infty\).

**Proof.** As \(X\) is ultrametric and the sequence \(\langle x_i \rangle_{i=0}^\infty\) is Cauchy-convergent, for every \(y \in X\) the sequence
\[\rho(x_i, y)\]
is eventually constant. Let this constant be \(\rho'(y, t)\).

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**Theorem 2.** Assume that \((X, d)\) is an ultrametric space with \(X = \{x_i: i \leq \omega\}\), \(d(x_i, x_j) = \varepsilon_i\) for \(i < j \leq \omega\), where \(\varepsilon_i > 0\). Assume moreover that \(\kappa\) is a cardinal. Then there exists an ultrametric \(Y\) such that whenever the points of \(Y\) are \(\kappa\)-colored, there exists a monocolored isometric copy of \(X\).

**Proof.** We assume that \(\kappa\) is infinite. By transfinite recursion on \(\alpha < \kappa^+\) we build an increasing sequence of ultrametric spaces, \((Y_\alpha, \rho_\alpha)\). Let \((Y_0, \rho_0)\) be the one-point metric space. If \((Y_\beta, \rho_\beta)\) have already been constructed for \(\beta < \alpha\) and \(\alpha\) is a limit ordinal, let \((Y_\alpha, \rho_\alpha)\) be the union of \((Y_\beta, \rho_\beta)\). To extend \((Y_\alpha, \rho_\alpha)\) to \((Y_{\alpha+1}, \rho_{\alpha+1})\) first add a limit point to every Cauchy-convergent but not convergent sequence in \(Y_\alpha\) (possible by Propositions 4 and 2) then extend all simplexes of size \(\leq \kappa\) which are in \(Y_\alpha\), one after the other, using Propositions 3 and 2. As in the 2\(\text{nd}\) step we possibly add \(|Y_\alpha|^\kappa\) new points, our \(Y_\alpha\)'s and therefore \((Y, \rho) = (\bigcup \{Y_\alpha: \alpha < \kappa^+\}, \bigcup \{\rho_\alpha: \alpha < \kappa^+\})\) will have size at most \(2^\kappa\). As every simplex of size at most \(\kappa\) and every Cauchy-convergent sequence appears in some \(Y_\alpha\), \((Y, \rho)\) is complete and no simplex of size \(\leq \kappa\) is maximal. In the following we will only use these properties.

We show that \((Y, \rho)\) works. Otherwise, there is a coloring \(f\) with \(\kappa\) colors, such that for every \(y \in Y\) there is an \(i < \omega\) such that for no \(y' \in Y\), \(\rho(y, y') = \varepsilon_i\) and \(f(y') = f(y)\) both hold. Changing this to a coloring with \(\kappa \omega = \kappa\) colors (here we used that \(\kappa\) is infinite), we can assume that for every color \(\xi\) there exists an \(i < \omega\) such that for \(f(y) = f(y') = \xi\), \(\rho(y, y') \neq \varepsilon_i\) holds. In this case we call \(\xi\) a color of type \(i\).

Let \(n(0)\) be the smallest \(n < \omega\) such that there exists a point with color of type \(n\). Clearly, \(n(0)\) exists. Let \(W_0\) be a maximal simplex of diameter \(\varepsilon_{n(0)}\) consisting only of points with colors of type \(n(0)\). \(W_0\) exists by Zorn's lemma. Obviously, the points in \(W_0\) have different colors, so we have \(|W_0| \leq \kappa\).
By the way $Y$ was constructed, there is a point $y \in Y$ such that $y$ extends $W_0$. Let $n(1)$ be the smallest $n < \omega$ such that a point of color of type $n$ extends $W_0$. Obviously, $n(1) > n(0)$. Now, let $W_1$ be a maximal simplex of diameter $e_{n(1)}$ consisting of points of colors of type $n(1)$, and containing a $y$ as above. Note that, as $\rho$ is an ultrametric, for $y_0 \in W_0, y_1 \in W_1$, $\rho(y_0, y_1) = e_{n(0)}$ holds. This $W_1$ exists, again by Zorn’s lemma, and the colors of $W_1$ will be different, so $|W_1| \leq \kappa$.

Continuing this way, we get natural numbers $n(0) < n(1) < \cdots$, and non-empty simplexes $W_0, W_1, \cdots \subseteq Y$ with the following properties:

1. $|W_i| \leq \kappa$, $W_i$ is of diameter $e_{n(i)}$;
2. $W_i$ consists of points of colors of type $n(i)$;
3. if $j < i$, $x \in W_j, y \in W_i$, then $\rho(x, y) = e_{n(j)}$;
4. when $W_0, \ldots, W_{i-1}$ defined, $n(i)$ is minimal such that a $W_i \neq 0$ exists, then $W_i$ is maximal with respect $W_0, \ldots, W_{i-1}$, $n(i)$ satisfying (6)–(8).

By the remarks above and by the construction of $(Y, \rho)$ this process will never break down. Choose $w(i) \subseteq W_i$. As $\rho(w(i), w(j)) = e_{n(i)}$ for $i < j < \omega$, this is a Cauchy-sequence, so, by the construction of $Y$, it converges to a $y \in Y$. Assume that $f(y)$ is of type $n$. If $n(i-1) < n < n(i)$, or $n < n(0)$, the existence of $y$ contradicts the choice of $n(i)$ (or $n(0)$). If $n = n(i)$, then for every $x \in W_i$, $\rho(x, y) = \lim \rho(x, w(j))$ for $j \to \infty$ which is $e_{n(i)}$, i.e., $y$ would extend $W_i$, again a contradiction.

**REFERENCES**